Teaching Deflection of Stepped Shafts: Castigliano's Theorem, Dummy Loads, Heaviside Step Functions and Numerical Integration

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Abstract - **The need for finding the deflections of shafts, many of which are stepped or varying cross-sectional areas is timeless. Each generation of engineers has used that part of mechanics of materials theory that fit the calculating capability available to them. The method presented here is offered in that vein. The method uses an engineer's ability to construct free body diagrams, derive moment equations, and knowledge of energy methods. The problem solution is kept general until the last step which is a digital numerical integration. The digital numerical integration can be performed on a wide variety of software to include TKSolver™, MatLab®, MathCad®, EES© and spread sheets. This method keeps the section properties independent of the moment equations making it straightforward to include scaling and shape factors on the cross-sectional dimensions. This allows an engineer to run any number of "what if" scenarios during a design process. Additionally, this method provides intermediate opportunities to validate the solution path by a) plotting the moment equation and comparing it against shear and moment diagram developed by hand, or b) plotting the cross section and comparing it against the drawings. Thus far, this approach to solving for the deflection of stepped shafts has been presented to nearly 300 junior Mechanical Engineering students.**

Index Terms – Deflection, Equation-solving software, Non-uniform diameter, Scaling factor, Shaft design.

INTRODUCTION

At the present time, there are wind energy farms being proposed and built, products are shipped around the world via ships, trucks and airplanes, geared turbofan engines are being developed and automotive hybrid power trains are being designed. Deep inside of these macro technologies, there resides a shaft doing the job as it was designed. In the design of these shafts there is typically a strength and a deflection/slope design specification. This paper presents an approach to calculating shaft deflection that is accessible to undergraduate engineering students.

A literature review of methods for calculating shaft deflections indicates numerous methods have been developed. For example, in 1960 Bert listed eight different ways, ranging from graphical to Laplace transforms. [1] Scholars and practitioners reflect in their writings the evolution of shaft deflection calculations.[2]-[5] We have previously taught a structured approach. [6] Textbooks have taken several approaches to account for non-uniform diameters, from using an equivalent diameter [7], advocating commercial software [8], or avoiding the topic altogether [8]-[10]. More recently, textbooks [11]-[13] seem to be gravitating towards integrating the moment curvature relationship

$$\frac{d^2y}{dx^2} = \frac{M}{EI}$$

with the moment equation expressed using discontinuity functions. While this method is quite correct, completing the calculus and then accurately computing the integration constants using the boundary conditions can be algebraically tedious, time-consuming and fraught with opportunities for error. The method proposed in this paper applies a numerical integration step to the approach presented by Professor Ju in his course notes and in an article [14]. Professor Ju's approach is based on Castigliano's Theorem and the use of Heaviside step functions to write the moment equation. If performed by hand, this approach is algebraically intense. However, the boundary conditions are embedded in the formulation, and once created, the formulation is ready for numerical integration.[15]

Here, we present a brief description, an example solution augmented by the introduction of size and shape factors to improve the design.

DESCRIPTION OF APPROACH

The method uses an engineer's ability to construct free body diagrams, develop moment equations, and knowledge of energy methods. Following the theory of Castigliano, the internal energy due to bending is

$$U = \int_0^L \frac{M^2}{2EI} dx \tag{1}$$

where M is the moment along the length of the beam as a function of x, E is Young's modulus, and I is the area moment of inertia (which for a stepped shaft is also a function of x). For this type of problem, Castigliano's second theorem indicates that the deflection (or slope) at a point is equal to the partial derivative of the strain energy with respect to a load (or moment) applied at that point. If a load (or moment) is not applied at the point of interest, then a dummy-load (or dummy-moment) can be applied at that point. After the partial derivative with respect to the dummy-load (or dummy-moment) is performed, the dummy load (or moment) is set to zero, i.e.

$$\delta_Q = \frac{\partial U}{\partial Q} = \int_0^L \frac{M_{Q=0}}{EI(x)} \frac{\partial M}{\partial Q} dx \tag{2}$$

$$\theta_m = \frac{\partial U}{\partial m} = \int_0^L \frac{M_{m=0}}{EI(x)} \frac{\partial M}{\partial m} dx \tag{3}$$

The problem solution is kept general until the last step which is numerical integration. The numerical integration can be performed using a wide variety of software products including TKSolver™, MatLab®, MathCad®, EES© and spread sheets.

Additionally, this method provides intermediate opportunities to validate the solution path by (a) plotting the moment equation and comparing it against the shear and moment diagrams developed by hand, (b) plotting the cross section and comparing it against the drawings, and (c) changing the stepped shaft into one with a convenient constant diameter along the length and then comparing the results to a closed-form analytical solution. The simple solution steps are listed below:

- Draw a FBD and apply a dummy-load (and moment) at an arbitrary location, ξ. Use statics to solve for reactions.
- Write a moment equation for the entire length using Heaviside functions as needed to serve the purpose of discontinuity function brackets
- Take the partial derivative of the Moment equation with-respect-to the dummy-load (and dummy-moment).
- Set dummy-load, Q=0, the dummy-moment m=0 and re-write $M(x)_{Q,m=0}$.
- Write Castigliano's integral for deflection, δ_Q, inserting M(x) and $\partial M/\partial Q$. Input this equation into an equation-solver and calculate δ at any number of user-selected ξ locations. Similarly, write Castigliano's integral for slope, θ_Q, inserting M(x) and $\partial M/\partial m$. Input this equation into an equation solver and calculate θ at any number of user-selected ξ locations.

EXAMPLE SOLUTION

The example problem shaft (Figure 1) is supported by bearings near the ends and loaded transversely with a 7-kN concentrated load positioned at 140-mm from center of the left bearing support. The material is steel and the initial shaft geometry is provided. The analyst is to examine the shaft for deflection [16]. The largest allowable slope at the bearings is specified as 0.001 radians.

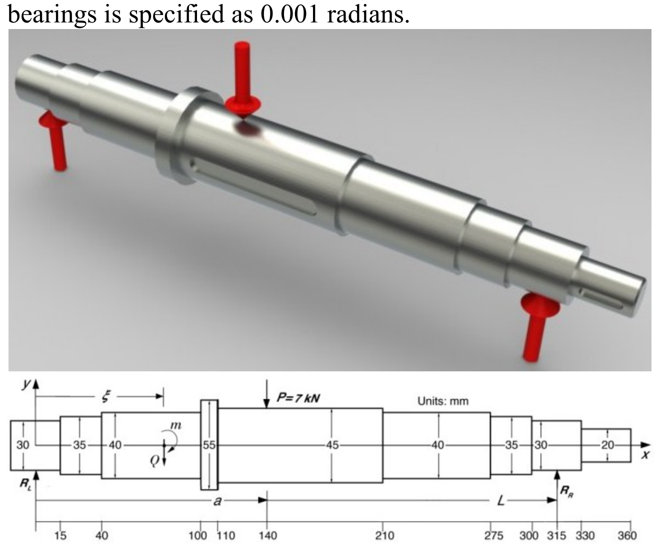


FIGURE 1
SHAFT WITH MULTIPLE STEP CHANGES IN DIAMETER (TOP). FREE BODY DIAGRAM SHOWING THE EXTERNAL LOAD (P=7-KN), DUMMY LOAD (Q), DUMMY MOMENT (m), REACTIONS (R_L, R_R) AND DIMENSIONS.

We begin by drawing the free body diagram as shown, which has been constructed to include a dummy-load, Q, and dummy-moment, m, at a distance, ξ, from the left support.

Using statics, solve for the reactions. These reactions include an expression which contains the dummy-load, Q, the dummy-moment, m, the beam length, L, and the point of interest coordinate position, ξ. The other variable terms include the distance, a, of the load, P, from the left bearing support R_L.

$$R_L = P\frac{(L-a)}{L} + Q\frac{(L-\xi)}{L} + \frac{m}{L} \tag{4}$$

$$R_R = P\frac{a}{L} + Q\frac{\xi}{L} - \frac{m}{L} \tag{5}$$

It may be useful to point out to students how the loads, P and Q, are apportioned to each reaction, and that the m/L terms comprise a couple whose influence on each reaction is independent of the location of m.

Before writing the moment equation, the nomenclature and definition of a Heaviside step function is introduced.

$$H(a,b)=0 \text{ if } a<b$$
$$H(a,b)=1 \text{ if } a\geq b \tag{6}$$

We write the moment equation as a function of x for the entire beam. Include dummy-loads and dummy-moments.

$$M = R_L x - Q(x-\xi)H(x,\xi) - mH(x,\xi) - P(x-a)H(x,a) \quad (7)$$

Depending on the instructor, various portions of this equation can be commented upon. In this form, (7), a brief review of chain-rule differentiation for the $R_L x$ term, or the expression for R_L may be helpful to students. The second term, $Q(x\text{-}\xi)$, may need explanation to help students remember there are two, complementary axes at work here, that ξ, is user-selected first, then x- is used for integration.

Next, write the partial derivative with respect to the dummy load, Q, and with respect to the dummy-moment, m.

$$\frac{\partial M}{\partial Q} = \frac{L-\xi}{L}x - (x-\xi)H(x,\xi) \quad (8)$$

$$\frac{\partial M}{\partial m} = \frac{x}{L} + H(x,\xi) \quad (9)$$

Write the Moment Equation setting the dummy term(s) to zero.

$$M_{Q,m=0} = P\frac{(L-a)}{L}x - P(x-a)H(x,a) \quad (10)$$

Using Castigliano's Theorem, substitute (8) or (9) along with (10) into (2) or (3) to construct the correct integral to be solved as detailed in (11)and (12) below.

$$\delta_Q = \frac{\partial U}{\partial Q} = \int_0^L \frac{M_{Q,m=0}}{EI(x)}\frac{\partial M}{\partial Q}dx \quad (11)$$

$$\theta_m = \frac{\partial U}{\partial m} = \int_0^L \frac{M_{Q,m=0}}{EI(x)}\frac{\partial M}{\partial m}dx \quad (12)$$

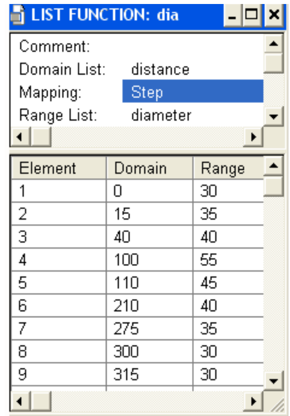
Solve. There are a couple of items to code into the software.

First, the diameter as a function of shaft position will need to be defined to calculate the area moment of inertia (13) during integration.

$$I(x) = \frac{\pi\,[dia(x)]^4}{64} \quad (13)$$

This can be accomplished by using a look-up table or a series of if-then-else statements in a user-defined sub-routine, depending on the equation-solving software options and the user's programming skills [17]. Figure 2A shows the list function used by TKSolver™ which serves as a look-up table and Figure 2B shows EES© code for the user-defined function which produces the same result.

We are aware that we are integrating across a discontinuity. However, we have found numerical convergence to be very rapid compared to closed-form solutions.



LIST FUNCTION: dia

Comment:
Domain List: distance
Mapping: Step
Range List: diameter

Element	Domain	Range
1	0	30
2	15	35
3	40	40
4	100	55
5	110	45
6	210	40
7	275	35
8	300	30
9	315	30

(A)

```
"Define the diameter as a function of x"
function dia(x)
if(x<15) then dia:=30 else;
  if(x<40) then dia:=35 else;
    if(x<100) then dia:=40 else;
      if(x<110) then dia:=55 else
        if(x<210) then dia:=45 else;
          if(x<275) then dia:=40 else;
            if(x<300) then dia:=35 else;
              if(x<315) then dia:=30 else
              endif;
            endif;
          endif;
        endif;
      endif;
    endif;
  endif;
endif
end
```

(B)

FIGURE 2
(A) USER-CREATED TKSOLVER™ LIST FUNCTION DEFINES THE SHAFT DIAMETER ALONG THE LENGTH. (B) USER-CREATED EES© CODE ALSO DEFINES SHAFT DIAMETER. BOTH SERVE AS LOOK-UP TABLES, DETERMINING DIAMETER FOR ANY LOCATION, X.

Second, the geometry and moment are checked by creating plots such as the one shown in Figure 3.

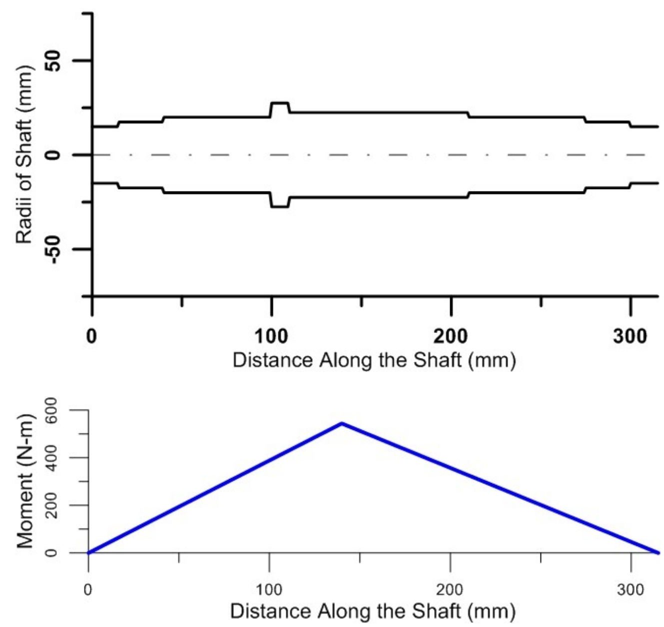


FIGURE 3
EQUATION-SOLVER GRAPH USED TO ENSURE THE GEOMETRY FUNCTION WORKS AS INTENDED. SHAFT MOMENT DIAGRAM SERVES AS A CHECK BEFORE PROCEEDING.

This prompt check of the shaft profile ensures that the problem is proceeding as expected; similarly, a check of the moment diagram would support confidence in the solution if done prior to attempting the repeated integrations.

Next, the crucial action is to integrate with respect to x but then create a series of solutions (such as in a table) using ξ as the indexing variable. Selecting 316 positions of ξ for this 315-mm shaft is sufficient to obtain a smooth deflection curve.

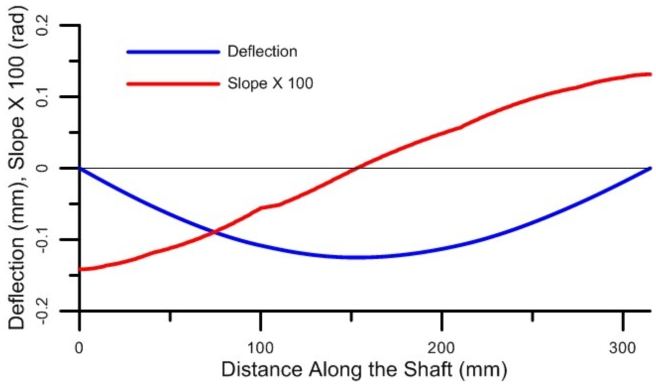


FIGURE 4
INITIAL SOLUTION FOR SHAFT DEFLECTION AND SLOPE.

With the equation-solver solution for deflection and slope at every 1-mm along the beam (Figure 4) we find that the loaded shaft does not yet meet the slope limitation of 0.001 rad at the bearing supports (read left bearing slope 0.0014 rad on graph). Therefore, we elect to apply a scaling factor to adjust the shaft geometry (diameter) in order to mitigate the excessive bearing slopes.

SCALING FACTOR

If either deflection or slope is deemed unsatisfactory, it is simple to modify the *I(x)* equation by inserting a scaling factor, *sc*, as shown in (14). Then one can solve a new list of ξ values to obtain a new deflection curve.

$$I(x) = \frac{\pi\,[sh(x) * sc * dia(x)]^4}{64} \tag{14}$$

In the case of this textbook problem, a scaling factor of about *sc*=1.1, when applied to the given shaft profile, results in acceptable slope conditions at the bearings (<0.001 radians) and deflection much less than the solution shown initially (deflection itself was not specifically limited in the original problem statement).

With *sc*=1.1, both the left support slope and the maximum deflection are lessened. The determination of the “precise” scaling factor, in this case *sc*=1.1058, can be accomplished using built-in optimization or “goal-seek” functions available within individual equation-solvers. We have found, however, that a guess-and-check approach will result in finding the scaling factor as close as 99% in only 3-6 guesses, quickly enough for even a novice user.

SHAPE FACTOR

The method as presented keeps the section properties independent of the moment equation, making it straightforward to include scaling and shape factors on the cross-sectional dimensions. This allows an engineer to run any number of "what if" scenarios during the design process.

It is easily seen (Figure 4) that initially, the maximum deflection occurs at about 155-mm from the left bearing support. Since the loading of 7-kN is located at 140-mm from the left bearing, and the 7-kN load may represent a gear meshing at that location, it may be desirable to design the shaft such that the maximum deflection occurs more precisely at the location of the 7-kN load. This will help ensure that the gear teeth will mesh properly and that no unintended loading results from a shaft which is sloped at that important location.

To shift the location of maximum deflection, we introduce a shape factor, *sh(x)*, and we use experience to implement this factor. By inspecting the shaft geometry and the initial deflection curve, we imagine that by judiciously adding more material to the shaft, we may impact the deflection in a favorable way. We don’t need additional material everywhere (e.g. not at the left end). We select a special “shape point” at x_{sh}=110-mm, just to the right of the largest-diameter section at the collar. We want to increase material to the right of x_{sh} and decrease material to the left of x_{sh}. Accordingly, we define *sh(x)* as in (15).

$$\begin{aligned} &\text{If } x < x_{sh}, \quad sh(x) = |sh| \\ &\text{If } x \geq x_{sh}, \quad sh(x) = \frac{1}{|sh|} \end{aligned} \tag{15}$$

Defining the Shape Factor in this way, we then proceed to the guess-and-check strategy, checking the location of maximum deflection until we are satisfied that it occurs at 140-mm from the left bearing support (i.e. at the location of the 7-kN load), which results in a Shape Factor of about *sh*=0.924.

FINAL DESIGN

In the example, the slope at the ends is greater than recommended, meaning the shaft is too compliant. A direct method to increase the stiffness of the shaft is to increase the diameter, dia(x), which will increase the area moment of inertia, I(x). Also, in Figure 4 it can be observed that the maximum deflection occurs at about 155-mm from the left end whereas the 7-kN load is applied at 140-mm. During redesign, it could be advantageous to not only add stiffness to the shaft but to do so in a manner that could move the maximum deflection (zero slope) to the location where the 7-kN load is applied.

The method of solution presented here can be modified by changing (13) to include a scale factor (sc) and a shape factor (sh) as shown in (14). The scale factor (sc) is independent of the distance along the shaft; if it is greater than one it enlarges the diameter everywhere, if less than one it shrinks the diameter everywhere.

The shape factor (sh) is dependent on the location along the shaft. Mathematically, it is similar to the Heaviside step function and can be explained by exploring how it is coded as shown in (15). In use, the designer must make two decisions. First, a location on the shaft, x, has to be selected about which the shaping will pivot, typically at an existing change in diameter; closer to the mid-length of the shaft is better. For this example, x_{sh}=110-mm was chosen. Second, the amount to change (magnification) the diameter must also be chosen. The magnification can be greater than or less than one. If the magnification is greater than one, then the shape factor (sh) would be greater than one for the shaft diameters to the left of x_{sh} thereby increasing those diameters. The diameters to the right of x_{sh} would be decreased by the reciprocal of the magnification.

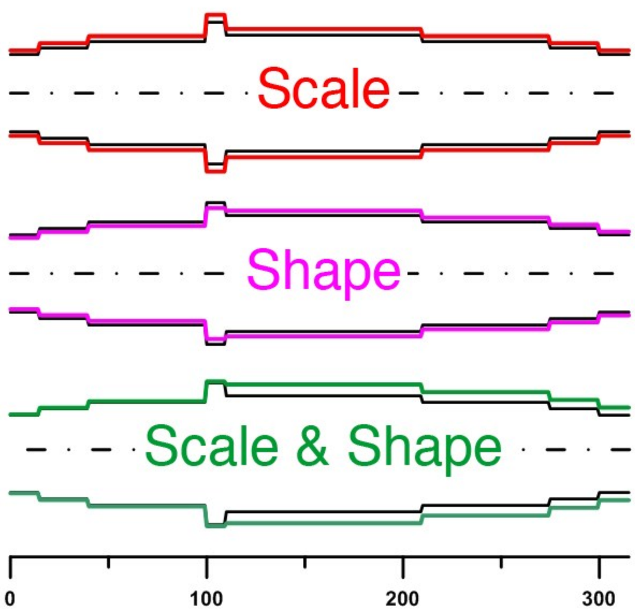


FIGURE 5
SUMMARY OF SHAFT GEOMETRY AT EACH STEP OF THE DESIGN PROCESS. TOP: APPLICATION OF ONLY THE SIZE FACTOR, SC. MIDDLE: APPLICATION OF ONLY SHAPE FACTOR, SH. BOTTOM: RESULT OF COMBINED FACTOR.

The shaft geometry at intermediate check-points is shown in Figure 5. The top image shows "before" (black) and "after" (red) the application of the scaling factor, *sc*. The middle image shows the effects of only the shape factor, *sh*, which produces the effect of decreasing the diameter to the left of x_{sh} =110 mm and increasing the diameter to the right of x_{sh} =110 mm.

When both factors are applied, as shown in the bottom image, the combined effect is barely visible to the left of x_{sh}=110 mm but to the right the effect is greater.

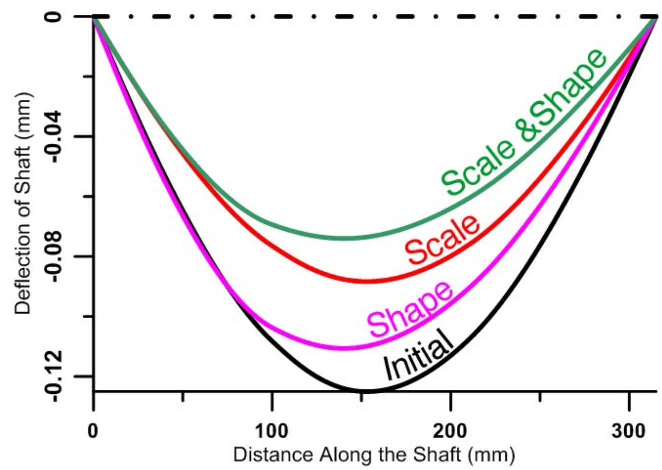


FIGURE 6
SUMMARY OF SHAFT DEFLECTIONS AT EACH STEP OF THE DESIGN PROCESS.

Finally, we summarize the shaft size and shape design process in Figure 6 showing deflection curves for initial calculations, with application of *only* the scale factor, with application of *only* the shape factor, and finally with application of *both* size and shape factors.

The final design shaft not only meets the slope specified in the problem statement but also produces a deflection nearly half that in the initial configuration. Additionally, the location of the maximum deflection has been adjusted using the shape factor such that the slope is zero at the 140-mm (Figure 7).

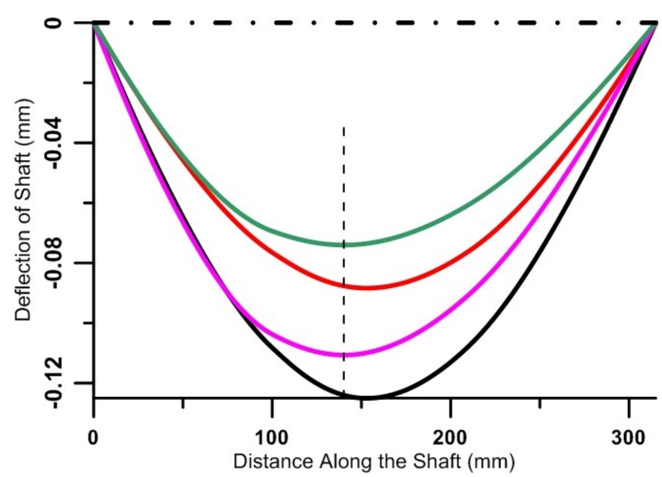


FIGURE 7
SUMMARY OF SHAFT DEFLECTIONS DURING SHAFT DESIGN, NOTING THE LOCATION OF MAXIMUM DEFLECTION IS SHIFTED TO 140 IN FINAL DESIGN CONFIGURATION.

The final design using the combined effects of both factors results in the smallest maximum deflection. The slope at each bearing is less than the maximum allowed 0.001 radians.

Listed in Table I are the salient numerical values for initial solution, applying *only* the scaling factor, applying *only* the shape factor and the final design which applies both factors.

TABLE I
SHAFT SLOPE AND DEFLECTIONS BEFORE AND AFTER SCALING AND SHAPE FACTORS ARE APPLIED FOR DESIGN

Value After	R_L Slope (rad)	R_R Slope (rad)	Maximum deflection (mm)	Location Maximum deflection (mm)
Initial Calculation	-0.00142	0.00133	-0.125	153
Application of Scaling Factor	0.001	0.00093	-0.092	151
Application of Shape Factor	-0.0011	0.0011	-0.111	140
Final Design	-0.001	0.000717	-0.074	140

CONCLUSIONS

The design of shafts of non-uniform cross-section is an ongoing need. We have reviewed the textbook treatment of this problem-type and presented a method of solving these challenging problems using fundamental engineering skills and modern engineering tools.

The method presented here relies on Castigliano's Theorem, the Heaviside Function and Discontinuity Equations in conjunction with equation-solving software. This method helps to mitigate the difficulties experienced using other methods of finding shaft deflection "everywhere" along the length, such as writing sets of governing equations for numerous sections, applying multiple continuity relationships to those sets of governing equations, and solving accurately for multiple constants of integration. The method presented keeps the process general until the equation-solver is introduced.

By way of example, we show how the shaft geometry and moment diagrams are easily checked early in the equation-solving process, thus keeping the process grounded in fundamentals and accessible to undergraduate students. The shaft slope and deflection are numerically calculated using any of several equation-solving tools. We show how a Scaling Factor, *sc*, and Shape Factor , *sh*, can be applied directly to the shaft diameter to meet deflection and slope design criteria. We note that historically finding slope and deflection for the initial problem would have been difficult. Using this method we not only find slope and deflection, but we competently design and refine quickly. Using equation-solving software we show how to distribute the shaft material from end to end by designing the complete shape of the shaft. Thus far, this approach to solving for the deflection of stepped shafts has been presented to nearly 300 junior Mechanical Engineering students.

ACKNOWLEDGMENTS

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17 We have successfully used TK Solver™, EES©, Mathematica© , Matlab® and MathCad® using the approach described in this paper