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Post-'87 Crash Fears in the S&P 500 Futures Option Market

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Abstract

Post-crash distributions inferred from S&P 500 future option prices have been strongly negatively skewed. This article examines two alternate explanations: stochastic volatility and jumps. The two option pricing models are nested, and are fitted to S&P 500 futures options data over 1988-1993. The stochastic volatility model requires extreme parameters (e.g., high volatility of volatility) that are implausible given the time series properties of option prices. The stochastic volatility/jump-diffusion model fits option prices better, and generates more plausible volatility process parameters. However, its implicit distributions are inconsistent with the absence of large stock index moves over 1988-93.

JEL classification: G13
Key words: stock index options, stochastic volatility, jump-diffusions, stock market crash, specification error

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Deviations of stock index option prices from the benchmark Black-Scholes model have been extraordinarily pronounced since the stock market crash on October 19, 1987. Out-of-the-money (OTM) put options that provide explicit portfolio insurance against substantial downward movements in the market have been trading at high prices (as measured by implicit volatilities) relative to at-the-money options. The OTM puts have been even more "overpriced" relative to OTM calls that will pay off only if the market rises substantially. The pronounced implicit volatility patterns emerged immediately after the stock market crash, and have since been a permanent feature of the S&P 500 futures options market. These patterns differ fundamentally from those observed prior to the crash.

The implication is that the distribution implicit in option prices since the crash of 1987 is substantially negatively skewed, in contrast to the essentially symmetric and slightly positively skewed lognormal distribution underlying the Black-Scholes model. There are three possible explanations for such a shift in implicit distributions: a change in investors' assessment of the S&P 500's stochastic process, a change in investors' aggregate risk aversion, and mispricing of post-crash options due, e.g., to option market frictions or market organization. The second explanation could be justified by a crash-related relative wealth redistribution between less and more risk-averse investors. The third explanation is sometimes used by option market practitioners, who argue that heavy demand for out-of-the-money put options has driven up prices.

This article examines the first explanation: that investors' assessment of the S&P 500 stochastic process has changed. There are two different ways of modeling the post-crash emergence of negatively skewed implicit distributions: crash fears, and time-varying volatility inversely related to market returns. The former approach assumes that the stock market crash sharply increased option market participants' assessed probability of *further* stock market crashes -- a view somewhat validated by the subsequent 5-8% drops on January 11, 1988 and October 13, 1989. Option pricing models exploring a crash fears explanation typically employ variants of Merton's (1976) jump-diffusion model; e.g., Bates (1991) and Bakshi, Cao, and Chen (1997).

The second approach draws upon the inverse relationship between the level of underlying equity prices and the instantaneous conditional volatility observed empirically for individual firms (Black, 1976) and for broad market indices (Nelson, 1991). Theoretical explanations for the phenomenon include the “leverage” effects of Black (1976), whereby lower overall firm values increase the volatility of equity returns, and the “volatility feedback” effects of Poterba and Summers (1986) and Campbell and Hentschel (1992), among others, whereby higher volatility assessments lead to heavier discounting of future expected dividends and thereby lower equity prices. However, the explanations of time-varying stock market volatility most relevant to the rapid emergence of substantial post-crash negative implicit skewness in stock index options appear to be those that focus on the demand by investors for option-like payoffs, or portfolio insurance. In Platen and Schweizer (1995), the impact of option writers’ dynamic hedges upon an imperfectly liquid underlying asset market alters the underlying asset price process, and therefore alters option prices. Their simulations show that relatively heavy hedging of out-of-the-money put options (which have in fact been more heavily traded since the crash) accentuates volatility/level feedbacks and makes implicit distributions more negatively skewed. Grossman and Zhou (1996) have a somewhat analogous but more explicit equilibrium model of risk-sharing between portfolio insurers and other investors, and also generate negatively skewed implicit distributions.[1]

Platen and Schweitzer (1995) and Grossman and Zhou (1996) can be viewed as theoretical justifications for the *implied binomial trees* models of Dupire (1994), Derman and Kani (1994) and Rubinstein (1994). These models postulate a flexible but deterministic functional form for instantaneous conditional volatility in terms of the underlying asset price and time, and typically require a strong inverse relationship to match observed stock index option prices (Rubinstein, 1994; Dumas, Fleming and Whaley, 1998). *Stochastic volatility* models such as Hull and White (1987) and Heston (1993) relax the assumption that volatility and equity return innovations are purely deterministically related, but impose more structure

[1]Grossman and Zhou do not explicitly address the divergence between pre- and post-crash option price patterns. However, Grossman (1988), Gennotte and Leland (1990), and Jacklin, Kleidon, and Pfleiderer (1992) suggest that market participants may have underestimated the extent of portfolio insurance prior to the crash. Such underestimation would affect pre-crash option price patterns.

on the volatility process in other dimensions.

This article explores which of the crash fears and volatility evolution explanations better explains the negative skewness implicit in post-'87 stock index option prices. Two diagnostics are proposed. First, it is noted that the two hypotheses have alternate implications for the relationship between option maturity and implicit skewness. Standard stochastic volatility models postulate that the stock market follows a diffusion, with the implication that the conditional distribution is instantaneously normal. Such models consequently imply an initially *direct* relationship between option maturity and the magnitude of implicit skewness, with little implicit skewness for extremely short-maturity options. By contrast, jump models such as Merton (1976) postulate finite-variance shocks that are independent and identically distributed. By the law of large numbers, such models imply an *inverse* relationship between option maturity and the magnitude of implicit skewness, with little implicit skewness for long-dated options. This article therefore examines the ability of a stochastic volatility model with and without jumps to match option price patterns at various maturities. The overall approach is similar to that in Bates (1996b), but with some refinements (multifactor models, time-varying jump risk) not previously employed in option pricing models. A nonlinear generalized least squares/Kalman filtration methodology is developed for estimating implicit parameters and factor realizations, and for assessing specification error.

Second, this article evaluates the consistency of the distributions inferred from option prices with the time series properties of S&P 500 futures and futures option prices. Any model fitted to observed option prices has strong and testable implications for the joint conditional and unconditional distributions of underlying asset returns and all option returns -- so many implications, in fact, that different papers examine different facets of these joint distributions. For instance, Day and Lewis (1992), Canina and Figlewski (1993), and Fleming (1994) test whether S&P 100 index implicit volatilities predict future volatility in subsequent index returns. Stein (1989) and Diz and Finucane (1993) examine whether the term structure of at-the-money implicit volatilities from S&P 100 index options predicts subsequent at-the-money implicit volatilities -- a particular prediction regarding option price evolution. Galai (1983) and

Dumas, Fleming and Whaley (1998) note that "market efficiency" tests examine whether the average return on particular options *conditional* upon the underlying asset return matches the model's predictions. That conditional distribution is equal to the ratio of the joint option/asset return distribution and the marginal asset return distribution. Hedge ratio tests examine this conditional distribution's standard deviation. Examples of market efficiency and hedge ratio tests on stock index options data include Whaley (1986), Nandi (1996), Bakshi, Cao, and Chen (1997), and Dumas, Fleming and Whaley (1998).

This article employs an "in-sample" methodology when testing the consistency between cross-sectional option price patterns and the dynamic evolution of S&P 500 futures and futures option prices. Distribution-specific parameters and factor realizations are inferred from option prices over 1988-93, and are used to generate conditional distributions that are tested using the dynamic evolution of S&P 500 futures and futures option prices over the same time interval. This methodology is consistent with the fundamental underlying premise used in option pricing models and time series analysis: that the underlying asset price follows an identifiable and stable data generating process that determines how options are priced and how asset and option prices evolve.

This approach does, however, differ from the "out-of-sample" methodology originally used in Whaley (1982), and implemented more recently in Bakshi, Cao, and Chen (1997) and Dumas, Fleming and Whaley (1998), among others. These papers infer model-specific parameters from option prices over a short time interval (e.g., daily), examine model-specific out-of-sample pricing and hedging errors, and repeat for subsequent periods. While the pricing error tests do in essence test the stability of model-specific implicit parameters over short horizons, the repeated recalibrations used in the out-of-sample approach imply that the model is never taken seriously as a genuine data generating process. Implicit distributions are persistent over time, as will be evident in the substantial positive serial correlations in option price residuals reported in this article. Repeatedly recalibrated models can consequently do well in short-horizon out-of-sample tests, and yet fail to pick up longer-horizon parameter instabilities. For example, implied binomial trees models erroneously predict that both instantaneous conditional volatilities and at-the-money implicit volatilities are nonstationary. These models consequently *must* fit asset return volatility and option prices poorly over a long time interval if not repeatedly recalibrated.

Section 1 describes the data, and documents the strong post-crash shift in S&P 500 futures option price patterns using two diagnostics: implicit volatility patterns and the Bates (1991, 1997) skewness premium. Section 2 presents the postulated multifactor stochastic volatility/jump-diffusion process and the option pricing methodology. Section 3 describes the implicit parameter estimation methodology, and presents estimates. Section 4 examines the consistency of the parameters implicit in option prices with those estimated from the time series properties of implicit factors and S&P 500 futures prices. Section 5 concludes.

1. Pre- and post-crash option price patterns

American options on the Chicago Mercantile Exchange’s S&P 500 futures contract have been traded at the CME since January 28, 1983. While only quarterly options maturing in March, June, September and December were initially available, serial options written on the quarterly futures contracts and maturing in the nearest other two months were subsequently introduced in 1987. The quarterly options' last trading day was initially the third Friday of the month, the expiration date of the underlying futures contract, but was changed in the second quarter of 1986 to the day before because of "triple witching hour" concerns. Serial options trade up through the third Friday of their expiration month.

Intradaily transactions data for futures options and the underlying futures contracts were obtained from the CME over 1983-93. The data consist of the time and price of every transaction for which the price changed from the previous transaction. The CME also reports any bid (ask) quotes above (below) the preceding transaction price, but otherwise does not report bid and ask data. The bid and ask data were discarded, since they did not reflect actual transactions. All options transactions were matched with the nearest preceding futures price of comparable maturity, provided the lapsed time was less than 5 minutes and no trading halt was in effect.[2]

Preliminary diagnostics of observed moneyness biases were run using the quarterly options traded throughout 1983-93, for two purposes. First, these diagnostics document the extraordinary shifts in S&P 500 futures option prices that occurred following the stock market crash in 1987. Second, the use of relatively familiar diagnostics such as implicit volatility patterns allow the model-specific results in sections 3 and 4 to be interpreted within a commonly understood framework.

All intradaily transactions for 1- to 4-month quarterly options were selected over 1983-93 for those days with at least 4 call strikes and 4 put strikes traded per day, and at least 20 call transactions and 20 put transactions per day. *Representative* daily option prices were then constructed using the constrained cubic spline methodology of Bates (1991). Cubic splines were fitted daily to pooled intradaily option price/futures price ratios, as a function of the strike price/futures price ratio. Option-specific no-arbitrage constraints (convexity, monotonicity, and intrinsic value) were imposed when fitting the splines to calls and to puts, as described in Appendix 1. Implicit volatility curves were then computed from the estimated splines using 3-month Treasury bill rates and the Barone-Adesi and Whaley (1987) American option pricing formula.[3]

Figure 1 shows the typical post-'87 "volatility smirk" for implicit volatilities across strike prices: high implicit volatilities for out-of-the-money (OTM) put options relative to those from at-the-money (ATM)

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[2]The only trading halts over 1983-93 occurred on October 13, 1989. In principle, all options trading stops when a trading halt in the underlying S&P 500 futures is declared. However, option trades were recorded on October 13, 1989 *after* the S&P 500 futures had hit its first downward price limit.

[3]While Shimko (1993) fits polynomials through implicit volatilities, fitting curves through option prices and then computing implicit volatilities permits easier imposition of no-arbitrage constraints. Constrained cubic splines fitted to European option prices generate a continuous, piecewise linear representation of the risk-neutral probability density function over the available strike price range.

calls and puts, which are in turn higher than out-of-the-money call options' implicit volatilities.[4] The volatility spreads ($\hat{\sigma}_{OTM} - \hat{\sigma}_{ATM}$) shown in the lower panel of Figure 2 indicate this has been the pattern virtually without exception throughout the 1988-93 period. The implicit volatilities from representative 4% OTM put options were on average 2.4% higher than ATM implicit volatilities during 1988-93, while 4% OTM call implicit volatilities were on average 1.6% lower. The magnitudes varied over time, with major shocks (the stock market mini-crashes in January 1988 and October 1989, the Kuwait crisis of 1990-91) substantially increasing volatility spreads as well as the at-the-money implicit volatilities shown in the upper panel of Figure 2.

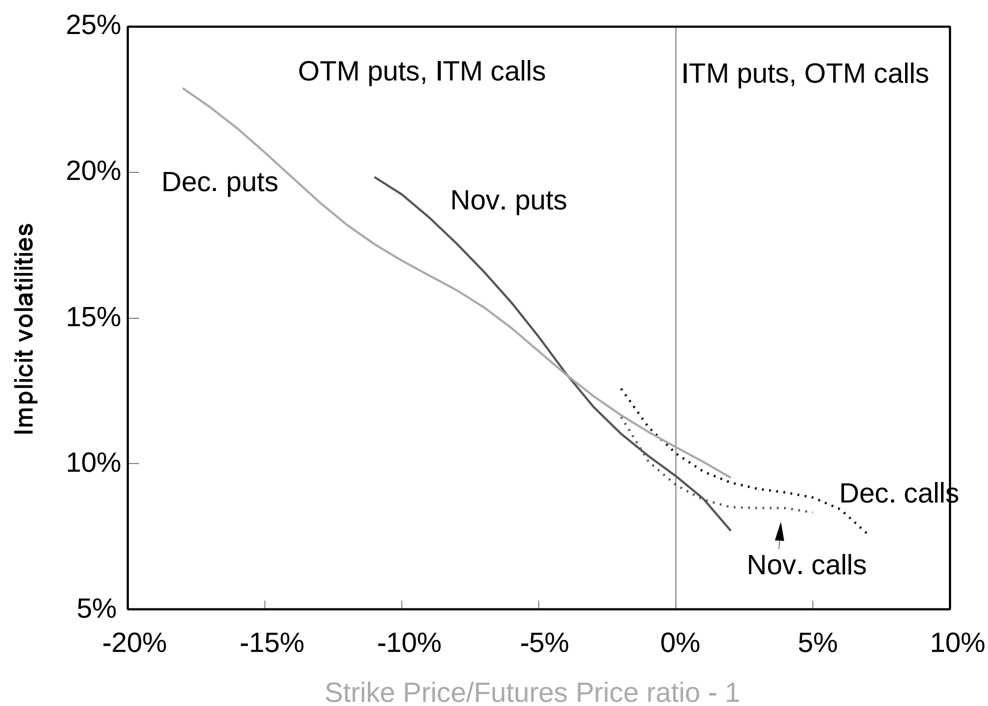

Figure 1. . Implicit volatility patterns on October 22, 1993.

[4]Out-of-the-money puts and in-the-money (ITM) calls have strike prices below the underlying futures price, while ITM puts and OTM calls have strike prices above the futures price. Figure 1 implicitly illustrates the fact that near-the-money and out-of-the-money call and put options are predominantly traded, with little trading of in-the-money options. While call and put options of comparable strike prices and maturities typically have comparable implicit volatilities, the comparison is only feasible for near-the-money strike prices.

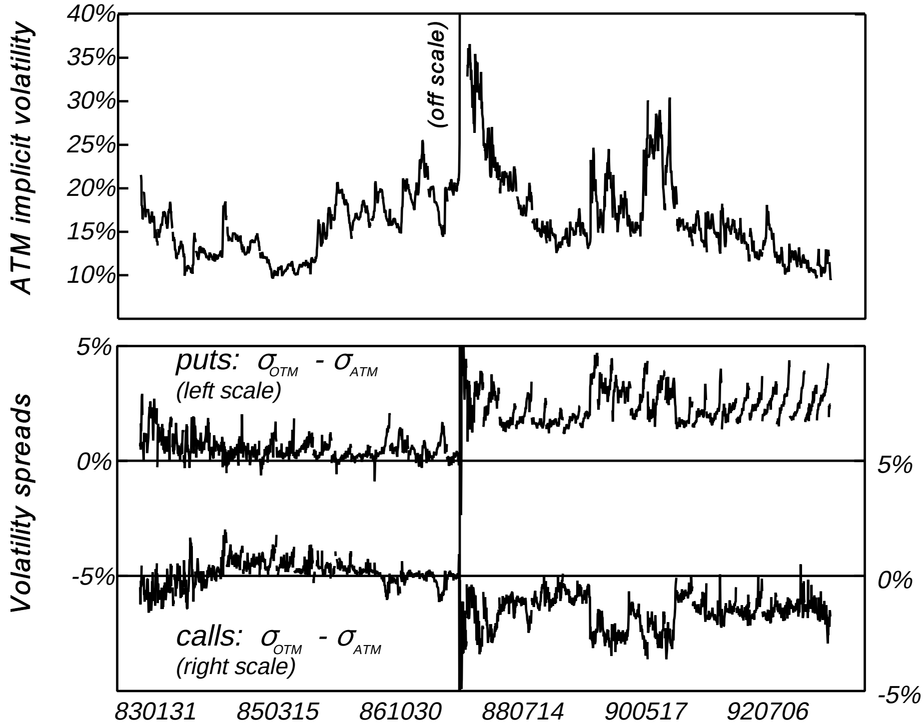

Figure 2. Implicit volatilities and volatility spreads ($\hat{\sigma}_{OTM} - \hat{\sigma}_{ATM}$), 1983-93.

The persistence and magnitudes of the post-crash implicit volatility patterns are in sharp contrast to those of the pre-crash period. Pre-crash OTM put implicit volatilities were almost invariably higher than those from ATM options. However, pre-crash implicit volatilities from OTM calls were sometimes below ATM implicit volatilities (generating a “smirk” pattern) and sometimes above (generating a “smile”). Each pattern lasted between 3 months and 1½ years before switching to the alternate pattern. The substantially smaller magnitudes of the pre-crash smirks and smiles relative to the post-crash smirks is also evident in the lower panel of Figure 2.

An alternate and substantially equivalent measure of moneyness biases is given by the "skewness premium," or percentage deviation between call and put prices for options comparably out-of-the-money:

$$SK(x) \equiv \frac{C(F;\ T,\ X_{call})}{P(F;\ T,\ X_{put})} - 1 \qquad \textbf{(1)}$$

where $X_{put}/F = (1+x)^{-1}$, $X_{call}/F = (1+x)$, and $x > 0$. Intuitively, since out-of-the money call (put) options pay off only upon realizations in the upper (lower) tail of the distribution of the underlying asset, comparing call and put prices is a direct gauge of the relative (risk-neutral) tail distributions, and therefore assesses implicit skewness. As discussed in Bates (1991, 1997), the skewness premium $SK(x)$ equals x for most standard and slightly positively skewed distributional hypotheses: Black and Scholes' lognormal model, Merton's (1976) jump-diffusion with mean-zero jumps, and Hull and White's (1987) stochastic volatility model. "Leverage" models such as the constant elasticity of variance model with standard parameterization, Geske's (1979) compound option model, and Rubinstein's (1983) displaced diffusion model imply roughly a $[0, x]$ range for the skewness premium. Values above (below) the $[0, x]$ range require a distribution more positively (negatively) skewed than the standard theoretical models.

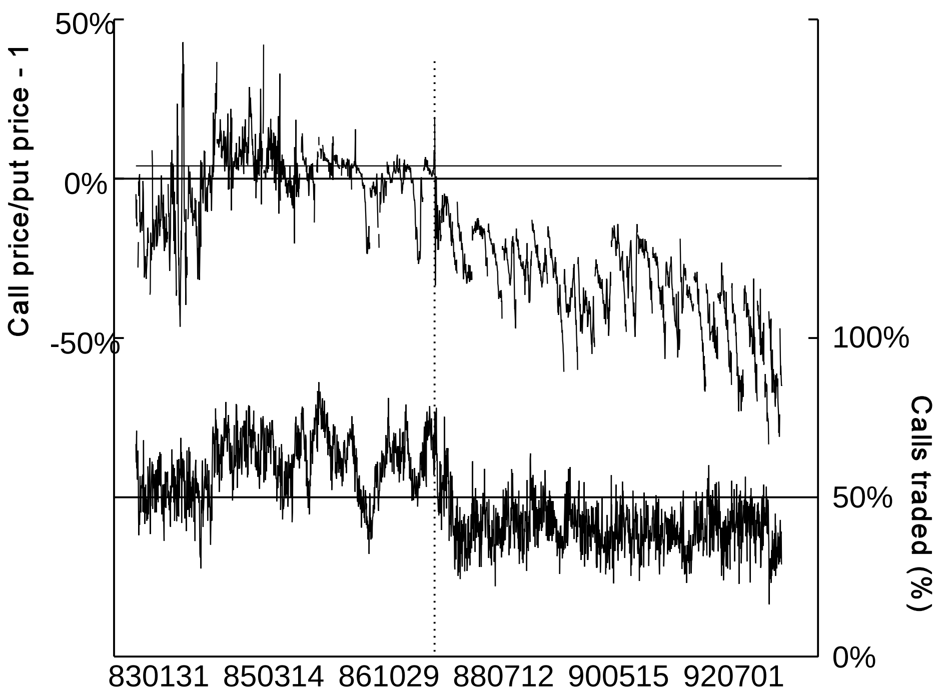

Figure 3. 4% OTM skewness premia (upper line) and relative trading activity (lower line)

The skewness premium computed using 4% OTM spline-interpolated option prices and shown in the upper panel of Figure 3 confirms that the post-crash moneyness biases have been enormous relative to standard distributional hypotheses. Whereas such hypotheses imply that 4% OTM American call options on

S&P 500 futures should be roughly 0-4% more expensive than correspondingly OTM put options, these calls have invariably been substantially *cheaper* than the puts -- 35% cheaper on average over 1988-93. Equivalently, the puts have been on average 64% more expensive than the calls. To put these magnitudes in perspective: a 2-month 4% OTM call or put option with a typical implicit volatility of 16% costs roughly 1% of the underlying asset price. If we made the extremely strong assumptions that standard theoretical models are correct and that OTM calls are correctly priced, the "overpricing" of 4% OTM puts would imply a 3.8% per year reduction in returns for equity funds that use rollover portfolio insurance. Dumas, Fleming and Whaley (1998) show that the comparable biases in the S&P 500 index options market are far too large to be attributable to bid-ask spreads.

A further interesting observation is that the skewness premium is strongly and directly related to the relative trading activity in calls *versus* puts of all strike prices shown in the lower half of Figure 3.[5] The correlation between the two variables is 73%, while sorting days based on whether call trades constitute more or less than 65% of all intradaily trades correctly identifies whether the skewness premium will be more or less than the x% benchmark for 86% of the days in the sample. The relationship is most apparent in the pre-crash period, given substantial parallel fluctuations in relative trading activity and skewness premia over that period, but also holds post-crash. Since in-the-money S&P 500 futures options are thinly traded, the relationship indicates that periods of substantial positive (negative) implicit skewness were typically periods in which OTM calls were more (less) heavily traded relative to OTM puts. Throughout 1988-93, puts have been heavily traded relative to calls, and negative skewness premia have been consistently observed.[6]

2. A proposed stochastic volatility/jump-diffusion model

Given the pronounced and persistent negative skewness implicit in post-'87 S&P 500 futures options, the

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[5]Bates (1996a) finds a similar relationship between the skewness premium and relative call/put trading activity for DM and yen futures options.

[6]Platen and Schweitzer (1995) predict this relationship between options' trading volume and implicit skewness. Whether their predicted shifts in the S&P 500 futures price process also exist is an open question. The observed volume/skewness premium correlation is also consistent with the claim that heavy demand for OTM put options has driven up post-crash prices.

following model will be used to nest the two major competing explanations.

Assumption A1: The S&P 500 futures price F is assumed to follow a two-factor geometric jump-diffusion of the following form:

$$
\begin{aligned}
dF/F &= (\mu - \lambda_t \bar{k} + c_{v1} V_{1t} + c_{v2} V_{2t})\, dt + \sqrt{V_{1t}}\, dZ_1 + \sqrt{V_{2t}}\, dZ_2 + k\, dq; \\
dV_{it} &= (\alpha_i - \beta_i V_{it})\, dt + \sigma_{vi} \sqrt{V_{it}}\, dZ_{vi}, \quad i = 1, 2; \\
Cov(dZ_i, dZ_{vi}) &= \rho_i\, dt, \quad i = 1, 2; \\
Cov(dZ_1, dZ_2) &= Cov(dZ_{v1}, dZ_{v2}) = 0;
\end{aligned}
\qquad \textbf{(2)}
$$

where

Z_i and Z_{vi}, $i = 1, 2$, are Wiener processes with the correlation structure specified above;

$\lambda_t = \lambda_0 + \lambda_1 V_{1t} + \lambda_2 V_{2t}$ is the instantaneous conditional jump frequency;

k is the random percentage jump conditional on a jump occurring, with time-invariant lognormal distribution $\ln(1+k) \sim N[\ln(1+\bar{k}) - \frac{1}{2}\delta^2, \delta^2]$; and

q is a Poisson counter with instantaneous intensity λ_t: $\text{Prob}(dq = 1) = \lambda_t\, dt$.

The postulated process nests both the stochastic volatility and jump explanations of the negatively skewed distributions implicit in observed S&P 500 futures option prices since the 1987 stock market crash. Negative skewness can arise either because of negative correlations between stock index and volatility shocks ($\rho < 0$), or because of negative-mean jumps ($\bar{k} < 0$). Similarly, conditional and unconditional excess kurtosis can arise either from volatile volatility, or from a substantial jump component. The two explanations differ in maturity effects. Jumps primarily affect short-maturity options, whereas stochastic volatility primarily affects longer-maturity options.

The postulated process extends the Bates (1996b) model in several directions potentially consistent with observed prices of S&P 500 futures options . First, the jump frequency λ_t can potentially be time-varying rather than constant. Given that implicit volatilities ranged from 40% to 10% over 1988-93, the assumption of constant jump risk throughout the period is implausible.

Second, a multifactor specification is used. Multifactor models have been extensively used in the bond pricing literature, but perhaps the only previous application to options is in Taylor and Xu (1994). However, the need for such models is potentially even greater than for bonds. On any given day, stock index options for a broad array of strike prices and up to 4 maturities are trading simultaneously. Bonds by contrast vary only by maturity. While there is no explicit evidence calling for a multifactor stock index option model, evidence from currency options (Taylor and Xu, 1994; Bates, 1996b) indicates that one-factor models can do a poor job in capturing the term structures of implicit volatilities over time, and that two-factor models would do better.

One risk is that using too many factors may overfit the options data, and start “explaining” what is in fact white noise from bid-ask bounce, data synchronization errors, or other phenomena. The appropriate number of factors must consequently be judged not only by how well one can fit option prices but by other criteria as well. Potential criteria include whether using more factors substantially modifies implicit conditional distributions, whether those modifications are in fact evident in the dynamic properties of asset and option returns, and whether implicit factors appear excessively noisy over time.

Options are priced not off the true process, but off the corresponding "risk-neutral" process that incorporates the appropriate compensation for volatility risk and jump risk:

$$
\begin{aligned}
&dF/F = -\lambda_t^* \bar{k}^* \, dt + \sqrt{V_{1t}} \, dZ_1^* + \sqrt{V_{2t}} \, dZ_2^* + k^* \, dq^*; \\
&dV_{it} = (\alpha_i - \beta_i V_{it} - \Phi_{vi}) \, dt + \sigma_{vi} \sqrt{V_{it}} \, dZ_{vi}^*, \quad i = 1, 2; \\
&Cov(dZ_i^*, dZ_{vi}^*) = \rho_i \, dt, \quad i = 1, 2; \\
&Cov(dZ_1^*, dZ_2^*) = Cov(dZ_{v1}^*, dZ_{v2}^*) = 0; \\
&Prob(dq^* = 1) = \lambda_t^* \, dt \quad .
\end{aligned} \tag{3}
$$

The volatility risk premia Φ_{iv} are constrained to be of the form[7]

$$
\begin{aligned}
\Phi_{vi} &\equiv -Cov(dV_{it}, \, dJ_w/J_w)/dt \\
&= \xi_i V_{it} \\
&\equiv (\beta^* - \beta) V_{it}
\end{aligned} \tag{4}
$$

where J_w is the marginal utility of nominal wealth for the representative investor and the ξ_i's are free "risk premium" parameters estimated by the divergence between the β_i^* parameters inferred from option prices and the β_i parameters from time series analysis.

Since unconstrained "risk premia" can potentially explain *any* deviations between actual and risk-neutral distributions,[8] it is important to have some idea of plausible values for signs and magnitudes. Under log utility,

$$
(\beta^* - \beta) dt = -Cov(dV/V, \, dJ_w/J_w) = Cov\left(dV/V, \, dW/W\right); \tag{5}
$$

see Cox *et al* (1985). Since volatility shocks are negatively correlated with shocks to the S&P 500 index,

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[7]This specification satisfies no-arbitrage constraints (Ingersoll, 1987; ch. 18), and can be derived assuming log utility for the representative investor (Cox, Ingersoll, and Ross, 1985).

[8]For instance, Stein (1989) observed that implicit volatilities from stock index options mean-revert faster than predicted by the term structure of implicit volatilities. The comparable observation for implicit variances would imply $\beta > \beta^* > 0$.

which represents a substantial fraction of nominal wealth W, $\beta^* - \beta$ is presumably negative.[9] Conversely, a lower bound on the volatility risk premium can be obtained under log utility if volatility shocks are assumed to covary more negatively with the equity than with the non-equity return components of nominal wealth returns. Proxying the former by the index futures return implies that $\beta^* - \beta > Cov(dV/V,\ dF/F) = \rho\,\sigma_v$, a small negative number inferable from option prices.[10]

The jump risk premia λ_t/λ_t^* and $\bar{k} - \bar{k}^*$ similarly reflect the compensation required for bearing systematic jump risk:

$$\lambda_t^* = \lambda_t\, E\left(1 + \frac{\Delta J_w}{J_w}\right)$$
$$\bar{k}^* = \bar{k} + \frac{Cov(k,\ \Delta J_w/J_w)}{E[1 + \Delta J_w/J_w]}\ . \tag{6}$$

where ΔJ_w is the change in the marginal utility of nominal wealth conditional upon a jump occurring. Under systematic jump risk the cost λ_t^* per unit time of Arrow-Debreu crash insurance will diverge from the actuarial rate λ_t at which jumps arrive. Assessing this divergence requires an assessment of how stock market jumps affect other investments. If jumps are assumed to occur only in stock markets and log utility is again assumed, then $\Delta \ln J_w = -\Delta \ln W \approx -f\,\Delta \ln F$ and

$$\lambda_t^* \approx \lambda_t\, E\,\exp(-f\ \Delta \ln F) = \lambda_t (1+\bar{k})^{-f}\, e^{½\delta^2(f^2+f)}$$
$$\ln(1+\bar{k}^*) \approx \ln(1+\bar{k}) - f\,\delta^2 \tag{7}$$

where f is the fraction of nominal wealth held in equity. When average jumps are negative, the "risk-neutral"

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[9]Intuitively, volatility-sensitive investments such as straddles are “negative-beta” investments that typically pay off in adverse states when the marginal utility of wealth is high. They therefore have a lower conditional mean under the actual than under the risk-neutral distribution -- a negative risk premium. This is in contrast to the positive conditional mean differential, or equity premium, of “positive-beta” investments such as the S&P 500.

[10]Cochrane and Saá-Requejo (1996) use “reasonable” values of maximal feasible Sharpe ratios to place bounds on the volatility risk premium.

jump frequency and average drop size will tend to exaggerate the downside risk: $\lambda^* > \lambda$, $\bar{k}^* < \bar{k}$. However, the $\bar{k}^*$ and δ values estimated below give little reason to believe that jump risk premia introduce a substantial wedge between the "risk-neutral" parameters implicit in option prices and the true parameters.

The above assumptions generate an analytically tractable method of pricing options without sacrificing accuracy or requiring undesirable restrictions (such as ρ=0) on parameter values. European call options that can be exercised only at maturity are priced as the expected value of their terminal payoffs under the "risk-neutral" probability measure:

$$\begin{aligned} c &= e^{-r(T+\Delta t_1)} E^* \max(F_T - X,\ 0) \\ &= e^{-r(T+\Delta t_1)} \left[\int_X^\infty F_T\, p^*(F_T)\, dF_T \ - \ X \int_X^\infty p^*(F_T)\, dF_T \right] \\ &= e^{-r(T+\Delta t_1)} (F P_1 - X P_2) \end{aligned} \tag{8}$$

where

Δt_1 is the one business day lag in settlement at option expiration;

E^* is the expectation with respect to the risk-neutral conditional probability density p^*;

$F = E^*(F_T)$ is the current futures price;

$P_2 = Prob^*(F_T > X)$ is one minus the risk-neutral distribution function; and

$P_1 = \int_X^\infty (F_T/F)\, p^*(F_T)\, dF_T$ is an alternate probability measure.

The relevant probabilities can be evaluated by Fourier inversion of the underlying characteristic functions:

$$P_j = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{Imag[\varphi_j(i\Phi)\, e^{-i\Phi x}]}{\Phi}\, d\Phi \ , \tag{9}$$

where $\varphi_1(\Phi)$ and $\varphi_2(\Phi)$ are the associated real-valued moment generating functions of $\ln(F_T/F)$ and $x \equiv \ln(X/F)$. φ_1 and φ_2 can be solved using the methodology described in Heston (1993) and Bates

(1996b), with straightforward extensions for multiple independent factors and time-varying jump risk:

$$\begin{aligned}\ln \varphi_j(\Phi \mid V_1, V_2, T) &\equiv \ln E^*[e^{\Phi \ln(F_T/F)} \mid P_j] \qquad (j = 1, 2)\\ &= \sum_{i=1}^{2}\left[A_{i,j}^*(T;\Phi) + B_{i,j}^*(T;\Phi)\,V_i\right] + \lambda_0^* T\, C_j^*(\Phi)\end{aligned} \tag{10}$$

where

$$\begin{aligned}A_{i,j}^*(T;\Phi) = &-\frac{\alpha_i T}{\sigma_{vi}^2}(\rho_i\sigma_{vi}\Phi - \beta_{i,j} - \gamma_{i,j}^*)\\ &-\frac{2\alpha_i}{\sigma_{vi}^2}\ln\left[1 + \tfrac{1}{2}(\rho_i\sigma_{vi}\Phi - \beta_{i,j} - \gamma_{i,j}^*)\frac{1-e^{\gamma_{i,j}^* T}}{\gamma_{i,j}^*}\right],\end{aligned} \tag{11}$$

$$B_{i,j}^*(T;\Phi) = -2\frac{\tfrac{1}{2}[\Phi^2 + (3-2j)\Phi] + \lambda_i^* C_j(\Phi)}{\rho_i\sigma_{vi}\Phi - \beta_{i,j} + \gamma_{i,j}^*\dfrac{1+e^{\gamma_{i,j}^* T}}{1-e^{\gamma_{i,j}^* T}}}, \tag{12}$$

$$C_j^*(\Phi) = (1+\bar{k}^*)^{2-j}\left[(1+\bar{k}^*)^{\Phi} e^{\frac{1}{2}\delta^2[\Phi^2 + (3-2j)\Phi]} - 1\right] - \bar{k}^*\Phi \tag{13}$$

$$\gamma_{i,j}^* = \sqrt{(\rho_i\sigma_{vi}\Phi - \beta_{i,j})^2 - 2\sigma_v^2\{\tfrac{1}{2}[\Phi^2 + (3-2j)\Phi] + \lambda_i^* C_j(\Phi)\}} \tag{14}$$

and $\beta_{i,j} = \beta_i^* + \rho_i\sigma_{vi}(j-2)$.

The above procedure gives the price of a *European* call option as a function of state variables and parameters, while European put prices can be computed using put-call parity. Estimates of the early-exercise premium component of *American* S&P 500 futures option prices were generated analogously to Bates (1996b). The expected average variance and expected average jump frequency were inserted into the Bates (1991) jump-diffusion early-exercise approximation, and appropriate "smooth-pasting" conditions were determined based upon the European option pricing formula above.

While this method of computing early-exercise premia introduces some approximation error that can influence implicit parameter estimates, the induced bias is not likely to be substantial. The options data set considered below contains predominantly at- and out-of-the-money options, with relatively few in-the-money options with substantial early exercise premia. Furthermore, Chaudhury and Wei (1994) show that American futures option prices C and P are bounded above by the *future value* of the European option price:

$$\begin{aligned} \max[F - X,\, c] &\leq C \leq e^{rT}c \\ \max[X - F,\, p] &\leq P \leq e^{rT}p \end{aligned} \qquad \textbf{(15)}$$

This implies that the proportional markup of American over European futures option prices is within the narrow range $[1,\ e^{rT}]$. Consequently, accurately evaluating the early-exercise premium is not a major issue for the 0-6 month futures options examined here.[11]

3. Implicit Parameter Estimation

3.1 Data and methodology

A data subset different from that of section 1 was used for the estimates of post-crash stochastic volatility/jump-diffusion processes. All reported transactions involving quarterly and serial options with at least one week to maturity were used, since serial options were available throughout 1988-93. However, only trades on Wednesday mornings (9-12 AM) were considered. Using daily data was ruled out partly because of the resulting extreme demands on computer memory and time, and partly to avoid modeling day-of-the-week volatility effects. Wednesday was selected as having the fewest trading holidays. The use of morning trades reflected a tradeoff between shortening the intradaily interval for greater cross-sectional option price synchronization, and lengthening it to get more observations. Linear interpolations of 3- and 6-month

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[11]Bates (1996b) finds that this method of computing early-exercise premia generates maximal approximation errors for 3-month in-the-money American *spot* options of approximately .025% - .035% of the underlying asset price, with much smaller errors for at- and out-of-the-money options. The errors will be smaller for American futures options.

Treasury bill yields were used for the corresponding risk-free discount rates.

The resulting 1988-93 data set consists of 39,607 transactions in up to 4 option maturities per day on 310 Wednesday mornings over January 6, 1988 to December 29, 1993; an average of 128 trades per morning. On average, 3.3 different maturities, 16 different strike prices, and 30 different strike price/maturity combinations were represented on any given morning. Table 1 summarizes characteristics of the data set. Overall, puts were more heavily traded than calls, the shortest-maturity options were most heavily traded, and there was virtually no trading of deep in-the-money call and put options.

[Table 1 about here]

The basic approach of this article is to infer parameter values and factor realizations from observed option prices, and to test whether the inferred distributions are consistent with the observed evolution of S&P 500 futures and futures option prices. A fundamental difficulty with implicit parameter estimation is the absence of an appropriate statistical theory of option pricing errors. Standard option pricing models assume that market participants know with certainty the underlying structure and parameter values of the data generating process that generates option prices. Given the abundance of option data, the overidentifying restrictions that all options be priced exactly by a parsimoniously parameterized model will almost assuredly be rejected.

One reason why observed and model option prices might deviate is microstructure-related measurement error; e.g., bid-ask spreads or imperfect synchronization between option data and the underlying futures data. *A priori,* one would expect such measurement error to be heteroskedastic, but the appropriate functional form across different strike prices, maturities, and dates is an open question.[12] This article follows Bates (1996b) in assuming that the option price/futures price *ratio* deviates from its theoretical value by a

[12]George and Longstaff (1993) find that market makers' bid-ask spreads in the S&P 100 index options market vary by strike price and maturity, suggesting a heteroskedastic impact from bid-ask bounce. Imperfect synchronization between options and futures data affects in-the-money options more than out-of-the-money options.

random, heteroskedastic, additive option price residual:

$$e_{i,t} \equiv \left(\frac{O}{F}\right)_{i,t} - O\left(1,\ \boldsymbol{V}_t,\ T_{i,t};\ \left(\frac{X}{F}\right)_{i,t},\ \theta\right) \qquad \textbf{(16)}$$

where

t is an index over the 310 Wednesday mornings in the sample;

i is an index over transactions (calls and puts of assorted strike prices and at most four maturities) on a given Wednesday morning;

$(O/F)_{i,t}$ is the observed call or put option price/futures price ratio for a given transaction;

$O(\bullet)$ is the theoretical American option price/futures price ratio given the contractual terms of the option (call/put, time to maturity $T_{i,t}$, strike price/ futures price ratio $(X/F)_{i,t}$) and given that Wednesday morning's vector of factor realizations $\boldsymbol{V}_t$, interest rate r_t and the time-invariant parameters θ of the model.

Option/futures price ratios were used to impose stationarity, since most option pricing models are homogeneous of degree one in the asset price and strike price.[13] The form of heteroskedasticity across calls and puts of different moneynesses and maturities was determined by the data, as described below.

A major issue for implicit parameter estimation is, however, the specification error that assuredly arises in any parsimonious model of the data generating process. Specification error implies that option pricing residuals of comparable moneyness and maturity will be contemporaneously correlated, and serially correlated as well if the conditional "risk-neutral" distribution evolves gradually over time in fashions not

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[13]Exceptions include the constant elasticity of variance model, binomial tree models, and the nonparametric approach of Aït-Sahalia and Lo (1998). Non-homogeneous models imply nonstationary options/futures price ratios, and therefore counterfactually imply nonstationary at-the-money implicit volatilities. Bates (1996c, pp. 591-2) discusses the evidence of stationarity in implicit volatilities; see also Figure 2 above.

captured by the model. Ignoring specification error affects the relative weighting of option prices used in implicit parameter estimation, and overestimates the amount of truly independent information. For instance, the nonlinear ordinary least squares approach used by Whaley (1986) and Bates (1991) weights the substantially redundant information provided by actively traded at- and slightly out-of-the-money options too heavily, while downweighting other option prices.

While it is not possible to fully eliminate specification error using these models, it is possible to gauge its importance. Implicit parameters were estimated using a nonlinear generalized least squares/Kalman filtration methodology that takes into account the heteroskedasticity, contemporaneous correlation, and serial correlation properties of option residuals. Option transactions were sorted into 64 groups, based on contract type (call/put), maturity (0-1, 1-2, 2-3, or 3-6 months), and moneyness criteria: whether the option was in-the-money by 0-1%, 1%-2%, or >2%, or out-of-the-money by 0-1%, 1%-2%, 2%-4%, 4%-8%, or >8%. The asymmetric moneyness criteria reflect the asymmetric trading activity reported in Table 1. The set of groups for which transactions were observed on any given day was constantly changing, with 27.7 out of 64 groups represented on an average Wednesday.

Option price residuals were assumed to include both group-specific and idiosyncratic shocks:

$$\begin{cases} e_{i,t} = \varepsilon_{I,t} + \sigma_I \eta_{i,t} & \text{for } i \in G(I,\ t) \\ \varepsilon_{I,t} = \rho_I \varepsilon_{I^*,t-1} + \upsilon_{I,t} & \end{cases} \tag{17}$$

where

$G(I,\ t)$ is the set of observations in group I at date t;

$\upsilon_{I,t}$ is a mean-zero, normally distributed shock term common to all option prices in group I at time t, with $E_{t-1} \upsilon_t \upsilon_t' = \boldsymbol{Q}$ for positive semidefinite $\boldsymbol{Q}$;

$\eta_{i,t} \sim N(0,1)$ is an idiosyncratic shock to transaction i at time t, uncorrelated with υ_t; and

I^* identifies lagged option residuals of the same moneyness and *delivery month* for 0-3 month quarterly and serial options, and of the same *maturity* ($I^* = I$) for 3-6 month quarterly options.[14] An absence of specification error is equivalent to observing heteroskedastic white noise in option residuals ($\boldsymbol{Q} = \mathbf{0}$; $\rho_I = 0$ for all I), which can be tested using standard tests of parameter restrictions.

This generalized heteroskedasticity specification allows the effective relative weighting of options in implicit parameter estimation to be determined empirically by the relative noisiness of the various categories of options, rather than relying on specific models of heteroskedasticity.[15] A possible disadvantage is that assuming additive shocks ignores both intrinsic value and nonnegativity constraints on option prices. However, apparent intrinsic value violations can arise from option synchronization error with the underlying futures price.[16] The nonnegativity constraints are partly captured through heteroskedasticity adjustments; in particular, through lower volatility estimates for out-of-the-money residuals.

The contract-, moneyness-, and maturity-related common shock estimates $\hat{\varepsilon}_t \equiv \{\hat{\varepsilon}_{I,t}\}_{I=1}^{64}$ are interpreted below as entirely attributable to specification error. The η's reflect idiosyncratic noise from bid-ask bounce and synchronization error, as well as any intra-morning variation in the underlying volatility state variables. Specification error can also affect the "idiosyncratic" noise estimates σ_I for those groups with substantial intra-group heterogeneity in strike price/futures price ratios.

Given the above specification for option residuals, the appropriate loss function for implicit parameter

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[14]Analytically simpler dynamics could have been generated by assuming $I^* = I$ throughout. However, time decay in option prices suggests a closer relationship between option residuals of 8-week (1-2 month) maturity and the preceding week's 9-week (2-3 month) maturity option residuals than between 8- and 5-week option residuals.

[15]Bates (1996c, pp.587-589) surveys and discusses alternate methods of implicit parameter estimation.

[16]Canina and Figlewski (1993) point out that the common practice of throwing out intrinsic value violations constitutes one-sided data censoring, biasing upward in-the-money option prices and affecting implicit parameter estimation. The 24 observations (out of 39,607) that violated intrinsic value constraints were consequently retained.

estimation is

$$\max_{\{V_t\},\,\theta} \ \ln L_{options} = -\tfrac{1}{2}\sum_t \ln|\Omega_{t|t-1}| + (e_t - E_{t-1}e_t)'\,\Omega^{-1}_{t|t-1}\,(e_t - E_{t-1}e_t)\,. \tag{18}$$

$E_{t-1}\boldsymbol{e}_t$ is a Kalman filtration-based forecast of option residuals conditional upon estimated dynamics (17) and lagged option residuals. The conditional covariance matrix $\Omega_{t|t-1}$ is also estimated using Kalman filtration methods.

The log likelihood in (18) is optimized by an alternating two-step procedure. Conditional upon the Kalman filtration parameters in (17), (18) is optimized via the nonlinear weighted least squares over the parameters $<\{\boldsymbol{V}_t\}_{t=1}^{310},\ \theta>$ that directly determine option residuals $\boldsymbol{e}_t$. For the 1-factor models, this involves estimating 310 daily factor realizations and up to 8 time-invariant stochastic volatility/jump-diffusion parameters, while the two-factor models involve 620 factor realizations (two per day) and up to 13 time-invariant parameters. Various submodels were estimated, with specific parameters zeroed out, to identify which features of the general models were important in matching observed option prices.[17] *Conditional* upon parameter estimates, optimization of (18) over $<\{\rho_I,\ \sigma_I\}_{I=1}^{64},\ \boldsymbol{Q}>$ involves estimating a high-dimensional linear Kalman filtration representation of the option residuals, with 2208 filtration parameters. Alternating between the two optimization steps until joint convergence yields estimates of implicit parameters and factor realizations for a specific model, estimates of the relative importance of idiosyncratic and common shocks in the observed option price residuals, and a full dynamic description of the specification error (as captured by the vector of common shocks). A slightly improved variant of the Shumway and Stoffer (1982) and Watson and Engle (1983) EM algorithm approach to estimating Kalman filtrations is developed in the

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[17]The Davidon-Fletcher-Powell quadratic hill-climbing algorithm was used (GQOPT subroutine DFP) for the parameter/factor realizations optimization. The score was computed numerically, exploiting specific features of the log likelihood function to increase efficiency. For instance, computing $\partial \ln L/\partial V_{it}$ required perturbing only date-*t* options and measuring subsequent propagation effects. Nonnegativity constraints were enforced through log transformations of parameters and factors, while correlations were constrained via a cumulative normal transformation.

appendix for this particular application.[18]

Optimization yields the estimates of parameters and state variable realizations that best fit observed option prices. However, such an optimization does not constrain the state variable estimates to evolve consistently with the assumed processes underlying the option pricing model. For instance, the implicit variances estimated under the Black-Scholes submodel are not constrained to be identical, contrary to the assumptions of that model. Consequently, the stochastic volatility and stochastic volatility/jump-diffusion models were also estimated using the likelihood function

$$\ln L(\{\boldsymbol{V}_t\};\ \theta,\ \beta_1,\ \beta_2) \;=\; \ln L_{options} + \ln L_{V1} + \ln L_{V2} \qquad \textbf{(19)}$$

where

$\ln\ L_{options}$ is the function of option price residuals given above in equation (18), and

$\ln L_{Vi} = \sum_{t=2}^{310} \ln p_V(\ln V_{it} \mid \alpha_i,\ \beta_i,\ \sigma_{vi};\ V_{i,t-1})$, i =1,2, is the log likelihood of an estimated $\{\ln V_{it}\}$ sample path under the postulated square root process (2), given the *actual* (as opposed to risk-neutral) rate of variance mean reversion β_i.

Factor transition densities over a discrete time interval are related to the noncentral chi-squared density; see, e.g., Cox *et al* (1985) and Bates (1996b). Consequently, the factor transition densities of $\ln V_{it}$ are also related to the noncentral chi-squared density. The series representation is

$$p_V(\ \ln V_{i,t+\Delta t}\ \mid\ \alpha_i,\ \beta_i,\ \sigma_{vi};\ V_{it}\ ) \;=\; \frac{2}{\kappa}\ \frac{e^{-\frac{1}{2}(y+\Lambda)}\ y^{\nu/2}}{2^{\nu/2}}\ \sum_{j=0}^{\infty}\ \frac{(\frac{1}{4}\,y\,\Lambda)^j}{\Gamma(\,j+\nu/2)\ j!}\ , \qquad \textbf{(20)}$$

where $\kappa = \frac{1}{2}\sigma_{vi}^2(1-e^{-\beta_i\Delta t})/\beta_i$, $y = 2V_{t+\Delta t}/\kappa$, $v = 4\alpha_i/\sigma_{vi}^2$, $\Lambda = 2V_{it}\,e^{-\beta_i\Delta t}/\kappa$, and $\Gamma(\cdot)$ is the gamma function.[19]

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[18]The procedure is superior to that in Bates (1996b) in two regards. First, it copes better with missing observations for particular groups. Second, the EM algorithm approach to estimating high-dimensional common shock vectors is substantially faster.

[19]The transition densities of $\ln V_{i,t+\Delta t}$ were used rather those of $V_{i,t+\Delta t}$ because the former density is strictly finite over the relevant $(-\infty,\ +\infty)$ domain of $\ln V_{i,t+\Delta t}$ whereas the latter density is infinite at V_{it} = 0 when the reflecting barrier at zero is attainable $(2\alpha < \sigma_v^2)$; see Bates (1996b). While both density

The constrained estimates serve three functions. First, since the underlying hypothesis is that the state variables follow a diffusion, the constrained estimates yield *smoothed* state variable sample paths that are useful in assessing major and persistent developments in S&P 500 futures option prices. The appropriate degree of smoothing is determined endogenously, based upon the estimated volatility of volatility σ_v. For instance, optimization of (19) under the Black-Scholes assumption $\alpha = \beta^* = \sigma_v = 0$ would involve estimating a single implicit variance over the entire 1988-93 period. Second, the constrained parameter estimates are somewhat more plausible relative to the time series properties of the state variable estimates. Finally, a comparison of the constrained and unconstrained parameter estimates can be used to test the option pricing models.

3.2 Results

For the full two-factor stochastic volatility/jump-diffusion model, the time-invariant parameter vector θ was the set of jump and stochastic volatility parameters: $< \lambda_0^*, \lambda_1^*, \lambda_2^*, \bar{k}^*, \delta, \alpha_1, \beta_1^*, \sigma_{v1}, \rho_1, \alpha_2, \beta_2^*, \sigma_{v2}, \rho_2 >$. The Wednesday morning factor realizations $\{\boldsymbol{V}_t\}$ were also estimated for all Wednesdays in the 1988-93 data set, entailing 310 (620) additional parameters for the 1- (2-)factor models. Intradaily movements in implicit factors were ignored in the estimation procedure.

Various 1- and 2-factor nested subcases of the general model were estimated, to see which features of that model were important in explaining option price patterns. The American futures option version of the Black-Scholes (BS) model imposes identical implicit variances V_{1t} on options of all strike prices and maturities on a given day, but allows different implicit variances on different days. The "deterministic volatility" models also assume lognormal implicit distributions, but relax the Black-Scholes assumption of the same implicit variance for all maturities.[20] The one-factor model (DV1) allows daily a monotonically

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functions yield identical maximum likelihood estimates of stochastic volatility parameters from a given set of $\{V_{it}\}$ (or $\{\ln V_{it}\}$) data, only the former is well-behaved as a smoothing function when *estimating* implicit factors from option prices.

[20]Dumas, Fleming, and Whaley (1998) use "deterministic volatility" to indicate dependency of the instantaneous conditional volatility upon the underlying asset price and time. Here, the term is used to indicate strictly time-dependent evolution of instantaneous variances, of a particular form.

upward or downward sloping term structure of implicit variances, depending on whether the daily implicit spot variance V_{1t} is greater or less than the time-invariant ratio α_1/β_1^*, while the two-factor model (DV2) allows more complicated term structures. The 1-factor stochastic volatility model (SV1) relaxes the lognormality assumption of the DV models, with hump-shaped magnitudes of implicit skewness and kurtosis across option maturities that are discussed in Das and Sundaram (1997). The 1-factor stochastic volatility/jump-diffusion model (SVJD1) allows more general patterns of implicit skewness and kurtosis across maturities. The 2-factor SV2 and SVJD2 models offer yet richer instantaneous moneyness/maturity option price patterns relative to their 1-factor counterparts. Furthermore, the second factor can potentially capture certain forms of "parameter drift" over time in 1-factor implicit distributions.

Several criteria are used in assessing how well the models fit, and the relative importance of moving to more general models. Log likelihood is one criterion. However, with almost 40,000 option observations, any nested submodel is invariably rejected in favor of a more general model at extreme levels of statistical significance. A further issue is that while the log likelihood function allows group-specific serial correlations in option residuals, substantial serial correlation is in fact evidence of specification error. Consequently, additional diagnostics are used to assess the *economic* impact of moving to more general models: the magnitudes and importance of autocorrelation in option residuals, group-specific average option pricing residuals both before and after serial correlation corrections, etc. Overall root mean squared error (RMSE) of option residuals across all strike prices and maturities is used as a broad summary measure of model performance, both with and without a serial correlation correction. While this measure ignores the substantial moneyness- and maturity-related heteroskedasticity that the estimation procedure explicitly takes into account, RMSE is relatively intuitive and is useful for comparison with other work in the area.

3.2.1 Relative model performance

Over 1988-93, the *ad hoc* Black-Scholes (BS) model's assumption of the same implicit variance for options of all strike prices and maturities on any given day does a very poor job of fitting observed option prices. The overall RMSE reported in Table 2 amounts to .224% of the underlying futures price -- a substantial fraction of the intrinsic value of observed option prices.[21] Taking account of the serial correlation in pricing errors reduces RMSE to .113%. Relaxing the assumption of identical implicit variances at all maturities under the lognormal deterministic volatility models (DV1, DV2) has little impact upon reducing RMSE relative to the BS model.

[Table 2 about here]

The 1-factor stochastic volatility and stochastic volatility/jump-diffusion models substantially reduce RMSE relative to the conditionally lognormal models (BS, DV1, DV2), both with and without the serial correlation correction. The major improvement in fit clearly originates in generating negatively skewed distributions to match the "volatility smirk," through negative-mean jump processes and/or through negative correlations between index returns and volatility shocks. The 2-factor models reduce RMSE yet further, with the most general SVJD2 model achieving overall RMSE of .078% (.068%) of the underlying futures price without (with) the correction for serial correlation in option pricing residuals. With an average futures price over 1988-93 of 362 and a option tick size of .05, this amounts to an overall RMSE of 5.6 and 4.9 price ticks, respectively.

Group-specific autocorrelation estimates summarized by minimum, maximum, and median values in Table 2 indicate severe persistence in option pricing residuals for all models and all moneyness and maturity categories. The problem is most pronounced for residuals from the conditionally lognormal models (BS, DV1, DV2), reflecting the presence of the "volatility smirk" throughout 1988-93. The filtration-based serial correlation correction cuts option residuals' RMSE in half for those models, implying an associated R^2 of roughly 75% in "explaining" option residuals. Moving to more complicated models does not especially reduce group-specific autocorrelation estimates, which still range from .16 to .78 for the SVJD2 model, but the serial correlation correction becomes less important in reducing RMSE for the more general models.

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[21] A 2-month, 4% out-of-the-money put option on futures with an implicit volatility of 16% costs about 1% of the underlying futures price, while an at-the-money option costs about 2.6%.

[Table 3 about here]

The option pricing models attempt to match observed option prices across three dimensions: across maturities, across strike prices, and across time. To assess the first two dimensions, group-specific average residuals are reported in Table 3 using three measures:

1. the average Kalman-smoothed systematic errors $\frac{1}{310}\sum_{t=1}^{310}\hat{\varepsilon}_{t\,|\,T}$ estimated conditional upon the full data set;
2. the average group-specific option pricing residuals; and
3. the average option pricing residual after serial correlation correction.

The first two measures are conceptually similar. However, the first facilitates cross-group comparisons by weighting all days equally, whereas the last two averages are affected by variations over time in group-specific trading activity. Because calls and puts with comparable strike prices and maturities have comparable implicit volatilities (and comparable $\hat{\varepsilon}_{I,\,t\,|\,T}$ estimates), the values in Table 3 are averages across both call and put residuals for comparable groups, using the call/put pairings in Table 1.[22]

Estimates of average systematic errors in Table 3 indicates that the stochastic volatility (SV2) model has some difficulty in matching option price patterns at both short and long maturities. The model tends to underprice low-strike 0-3 month options, and overprice high-strike 2-6 month options. The serial correlation correction reduces but doesn't eliminate this pattern. The more general SVJD2 model has less pronounced moneyness/maturity patterns in option residuals both before and after the serial correlation correction, but perceptible error patterns remain.

It is important to note that the SVJD2 model is relying on both the crash fears *and* the volatility feedback explanations of implicit negative skewness to match the volatility smirk at different option maturities. To see this, Figure 4 decomposes SVJD2's estimated volatility smirk for 1- and 6-month options into a component directly attributable to jump risk (SVJD2 model with $\sigma_{v1} = \sigma_{v2} = 0$), and a remainder

[22]The reported average $\hat{\varepsilon}_{I,\,t\,|\,T}$ are simple averages of corresponding call and put estimates when both exist. Other average residuals depend upon relative trading activity in calls and puts.

attributable to the degree to which the implicit factors that determine jump and non-jump risk are volatile and are negatively correlated with index returns. Average factor realizations were used (V_1 = .00963, V_2 = .01352), and the Black-Scholes implicit volatilities associated with estimated option prices were graphed over a moneyness range of ±2 standard deviations, measured using expected average variances at different horizons.[23]

The estimated volatility smirk in Figure 4 is as pronounced at long horizons as at short horizons after appropriately scaling the moneyness range. Neither jumps nor stochastic volatility alone can match this pattern, so the general stochastic volatility/jump-diffusion model relies on both explanations: jumps to capture most of the short-horizon moneyness biases, distributional randomization to capture most of the long horizon

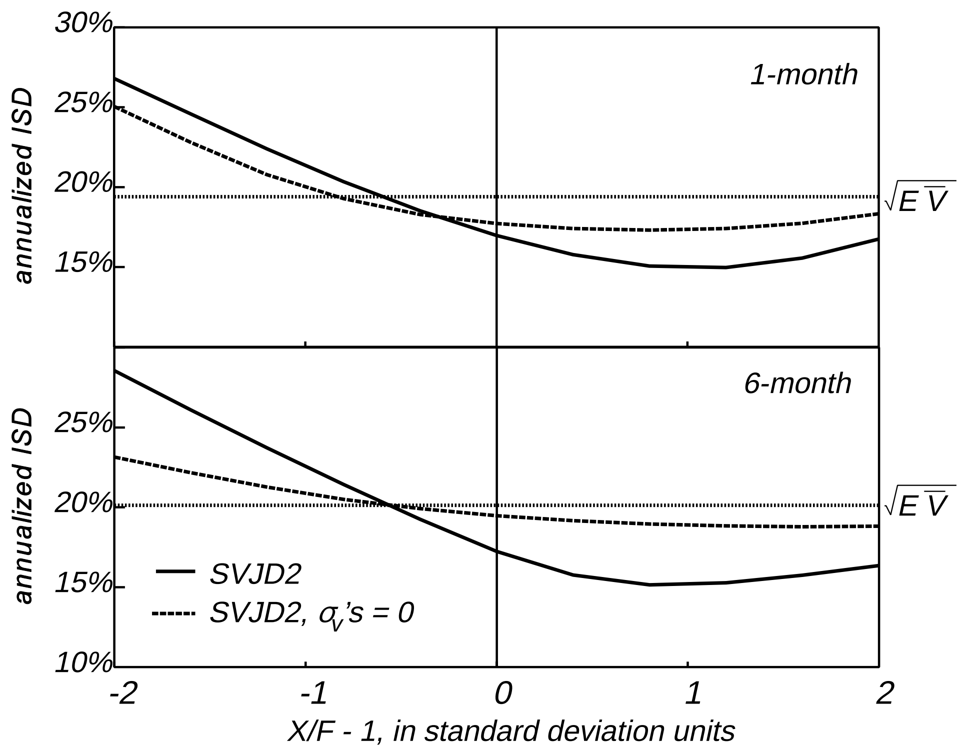

Figure 4. 1- and 6-month implied standard deviation (ISD) patterns from SVJD2 estimates, with and without σ_v's set to zero.

[23]Average factor values imply a roughly flat term structure of expected average variances -- about $(.20)^2$ per annum at all maturities. Consequently, the range of $(X/F - 1)$ was roughly $\pm 2 \times .20\sqrt{T}$ for options of maturity T.

biases. In contrast to stochastic volatility models such as SV2 that randomize the variance of instantaneous conditionally Gaussian returns, allowing σ_v's different from zero in the SVJD models involves randomizing conditionally fat-tailed distributions.

[Table 4 about here]

Table 4 provides additional detail on the stochastic properties of option pricing residuals from the SVJD2 model. Idiosyncratic noise generally decreases for lower-priced, farther out-of-the-money options, as would be expected from either synchronization error or bid-ask bounce. The volatility of common shocks follows a somewhat similar moneyness pattern. Common shock noise has a very pronounced maturity effect, with much higher levels for the thinly traded 3-6 month quarterly options than for 0-1, 1-2, or 2-3 month serial and quarterly options. Much of this higher volatility is observed in the first week that contracts with a new long-term maturity begin trading. A likelihood ratio test strongly rejected the hypothesis of no common shocks ($\boldsymbol{Q} = \mathbf{0}$, $\rho_I \equiv 0$ for all I; 2144 parameter restrictions), with log likelihood dropping from 258,151 to 239,868 (P-value less than 10^{-16}) after re-estimating SVJD2 model parameters and implicit factor realizations.

The weaker hypothesis that common shocks occur but are transient ($\rho_I \equiv 0$ for $I = 1, ..., 64$) was also strongly rejected, with a constrained log likelihood value of 257,555 and a P-value less than 10^{-16}. The estimated serial correlations were substantially positive for all groups, with a median value of .52. The somewhat lower autocorrelations for 0-3 month than for 3-6 month option residuals probably reflect the dynamics postulated in equation (17). 0-3 month autocorrelation estimates include a time decay component, since the preceding week's estimated residuals are from options with a maturity exactly one week longer. By contrast, the 3-6 month autocorrelation estimates always use the preceding week's 3-6 month estimated option residuals, inducing a "sawtooth" option maturity structure over time.

The substantial and persistent common shock components of SVJD2 residuals suggest that it is possible to do better in terms of cross-sectional fit; through adding more factors, or through an alternate specification. How much better could one reasonably expect to do? While a better option pricing model could conceivably price options perfectly, yielding zero overall RMSE, the existence of idiosyncratic noise from bid-ask bounce, synchronization error, and intra-morning pooling error suggest such a perfect fit is unattainable on this data set. It should, however, be feasible to reduce or eliminate the common shock component ε_t with a better model.

One measure of the best attainable fit comes from various profligately parameterized approaches with daily parameter re-estimation. Examples include the implied binomial trees approach of Dupire (1994), Derman and Kani (1994), and Rubinstein (1994); the constrained cubic splines approach of Bates (1991) and of Section 1; and the 4-parameter jump-diffusion model of Bates (1991). Such approaches impose option-specific no-arbitrage constraints, and therefore constitute valid daily representations of risk-neutral probability densities from a deeper, unspecified data generating process. Bates (1991, Figure 4 and p. 1035) found that the latter two approaches applied to pre-crash intradaily data on S&P 500 futures options of a single 1-4 month quarterly maturity yielded overall RMSE of approximately .04% of the underlying futures prices. Comparable results were achieved over 1988-93 by the constrained cubic spline estimates of Section 1. By this benchmark, the fact that the SVJD2 model has achieved overall RMSE of .078% of the underlying futures price in Table 2 (excluding the serial correlation correction) represents a substantial reduction relative to the .221% overall RMSE of the Black-Scholes model, but there is some room for further improvement.

3.2.2 Interpretation of factor estimates

The linearity of the log moment generating function (10) in implicit factors implies that the risk-neutral cumulants of log-differenced futures prices are also linear functions of implicit factor realizations:[24]

$$K_n = \sum_{i=1}^{2} \left[\frac{\partial^n A_{i,2}^*(T;\ \Phi)}{\partial \Phi^n} + \frac{\partial^n B_{i,2}^*(T;\ \Phi)}{\partial \Phi^n} V_{it} \right]_{\Phi=0} + \lambda_0^* T \left[\frac{\partial^n C_2^*(\Phi)}{\partial \Phi^n} \right]_{\Phi=0} . \qquad \textbf{(21)}$$

K_2 is the implicit risk-neutral variance of log-differenced futures prices over a holding period of length T, $K_3 / K_2^{3/2}$

[24]Cumulants are defined in terms of the coefficients of a Taylor series expansion in Φ of the log moment generating function; see Kendall, Ord, and Stuart (1987, Ch. 3). (21) follows directly.

is the coefficient of skewness, and K_4 / K_2^2 is the coefficient of excess kurtosis. An examination of the cumulant factor loadings in Table 5 for the SVJDC2 model indicates that the two factors play fundamentally different roles.[25] V_1 is a "volatility-and-skewness" factor that heavily affects implicit skewness and leptokurtosis at all maturities through two channels: its almost total determination of the instantaneous risk-neutral jump frequency λ_t^*, and an assessed strong “volatility feedback” channel ($\hat{\rho}_1 = -.848$) that predicts a strong tendency for jump and non-jump risk to rise whenever the market falls. V_2 by contrast primarily affects implicit variances, with relatively little impact upon higher cumulants. Because of the jump risk channel, V_1 has roughly twice the impact of V_2 on instantaneous and longer-maturity expected average conditional variances.

[Table 5 about here]

Model-specific estimates of implicit distributions indicate extremely turbulent conditions in the S&P 500 futures option market over 1988-91, with somewhat quieter conditions over 1991-93 following the end of the Gulf war. The smoothed SVJDC2 estimates of these factors in Figure 5 indicate the major shocks that affected the options market over 1988-93: an 8% intradaily drop in S&P 500 futures prices on January 8, 1988, the mini-crash of October 13, 1989, and the Kuwait crisis from Iraq's invasion on August 2, 1990 through the conclusion of the Gulf war on March 3, 1991. Smaller shocks also appear, such as the Clinton tax increase announced on February 17, 1993. The shocks show up primarily as movements of the V_1 factor, indicating that both variance and higher cumulants are heavily affected.

[25]The derivatives in (21) were computed analytically using Mathematica, and evaluated using the SVJDC2 parameter values in Table 2.

The 2-factor models are able to distinguish between shifts in implicit volatilities that do and do not accompany distributional shifts in higher moments. One striking illustration of this is the evolution in implicit distributions that followed the end of the Gulf war. In Figure 2, at-the-money implicit volatilities fell rapidly from their war-time levels near 30% to a level around 15%, and then declined gradually over the next two years to a level below 10%. One-factor models perforce rely on the single factor to explain all cumulants, and consequently interpret the post-war 1991-93 period as a period of gradually declining higher cumulants. The two-factor estimates in Figure 5 by contrast indicate extremely rapid reduction (a V_1 drop) in higher cumulants following the end of the Gulf war to a relatively stable post-war level, followed by a more gradual reduction in non-jump volatility (a V_2 downward trend) that did not especially affect higher cumulants. Because skewness and excess kurtosis are ratios of powers of cumulants, the post-war assessed declines in non-jump volatility and increasing relative importance of implicit jump risk implies implicit skewness and excess kurtosis steadily increased in magnitude over the post-war 1991-93 period.

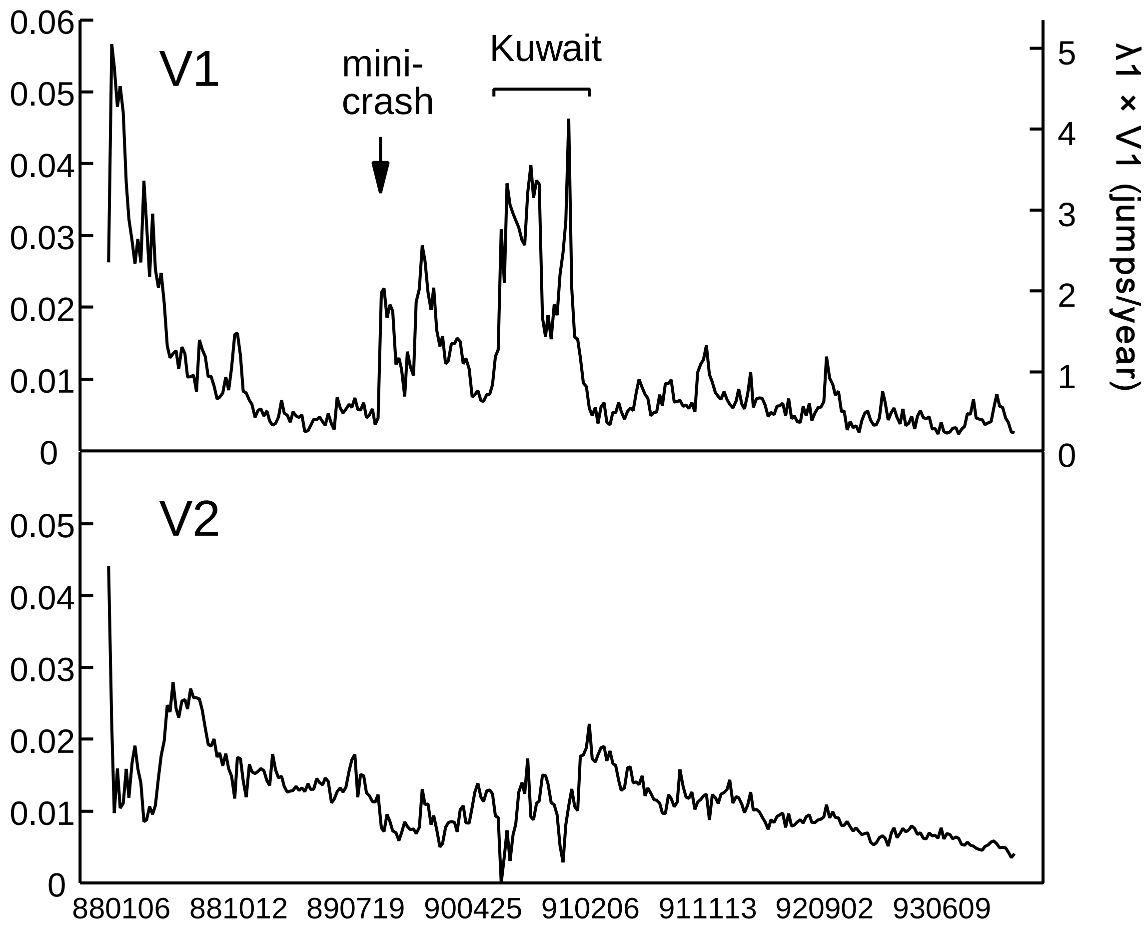

Figure 5. Implicit factor estimates from constrained (smoothed) SVJDC2 model

Implicit conditional variances were quite similar for all two-factor models, indicating that implicit volatilities are relatively insensitive to model specification.[26] The estimates of higher implicit cumulants were more affected by model choice. Nevertheless, SVC2 estimates evolved over 1988-93 similarly to SVJDC2 estimates. Third and fourth cumulants increased synchronously in magnitude with the stock market mini-crashes and the Kuwait crisis, and fell rapidly to a relatively stable post-war level following the end of the Gulf war.

4. Dynamic Implications

The implicit parameter estimates and factor realizations above essentially describe distributions substantially more consistent with post-1987 S&P 500 futures option prices than the lognormal distribution underlying Black-Scholes, and how those implicit distributions have varied over time. While an ability to reduce or eliminate systematic option pricing errors is an important attribute of any option pricing model, such models are also important for their purported ability to predict the future evolution of asset and option prices. The sections below consequently test whether observed S&P 500 futures and futures option prices have in fact evolved consistently with the distributions inferred from option prices under the models considered.

4.1 Tests of the stochastic evolution of option prices

While there is no presumption under jump or stochastic volatility models that a delta-hedged option position will be riskfree, the models do specify the distribution from which option price changes should be drawn. For the postulated stochastic volatility/jump-diffusion process, the stochastic component of call or put option price changes can be roughly decomposed into "moneyness" and "implicit factor" effects:

$$\Delta C = [C(F+\Delta F, \boldsymbol{V}, t+\Delta t) - C(F, \boldsymbol{V}, t)] + C_{\boldsymbol{V}} \Delta \boldsymbol{V} + O(\Delta t) \qquad \textbf{(22)}$$

where $\boldsymbol{V}' = (V_1, V_2)$ and $O(\Delta t)$ captures deterministic terms of order Δt. The "moneyness" effect reflects

[26] Annualized implicit volatilities inferred using the SV2 and SVC2 models averaged roughly 1% higher than DV2 estimates at 1-6 month horizons. Those inferred using the SVJD2 and SVJDC2 models ranged from 1.5% to 0.5% higher than DV2 estimates on average, depending on maturity. Bates (1996c, pp. 589-91) discusses the (limited) impact of specification error on implicit volatility estimation.

the option pricing impact from the option moving deeper in- or out-of-the-money as the underlying asset price changes. The "implicit factor" effect captures how option prices of a standardized moneyness and maturity evolve. Models with time-invariant return distributions such as Black and Scholes (1973) and Merton (1976) attribute all stochastic option price variation to the moneyness effect.

The unconstrained stochastic volatility (SV) and stochastic volatility/jump-diffusion (SVJD) models examined above fit option prices somewhat comparably. Equivalently, the two models yield somewhat similar predictions regarding the risk-neutral distributions of underlying asset prices and option prices at option maturity. However, the models yield quite different predictions regarding option price evolution. The SV models attribute the substantial negative skewness implicit in S&P 500 futures options to highly volatile stochastic volatility factors that typically rise as the market falls. The SVJD models by contrast assign less weight to implicit factor movements and more weight to the moneyness impact of occasional large and predominantly downward changes in the market. Both models predict a substantial negative correlation between moneyness and implicit factor shocks.

In contrasting the two models, therefore, it is useful to single out the second component and examine the models' predictions for the stochastic evolution of option prices of a standardized moneyness and maturity. That evolution is conveniently summarized across options of *all* moneynesses and maturities by the estimates of the stochastic factors V_1 and V_2. Whether those estimates evolve consistently with the postulated square-root diffusions and with the parameters inferred from option prices can be tested. The procedure is analogous to using implicit volatilities inferred daily under Black-Scholes model to describe option price evolution, and testing the Black-Scholes model based upon its prediction that such implicit volatilities should not change over time.

Table 6 contains maximum likelihood estimates of the $\{\alpha_i, \beta_i, \sigma_{vi}\}$ parameters of the $\{V_{it}\}$ processes under the assumption that implicit factors follow mean-reverting square root diffusions. The parameters were estimated using the log-likelihood function (20) for transition densities of discrete-time $\{\ln V_{it}\}$ data, and using factor realizations inferred from option prices under unconstrained and constrained estimation. The estimation procedure is equivalent to estimating the AR(1) processes followed by implicit factors conditional upon factor heteroskedasticity of a particular form. Table 6 also reports the impact on log likelihoods of constraining parameters to the values inferred from option prices, as a test of the consistency between the distributions implicit in option prices and the dynamic evolution of option prices.

[Table 6 about here]

The original estimation of implicit parameters and implicit factor realizations using (19) takes the time series plausibility of the resulting implicit factors into account using likelihood-based weighting. For the 1-factor models, comparison of the implicit factor processes in Table 6 from unconstrained and constrained models (e.g., SV1 versus SVC1) indicates that this plausibility constraint has little impact relative to the objective of matching observed option prices. The constrained and unconstrained factor processes are virtually identical, while the small reductions in σ_v indicate that implicit factor realizations are only slightly smoothed over time.

By contrast, the 2-factor models are heavily affected by the constraint of time series plausibility of implicit factors. The volatilities (σ_v's) of the SVC2 model's implicit factor realizations are substantially reduced relative to SV2's σ_v values in Table 6, and the second factor takes on somewhat of a low-volatility "parameter drift" role. The fact that this smoothing occurs at relatively little cost to the option fits reported in Table 2 suggests that the SV2 model is overfitted, with the two implicit factors responding excessively to trivial daily twitches in option prices. Comparison of the SVJD2 and SVJDC2 implicit factor processes in Table 6 indicates a less pronounced but still substantial impact from the constraint of time series plausibility.

Comparison of implicit and observed drift in factor processes should in principle identify the risk premia (4) on the underlying factors:

$$\hat{\Phi}_{vi} \equiv [E\,dV_i - E^*dV_i]/dt = (\hat{\alpha}_i - \hat{\beta}V_i)_{time\ series} - (\hat{\alpha}_i - \hat{\beta}_i^*V_i)_{options} \qquad \textbf{(23)}$$

For the SVC1 model there are three alternate estimates of expected factor drift, depending upon whether α is or is not constrained by observed option prices:

$$\begin{aligned} E(dV) &= (.120 - 4.62V)dt \quad (\textit{time series}:\ \hat{\alpha}, \hat{\beta} \textit{ from Table } 6) \\ E(dV) &= (.090 - 3.54V)dt \quad (\textit{options \& time series}:\ \hat{\alpha}, \hat{\beta} \textit{ from Table } 2) \\ E^*(dV) &= (.090 - 1.26V)dt \quad (\textit{options}:\ \hat{\alpha}, \hat{\beta}^* \textit{ from Table } 2) \end{aligned} \tag{24}$$

For this model, $\hat{\alpha}_{time\ series} \approx \hat{\alpha}_{options}$, so the greater observed than implicit rates of factor mean reversion ($\beta > \beta^*$) that is also observed for most other models implies an appropriately negative volatility risk premium. However, the divergence between β and β^* is large relative to the log utility calibration based on (5) above.

For other models, the α estimates based on factor processes in Table 6 deviate significantly from the supposedly identical values implicit in option prices, violating a no-arbitrage constraint on risk premia.[27] Furthermore, the risk-neutral factor drift E^*dV inferred essentially from the term structures of option prices appears sensitive to model specification, as is evident in the varying (α, β^*) estimates in Table 2. Consequently, interpreting $\beta^* - \beta$ as a volatility risk premium appears open to question. Alternate explanations such as post-crash expectational error should certainly be kept in mind.

The major difference between risk-neutral and actual factor processes is, however, in divergent estimates of the volatility of volatility parameter σ_v. This divergence is primarily responsible for the extremely strong rejections of option-based parameter constraints in the final column of Table 6. Stochastic volatility models require high values of σ_v to generate substantial implicit skewness and leptokurtosis; and those values are not justified by the observed volatility of factor realizations and standardized option prices. Models with jumps attribute much of the implicit abnormalities to jump risk, and consequently generate smaller and more plausible σ_v values. Nevertheless, those models still rely on distributional randomization to match longer-maturity option prices; and the necessary σ_v values are too large relative to observed factor/option price evolution. Constrained estimation using (19) to induce time series plausibility upon

[27] See Ingersoll (1987), chapter 18. Different α's could be justified as a locally linear approximation to a nonlinear volatility risk premium that appropriately attains zero when $V_i = 0$.

implicit parameter estimates cannot reconcile a model-specific incompatibility between how options are priced and how option prices evolve.

Some intuition for the implausibility of the σ_v values from the 1-factor SV1/SVC1 models is generated by recognizing that $\frac{1}{2}\sigma_v$ should roughly be the volatility of first-differenced Black-Scholes implied volatilities from at-the-money options. Black-Scholes implied variances are roughly equal to expected average variances,[28] which are linear functions of implied factors. By Ito's lemma,

$$d(\bar{V})^{\frac{1}{2}} = O(dt) + \tfrac{1}{2}\sigma_v \sqrt{w(T)}\, dW_v \tag{25}$$

where the variance factor loading $w(T) \equiv \partial\bar{V}/\partial V = (1 - e^{-\beta^* T}) / \beta^* T$ is roughly 1 given β^* estimates. Consequently, the SVC1 σ_v value of .693 implies roughly a 34.7% per annum volatility for changes in Black-Scholes implied volatilities -- a weekly standard deviation of 4.8%. The implied volatilities in Figure 2 have not been that volatile.

A more general diagnostic for the misspecification of the factor processes is generated by "normalizing" factor transitions using the monotonic transformation

$$z_{t+1} \equiv N^{-1}\Big[P_V(V_{t+1} \mid V_t;\ \hat{\alpha},\ \hat{\beta},\ \hat{\sigma}_v\ ) \Big],\ t = 1,\ ...,\ 309, \tag{26}$$

where $P_V(V_{t+1} \mid V_t;\ \bullet)$ is the conditional distribution function of factor transitions and $N^{-1}(\bullet)$ is the inverse of the cumulative normal distribution function. If the conditional distribution function is correctly specified with correct parameters, the z's should be independent and identically distributed draws from a normal $N(0, 1)$ density -- a testable hypothesis. Conversely, if the conditional distribution is *not* correctly specified, analysis of the z's summarizes the overall misspecification of conditional distributions.[29]

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[28]See Bates (1996c, p. 590-91) for a discussion of this relationship.

[29]This transformation has been used independently by Palm and Vlaar (1997), who also cite Pearson (1933) and Smith (1985). It is related to the "calibration" approach of Fackler and King (1990) and Silva and Kahl (1993), who use the fact that realized distribution function values should be independently and uniformly distributed under correct model specification (Pearson, 1933). The additional transformation into normally distributed residuals appears desirable for highlighting outliers and for permitting use of

The upper histogram in Figure 6 illustrates the extent of the misspecification of the 1-factor constrained stochastic volatility (SVC1) model under parameters inferred predominantly from option prices. Factor realizations are substantially less volatile than predicted by implicit σ_v values, yielding a concentration of normalized residuals well within the standard normal curve. This distribution is also reflected in the summary quartile values in the upper half of Table 7. 50% of the normalized factor changes from the SVC1 model fall within a narrow [-.08, .30] range, whereas they should be spread over a ±.67 range if the square root process and the implicit parameters of Table 2 correctly captured the distribution of implicit factor changes. While most pronounced for the SVC1 model, the excessively narrow quartile ranges in the upper half of Table 7 indicate similar problems with overestimation of σ_v values for other models.

[Table 7 about here]

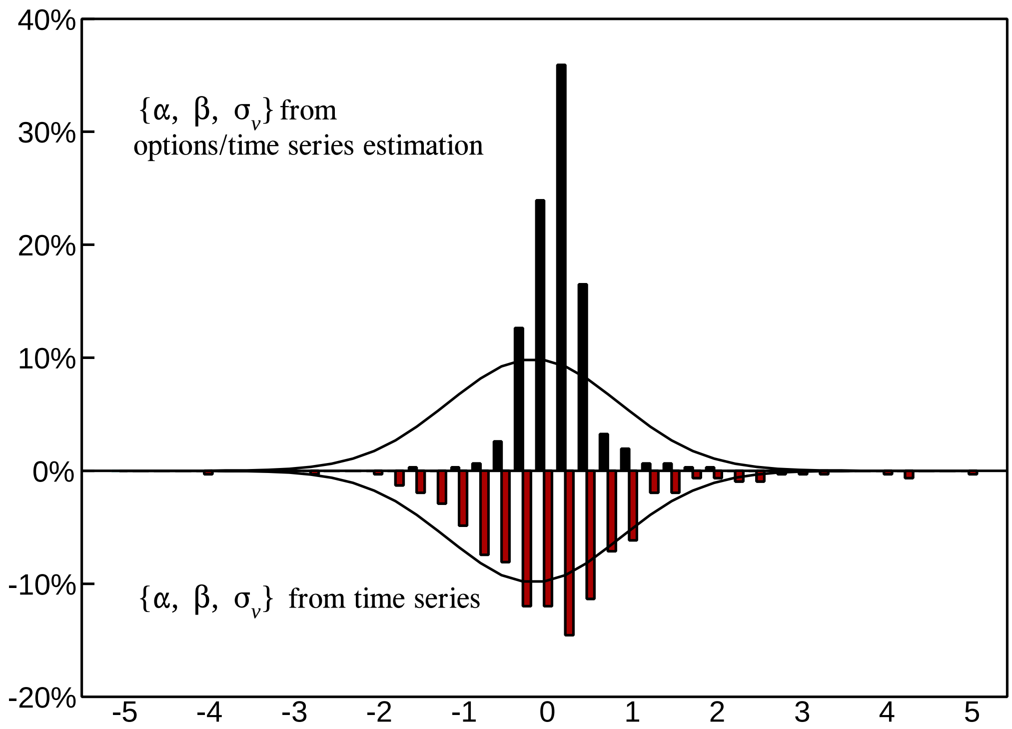

Figure 6. Distribution of "normalized" implicit factor shocks: 1- factor constrained stochastic volatility model (SVC1).

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standard normality tests. If V_{t+1} were drawn from a conditionally normal distribution, the transformation (26) would be equivalent to the standard normalization $\tilde{z}_{t+1} = [V_{t+1} - E_t(V_{t+1})] / \sqrt{Var_t(V_{t+1})}$.

The constrained two-factor models have even greater difficulties than the one-factor models in generating factor sample paths consistent with the postulated process and the implicit parameters. A major outlier for these models is August 8, 1990, when Iraq's invasion of Kuwait created a pronounced inversion of the term structure of implicit volatilities that lasted less than a week. The two-factor models capture this through large gyrations in implicit factors, including factor reflections off the barrier at zero, that are possible but highly implausible under diffusion assumptions.

Normalizing implicit factor transitions using instead the time series-based parameter estimates of Table 6 reveals that the postulated square root process for implicit factor evolution is fundamentally misspecified. While fitting such a process to implicit factor realizations successfully centers the distribution and captures much of the 50% central probability mass (Figure 6, lower histogram and Table 7, lower half quartile values), there are far too many large outliers. These outliers are almost all positive for the one-factor models, and correspond to the sharp increases in implicit volatilities evident in Figure 2 that accompanied events such as Kuwait-related shocks and the mini-crashes of January 8, 1988 and October 13, 1989.[30] Two-factor models' outliers are more complicated, and include occasional substantial shifts in the relative importance of the two factors -- most notably that of August 8, 1990. The high improbability of those outliers given diffusion assumptions indicates that the true conditional transition densities are far more leptokurtic than hypothesized, and suggests that the factors underlying option prices follow jump processes.

A potential criticism of using implicit factors to describe option price evolution is an "errors-in-variables" problem addressed in Merville and Pieptea (1989): noisy option prices introduce transient noise into factor estimates. However, the Kalman filtration methodology used here takes idiosyncratic noise in option prices explicitly into account. Smoothing also reduces transient implicit factor shocks somewhat,

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[30]For instance, the +4.9 outlier in the lower half of Figure 5 occurred during October 11-18, 1989, when instantaneous conditional volatility estimates rose from 13.7% to 22.3%. This move is probabilistically equivalent to observing a 4.9 standard deviation move in a standard normal random variable -- something that should occur only once every 40,000 years if the diffusion specification and parameter estimates were correct. The one large negative outlier in the lower half of Figure 6 occurred on January 16-23, 1991, following the start of the Gulf war, and reversed the previous week's increase.

although implicit factor realizations are still primarily determined by what best fits observed option prices. Furthermore, the impact of any remaining transient shocks to implicit factors is to bias *upwards* estimates of the volatility of implicit factor changes. The fundamental problem for the square root diffusion specification is that the time series-based estimates of σ_v are too low relative to the values inferred from option prices, and relative to the size of the largest shocks to implicit factors.

4.2 Tests of consistency with the time series properties of futures prices

The central empirical question is whether the substantial negative implicit skewness that appeared in stock index options following the crash was in fact justified by the post-crash distribution of S&P 500 futures prices. The unconditional moments reported in Table 8 suggest that it was not. The unconditional distribution of weekly log-differenced short-maturity futures prices from contracts with at least 1 week to maturity was indistinguishable from a normal distribution both before and after the crash. There was somewhat more evidence of greater post-crash abnormalities from daily returns. However, while there were four large daily moves over 1988-93 of 4-6% in magnitude that might be interpreted as jumps,[31] these returns are not especially compatible with the large negative means (-9.5%, -5.7%) and substantial jump standard deviations (10-11%) inferred from option prices under the 1- and 2-factor SVJD models.

[Table 8 about here]

One possible explanation is a "peso problem:" options may have incorporated fears of a 1987-like rare event that did not recur over 1988-93. But according to the SVJD models' implicit parameter estimates, there *should* have been additional large moves observed over 1988-93. For instance, the risk-neutral probability of observing a weekly futures price move greater than 10% in magnitude over 1988-93 is

$$1 - \prod_{t=1}^{310} Prob^* \left[-10\% < \ln[F(t+1\,week)/F(t)] < 10\% \mid \hat{\boldsymbol{V}}_t,\ \hat{\theta} \right], \qquad \textbf{(27)}$$

where $Prob^*(\bullet)$ can be computed using the distribution function $1 - P_2$ from equation (9) at weekly horizons.

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[31]On a noon-to-noon basis, the four largest outliers were January 8-11, 1988 (-5.8%), October 13-16, 1989 (-4.8%), August 24-27, 1990 (+5.0%), and September 28 - October 1, 1990 (+4.4%).

No such move was observed, despite an assessed 90% probability according to risk-neutral distributions inferred under the SVJD 1- and 2-factor models. It is conceivable that the divergence reflects a jump risk premium. However, assuming jumps occur at only half the rate inferred from option prices ($\lambda_t = ½ \lambda_t^*$) still implies a 70% probability of observing a 10% weekly move over 1988-93.[32] Figure 5 (right-hand scale) shows that the risk-neutral jump risk $\lambda_t^* \approx \lambda_1^* V_{1t}$ was highest during the Kuwait crisis, and following the mini-crashes of January 8, 1988 and October 13, 1989

The SV models generate less abnormal weekly conditional distributions, and predict only a 40-45% probability of observing a 10% move over 1988-93. However, the U-shaped 0-1 month option pricing errors reported in Table 3 indicate that implicit SV2 distributions underestimate the short-horizon risk-neutral probabilities of large moves. The SVJD2 model does much better at fitting 0-1 month option prices and implicit risk-neutral distributions.

To examine whether implicit distributions contain *any* information for conditional distributions, including higher moments, the short-maturity (0-3 month) futures price process was modeled as in (2). The conditional mean μ was specified as

$$\mu_t \equiv E(dF/F)/dt = c_0 + c_1 r_t + c_2 y_t + c_{v1} V_{1t} + c_{v2} V_{2t} \qquad \textbf{(28)}$$

where r_t is the preceding day's (Tuesday's) 3-month Treasury bill yield and y_t is the previous day's implicit dividend yield from synchronous futures prices of different maturities.[33] The last two terms generate instantaneous "GARCH-in-mean" effects, although higher moments are also affected in discrete time.

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[32]If one-half of investors' wealth is exposed to equity jump risk, the log utility calibration in equation (6) indicates the true jump frequency should be 95 - 97% of the risk-neutral jump frequency, given implicit $\bar{k}^*$ and δ values.

[33]The average cost-of-carry was computed as $COC = \frac{1}{N}\sum_{n=1}^{N} \ln[F_n^{MT}/\bar{F}_n^{ST}]$, where $\bar{F}_n^{ST}$ was the average of all short-term futures prices observed within a ±20 second window around the corresponding medium-term futures price F_n^{MT}. The shortest two futures maturities available were used. The implicit dividend yield is $y_t = r_t - COC_t$.

The futures data were short-maturity (typically 0-3 month) noon quotes on Wednesdays for which there were options data available. The typical time interval was therefore one week, although holidays twice induced a longer time interval. To avoid maturity shifts, the futures contract maturity was the shortest maturity such that futures contracts with identical delivery date existed at the next available Wednesday.

To examine whether implicit variances are biased forecasts of future variance, the instantaneous variance conditional upon no jumps was modeled as a linear transformation of the factor realizations inferred from option prices:

$$Var_t(dF/F) = V_0 + d_1 V_{1t} + d_2 V_{2t} \tag{29}$$

where V_0, d_1, and d_2 are constants. Similarly, the actual jump frequency was modeled as a linearly transform of the implicit jump frequency, and therefore of the implicit factors:

$$\begin{aligned} \lambda_t &= l_0 + l_1 \lambda_t^* = (l_0 + \lambda_0^*) + (l_1 \lambda_1^*) V_{1t} + (l_2 \lambda_2^*) V_{2t} \\ &\equiv \lambda_0 + \lambda_1 V_{1t} + \lambda_2 V_{2t} . \end{aligned} \tag{30}$$

This second transformation has two purposes. First, l_1 can conceivably be different from 1 because of a jump risk premium. Second, using the two linear transforms (29) and (30) allow separate consideration of the informational content of implicit factors for conditional variances, and the informational content for higher moments. The resulting log-likelihood function of log-differenced futures prices can be computed by Fourier inversion and optimized over the parameter space $\theta_F = \langle l_0, l_1, \bar{k}, \delta, c_0, c_1, c_2, c_{v1}, c_{v2} \rangle$ *conditional* upon variance process parameters $\theta_{SV} = \langle \alpha_i, \beta_i^*, \sigma_{vi}, \rho_i \rangle$ and factor realizations $\{\boldsymbol{V}_t\}$ inferred from option prices:

$$\begin{aligned} \ln L_{\{F\}}(\theta_F) &= \sum_{t=2}^{310} \ln p_F[\ln(F_t/F_{t-1}) \mid \boldsymbol{V}_{t-1}, \theta_F, \theta_{SV}, \Delta t] \\ &= \sum_{t=2}^{310} \ln\left[\frac{1}{\pi}\int_{\Phi=0}^{\infty} Real\left[\varphi_F(i\Phi; \Delta t)\, e^{-i\Phi\ln(F_t/F_{t-1})}\right] d\Phi\right] . \end{aligned} \tag{31}$$

The relevant log moment generating function for predominantly weekly log-differenced futures prices is

$$\ln \varphi_F(\Phi,\ \Delta t) \;=\; (c_0 + c_1 r_t + c_2 y_t)\,\Phi\,\Delta t \;+\; \tfrac{1}{2} V_0 \Phi^2 \Delta t \;+\; \lambda_0 \Delta t\, C_2(\phi) \\ +\; \sum_{i=1}^{2} \left[A_{i,\,2}(\Delta t;\ \Phi) \;+\; B_{i,\,2}(\Delta t;\ \Phi)\ (d_i V_i) \right], \tag{32}$$

where A, B, and C are essentially the same as in equations (11)-(14) but with actual rather than risk-neutral parameters.[34]

Maximum-likelihood estimates of the optimal linear transforms of implicit factors and other parameter estimates are presented in Table 9. Variables affecting the conditional mean exhibited no significant ability to forecast futures prices (R^2 typically around 1%), and are consequently not shown. By contrast, implicit factors generated informative but biased forecasts of subsequent S&P 500 volatility, with d_i estimates significantly positive but significantly less than 1. The biases were most pronounced for implicit variances from the stochastic volatility models; more so than just using Black-Scholes (BS) implied volatilities to forecast future weekly returns. Log likelihoods indicate that 2-factor models do somewhat better than 1-factor models and that SV and SVJD models do somewhat better than BS in forecasting return distributions. The hypotheses are not nested, however, precluding formal tests.

[Table 9 about here]

There is virtually no evidence that implicit conditional weekly distributions contained any information over 1988-93 regarding abnormalities in S&P 500 futures returns -- not surprising given unconditional weekly returns were essentially normally distributed. Unconstrained estimates of l_1 of approximately zero indicate no ability of implicit factors to identify when large moves are likely to occur, beyond the information contained in conditional variances. The infrequent *positive* jump component estimated under the SVJD1 and SVJD2 unconstrained models is an artifact of the slight negative skewness in weekly conditional distributions induced by "volatility feedback" effects ($\rho_i < 0,\ \sigma_{vi} > 0$) inferred from option prices, which makes the

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[34]The higher-moment effects of c_{vi} involve replacement of $(3 - 2j)\Phi$ in (13)-(14) by $(2c_{vi} - 1)\Phi$; see Bates (1996b, equations 34-35). Since β_i values are not inferable from option prices, implicit β_i^* parameters were used instead. All β and β^* estimates in Tables 2 and 6 imply relatively long half-lives of volatility mean reversion (months or years), indicating weekly futures return distributions are insensitive to the distinction between β and β^*.

largest positive weekly return of 5.5% look like an outlier.

4.4 Correlation tests

While it is possible using Fourier inversion techniques to evaluate and estimate joint transition densities of S&P 500 futures prices and factor realizations, the inconsistencies evident above between implicit and observed marginal densities suggests little to be gained from the exercise. However, both the stochastic volatility and stochastic volatility/jump-diffusion models attribute some of the negative implicit skewness to negative correlations between market and volatility shocks. That there exists such a correlation between stock market returns and *actual* conditional volatility changes is fundamental to the EGARCH approach of Nelson (1991), while corresponding negative correlations between returns and *implicit* volatility changes have been found for individual stocks (Schmalensee and Trippi, 1978) and for the British stock market (Franks and Schwartz, 1991). Simple correlation computations on weekly data reported in Table 10 confirm that the underlying assumption of substantial negative correlations between S&P 500 futures returns and implicit factor changes are in fact observed.

[Table 10 about here]

In the 2-factor models, the correlation is most pronounced for innovations in the factor (V_1) that most influences higher cumulants -- a result qualitatively consistent with implicit ρ_1 estimates. Consequently, the typical option price shift accompanying substantial market drops was higher implicit downside risk -- not just through higher implicit volatilities, but through higher implicit cumulants as well. By contrast, the substantial run-up in the market over 1991-93 was largely accompanied by declining assessments of non-jump volatility and correspondingly lower downside risk; see Figure 4. Judging from option prices, market participants did *not* view the stock market as overvalued and more prone to crash following the run-up; quite the contrary. It appears that crashes accompanied increased crash fears in the S&P 500 futures options market, while an absence of crashes reduced crash fears to an assessed biannual frequency shown in Figure 5.[35]

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[35]The experience of other countries appears to be quite different from that of the U.S. Gemmill (1996) found little change in implicit skewness from British stock index options following the British stock market crash in 1987, while Trautmann and Beinert (1994) found that German stock options exhibited

The two-factor model was premised upon the assumption of independent factors, whereas innovations in the two unconstrained implicit factor estimates are in fact substantially negatively correlated: -.58 for the SV2 model, -.50 for the SVJD2 model. This is symptomatic of overfitting, with implicit factors responding excessively to minor twitches in option prices. The correlations between factor innovations are less pronounced for the constrained estimates (-.157 and -.325 for SVC2 and SVJDC2, respectively), in which the second factor plays more of a "parameter drift" role. These reduced correlations between smoothed factor innovations suggest that likelihood-based smoothing can mitigate overfitting problems that arise in multifactor models with a large number of free parameters.

5. Summary and conclusions

This article has presented evidence that post-'87 distributions implicit in S&P 500 futures options are strongly negatively skewed, and has examined two competing hypotheses: a stochastic volatility model with negative correlations between index and volatility shocks, and a stochastic volatility jump-diffusion model with time-varying jump risk. The fundamental premise underlying the stochastic volatility model is confirmed: index and implicit volatility shocks are in fact strongly negatively correlated. However, this negative correlation is not sufficient of itself to generate sufficiently negative implicit skewness. An extremely high volatility of volatility is also necessary -- implausibly high when judged against the time series properties of option prices. The stochastic volatility model also has some difficulty in matching observed option prices even with implausible parameter values, with an tendency to underprice 0-2 month options with low strike prices and overprice 2-6 month options with high strike prices.

This article has also presented strong evidence against the hypothesized square root diffusion processes driving instantaneous volatility and jump risk. Such processes possess many desirable features (nonnegativity, leverage effects, analytic tractability), but cannot account for the large and typically positive implicit volatility shocks observed in the S&P 500 futures options market. Implicit factor evolutions appear better described by an asymmetric jump-diffusion. Whether such behavior is also observed for conditional volatilities of S&P

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increased *positive* skewness (rebound expectations) following the German 1987 crash.

500 futures returns is an open question. Standard GARCH models substantially rule out conditional volatility jumps, while regime-switching models typically assume *all* volatility changes are jumps.

The stochastic volatility/jump-diffusion model is more compatible with observed option prices, and generates more plausible stochastic volatility parameter values. However, the model still relies on a substantial amount of distributional randomization to match longer-maturity option prices; more so than justified by the observed volatility of implicit factors. The difficulty is that the volatility smile is equally pronounced for short- and long-maturity options, after scaling by the appropriate standard deviation at different maturities. Finite-variance jump explanations cannot match this pattern, because of rapid convergence towards lognormal distributions at longer maturities. Alternate infinite-variance processes that don't have this property are consequently worth exploring, such as McCulloch's (1995) stable Paretian model.

The substantial negative skewness and leptokurtosis implicit in actively traded short-maturity option prices appear fundamentally inconsistent with an absence of large weekly movements in S&P 500 futures returns over 1988-93. This incompatibility is most apparent in implicit risk-neutral distributions inferred from models with jumps, which fit short-maturity option prices quite well on average. These models assign a 90% risk-neutral probability of observing at least 1 weekly move of 10% in magnitude over 1988-93; none was observed. And while the stochastic volatility models yield a lower probability of large moves over 1988-93, the systematic option pricing errors of these models indicates they are understating the implicit risk-neutral probabilities of large moves at short horizons.

Alternate explanations for the divergence between risk-neutral distributions and observed returns include peso problems, risk premia, and option mispricing. A log utility representative agent calibration indicates that peso problems alone cannot explain the divergence; large moves *should* have been observed over 1988-93 according to distributions inferred from option prices. However, more extreme risk aversion would presumably decrease the true/risk-neutral jump frequency ratio to the point where a peso problem explanation becomes viable. Whether this can be achieved under plausible levels of risk aversion is an open question.

A more promising explanation lies in the industrial organization of stock index option markets. These markets have been functioning since the crash substantially as an insurance market for crash risk. Relatively few option marketmakers apparently have been writing crash insurance for a broad array of money managers, which may pose institutional difficulties for the risk-sharing assumptions underlying representative agent models. On the demand side, it is conceivable that especially risk-averse money managers have been willing to buy crash insurance that never seems to pay off. The puzzle is on the supply side. Why have new entrants not undercut the marketmakers, given no formal barriers to entry in option markets? Or, alternatively, why have existing marketmakers not devoted more capital to the profitable business of writing options? Perhaps the largest profit opportunities did not last long enough for new entry. While implicit crash fears were present throughout 1988-93, the largest implicit abnormalities declined substantially with the end of the Gulf war.

Appendix A: Constrained Cubic Splines

An unconstrained cubic spline fitted through observed option price/future price ratios $y_j \equiv O_j / F_j$ as a function of observed strike price/futures price ratios $x_j = X_j / F_j$ is of the form

$$y(x) = a_i + b_i x + c_i x^2 + d_i x^3 \qquad for\ \bar{x}_i \leq x < \bar{x}_{i+1} \tag{A.1}$$

where

$\bar{x}_1 = \min\{x\} \equiv x_{\min}$,

$\bar{x}_i$ for $i = 2, ..., N$ is the average *X/F* ratio observed for option transactions in the *i*th strike class, and

$\bar{x}_{N+1} = \max\{x\} + \varepsilon = x_{\max} + \varepsilon$, for arbitrarily small ε (e.g., $x_{\max} \times 10^{-8}$).

For $N+1$ strike classes, there are $N-1$ interior breakpoints and N cubic polynomials. The *4N* cubic coefficients are constrained by the linear restrictions that $y(x)$, $y'(x)$, and $y''(x)$ be continuous across breakpoints:

$$\begin{pmatrix} 1 & \bar{x}_{i+1} & \bar{x}_{i+1}^2 & \bar{x}_{i+1}^3 \\ 0 & 1 & 2\bar{x}_{i+1} & 3\bar{x}_{i+1}^2 \\ 0 & 0 & 2 & 6\bar{x}_{i+1} \end{pmatrix} \begin{pmatrix} a_{i+1} \\ b_{i+1} \\ c_{i+1} \\ d_{i+1} \end{pmatrix} = \begin{pmatrix} 1 & \bar{x}_{i+1} & \bar{x}_{i+1}^2 & \bar{x}_{i+1}^3 \\ 0 & 1 & 2\bar{x}_{i+1} & 3\bar{x}_{i+1}^2 \\ 0 & 1 & 2 & 6\bar{x}_{i+1} \end{pmatrix} \begin{pmatrix} a_i \\ b_i \\ c_i \\ d_i \end{pmatrix} \tag{A.2}$$

for i = 1, ..., $N-1$. Thus, for $N+1$ strike classes there are $4N - 3(N-1) = N+3$ free parameters, which for maximum convenience were chosen to be $\Theta = [a_1,\ b_1,\ c_1,\ d_1,\ d_2,\ \ldots,\ d_N]'$.

Cubic splines are typically fitted using the assumption that actual and spline values differ by additive homoskedastic white noise:

$$y_j = \boldsymbol{X}_j{}'\Theta + \eta_j, \; \eta_j \sim (0, \sigma^2) \qquad \textbf{(A.3)}$$

where

$$\boldsymbol{X}_j \equiv \frac{\partial y(x_j)}{\partial \Theta} = \frac{\partial a_i}{\partial \Theta} + \frac{\partial b_i}{\partial \Theta} x_j + \frac{\partial c_i}{\partial \Theta} x_j^2 + \frac{\partial d_i}{\partial \Theta} x_j^3 \qquad \textbf{(A.4)}$$

is an $(N+3) \times 1$ vector, $x_j \in [\bar{x}_i, \bar{x}_{i+1})$, and $\partial a_i / \partial \Theta, \ldots, \partial d_i / \partial \Theta$ are $(N+3) \times 1$ vectors with constant coefficients, computed most simply by computing $[a_i, b_i, c_i]$ recursively from (A.2) and taking numerical derivatives. The parameters Θ_t that minimized root mean squared error

$$\sigma(\Theta_t) = \sqrt{\frac{\sum_j [y_j - \hat{y}(x_j \mid \Theta_t)]^2}{NOBS_t - N - 3}} \qquad \textbf{(A.5)}$$

for a given day's data were estimated using GQOPT subroutine DFP, yielding the ordinary least squares estimate $\hat{\Theta} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{y}$. Point estimates of interpolated values from the unconstrained cubic spline were given by $\hat{y}(x_j) = \boldsymbol{X}_j{}'\hat{\Theta}$ with associated standard error $\sigma(\hat{\Theta})[\boldsymbol{X}_j{}'(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}_j]^{1/2}$. Separate cubic splines were of course fitted to call data and to put data.

Roughly one-half of the cubic splines estimated in this fashion mildly violated arbitrage-based constraints on option prices somewhere within the domain $[x_{min}, x_{max}]$. These violations were typically in interior regions where no transactions were observed, and consequently did not represent actual arbitrage opportunities. In cases where such violations were observed, $N+5$ additional linear Kuhn-Tucker constraints were imposed on the $4N$ cubic coefficients to ensure fitted values satisfied no-arbitrage constraints.

1. *Convexity*. Since the second derivative $y''(x)$ is continuous and piecewise linear, convexity was ensured by $N+1$ nonnegativity constraints at endpoints and breakpoints:

$$2c_i + 6\bar{x}_i d_i \geq 0, \quad i = 1, \ldots, N$$
$$2c_N + 6\bar{x}_{N+1} d_i \geq 0. \tag{A.6}$$

2. *Monotonicity with slope between 0 and 1 in absolute value.* Given (A.6), two endpoint restrictions sufficed:

$$b_1 + 2c_1 x_{\min} + 3d_1 x_{\min}^2 \geq -1 \textit{ for calls}, \geq 0 \textit{ for puts}$$
$$b_1 + 2c_1 x_{\max} + 3d_1 x_{\max}^2 \leq 0 \textit{ for calls}, \leq 1 \textit{ for puts}. \tag{A.7}$$

3. *Intrinsic value constraints*. Given (A.6) and (A.7), two endpoint restrictions sufficed:

$$a_1 + b_1 x_{\min} + c_1 x_{\min}^2 + d_1 x_{\min}^3 \geq f(x_{\min})$$
$$a_1 + b_1 x_{\max} + c_1 x_{\max}^2 + d_1 x_{\max}^3 \geq f(x_{\max}) \tag{A.8}$$

where $f(x) = \max(1 - x,\ 0)$ for calls, $\max(x - 1,\ 0)$ for puts.

The cubic splines satisfying these constraints that minimized (A.5) were estimated using the constrained optimization subroutine CONOPT in GQOPT. Imposing the constraints when necessary increased standard errors by only 0.004% of the futures price on average.

Computing standard errors for an interpolated option value $\hat{y}(x_j) = \boldsymbol{X_j}'\hat{\Theta}$ when one or more Kuhn-Tucker constraints is expected to bind with estimated probability ½ is difficult. Since the constraints did not bind severely, in the sense that the constrained and unconstrained fits were similar, standard errors were estimated using an appropriately scaled version of the unconstrained estimated standard errors:

$$SD[\hat{y}(x_j)] = \hat{\sigma}[\boldsymbol{X}_j'(\boldsymbol{X}_j'\boldsymbol{X}_j)^{-1}\boldsymbol{X}_j]^{1/2} \tag{A.9}$$

where $\hat{\sigma}$ was the root mean squared error from the constrained (unconstrained) cubic spline when some (no) constraint was binding.

Appendix B. Estimating the dynamic properties of option residuals

Conditional upon particular estimates $<\{\boldsymbol{V}_t\},\ \theta>$ of the factor realizations and implicit parameters, option pricing residuals are assumed to satisfy

$$\begin{cases} e_{i,t} = \varepsilon_{I,t} + \sigma_I \eta_{i,t} & \text{for } i \in G(I,t) \\ \varepsilon_t = \boldsymbol{F}_{t-1}\varepsilon_{t-1} + \upsilon_t \end{cases} \tag{B.1}$$

where

$\eta_{i,t} \sim N(0, 1)$ is an idiosyncratic shock to transaction i at time t, uncorrelated with all other shocks;

$G(I, t)$ is the set of residuals in group I at time t;

ε_t is the N-dimensional vector of common shocks, with Ith entry $\varepsilon_{I,t}$;

$\boldsymbol{F}_t = \boldsymbol{D}_\rho \boldsymbol{A}_t$ is the product of an $N \times N$ diagonal matrix $\boldsymbol{D}_\rho$ with serial correlations $\rho = \{\rho_I\}$ along the diagonal, and a matrix $\boldsymbol{A}_t$ of ones and zeros that captures the assumed maturity-related serial dependency of common shocks; and

υ_t is a mean-zero, normally distributed vector with $E_{t-1}\upsilon_t\upsilon_t' = \boldsymbol{Q}$ for positive semidefinite **Q**.

It is useful to orthogonalize option residuals by dividing into group-average and deviation from group-average components:

$$e_{i,t} = \bar{e}_{I,t} + u_{i,t} \tag{B.2}$$

where $\bar{e}_{I,t} = \frac{1}{N_{I,t}} \sum_{i \in G(I,\ t)} e_{i,t} \sim N\left(\varepsilon_{I,t},\ \sigma_I^2 / N_{I,t}\right)$ is a reduced-noise signal that summarizes all relevant date t, group I information about the level of the underlying common shocks. The precision of the signal varies observably with the group-specific number of observations $N_{I,t}$; frequently no information is available for particular groups. The deviations from group-average $u_{i,t}$ collected into the vector $\boldsymbol{u}_t$ are useful in identifying the magnitudes of idiosyncratic noise σ_I but are otherwise orthogonal to the Kalman filtration. The covariance matrix $\boldsymbol{S}_t = E\,\boldsymbol{u}_t\boldsymbol{u}_t'$ is block-diagonal, and depends only upon $\{\sigma_I,\ N_{I,t}\}$.

Let $\boldsymbol{x}_t$ represent the n_t-sized subset of $\bar{\boldsymbol{e}}_t$ observed on date t, where the number $n_t \le N$ of groups represented changes constantly over time. Estimating the parameters $<\{\rho_I, \sigma_I\}_{I=1}^{64}, \boldsymbol{Q}>$ of (B.1) is a standard Kalman filtration problem with missing observations. Let $\hat{\varepsilon}_{t|s}$ and $\boldsymbol{P}_{t|s}$ be the mean and variance of the unobserved vector ε_t conditional on information through time *s*. By linear projection, the observed $\boldsymbol{x}_t$ can be used to update the conditional distribution of ε_t:

$$\begin{aligned} \hat{\varepsilon}_{t|t} &= \hat{\varepsilon}_{t|t-1} + \boldsymbol{P}^{\varepsilon x}_{t|t-1} (\boldsymbol{P}^{xx}_{t|t-1} + R^{xx}_t)^{-1} (\boldsymbol{x}_t - \hat{\boldsymbol{x}}_{t|t-1}) \\ \boldsymbol{P}_{t|t} &= \boldsymbol{P}_{t|t-1} - \boldsymbol{P}^{\varepsilon x}_{t|t-1} (\boldsymbol{P}^{xx}_{t|t-1} + R^{xx}_t)^{-1} \boldsymbol{P}^{x\varepsilon}_{t|t-1} \end{aligned} \qquad \textbf{(B.3)}$$

where

$\boldsymbol{P}^{\varepsilon x}_{t|t-1} = Cov_{t-1}(\varepsilon_t, \boldsymbol{x}_t)$ is an $N \times n_t$ matrix consisting of columns of $\boldsymbol{P}_{t|t-1}$ corresponding to observed $\boldsymbol{x}_t$;

$\boldsymbol{P}^{x\varepsilon}_{t|t-1}$ is its transpose;

$\boldsymbol{P}^{xx}_{t|t-1}$ is an $n_t \times n_t$ submatrix of $\boldsymbol{P}_{t|t-1}$ based on selecting rows and columns corresponding to observed $\boldsymbol{x}_t$;

$\boldsymbol{R}^{xx}_t$ is an $n_t \times n_t$ diagonal matrix with $\boldsymbol{x}$-specific diagonal entries $\sigma_I^2 / N_{I,t}$; and

$\hat{\boldsymbol{x}}_{t|t-1} = \hat{\varepsilon}^{x}_{t|t-1}$ is the $n_t \times 1$ $\boldsymbol{x}$-specific subvector of $\hat{\varepsilon}_{t|t-1}$.

If there were no idiosyncratic noise ($\boldsymbol{R}^{xx}_t = 0$), the $\boldsymbol{x}$-specific components of $\hat{\varepsilon}_{t|t}$ would be known exactly, and the corresponding rows and columns of $\boldsymbol{P}_{t|t}$ would be zero. However, it would still be necessary to estimate unobserved components of ε_t based on the conditional covariance structure in (B.3).

The conditional distribution of next period's ε_{t+1} is given by

$$\begin{aligned} \hat{\varepsilon}_{t+1|t} &= \boldsymbol{F}_t \, \hat{\varepsilon}_{t|t} \\ \boldsymbol{P}_{t+1|t} &= \boldsymbol{F}_t \boldsymbol{P}_{t|t} \boldsymbol{F}_t' + \boldsymbol{Q} \end{aligned} \qquad \textbf{(B.4)}$$

while next period's observed group-average residuals $\boldsymbol{x}_{t+1}$ are conditionally distributed $N[\hat{\varepsilon}^{x}_{t+1|t}, \boldsymbol{P}^{xx}_{t+1|t} + \boldsymbol{R}^{xx}_{t+1}]$. The log-likelihood of observed option pricing residuals is consequently

$$\ln L_{options} = -\frac{1}{2}\sum_t \Big[\ln|\boldsymbol{P}^{xx}_{t|t-1} + \boldsymbol{R}^{xx}_t| + (\boldsymbol{x}_t - \hat{\boldsymbol{x}}_{t|t-1})'(\boldsymbol{P}^{xx}_{t|t-1} + \boldsymbol{R}^{xx}_t)^{-1}(\boldsymbol{x}_t - \hat{\boldsymbol{x}}_{t|t-1}) + \ln|\boldsymbol{S}_t| + \boldsymbol{u}_t'\boldsymbol{S}_t^{-1}\boldsymbol{u}_t \Big] \qquad \textbf{(B.5)}$$

where $\boldsymbol{P}_{1|0}$, the unconditional covariance matrix of ε_1, depends upon ρ and $\boldsymbol{Q}$,[23] and $\hat{\varepsilon}_{1|0} = \mathbf{0}$.

The log likelihood function could in principle be optimized with regard to the option pricing parameters $<\{V_t\}, \theta>$ and the parameters $<\{\rho_I, \sigma_I\}_{I=1}^{64}, \boldsymbol{Q}>$ governing the volatility and dynamics of option pricing residuals. As discussed in Watson and Engle (1983) in a strictly linear framework, sequential optimization over subsets of the parameters is convenient and reasonably efficient. Conditional upon the Kalman filtration parameters, optimization of (B.5) over $<\{\boldsymbol{V}_t\}, \theta>$ involves nonlinear weighted least squares, and can be achieved by quadratic hill-climbing. Conditional on option parameters and the resulting option residuals $\{\boldsymbol{x}_t, \boldsymbol{u}_t\}_{t=1}^T$, optimization of (B.5) over $<\{\rho_I, \sigma_I\}_{I=1}^{64}, \boldsymbol{Q}>$ could in principle also be optimized by quadratic hill-climbing.

In practice, the high dimensionality of ε makes direct optimization of (B.5) with regard to filtration parameters quite slow. The major problem is estimating $\boldsymbol{Q}$, which has $½(N^2 + N) = 2080$ free parameters in this case. Furthermore, nonlinear parameter transformations (Cholesky factorization) must be applied to ensure positive semidefinite $\boldsymbol{Q}$, further slowing direct parameter optimization when the likelihood gradient is evaluated numerically. It is far more efficient in this case to estimate filtration parameters via an EM algorithm approach.

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[23] Given that serial persistence of common shocks is assumed to depend on delivery month whereas groups are categorized by maturity (0-1, 1-2, 2-3 and 3-6 months), computing $\boldsymbol{P}_{1|0}$ is slightly tricky. The longest-maturity (3-6 month) groups have unconditional covariances $q_{IJ}/(1 - \rho_I \rho_J)$, where q_{IJ} is the (I, J)-th entry of $\boldsymbol{Q}$. Unconditional covariances involving shorter-maturity groups are computed recursively off longer-maturity covariances based on an assumption of maturity shifts every 4 weeks, with the first shift occurring (for this data set) 2 weeks prior to January 4, 1988.

The EM algorithm of Dempster, Laird and Rubin (1977) proceeds in two alternating steps. First, the expectation of the *joint* log likelihood of option residuals and of the (unobserved) ε_t's is computed *conditional* upon observed residuals and an initial guess of the filtration parameters. Second, the expected log likelihood is maximized -- or at least increased -- with regard to its direct dependence upon filtration parameters, yielding new parameter values to be used in the first step.[24] The algorithm always increases the true log likelihood of option residuals (B.5), and a (local) optimum is attained when parameter estimates are no longer revised at the maximization step. For exponential distributions such as the one considered here, the expectation step is quite tractable and the steps that increase expected log likelihood can be computed analytically.

In this problem, the joint log likelihood of observed option residuals and common shocks is[25]

$$\begin{aligned} \ln L \; = \; & -\tfrac{1}{2} \ln |\boldsymbol{P}_{0|0}| \; - \; \tfrac{1}{2} \varepsilon_0' \boldsymbol{P}_{0|0}^{-1} \varepsilon_0 \\ & - \frac{T}{2} \ln |\boldsymbol{Q}| \; - \; \tfrac{1}{2} \sum_{t=1}^{T} (\varepsilon_t - \boldsymbol{F}_{t-1} \varepsilon_{t-1})' \boldsymbol{Q}^{-1} (\varepsilon_t - \boldsymbol{F}_{t-1} \varepsilon_{t-1}) \\ & - \frac{1}{2} \sum_{t=1}^{T} \sum_{I=1}^{64} \sum_{i \in G(I,t)} \left[\ln \sigma_I^2 + \frac{(e_{i,t} - \varepsilon_{I,t})^2}{\sigma_I^2} \right] \end{aligned} \qquad \textbf{(B.6)}$$

where $\boldsymbol{P}_{0|0}$ is the unconditional covariance matrix of the initial common shocks ε_0.[26] The expectation of this conditional upon observed option residuals can be computed using a Kalman smoother:

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[24] Ruud (1991) notes that it is sufficient for the new parameter estimates to increase the expected log likelihood. Maximization is not necessary, and can slow the algorithm.

[25] It is not necessary to include missing option residuals in the log likelihood, since the relevant function is the joint log likelihood of *observed* data and underlying common shocks. Shumway and Stoffer's (1982) procedure of including missing observations and zeroing out relevant entries of vectors and matrices is equivalent to not including those missing data in the first place.

[26] $\boldsymbol{P}_{0|0}$ and $\boldsymbol{P}_{1|0}$ are related ($\boldsymbol{P}_{1|0} = \boldsymbol{F}_0 \boldsymbol{P}_{0|0} \boldsymbol{F}_0' + \boldsymbol{Q}$) but not identical because the unconditional covariance matrices have intramonthly seasonals determined by the timing of maturity shifts.

$$E_T \ln L = -½ \ln |\boldsymbol{P}_{0|0}| - ½\, trace\Big[\boldsymbol{P}_{0|0}^{-1}\Big(\hat{\varepsilon}_{0|T}\hat{\varepsilon}_{0|T}{}' + \boldsymbol{P}_{0|T}\Big)\Big]$$
$$- \frac{T}{2}\ln|\boldsymbol{Q}| - ½\sum_{t=1}^{T} trace\Big[\boldsymbol{Q}^{-1}E_T(\varepsilon_t - \boldsymbol{F}_{t-1}\varepsilon_{t-1})(\varepsilon_t - \boldsymbol{F}_{t-1}\varepsilon_{t-1})'\Big] \qquad \textbf{(B.7)}$$
$$- \frac{1}{2}\sum_{t=1}^{T}\sum_{I=1}^{64}\sum_{i\in G(I,t)}\left[\ln\sigma_I^2 + \frac{(e_{i,t} - \hat{\varepsilon}_{I,t|T})^2 + \boldsymbol{P}_{t|T}(I,I)}{\sigma_I^2}\right]$$

where

$$E_T(\varepsilon_t - \boldsymbol{F}_{t-1}\varepsilon_{t-1})(\varepsilon_t - \boldsymbol{F}_{t-1}\varepsilon_{t-1})'$$
$$= (\hat{\varepsilon}_{t|T} - \boldsymbol{F}_{t-1}\hat{\varepsilon}_{t-1|T})(\hat{\varepsilon}_{t|T} - \boldsymbol{F}_{t-1}\hat{\varepsilon}_{t-1|T})' + \boldsymbol{P}_{t|T} \qquad \textbf{(B.8)}$$
$$- Cov_T(\varepsilon_t, \varepsilon_{t-1})\boldsymbol{F}_{t-1}{}' - \boldsymbol{F}_{t-1}Cov_T(\varepsilon_{t-1}, \varepsilon_t) + \boldsymbol{F}_{t-1}\boldsymbol{P}_{t-1|T}\boldsymbol{F}_{t-1}{}'$$

and $\boldsymbol{P}_{t|T}(I,I)$ is the *I*th diagonal term of $\boldsymbol{P}_{t|T}$. Smoothed conditional means and variances are computed by updating filtration-based estimates recursively backwards from the terminal values $\hat{\varepsilon}_{T|T}$ and $\boldsymbol{P}_{T|T}$:

$$\hat{\varepsilon}_{t|T} = \hat{\varepsilon}_{t|t} + \boldsymbol{J}_t(\hat{\varepsilon}_{t+1|T} - \hat{\varepsilon}_{t+1|t})$$
$$\boldsymbol{P}_{t|T} = \boldsymbol{P}_{t|t} + \boldsymbol{J}_t(\boldsymbol{P}_{t+1|T} - \boldsymbol{P}_{t+1|t})\boldsymbol{J}_t{}' \qquad \textbf{(B.9)}$$

where $\boldsymbol{J}_t = \boldsymbol{P}_{t|t}\boldsymbol{F}_t{}'\boldsymbol{P}_{t+1|t}^{-1}$.

Shumway and Stoffer (1982, p. 263) give a recursion for evaluating the autocovariances in (B.8), while Watson and Engle (1983) similarly advocate augmenting the state vector to include lagged variables. However, a simpler expression can be derived. Hamilton (1994, p. 395) shows that if next period's vector ε_{t+1} were observed, the conditional expectation $E_T[\varepsilon_t \mid \varepsilon_{t+1}] = \varepsilon_{t|t} + \boldsymbol{J}_t(\varepsilon_{t+1} - \hat{\varepsilon}_{t+1|t})$. Consequently,

$$E_T[\varepsilon_t\varepsilon_{t+1}{}'] = E_T[E_T(\varepsilon_t|\varepsilon_{t+1})\varepsilon_{t+1}{}']$$
$$= E_T\{[\hat{\varepsilon}_{t|t} + \boldsymbol{J}_t(\varepsilon_{t+1} - \hat{\varepsilon}_{t+1|t})]\varepsilon_{t+1}{}'\} \qquad \textbf{(B.10)}$$

Similarly from (B.9) above,

$$\hat{\boldsymbol{\varepsilon}}_{t|T}\,\hat{\boldsymbol{\varepsilon}}_{t+1|T}{}' \;=\; [\,\hat{\boldsymbol{\varepsilon}}_{t|t} + \boldsymbol{J}_t(\hat{\boldsymbol{\varepsilon}}_{t+1|T} - \hat{\boldsymbol{\varepsilon}}_{t+1|t})]\,\hat{\boldsymbol{\varepsilon}}_{t+1|T}{}' \tag{B.11}$$

Subtracting (B.11) from (B.10) yields

$$\begin{aligned} Cov_T(\varepsilon_t,\ \varepsilon_{t+1}) &\equiv E_T[\varepsilon_t\,\varepsilon_{t+1}{}'] - \hat{\varepsilon}_{t|T}\hat{\varepsilon}_{t+1|T}{}' \\ &= \boldsymbol{J}_t[\,E_T(\varepsilon_{t+1}\,\varepsilon_{t+1}{}') - \hat{\varepsilon}_{t+1|T}\,\hat{\varepsilon}_{t+1|T}{}'\,] \\ &= \boldsymbol{J}_t\boldsymbol{P}_{t+1|T}\,. \end{aligned} \tag{B.12}$$

Direct substitution confirms that this solution satisfies Shumway and Stoffer's recursion.

Apart from the nonlinear dependency of the initial unconditional covariance matrix $\boldsymbol{P}_{0|0}$ upon parameter values, optimizing (B.7) with regard to $<\{\rho_I,\ \sigma_I\}_{I=1}^{64},\ \boldsymbol{Q}>$ is a relatively straightforward linear exercise in estimating a constrained vector autoregression (VAR) using modified moments. Improved estimates of idiosyncratic noise are generated by

$$\hat{\sigma}_I^2 \;=\; \frac{1}{N_I}\sum_{t=1}^{T}\sum_{i\in G(I,t)}\left[(e_{i,t} - \hat{\varepsilon}_{I,t|T})^2 + \boldsymbol{P}_{t,T}(I,I)\right], \tag{B.13}$$

where $N_I = \sum_t N_{I,t}$. Conditional on earlier $\boldsymbol{Q}$ estimates, improved $\{\rho_I\}$ estimates (which determine $\boldsymbol{F}_t = \boldsymbol{D}_\rho\,\boldsymbol{A}_t$) are given by the constrained VAR moment conditions in Hamilton (1994, p. 318):

$$\hat{\rho} \;=\; \begin{pmatrix} q^{11}\sum_t E_T(x_{1t}\,x_{1t}) & \ldots & q^{1n}\sum_t E_T(x_{1t}\,x_{nt}) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ q^{n1}\sum_t E_T(x_{nt}\,x_{1t}) & \ldots & q^{nn}\sum_t E_T(x_{nt}\,x_{nt}) \end{pmatrix}^{-1} \times \begin{pmatrix} \sum_t\sum_j q^{1j}E_T(x_{1t}y_{jt}) \\ \cdot \\ \cdot \\ \sum_t\sum_j q^{nj}E_T(x_{nt}y_{jt}) \end{pmatrix} \tag{B.14}$$

where

q^{ij} is the $(i,\ j)$-th element of $\boldsymbol{Q}^{-1}$;

$E_T(x_{it}\,x_{jt})$ is the $(i,\ j)$-th element of $\boldsymbol{A}_{t-1}(\hat{\varepsilon}_{t-1|T}\,\hat{\varepsilon}_{t-1|T}{}' + \boldsymbol{P}_{t-1|T})\boldsymbol{A}_{t-1}{}'$; and

$E_T(x_{it}\,y_{jt})$ is the $(i,\ j)$-th element of $\boldsymbol{A}_{t-1}(\hat{\varepsilon}_{t-1|T}\,\hat{\varepsilon}_{t|T}{}' + \boldsymbol{J}_{t-1}\boldsymbol{P}_{t|T})$.

Conditional upon the $\{\rho_I\}$ estimates, improved $\boldsymbol{Q}$ estimates are given by

$$\hat{\boldsymbol{Q}} = \frac{1}{T}\sum_{t=1}^{T}\Big[(\hat{\varepsilon}_{t|T} - \boldsymbol{F}_{t-1}\hat{\varepsilon}_{t-1|T})(\hat{\varepsilon}_{t|T} - \boldsymbol{F}_{t-1}\hat{\varepsilon}_{t-1|T})' + \boldsymbol{P}_{t|T} + \boldsymbol{P}_{t|T}\boldsymbol{J}_{t-1}{}'\boldsymbol{F}_{t-1}{}' + \boldsymbol{F}_{t-1}\boldsymbol{J}_{t-1}\boldsymbol{P}_{t|T} + \boldsymbol{F}_{t-1}\boldsymbol{P}_{t-1|T}\boldsymbol{F}_{t-1}\Big] \qquad \textbf{(B.15)}$$

Each step in the EM algorithm estimation of $<\{\rho_I,\ \sigma_I\}_{I=1}^{64},\ \boldsymbol{Q}>$ therefore consists of the following steps:

1. Applying a Kalman smoother to estimate $\{\hat{\varepsilon}_{t|t},\ \hat{\varepsilon}_{t|T},\ \boldsymbol{P}_{t|t},\ \boldsymbol{P}_{t|T},\ \boldsymbol{J}_t\}$ and other relevant moments conditional upon particular parameter values $<\{\rho_I,\ \sigma_I\},\ \boldsymbol{Q}>$;

2. Revising estimates of $\{\rho_I,\ \sigma_I\}_{I=1}^{64}$ based upon estimated summary statistics from the first step;

3. Revising estimates of $\boldsymbol{Q}$ based upon estimated summary statistics and upon revised estimates of $\{\rho_I\}$.

The estimation procedure ensures positive definite estimates for $\boldsymbol{Q}$, while the final parameter estimates $<\{\rho_I,\ \sigma_I\}_{I=1}^{64},\ \boldsymbol{Q}>$ from repeated applications of the algorithm approximately optimize (B.5).[27]

Conditional upon fixed-point estimates of $<\{\rho_I,\ \sigma_I\}_{I=1}^{64},\ \boldsymbol{Q}>$ from the EM algorithm, option-specific parameters $<\{V_t\},\ \theta>$ were estimated by optimizing the nonlinear weighted least squares function (B.5) via the Davidon-Fletcher-Powell (DFP) algorithm, using a numerically computed gradient. The DFP and EM optimizations were alternated until joint convergence. The bulk of the computer time was taken up in the DFP stage.

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[27]The maximization step ignores the direct dependency of $\boldsymbol{P}_{0|0}$ upon ρ and $\boldsymbol{Q}$, so that the EM algorithm approach is not exactly equivalent to optimizing (B.5). This difference arises from the difference between maximum likelihood- and regression-based estimation of VAR's, and is not an issue in large samples.

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Table 1

Number of call and put observations, by moneyness and maturity

	Moneyness range $(X/F - 1)$		**Number of calls** (by months to maturity)				**Number of puts** (by months to maturity)			
	calls	**puts**	**0-1**	**1-2**	**2-3**	**3-6**	**0-1**	**1-2**	**2-3**	**3-6**
1.		<-.08					300	244	130	75
2.		[-.08, -.04)					503	220	87	44
3.	≤-2%	[-.04, -.02)	477	277	173	94	1330	527	169	95
4.	(-2%, -1%]	[-.02, -.01)	551	200	77	58	2139	770	300	163
5.	(-1%, 0%]	[-.01, .00)	1260	480	181	104	1925	941	294	173
6.	(0%, 1%]	[.00, .01)	2063	747	261	134	2402	1493	579	284
7.	(1%, 2%]	[.01, .02)	1650	753	300	130	1564	1546	746	447
8.	(2%, 4%]	≥.02	1882	1190	476	236	762	1374	881	606
9.	(4%, 8%]		642	902	493	261				
10.	>8%		47	184	146	65				
	All moneyness classes:		8572	4733	2107	1082	10925	7115	3186	1887

Table 2
Implicit parameter estimates, autocorrelations of option residuals, and goodness-of-fit diagnostics

Model	**Factor**	**Stochastic volatility parameters** α	β^*	β	σ_v	ρ	**Autocorrelations** range	median	SE1[a] ($\times 10^4$)	SE2[a] ($\times 10^4$)	**ln** $L_{options}$
One-factor models											
BS	V1	0	0		0	0	[.35, .79]	.55	22.4	11.3	249,796.60
DV1	V1	.032	.55		0	0	[.36, .81]	.59	21.8	10.6	250,008.47
SV1	V1	.100	1.49		.742	-.571	[.28, .85]	.60	11.0	7.8	255,367.69
SVJD1	V1	.049	2.45		.378	-.545	[.25, .81]	.56	9.8	7.7	256,483.87
Two-factor models											
DV2	V1	.112	7.14		0	0	[.26, .80]	.56	20.9	10.2	250,437.91
	V2	.010	.00		0	0					
SV2	V1	.028	.00		1.039	-.775	[.25, .77]	.54	8.7	6.6	257,037.96
	V2	.130	5.58		.667	-.382					
SVJD2	V1	.010	.91		.582	-.848	[.16, .78]	.52	7.8	6.8	258,150.90
	V2	.040	1.76		.346	-.402					
Constrained estimates											
SVC1	V1	.090	1.26	3.54	.694	-.587	[.27, .85]	.59	11.2	7.9	255,352.69
SVJDC1	V1	.043	2.48	2.31	.322	-.599	[.24, .79]	.54	9.9	7.7	256,478.68
SVC2	V1	.064	.00	4.11	.946	-1.000	[.20, .76]	.59	9.0	7.1	256,883.61
	V2	.007	.01	1.10	.152	.279					
SVJDC2	V1	.019	1.65	3.21	.479	-1.000	[.10, .75]	.51	7.8	6.8	258,013.53
	V2	.017	.87	.85	.207	-.314					

Jump parameters

SVJD1: $\lambda_t^* = .0000 + 27.19\, V_t\,,\qquad \bar{k}^* = -.095,\ \delta = .109.$

SVJDC1: $\lambda_t^* = .0000 + 31.62\, V_t\,,\qquad \bar{k}^* = -.085,\ \delta = .113.$

SVJD2: $\lambda_t^* = .1143 + 81.56\, V_{1t} + .28\, V_{2t}\,,\qquad \bar{k}^* = -.057,\ \delta = .102.$

SVJDC2: $\lambda_t^* = .0000 + 88.63\, V_{1t} + .00\, V_{2t}\,,\qquad \bar{k}^* = -.054,\ \delta = .102.$

[a]SE1 (SE2) is the overall equally weighted root mean squared error of option residuals as a fraction of the underlying futures price, ignoring (adjusting for) estimated serial correlation in option residuals.

Table 3
Average option pricing residuals, by moneyness and maturity

Maturity (months)	**Low X/F**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**High X/F**	***All strikes***
SV2: average $\varepsilon_{t\|T}$, $\times\ 10^4$											
0-1	4.0	4.2	2.0	0.0	-1.1	-1.2	-.8	.1	1.1	.5	*.5*
1-2	4.4	3.1	1.4	0.4	.1	.0	.1	-1.0	-.8	-.5	*.5*
2-3	2.1	3.6	2.8	2.4	1.3	.7	.2	-1.1	-3.2	-2.9	*.8*
3-6	.0	2.8	1.3	.4	-.2	-1.0	-1.8	-4.3	-6.5	-8.5	*-1.5*
All	*2.6*	*3.4*	*1.9*	*.8*	*.0*	*-.4*	*-.6*	*-1.6*	*-2.4*	*-2.8*	*.1*
SV2: average residuals, $\times\ 10^4$, without serial correlation correction											
0-1	5.9	4.6	2.5	.7	.3	.1	.1	.2	1.5	1.9	*1.2*
1-2	7.1	3.7	1.9	.8	.1	.1	-.2	-1.1	-.7	2.8	*1.6*
2-3	4.8	4.6	3.5	3.1	2.5	.9	.4	.5	-2.3	1.6	*2.4*
3-6	-1.0	3.1	.1	-5.3	1.8	-3.2	-3.1	-2.5	-4.6	.4	*-1.0*
All	*4.9*	*4.1*	*2.3*	*.6*	*.5*	*.1*	*-.1*	*-.4*	*-.9*	*1.9*	*1.3*
SV2: average residuals, $\times\ 10^4$, with serial correlation correction											
0-1	2.4	2.6	2.1	1.0	.8	.4	.5	.4	1.2	1.7	*1.1*
1-2	2.5	1.4	.6	.1	.1	.1	-.2	-.8	-.5	1.3	*.5*
2-3	3.0	2.2	1.8	1.7	.6	.2	-.1	-.7	-2.6	-.1	*.9*
3-6	-.4	1.5	.2	-1.8	.7	.1	-1.0	-1.2	.4	-1.0	*-.1*
All	*2.1*	*2.0*	*1.5*	*.7*	*.6*	*.3*	*.2*	*-.2*	*-.4*	*.5*	*.8*
SVJD2: average $\varepsilon_{t\|T}$, $\times\ 10^4$											
0-1	-.6	.2	.6	.5	-.1	-.2	-.2	.1	.6	.1	*.1*
1-2	-.4	-.7	-1.4	-1.9	-2.1	-2.1	-1.9	-2.2	-1.7	-.9	*-1.7*
2-3	.0	.4	-.2	-.4	-.6	-.7	-1.2	-2.7	-2.9	-2.7	*-1.1*
3-6	.7	2.8	1.5	1.6	1.2	1.0	1.1	-1.9	-3.0	-6.3	*.2*
All	*-.1*	*.7*	*.1*	*.0*	*-.4*	*-.5*	*-.5*	*-1.7*	*-1.7*	*-2.5*	*-.6*
SVJD2: average residuals, $\times\ 10^4$, without serial correlation correction											
0-1	-.4	.4	1.7	1.1	.7	.7	.4	.6	1.2	-.5	*.8*
1-2	1.3	-.4	-.7	-1.6	-2.0	-2.0	-1.9	-2.6	-2.1	.8	*-1.2*
2-3	2.2	1.0	-.1	.1	-.8	-1.6	-1.6	-2.7	-4.3	.8	*-.5*
3-6	-1.6	.7	-1.8	-3.3	-.1	-1.8	-1.8	-3.3	-3.8	-2.7	*-1.7*
All	*.6*	*.3*	*.5*	*.1*	*-.1*	*-.2*	*-.5*	*-1.1*	*-1.9*	*.2*	*-.2*
SVJD2 average residuals, $\times\ 10^4$, with serial correlation correction											
0-1	-.4	.7	2.1	1.7	1.7	1.5	1.2	1.2	1.2	-.3	*1.5*
1-2	.4	-.2	-.4	-1.0	-1.2	-1.1	-1.2	-1.5	-.9	-.3	*-.7*
2-3	1.5	.3	-.3	.1	-1.4	-1.6	-1.4	-2.5	-3.3	1.5	*-.6*
3-6	-1.1	.9	-.7	-.8	.2	.7	-1.0	-1.7	.0	-2.0	*-.4*
All	*.2*	*.3*	*.8*	*.7*	*.7*	*.6*	*.2*	*-.3*	*-.7*	*.0*	*.4*

Table 4
Group-specific properties of SVJD2 residuals

call/ put	Maturity (months)	Low X/F	2	3	4	5	6	7	8	9	High X/F
					Idiosyncratic noise σ_I, $\times\ 10^4$						
calls	0-1			4.32	3.01	3.31	2.83	2.25	2.29	2.06	.88
	1-2			4.88	3.12	3.54	3.44	3.01	3.42	3.25	2.83
	2-3			4.97	3.28	4.31	3.43	4.80	4.39	4.38	5.93
	3-6			4.76	3.60	5.03	4.18	4.22	5.04	5.59	4.11
puts	0-1	2.19	2.47	2.64	2.75	2.78	3.42	3.77	5.03		
	1-2	3.14	3.31	3.44	3.88	3.20	3.55	3.88	6.10		
	2-3	3.58	3.96	3.87	4.12	3.72	4.79	4.12	6.56		
	3-6	3.89	3.84	4.39	4.12	4.58	5.27	4.51	6.58		
					Systematic noise estimates $\sqrt{\boldsymbol{Q}(I,\ I)}, \times 10^4$						
calls	0-1			5.40	6.26	5.36	5.46	4.73	3.48	2.52	1.09
	1-2			3.70	5.21	5.31	5.73	4.79	4.79	4.16	3.03
	2-3			6.05	6.74	5.25	6.47	5.72	6.49	5.60	4.31
	3-6			14.55	15.31	14.23	16.54	15.94	13.90	11.23	10.84
puts	0-1	2.16	3.66	5.26	6.23	5.47	5.02	4.04	3.50		
	1-2	2.85	3.62	4.23	4.86	5.18	5.63	4.83	6.74		
	2-3	3.61	5.74	5.66	6.03	7.84	5.41	6.23	11.00		
	3-6	11.22	14.49	14.13	15.90	16.08	15.19	14.00	16.56		
					Serial correlation estimates ρ_I						
calls	0-1			.506	.468	.562	.545	.493	.523	.426	.165
	1-2			.719	.552	.382	.448	.411	.449	.508	.657
	2-3			.410	.470	.552	.358	.416	.480	.555	.512
	3-6			.599	.642	.614	.619	.626	.639	.742	.781
puts	0-1	.296	.454	.462	.507	.560	.562	.591	.515		
	1-2	.656	.518	.490	.467	.401	.511	.461	.534		
	2-3	.615	.530	.424	.453	.446	.485	.223	.337		
	3-6	.599	.621	.674	.661	.596	.593	.657	.598		

Table 5
Implicit cumulants at 1- and 6-month horizons: SVJDC2 model.

1-month

$$K_2 = 2.03e{-4} + .1791 \quad V_{1t} + .0806\,V_{2t}$$

$$K_3 = -1.82e{-4} - .0255 \quad V_{1t} - .0007\,V_{2t}$$

$$K_4 = -4.36e{-6} + 6.34e{-3}\,V_{1t} + .03e{-3}\,V_{2t}$$

6-month

$$K_2 = 0063 + 8508\,V_{1t} + .4115\,V_{2t}$$

$$K_3 = -.0014 - .3562\,V_{1t} - .0203\,V_{2t}$$

$$K_4 = .0007 + .2205\,V_{1t} + .0045\,V_{2t}$$

Average factor realizations: $Avg(V_1) = .01101$; $Avg(V_2) = .01196$.

Table 6
Maximum likelihood estimates of the process followed by implicit factor realizations.

$$dV = (\alpha - \beta V)dt + \sigma_v \sqrt{V} dW_v$$

	Series	Stochastic volatility parameters: α	β	σ_v	$\sqrt{\alpha/\beta}$	**half-life**[a] (months)	**ln L_{Vi}**	**Constrained ln L_{Vi}** (P-value)[b]
Unconstrained models								
SV1	V1	.122 (.030)	4.65 (1.21)	.268 (.011)	.162 (.011)	1.8 (0.5)	18.69	-175.95 (0)
SVJD1	V1	.093 (.024)	4.35 (1.17)	.215 (.009)	.146 (.010)	1.9 (0.5)	58.62	-19.17 (0)
SV2	V1	.020 (.004)	1.32 (1.63)	.487 (.019)	.123 (.075)	6.3 (7.7)	-433.87	-610.00 (0)
	V2	.028 (.005)	3.37 (1.92)	.538 (.021)	.091 (.026)	2.5 (1.4)	-408.76	-438.12 (2e-13)
SVJD2	V1	.055 (.010)	6.02 (1.49)	.256 (.011)	.096 (.009)	1.4 (0.3)	-172.11	-313.33 (0)
	V2	.007 (.002)	1.26 (1.11)	.314 (.012)	.073 (.033)	6.6 (5.8)	-179.75	-201.09
Constrained models								
SVC1	V1	.120 (.030)	4.62 (1.20)	.262 (.011)	.161 (.011)	1.8 (0.5)	24.39	-157.60 (0)
SVJDC1	V1	.090 (.024)	4.32 (1.16)	.209 (.009)	.144 (.010)	1.9 (0.5)	63.61	12.78 (0)
SVC2	V1	.010 (.006)	.98 (1.00)	.333 (.013)	.099 (.051)	8.4 (8.6)	-193.64	-414.41 (0)
	V2	.071 (.016)	5.97 (1.30)	.139 (.006)	.109 (.005)	1.4 (0.3)	116.16	103.14 (9e-6)
SVJDC2	V1	.056 (.012)	5.45 (1.38)	.216 (.009)	.101 (.008)	1.5 (0.4)	-80.25	-231.76 (0)
	V2	.002 (.001)	.76 (.69)	.184 (.007)	.046 (.025)	11.0 (10.0)	-14.36	-30.55 (4e-7)

[a]The half-life $12 \ln 2/\beta$ is in months. All other parameters are in annualized units.

[b]Constrained log likelihood reflects the imposition of implicit parameter estimates from Table 1: (α, σ_v) estimates for the unconstrained models, $(\alpha, \beta, \sigma_v)$ estimates for constrained models. P-values are from corresponding χ^2_2 (χ^2_3) likelihood ratio tests for the unconstrained (constrained) models.

Table 7

Implicit factor evolution: summary statistics of normalized factor transitions

Model	Series	Quartiles[a] Q1	Median	Q3	Range	SW *P*-value[b]
Normalization using parameters inferred from option prices						
SVC1	V1	-.08	.13	.30	[-1.50, 1.96]	1e-10
SVJDC1	V1	-.32	.05	.35	[-2.80, 3.11]	3e-11
SVC2	V1	.12	.32	.50	[-2.91, 2.82]	0
	V2	-.35	.04	.43	[-10.40, 6.26]	0
SVJDC2	V1	-.04	.21	.41	[-1.70, 2.65]	0
	V2	-.33	.00	.33	[-∞, 3.62]	0
Normalization using maximum likelihood time series estimates						
SVC1	V1	-.58	-.06	.43	[-4.07, 4.90]	0
SVJDC1	V1	-.59	-.06	.46	[-4.17, 4.75]	0
SVC2	V1	-.47	.09	.51	[-5.43, ∞]	0
	V2	-.50	-.08	.43	[-10.66, 7.29]	0
SVJDC2	V1	-.61	-.10	.32	[-3.79, 5.75]	0
	V2	-.29	.07	.46	[-7.58, 4.98]	0

[a]The quartiles of a standard normal distribution are -.67, 0, and +.67.
[b]Shapiro-Wilks test of normality.

Table 8

Summary statistics for log-differenced noon S&P 500 futures prices.

Pre-crash: January 3, 1983 - October 16, 1987; Post-crash: January 2, 1988 - December 31, 1983.

	Daily returns		**Weekly returns** (Wednesday → Wednesday)	
	Pre-crash	Post-crash	Pre-crash	Post-crash
Number of observations	1211	1515	244	307
mean	.0004	.0003	.0023	.0011
standard deviation	.0098	.0089	.0210	.0183
skewness	-.278	-.316	-.076	-.381
Excess kurtosis	1.35	3.52	.45	.70
minimum	-.043	-.058	-068	-.062
maximum	+.034	+.050	+.068	+.055
Shapiro-Wilks normality test -- *P*-value	.000	.000	.961	.702

Table 9

Maximum likelihood estimates of the S&P 500 futures price process *conditional* on stochastic variance parameters and weekly factor realizations inferred from options prices.[a]

Jump frequency: $\lambda_t = l_0 + l_1 \lambda_t^*$

Instantaneous variance: $V_t = V_0 + d_1 V_{1t}^{options} + d_2 V_{2t}^{options}$

	Jump parameters				Variance parameters			
Model	l_0	l_1	$\bar{k}$	δ	V_0	d_1	d_2	**ln** $\boldsymbol{L_{\{F\}}}$
BS					.00000[c]	.651 (.052)		811.59
SV1					.00352 (.00192)	.465 (.108)		813.89
SVJD1	0	1	-.096[b]	.108[b]	.00120 (.00227)	.671 (.139)		812.47
SVJD1	.291 (.300)	.000 (.002)	.064 (.023)	.000[c]	.00000[c]	.707 (.072)		815.11
SV2					.00324 (.00194)	.447 (.111)	.470 (.165)	815.58
SVJD2	0	1	-.057[b]	.102[b]	.00000[c]	.629 (.178)	.730 (.121)	814.50
SVJD2	.354 (.297)	.000[c]	.063 (.017)	.000[c]	.00000[c]	.688 (.195)	.658 (.114)	817.55

[a]Wednesday noon log-differenced future prices, 1988-93, 309 observations. Asymptotic standard errors are in parentheses.

[b]Parameter set equal to value inferred from option prices.

[c]Nonnegativity constraint binding.

Table 10
Correlation estimates for weekly S&P 500 futures returns and implicit factor shocks

$$Corr(\Delta \ln F,\ \Delta V_1,\ \Delta V_2) \equiv \begin{pmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_{vv} \\ \rho_2 & \rho_{vv} & 1 \end{pmatrix}$$

Model	One-factor models (ρ_1)	Two-factor models		
		ρ_1	ρ_2	ρ_{vv}
Unconstrained $\{V_t\}$ estimates				
SV	-.613	-.522	.006	-.575
SVJD	-.615	-.563	.035	-.501
Smoothed $\{V_t\}$ estimates				
SVC	-.614	-.561	-.178	-.157
SVJDC	-.615	-.612	.033	-.325