THE CRITICAL SPECTRUM OF A STRONGLY CONTINUOUS SEMIGROUP

RAINER NAGEL AND JAN POLAND

Abstract. For a strongly continuous semigroup $(T(t))_{t\geq 0}$ with generator A we introduce its critical spectrum $\sigma_{crit}(T(t))$. This yields in an optimal way the spectral mapping theorem

$$\sigma(T(t)) = e^{t\sigma(A)} \ \cup \ \sigma_{crit}(T(t))$$

and improves classical stability results.

1. Introduction

Already in 1892, M.A. Liapunov showed that the asymptotic behavior as $t \to \infty$ of the exponential function $t \mapsto e^{tA}$ for a matrix $A \in \mathrm{M}_n(\mathbb{C})$ can be described by the location of the eigenvalues of A (see [6], p. 291). Later, E. Hille and R. Phillips ([4], Section 23.16) discovered that an analogous statement does not hold for strongly continuous semigroups $(T(t))_{t\geq 0}$ with unbounded generator A on Banach spaces. The reason is the failure of the spectral mapping theorem

$$\sigma(T(t_0)) = e^{t_0\sigma(A)}, \ t_0 > 0.$$

In fact, such an identity only holds for the point and for the residual spectrum (see [3], Theorem IV.3.6) or if we make additional assumptions on $(T(t))_{t\geq 0}$ (such as eventual norm continuity, see [3], Theorem IV.3.9). In general, only an inclusion holds, and we must write

$$\sigma(T(t_0)) = e^{t_0\sigma(A)} \ \cup \ \sigma_?(T(t_0))$$

for some set $\sigma_?(T(t_0))$.

It follows from the spectral mapping theorem for the point spectrum that we may take as $\sigma_?(T(t_0))$ the essential spectrum $\sigma_{ess}(T(t_0))$. This has been done by many authors mainly for the study of perturbed semigroups (see, e.g., [1]).

However, the essential spectrum is not related to the semigroup structure, and even for bounded A it is unnecessarily big in order to yield the above identity.

We therefore propose a new spectrum, called the *critical spectrum* $\sigma_{crit}(T(t))$, which yields in an optimal way a spectral mapping theorem of the form

$$\sigma(T(t_0)) = e^{t_0\sigma(A)} \ \cup \ \sigma_{crit}(T(t_0)) \text{ for } t_0 \geq 0.$$

In addition, we obtain stability theorems and characterizations of asymptotically norm continuous semigroups. These results will be applied to perturbed semigroups in a subsequent paper.

We thank S. Piazzera and F. Räbiger for helpful discussions.

2. The critical spectrum

For a given strongly continous semigroup $(T(t))_{t\geq 0}$ with generator $(A, D(A))$ on a Banach space X we perform a series of abstract, but natural constructions.

2.1. Definition. On the Banach space $\ell^\infty(X)$ of all bounded sequences in X, endowed with the sup-norm $\|(x_n)_{n\in\mathbb{N}}\| := \sup_{n\in\mathbb{N}} \|x_n\|$, consider the semigroup $\tilde{\mathcal{T}} := (\tilde{T}(t))_{t\geq 0}$ given by

$$\tilde{T}(t)(x_n)_{n\in\mathbb{N}} := (T(t)x_n)_{n\in\mathbb{N}} \text{ for } t \geq 0$$

and the operator $(\tilde{A}, D(\tilde{A})$ given by

$$\tilde{A}(x_n)_{n\in\mathbb{N}} := (Ax_n)_{n\in\mathbb{N}}$$

with domain

$$D(\tilde{A}) := \{(x_n)_{n\in\mathbb{N}} \in \ell^\infty(X) : x_n \in D(A),\ (Ax_n) \in \ell^\infty(X)\}.$$

Note that the semigroup $(\tilde{T}(t))_{t\geq 0}$ is strongly continuous only if $(T(t))_{t\geq 0}$ is uniformly continuous, hence if A is bounded. Moreover, one has $\sigma(T(t)) = \sigma(\tilde{T}(t))$ and $\sigma(A) = \sigma(\tilde{A})$ with $R(\lambda, \tilde{A})(x_n)_{n\in\mathbb{N}} = (R(\lambda, A)x_n)_{n\in\mathbb{N}}$ for $\lambda \in \rho(A)$. In particular, it follows that $\tilde{A}$ is a (nondensely defined) Hille-Yosida operator.
We now consider the space of strong continuity

$$\ell^\infty_T(X) := \{(x_n)_{n\in\mathbb{N}} \in \ell^\infty(X) : \lim_{t\to 0} \sup_{n\in\mathbb{N}} \|T(t)x_n - x_n\| = 0\} = \overline{D(\tilde{A}_0)},$$

where $\tilde{A}_0$ denotes the part of $\tilde{A}$ in $\ell^\infty_T(X)$. The space $\ell^\infty_T(X)$ is a closed and $(\tilde{T}(t))_{t\geq 0}$-invariant subspace of $\ell^\infty(X)$ and therefore allows the following quotient construction.

2.2. Definition. On the quotient space $\hat{X} := \ell^\infty(X)/\ell^\infty_T(X)$ we define the semigroup $(\hat{T}(t))_{t\geq 0}$ by

$$\hat{T}(t)\hat{x} := (T(t)x_n)_{n\in\mathbb{N}} + \ell^\infty_T(X) \text{ for } \hat{x} := (x_n)_{n\in\mathbb{N}} + \ell^\infty_T(X) \in \hat{X}.$$

This is a semigroup of bounded operators on $\hat{X}$. As we will see in Proposition 2.7, the only continuous orbit $t \mapsto \hat{T}(t)\hat{x}$ occurs for $\hat{x} = 0$. Moreover, there is no natural "generator" associated to this semigroup. Its growth bound and the spectra of the operators $\hat{T}(t)$, however, turn out to be of considerable interest.

2.3. Definition. For a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space X we call

$$\sigma_{crit}(T(t)) := \sigma(\hat{T}(t))$$

the *critical spectrum* of $T(t)$, $t \geq 0$, and denote by

$$\omega_{crit}(\mathcal{T}) := \omega_0(\hat{\mathcal{T}}) = \inf\{w \in \mathbb{R} :\ \exists\ M > 0 \text{ such that } \|\hat{T}(t)\| \leq Me^{wt}\}$$

its *critical growth bound.*

We point out that the critical spectrum $\sigma_{crit}(T(t_0))$ at $t_0 > 0$ is not determined by the operator $T(t_0)$ alone but by the entire semigroup $(T(t))_{t\geq 0}$. The proposed notation should, however, not lead to any confusion.
The following lemma will be of great help to compute the critical spectrum.

2.4. Lemma. *Let $Y \subset X$ be a closed, $(T(t))_{t\geq 0}$-invariant subspace such that the restriction $(T(t)|_Y)_{t\geq 0}$ is norm continuous and denote the quotient semigroup on $Z := X/Y$ by $(S(t))_{t\geq 0}$. Then*

$$\sigma_{crit}(T(t)) = \sigma_{crit}(S(t))$$

holds for all $t \geq 0$.

Proof. The space $\tilde{Y} := \ell^\infty(Y)$ is a closed, $(\tilde{T}(t))_{t\geq 0}$-invariant subspace of $\ell^\infty(X)$, and we have

$$\ell^\infty(Z) \cong \ell^\infty(X)/\tilde{Y}.$$

¿From the assumption follows $\tilde{Y} \subset \ell^\infty_T(X)$, hence we can identify the spaces $\hat{X}$ and $\hat{Z}$ associating to

$$\hat{x} = ((x_n)_{n\in\mathbb{N}} + \ell^\infty_T(X)) \in \hat{X} \text{ with } (x_n)_{n\in\mathbb{N}} \in \ell^\infty(X)$$

the element

$$(((x_n)_{n\in\mathbb{N}} + \tilde{Y}) + \ell^\infty_S(Z)) =: \hat{z} \in \hat{Z}.$$

Under this identification the semigroups $(\hat{T}(t))_{t\geq 0}$ and $(\hat{S}(t))_{t\geq 0}$ become equal, hence

$$\sigma_{crit}(T(t)) = \sigma_{crit}(S(t)) \text{ for all } t \geq 0.$$

□

2.5. Theorem. *For the critical and the essential spectrum of a strongly continuous semigroup $(T(t))_{t\geq 0}$ one has the inclusions*

$$\sigma_{crit}(T(t)) \subset \sigma_{ess}(T(t)) \subset \sigma(T(t)) \text{ for all } t \geq 0.$$

Proof. If $\lambda \in \rho(T(t))$, then the resolvent operator $R(\lambda, T(t))$ induces an operator $\hat{R}(\lambda)$ on $\hat{X}$ by

$$\hat{R}(\lambda)((x_n)_{n\in\mathbb{N}} + \ell^\infty_T(X)) := (R(\lambda, T(t))x_n)_{n\in\mathbb{N}} + \ell^\infty_T(X),$$

which is the resolvent of $\hat{T}(t)$ in λ. This proves that $\lambda \in \rho(\hat{T}(t))$, hence $\sigma_{crit}(T(t)) \subset \sigma(T(t))$.

Taking into account the definitions of the critical and the essential spectrum, it suffices to show that $\lambda \in \sigma(T(t)) \setminus \sigma_{ess}(T(t))$ implies $\lambda \notin \sigma_{crit}(T(t))$. By definition, λ is a pole of the resolvent map with finite algebraic multiplicity. Hence, by [5], Theorem III.6.17, there exists a spectral decomposition

$$X = X_1 \oplus X_2,$$

such that $\sigma(T(t)|_{X_1}) = \{\lambda\}$ and $\sigma(T(t)|_{X_2}) = \sigma(T(t))\setminus\{\lambda\}$. Since X_1 is finite dimensional, the restricted semigroup $(T(t)|_{X_1})_{t\geq 0}$ is uniformly continuous. Thus we can apply Lemma 2.4 and obtain

$$\sigma_{crit}(T(t)) = \sigma_{crit}(T(t)|_{X_2}) \subset \sigma(T(t)|_{X_2}),$$

hence $\lambda \notin \sigma_{crit}(T(t))$. □

Since the space $\hat{X}$ was defined by taking the quotient along the space of strong continuity of $(\hat{T}(t))_{t\geq 0}$, it can be expected that the only continuous orbit of $(\hat{T}(t))_{t\geq 0}$ occurs for $\hat{x} = 0$. However, the proof is not quite so simple, and we need to introduce another norm on $\hat{X}$.

2.6. Lemma. *The norm* $|||\cdot|||$*, defined by*

$$|||\hat{x}||| := \overline{\lim_{h\downarrow 0}} \sup_{n\in\mathbb{N}} \|T(h)x_n - x_n\| \text{ for } \hat{x} = (x_n)_{n\in\mathbb{N}} + \ell_T^\infty(X) \in \hat{X},$$

is equivalent to the quotient norm $\|\cdot\|$ *on* $\hat{X}$.

Proof. First of all, it is not difficult to verify, that $|||\cdot|||$ in fact defines a norm on $\hat{X}$. Now take

$$M := \sup_{0\le t\le 2} \|T(t)\|$$

and assume $|||(x_n)_{n\in\mathbb{N}} + \ell_T^\infty(X)||| \le 1$. We then find $0 \le \delta \le 1$ such that

$$\sup_{n\in\mathbb{N}} \|T(s)x_n - x_n\| \le 2 \text{ for all } 0 \le s \le \delta.$$

Defining

$$y_n := \frac{1}{\delta}\int_0^\delta T(t)x_n dt$$

we obtain that $(y_n)_{n\in\mathbb{N}} \in \ell_T^\infty(X)$, since

$$\begin{aligned} \|T(s)y_n - y_n\| &= \frac{1}{\delta}\left\|\int_s^{s+\delta} T(t)x_n dt - \int_0^\delta T(t)x_n dt\right\| \\ &= \frac{1}{\delta}\left\|\int_\delta^{s+\delta} T(t)x_n dt - \int_0^s T(t)x_n dt\right\| \le s\cdot\frac{2M}{\delta}\|x_n\| \end{aligned}$$

for all $0 \le s \le \delta$. Moreover, we have

$$\|y_n - x_n\| = \frac{1}{\delta}\left\|\int_0^\delta (T(t)x_n - x_n)dt\right\| \le \frac{1}{\delta}\int_0^\delta 2dt = 2$$

for all $n \in \mathbb{N}$, hence $\|(x_n)_{n\in\mathbb{N}} + \ell_T^\infty(X)\| \le 2$.
Assume now $\|(x_n)_{n\in\mathbb{N}} + \ell_T^\infty(X)\| < 1$. Then there is $(y_n)_{n\in\mathbb{N}} \in \ell_T^\infty(X)$ satisfying $\|x_n - y_n\| < 1$ for all $n \in \mathbb{N}$. For this $(y_n)_{n\in\mathbb{N}}$ we find $0 \le \delta \le 1$, such that $\|T(s)y_n - y_n\| < 1$ for all $n \in \mathbb{N}$ and $0 \le s \le \delta$. This implies

$$\|T(s)x_n - x_n\| \le \|T(s)x_n - T(s)y_n\| + \|T(s)y_n - y_n\| + \|y_n - x_n\| < M + 2,$$

hence $|||(x_n)_{n\in\mathbb{N}} + \ell_T^\infty(X)||| < M + 2$. □

We can now state the following result due to S. Brendle.

2.7. Proposition. *The map* $\mathbb{R}_+ \ni t \mapsto \hat{T}(t)\hat{x} \in \hat{X}$ *is continuous if and only if* $\hat{x} = 0$.

Proof. Suppose that the map $t \mapsto \hat{T}(t)\hat{x}$ is continuous for some $\hat{x} = (x_n)_{n\in\mathbb{N}} + \ell_T^\infty(X)$. For each $t > 0$ we have seen above that

$$\left(\frac{1}{t}\int_0^t T(s)x_n ds\right)_{n\in\mathbb{N}} \in \ell_T^\infty(X).$$

Therefore, using Lemma 2.6 (with the appropriate constant M), we get

$$\begin{aligned}
\|\hat{x}\| &= \left\| \left(\frac{1}{t} \int_0^t (T(s) - \mathrm{Id}) x_n ds \right)_{n\in\mathbb{N}} + \ell_T^\infty(X) \right\| \\
&\le 2 \cdot \overline{\lim_{h\downarrow 0}} \sup_{n\in\mathbb{N}} \left\| \frac{1}{t} \int_0^t (T(h) - \mathrm{Id})(T(s) - \mathrm{Id}) x_n \; ds \right\| \\
&\le 2 \cdot \overline{\lim_{h\downarrow 0}} \frac{1}{t} \int_0^t \sup_{n\in\mathbb{N}} \|(T(h) - \mathrm{Id})(T(s) - \mathrm{Id}) x_n\| \; ds \\
&\le 2 \cdot \frac{1}{t} \int_0^t \overline{\lim_{h\downarrow 0}} \sup_{n\in\mathbb{N}} \|(T(h) - \mathrm{Id})(T(s) - \mathrm{Id}) x_n\| \; ds \\
&\le 2(M+2) \cdot \frac{1}{t} \int_0^t \|(\hat{T}(s) - \mathrm{Id})\hat{x}\| ds.
\end{aligned}$$

Here, the penultimate inequality follows from Fatou's Lemma for the limit superior. By letting $t \to 0$, we conclude $\hat{x} = 0$. □

Before applying this new spectrum to the study of the asymptotic behavior of semigroups, we show how to compute it.

2.8. Examples.

(i) If the semigroup $(T(t))_{t\ge 0}$ is eventually norm continuous, then one has

$$\sigma_{crit}(T(t)) = \{0\} \text{ for all } t > 0.$$

This follows since, by definition, $(\hat{T}(t))_{t\ge 0}$ becomes a nilpotent semigroup, hence $\sigma(\hat{T}(t)) = \{0\}$, except in the trivial case $\hat{X} = \{0\}$.

(ii) On $X := \mathrm{C}_0(\Omega)$ for some locally compact space Ω take a multiplication semigroup $(T(t))_{t\ge 0}$ with generator $(A, D(A))$ given by

$$Af := q \cdot f, \;\; f \in D(A),$$

where $q : \Omega \to \mathbb{C}$ is a continuous function satistying $\sup_{s\in\Omega} \mathrm{Re}\; q(s) < \infty$. It is well known that the spectrum is given by

$$\sigma(A) = \overline{q(\Omega)} \text{ and } \sigma(T(t)) = \overline{e^{tq(\Omega)}}$$

(see [3], II.2.9). For the critical spectrum one has

$$\begin{aligned}
\sigma_{crit}(T(t)) \setminus \{0\} \;=\; &\{\lambda \in \mathbb{C} \setminus \{0\} : \exists \; (\mu_n)_{n\in\mathbb{N}} \subset q(\Omega), \\
&|\mathrm{Im}\; \mu_n| \to \infty \text{ such that } \lambda = \lim_{n\to\infty} e^{t\mu_n}\}.
\end{aligned}$$

To show the inclusion "$\supset$", one constructs an approximate eigenvector $(f_n)_{n\in\mathbb{N}}$ corresponding to λ using the $(\mu_n)_{n\in\mathbb{N}}$, such that $(f_n)_{n\in\mathbb{N}} \notin \ell_T^\infty(X)$. For the inclusion "$\subset$" we assume that λ is not contained in the set on the right hand side and consider $\mu_0 \in \mathbb{C}$, such that

$$\lambda = e^{t\cdot(\mu_0 + \frac{2\pi i k}{t})} \text{ for all } k \in \mathbb{N}.$$

Then there exists $\varepsilon > 0$, such that

$$\overline{q(\Omega)} \cap B_\varepsilon(\mu_0 + \frac{2\pi i k}{t}) = \varnothing$$

for all but finitely many $k \in \mathbb{N}$. (Here, $B_\varepsilon(x) := \{y \in \mathbb{C} : |y - x| < \varepsilon\}$.) We denote these finitely many indices by $k_1, k_2, \dots, k_l$ und set

$$B := \bigcup_{j=1}^{l} B_\varepsilon(\mu_0 + \frac{2\pi i k_j}{t}).$$

Then $G := \Omega \setminus q^{-1}(B)$ is a closed subset of Ω and

$$Y := \{f \in \mathrm{C}_0(\Omega) \ : \ f|_G \equiv 0\}$$

is a closed subspace of $\mathrm{C}_0(\Omega)$. The restriction $(T(t)|_Y)_{t\geq 0}$ ist uniformly continuous, since B is bounded. Therefore we can apply Lemma 2.4, and we obtain $\sigma_{crit}(T(t)) = \sigma_{crit}(S(t))$, where $(S(t))_{t\geq 0}$ denotes the semigroup induced by $(T(t))_{t\geq 0}$ on the quotient space $Z = X/Y$. Since $(S(t))_{t\geq 0}$ is isomorphic to the multiplication semigroup on $\mathrm{C}_0(G)$ generated by $q|_G$, we have $\lambda \notin \sigma(S(t))$ and therefore $\lambda \notin \sigma_{crit}(T(t))$.

(iii) Replacing $q(\Omega)$ by the essential range of a measurable function q, one shows the analogous statement for multiplication semigroups on $X = \mathrm{L}^p(\Omega, \mu)$ for $1 \leq p < \infty$.

(iv) As a typical example for translation semigroups, we mention that

$$\sigma_{crit}(T(t)) = \sigma(T(t)) = \{\lambda \in \mathbb{C} : |\mu| \leq 1\}, \ t > 0,$$

for the left translation semigroup

$$T(t)f(s) := f(s+t)$$

on $X := \mathrm{C}_0(\mathbb{R}_+)$. This can be seen by defining, for $t > 0$ and $\lambda \in \mathbb{C}$ with $\mathrm{Re}\ \lambda < 0$, functions $f_n \in \mathrm{C}_0(\mathbb{R}_+)$ by

$$f_n(s) := e^{\lambda s} \cdot \sin\left(\frac{2\pi n s}{t}\right).$$

Then $(T(t))_{t\geq 0}$ is not uniformly continuous on $(f_n)_{n\in\mathbb{N}}$, i.e. $(f_n)_{n\in\mathbb{N}} \notin \ell^\infty_T(X)$. On the other hand, we have

$$T(t)f_n(s) = e^{\lambda(t+s)} \cdot \sin\left(\frac{2\pi n(t+s)}{t}\right) = e^{\lambda t} \cdot f_n(s) \text{ for all } s \geq 0, \ n \in \mathbb{N}$$

and each fixed $t > 0$. This shows that $(f_n)_{n\in\mathbb{N}} + \ell^\infty_T(X)$ becomes an eigenvector of $\hat{T}(t)$ belonging to the eigenvalue $e^{\lambda t}$. Since this holds for each $\mathrm{Re}\ \lambda < 0$ and $\sigma(\hat{T}(t))$ is a closed subset of $\sigma(T(t))$, we obtain the assertion.

3. The Spectral Mapping Theorem

We recall from [3], Theorem IV.3.6, that the spectral mapping theorem for a strongly continuous semigroup $(T(t))_{t\geq 0}$ with generator A holds for the point and the residual spectrum. So the only part of the spectrum, for which the spectral mapping theorem can fail, is the approximate point spectrum $A\sigma(T(t))$. More precisely, if $\lambda \in \sigma(T(t))$, but $\lambda \notin e^{t\sigma(A)}$, then λ must be an approximate eigenvector of $T(t)$, i.e., there exists $(x_n)_{n\in\mathbb{N}} \in \ell^\infty(X)$, $\|x_n\| = 1$, such that

$$\|T(t)x_n - \lambda x_n\| \to 0 \text{ as } n \to \infty.$$

The following lemma characterizes the approximate eigenvalues λ of $T(t)$ already contained in $e^{t\sigma(A)}$.

3.1. Lemma. *For $\lambda \in \mathbb{C} \setminus \{0\}$ and $s \geq 0$ the following assertions are equivalent.*

(i) *There exists $\mu \in A\sigma(A)$ such that $e^{\mu s} = \lambda \in A\sigma(T(s))$.*

(ii) *There exists an approximate eigenvector $(x_n)_{n\in\mathbb{N}}$ of $T(t)$ for the approximate eigenvalue λ such that $(x_n)_{n\in\mathbb{N}} \in \ell_T^\infty(X)$.*

(iii) *There exists $(x_n)_{n\in\mathbb{N}}$, $\|x_n\| = 1$, and $\mu \in \mathbb{C}$ with $e^{\mu s} = \lambda$, such that*

$$\|T(t)x_n - e^{\mu t}x_n\| \to 0 \text{ as } n \to \infty \text{ for all } t \geq 0.$$

Proof. *(i)* $\Rightarrow$ *(iii)* follows from the spectral inclusion theorem for the approximate point spectrum (see [3], Theorem IV.3.5).
(iii) $\Rightarrow$ *(ii)*. Assume that $(T(t))_{t\geq 0}$ is not uniformly continuous on $(x_n)_{n\in\mathbb{N}}$. This means that

$$0 \neq \hat{x} := (x_n)_{n\in\mathbb{N}} + \ell_T^\infty(X) \in \hat{X}.$$

The condition in *(iii)* now implies

$$\hat{T}(t)\hat{x} = e^{\mu t}\hat{x} \text{ for all } t \geq 0.$$

By Proposition 2.7, the semigroup $(\hat{T}(t))_{t\geq 0}$ has no nontrivial continuous orbits, hence we must have $\hat{x} = 0$, contradicting the assumption.
(ii) $\Rightarrow$ *(i)* After a rescaling (cf. [3], II.2.2), may assume $s = 1$ and $\lambda = 1$. Take now the approximate eigenvector $(x_n)_{n\in\mathbb{N}}$ as in *(ii)*. The uniform continuity of $(T(t))_{t\geq 0}$ on $(x_n)_{n\in\mathbb{N}}$ implies that the maps $[0,1] \ni t \mapsto T(t)x_n$, $n \in \mathbb{N}$, are equicontinuous. Choose now $x_n' \in X'$, $\|x_n'\| \leq 1$, satisfying $\langle x_n, x_n' \rangle \geq \frac{1}{2}$ for all $n \in \mathbb{N}$. Then the functions

$$[0,1] \ni t \mapsto \xi_n(t) := \langle T(t)x_n, x_n' \rangle$$

are uniformly bounded and equicontinuous. Hence there exists, by the Arzela–Ascoli theorem, a convergent subsequence, still denoted by (ξ_n), such that $\lim_{n\to\infty} \xi_n =: \xi \in \mathrm{C}[0,1]$. From $\xi(0) = \lim_{n\to\infty} \xi_n(0) \geq \frac{1}{2}$, we obtain that $\xi \neq 0$. Therefore, this function has a non-zero Fourier coefficient, i.e., there exists $\mu_m := 2\pi i m$, $m \in \mathbb{Z}$, such that

$$\int_0^1 e^{-\mu_m t}\xi(t)dt \neq 0.$$

If we put

$$z_n := \int_0^1 e^{-\mu_m t}T(t)x_n dt,$$

we have $z_n \in D(A)$ by [3], Lemma II.1.3. In addition, the elements z_n satisfy

$$\begin{aligned}(\mu_m - A)z_n &= \big(1 - e^{-\mu_m}T(1)\big)x_n \\ &= \big(1 - T(1)\big)x_n \to 0\end{aligned}$$

and

$$\begin{aligned}\liminf_{n\to\infty} \|z_n\| &\geq \liminf_{n\to\infty} |\langle z_n, x_n' \rangle| \\ &\geq \liminf_{n\to\infty} \left| \int_0^1 e^{-\mu_m t} \langle T(t)x_n, x_n' \rangle \, dt \right| \\ &\geq \left| \int_0^1 e^{-\mu_m t}\xi(t)dt \right| > 0.\end{aligned}$$

This shows that $\left(\frac{z_n}{\|z_n\|}\right)_{n\in\mathbb{N}}$ is an approximate eigenvector of A with approximate eigenvalue $\mu_m = 2\pi i m$. □

Using the spectral mapping theorem for the point and the residual spectrum and the above equivalence $(i) \Leftrightarrow (ii)$, we can characterize the spectrum of $T(t)$ as

$$\sigma(T(t))\setminus\{0\} = e^{t\cdot\sigma(A)} \;\cup\; \{\lambda \in A\sigma(T(t)) : \overline{\lim}_{s\to 0} \sup_{n\in\mathbb{N}} \|T(s)x_n - x_n\| > 0 \text{ for all approximative eigenvectors } (x_n)_{n\in\mathbb{N}} \text{ corresponding to } \lambda\}\setminus\{0\}.$$

This clumsy extra set can now be replaced by the critical spectrum.

3.2. Theorem. *For a strongly continuous semigroup $(T(t))_{t\geq 0}$ with generator $(A, D(A))$ one has*

$$\sigma(T(t))\setminus\{0\} = e^{t\sigma(A)} \;\cup\; \sigma_{crit}(T(t))\setminus\{0\} \textit{ for all } t \geq 0.$$

Proof. By the spectral mapping theorem for the point and the residual spectrum and by Lemma 3.1, it suffices to show that each approximate eigenvalue $0 \neq \lambda \in A\sigma(T(t))$ with approximate eigenvector $(x_n)_{n\in\mathbb{N}} \notin \ell^\infty_T(X)$ belongs to $\sigma_{crit}(T(t))$. However, this follows since $\hat{x} := (x_n)_{n\in\mathbb{N}} + \ell^\infty_T(X) \in \hat{X}\setminus\{0\}$ and

$$(T(t)x_n - \lambda x_n)_{n\in\mathbb{N}} \in \ell^\infty_T(X), \text{ hence } \hat{T}(t)\hat{x} = \lambda\hat{x}.$$

□

As an immediate consequence of this theorem and of Example 2.8(i) we obtain the spectral mapping theorem for eventually norm continuous semigroups.

3.3. Corollary. *For an eventually norm continuous semigroup $(T(t))_{t\geq 0}$ with generator A one has*

$$\sigma(T(t))\setminus\{0\} = e^{t\sigma(A)} \textit{ for all } t \geq 0.$$

4. The critical growth bound

We now use the critical spectrum and Theorem 3.2 to describe the asymptotic behavior of the semigroup $\mathcal{T} = (T(t))_{t\geq 0}$. To that purpose we recall the definition of the growth bound (see [3], Definition IV.2.1), of the essential growth bound (see [5]), and of the critical growth bound (see Definition 2.3). Each of these bounds can be characterized in terms of the spectral radius $r(T(t))$, the essential spectral radius $r_{ess}(T(t))$, and the *critical spectral radius* $r_{crit}(T(t)) := r(\hat{T}(t))$, respectively. We state the result only for the critical growth bound.

4.1. Lemma. *For a strongly continuous semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ one has*

$$\omega_{crit}(\mathcal{T}) = \inf_{t>0} \frac{1}{t}\log\|\hat{T}\| = \lim_{t\to\infty} \frac{1}{t}\log\|\hat{T}\| = \frac{1}{t_0}\log r(\hat{T}(t_0)) \textit{ for each } t_0 > 0.$$

The proof is similar to the proof of Proposition IV.2.2 in [3]. This lemma (and the analogous version for $\omega_{ess}(\mathcal{T})$) and Theorem 2.5 immediately imply the following inequalities.

4.2. Proposition. *For the growth bounds of a strongly continuous semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ one has*

$$\omega_{crit}(\mathcal{T}) \leq \omega_{ess}(\mathcal{T}) \leq \omega_0(\mathcal{T}).$$

In addition, we obtain from Theorem 3.2 that

$$r(T(t_0) = \max\{e^{t_0 \cdot s(A)}, r_{crit}(T(t_0))\} \text{ for each } t_0 > 0.$$

Here, $s(A)$ denotes the *spectral bound* of the generator A, i.e.

$$s(A) = \sup\{\text{Re } \mu : \mu \in \sigma(A)\}.$$

Taking the logarithm and dividing by t_0, we obtain a characterization of the growth bound $\omega_0(\mathcal{T})$.

4.3. Proposition. *For a strongly continuous semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ with generator A one has*

$$\omega_0(\mathcal{T}) = \max\{s(A), \omega_{crit}(\mathcal{T})\}.$$

Moreover, the following partial spectral mapping theorem holds:

$$\sigma(T(t)) \cap \{\lambda \in \mathbb{C} : |\lambda| > r_{crit}(T(t))\} = e^{t\sigma(A)} \cap \{\lambda \in \mathbb{C} : |\lambda| > r_{crit}(T(t))\}$$

for each $t \geq 0$.

This result is useful only if $\omega_{crit}(\mathcal{T}) < \omega_0(\mathcal{T})$. Therefore, we introduce a special name for these semigroups.

4.4. Definition. A strongly continuous semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ satisfying $\omega_{crit}(\mathcal{T}) < \omega_0(\mathcal{T})$ is called *asymptotically norm continuous.*

For asymptotically norm continuous semigroups we obtain the spectral mapping theorem for the boundary spectrum as an immediate consequence of Theorem 3.2, i.e.

$$\sigma(T(t)) \cap \{\lambda \in \mathbb{C} : |\lambda| = r(T(t))\} = e^{t\sigma(A)} \cap \{\lambda \in \mathbb{C} : |\lambda| = r(T(t))\}$$

for each $t \geq 0$. In particular, one has

$$\omega_0(\mathcal{T}) = s(A)$$

for these semigroups. Clearly, each eventually norm continuous semigroup is asymptotically norm continuous.

In the final part of this paper we show that the asymptotically norm continuous semigroups defined above coincide with the semigroups introduced by Martínez-Mazón [7] (called *norm continuous at infinity*) and studied later by Thieme [10] and Blake [2]. To that purpose we recall a definition from [2].

4.5. Definition. For a strongly continuous semigroup $\mathcal{T} = (T(t))_{t\geq 0}$, we define a growth bound of non-norm-continuity by

$$\delta(\mathcal{T}) := \inf\{v \in \mathbb{R} : \exists M > 0 \text{ such that } \overline{\lim_{h\downarrow 0}} \|T(t+h) - T(t)\| \leq Me^{vt}\}.$$

For this new growth bound, one can show that

$$\delta(\mathcal{T}) = \inf_{t>0} \frac{1}{t} \log \overline{\lim_{h\downarrow 0}} \|T(t+h) - T(t)\| = \lim_{t\to\infty} \frac{1}{t} \log \overline{\lim_{h\downarrow 0}} \|T(t+h) - T(t)\|.$$

Using the norm $|\!|\!| \cdot |\!|\!|$ on $\hat{X}$ introduced in Lemma 2.6, we can show the following result.

4.6. Proposition. *For a strongly continuous semigroup $\mathcal{T} = (T(t))_{t\geq 0}$, one has*

$$\delta(\mathcal{T}) = \omega_{crit}(\mathcal{T}).$$

Proof. From the equivalence of the norm $\|\!| \cdot \|\!|$ and the quotient norm $\| \cdot \|$ on $\hat{X}$ (see Lemma 2.6), we obtain

$$\begin{aligned} \delta(\mathcal{T}) &= \lim_{t\to\infty} \frac{1}{t} \log(\overline{\lim_{h\downarrow 0}} \|T(t+h) - T(t)\|) \\ &= \lim_{t\to\infty} \frac{1}{t} \log \|\hat{T}(t)\| = \omega_{crit}(\mathcal{T}), \end{aligned}$$

□

Now it is easy to prove that Definition 4.4 of asymptotically norm continuous semigroups is equivalent to the definitions in [7] and [2].

4.7. Corollary. *For a strongly continuous semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ the following assertions are equivalent.*

(i) $(T(t))_{t\geq 0}$ *is asymptotically norm continuous, i.e.* $\omega_{crit}(\mathcal{T}) < \omega_0(\mathcal{T})$.

(ii) $\lim_{t\to\infty} \overline{\lim_{h\downarrow 0}}\, e^{-\omega_0(\mathcal{T})\cdot t}\ \|T(t+h) - T(t)\| = 0.$

Proof. Using Proposition 4.6 the implication $(i) \Rightarrow (ii)$ is obvious. To show $(ii) \Rightarrow (i)$ we observe that the map

$$t \mapsto \overline{\lim_{h\downarrow 0}}\, e^{-\omega_0(\mathcal{T})\cdot t}\ \|T(t+h) - T(t)\|$$

is submultiplicative (see [2], Proposition 3.5). Since by assumption (ii) it tends to 0, it decays exponentially fast, and again by Proposition 4.6 we conclude (i). □

References

[1] F. Andreu, J. Martínez, J. M. Mazón, A spectral mapping theorem for perturbed strongly continuous semigroups, *Math. Ann.* **291** (1991), 453-462.
[2] M. D. Blake, A spectral bound for asymptotically norm-continuous semigroups, Preprint 1998.
[3] K. Engel and R. Nagel, One-parameter Semigroups for Linear Evolution Equations, Graduate Text in Math., Springer-Verlag (to appear), June 1998 (preliminary version).
[4] E. Hille and R.S.Phillips, Functional Analysis and Semigroups, Amer. Math. Soc. Coll. Publ., vol. 31, Amer. Math. Soc., 1957.
[5] T. Kato, Perturbation Theory for Linear Operators, Springer-Verlag, 1966.
[6] M. A. Liapunov, Problème Général de la Stabilité du Mouvement (reprint), Princeton Univ. Press, 1949.
[7] J. Martínez and J. M. Mazón, C_0-semigroups norm continuous at infinity, *Semigroup Forum* **52** (1996), 213-224.
[8] R. Nagel (ed.), One-parameter Semigroups of Positive Operators, Lect. Notes in Math., vol. 1184, Springer-Verlag, 1986.
[9] M. Renardy, On the stability of differentiability of semigroups, *Semigroup Forum* **51** (1995), 343-346.
[10] H.R. Thieme, Balanced exponential growth of operator semigroups, Preprint.
[11] J. van Neerven, The Asymptotic Behaviour of Semigroups of Linear Operators, Operator Theory Adv. Appl., vol. 88, Birkhäuser Verlag, 1996.
[12] J. Voigt, Stability of the essential type of strongly continuous semigroups, *Trans. Steklov Math. Inst.* **203** (1994), 469-477.

Mathematisches Institut, Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany.
E-mail address: rana@michelangelo.mathematik.uni-tuebingen.de
E-mail address: japo@michelangelo.mathematik.uni-tuebingen.de