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Derivatives and the Role of the Drinfel'd Twist in Noncommutative String Theory

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Abstract

We consider the derivatives which appear in the context of noncommutative string theory. First, we identify the correct derivations to use when the underlying structure of the theory is a quasitriangular Hopf algebra. Then we show that this is a specific case of a more general structure utilising the Drinfel'd twist. We go on to present reasons as to why we feel that the low-energy effective action, when written in terms of the original commuting coordinates, should explicitly exhibit this twisting.

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1 Introduction

Recent works on theories of open strings and Dp -branes with a constant nonvanishing Neveu-Schwarz 2-form B_{ij} have suggested that the noncommutativity which appears is an underlying and very general property of such theories [1]–[14]. Since Hopf algebras (HAs) often lie at the root of noncommutative systems, we were motivated to look for a HA structure for these theories, and showed that the noncommutative $*$ -product [15]–[18] was in fact a specific case of a more general multiplication defined in terms of the R-matrix R of a quasitriangular HA \mathcal{H} ; furthermore, when $\mathcal{H} = \mathcal{F}$, where \mathcal{F} was the HA of functions on \mathbb{R}^{p+1} , as was the case for the aforementioned noncommutative string theories, we found an explicit form for R [19] which covers both the commutative ($B_{ij} = 0$) and noncommutative ($B_{ij} \neq 0$) cases.

However, it was not immediately apparent how we could introduce derivations on the algebra endowed with this multiplication, $\widehat{\mathcal{H}}$. One way to think of derivations on a HA \mathcal{H} is as elements of the dually paired HA \mathcal{H}^* , with the action of the latter on the former given in terms of the HA properties of both. The problem was that $\widehat{\mathcal{H}}$ was shown *not* to be a HA, and therefore neither was the dually paired space $\widehat{\mathcal{H}}^*$. This precluded the interpretation of the latter as local derivatives on $\widehat{\mathcal{H}}$, so it was not immediately obvious how one might define a gauge theory on the noncommutative space, since we needed a derivative in order to construct the (noncommutative) field strength tensor \widehat{F}_{ij} from the gauge field \widehat{A}_i , *i.e.* $\widehat{F}_{ij} = \partial_{[i}\widehat{A}_{j]} - i\widehat{A}_{[i} * \widehat{A}_{j]}$ [14]. The question was, what could we use for ∂_i ? We speculated that we might have to replace local derivatives by difference operators, but this guess seemed to be contradicted by the fact that regular derivatives were used consistently in [14].

In this follow-up note to [19], we explain why the usual derivatives are in fact the correct ones when dealing with noncommutative string theory: We show that even though $\widehat{\mathcal{H}}$ is not a HA, there exists a HA which has a well-defined action on $\widehat{\mathcal{H}}$ and plays the role of the space of derivations. This holds for arbitrary \mathcal{H} , and for the specific case where $\mathcal{H} = \mathcal{F}$, this HA is \mathcal{F}^* and the action is the same as that for the usual partial derivative. This is done in Section 2.

The ability to relate the commutative and noncommutative theories via the R-matrix, however, turns out to be a bit of a fluke, being true only if \mathcal{H}^* is cocommutative. While this is certainly true of the space of derivations \mathcal{F}^* , if we want to be as general as possible, we must relax this condition. In Section 3, we demonstrate how this can be done by using the Drinfel'd twist [20], which allows us to find a generalisation of the $*$ -product and the space of derivations on the algebra constructed with this $*$. (Related but more mathematical treatments of this construction may be found in [21, 22], and very recently [23], which covers much of the same in a broader context.)

However, using the Drinfel'd twist gives exactly the same derivatives and noncommutative product as if we had used an R-matrix approach, so why pick one over the other? In Section 4, we present two arguments why we think the

former is more appropriate: First, the R-matrix construction connects the commutative and noncommutative cases only when \mathcal{H} is commutative (*i.e.* \mathcal{H}^* is cocommutative), whereas the Drinfel'd twist includes both cases, and therefore does not require us to make any *a priori* assumptions about the algebraic structure of \mathcal{H} . Secondly, an R-matrix must satisfy two coproduct conditions, while the analogous element in the Drinfel'd twisting only has to fulfill one, and there may be a way of naturally implementing the latter using the Ward identity which must arise out of gauge-fixing the form of B_{ij} . These reasons lead us to think that the Drinfel'd twist plays a fundamental role in noncommutative string theory, specifically in helping to determine the form of the low-energy effective action of the theory in terms of the *commutative* coordinates, *e.g.* Born-Infeld.

Throughout this letter we use terms and notations described in our previous paper [19]; the reader is referred therein for the details.

2 The Leibniz Rule and the *-product

We begin with a HA \mathcal{H} and its dually paired HA \mathcal{H}^* . The (left) action of $x \in \mathcal{H}^*$ on $f \in \mathcal{H}$ is given by $x \cdot f := f_{(1)} \langle x, f_{(2)} \rangle$, and satisfies the Leibniz rule $x \cdot (fg) = (x_{(1)} \cdot f) (x_{(2)} \cdot g)$. The elements of \mathcal{H}^* (with the exception of the unit 1 and its multiples) thus may be thought of as derivations on \mathcal{H} .

In the case where \mathcal{H}^* is quasitriangular with R-matrix R , we can define a new multiplication between $f, g \in \mathcal{H}$ as

$$f * g := f_{(1)} g_{(1)} \langle R_{21}, f_{(2)} \otimes g_{(2)} \rangle. \quad (2.1)$$

$\widehat{\mathcal{H}}$ is taken to be the algebra equivalent to \mathcal{H} as a vector space and with the multiplication $*$. It is *not* a HA, so neither is the dually paired coalgebra $\widehat{\mathcal{H}}^*$. We therefore cannot think of $\widehat{\mathcal{H}}^*$ as derivations of $\widehat{\mathcal{H}}$.

However, let's go ahead and compute the action of $x \in \mathcal{H}^*$ on the product $f * g$:

$$\begin{aligned} x \cdot (f * g) &= x \cdot (f_{(1)} g_{(1)}) \langle R_{21}, f_{(2)} \otimes g_{(2)} \rangle \\ &= f_{(1)} g_{(1)} \langle x, f_{(2)} g_{(2)} \rangle \langle R_{21}, f_{(3)} \otimes g_{(3)} \rangle \\ &= f_{(1)} g_{(1)} \langle \Delta(x) \otimes R_{21}, f_{(2)} \otimes g_{(2)} \otimes f_{(3)} \otimes g_{(3)} \rangle \\ &= f_{(1)} g_{(1)} \langle \Delta(x) R_{21}, f_{(2)} \otimes g_{(2)} \rangle. \end{aligned} \quad (2.2)$$

Using $\Delta(x) = R_{21} (\tau \circ \Delta(x)) R_{21}^{-1}$, we obtain

$$\begin{aligned} x \cdot (f * g) &= f_{(1)} g_{(1)} \langle R_{21} (\tau \circ \Delta)(x), f_{(2)} \otimes g_{(2)} \rangle \\ &= f_{(1)} g_{(1)} \langle R_{21} \otimes (\tau \circ \Delta)(x), f_{(2)} \otimes g_{(2)} \otimes f_{(3)} \otimes g_{(3)} \rangle \\ &= f_{(1)} g_{(2)} \langle R_{21}, f_{(2)} \otimes g_{(2)} \rangle \langle x_{(2)} \otimes x_{(1)}, f_{(3)} \otimes g_{(3)} \rangle \end{aligned}$$

$$\begin{aligned}
&= (f_{(1)} * g_{(2)}) \langle x_{(2)}, f_{(2)} \rangle \langle x_{(1)}, g_{(2)} \rangle \\
&= (x_{(2)} \cdot f) * (x_{(1)} \cdot g).
\end{aligned} \tag{2.3}$$

So we see that the Leibniz rule is ‘reversed’: The first piece of the coproduct of x acts on the *second* function, and vice versa.

In [19], we reviewed the construction of the HA dually paired to \mathcal{H} , denoted \mathcal{H}^* . Recall that the coproduct and antipode for $x \in \mathcal{H}^*$ were defined by

$$\begin{aligned}
\langle \Delta(x), f \otimes g \rangle &:= \langle x, fg \rangle, \\
\langle S(x), f \rangle &:= \langle x, S(f) \rangle.
\end{aligned} \tag{2.4}$$

However, this is not the only HA which may be constructed to the vector space dual to \mathcal{H} : A different HA, called the *opposite dual* and denoted $\mathcal{H}^{\text{op}\dagger}$, can be defined by keeping all the relations between \mathcal{H} and \mathcal{H}^* except the above two, which are replaced by

$$\begin{aligned}
\langle \Delta'(x), f \otimes g \rangle &:= \langle x, gf \rangle, \\
\langle S'(x), f \rangle &:= \langle x, S^{-1}(f) \rangle.
\end{aligned} \tag{2.5}$$

We see that the coproduct on \mathcal{H}^{op} is the one on \mathcal{H}^* with the two spaces flipped, *i.e.* $\Delta' = \tau \circ \Delta$. From (2.3) we can see that although \mathcal{H}^* does not have an action on $\widehat{\mathcal{H}}$, \mathcal{H}^{op} does, because

$$x \cdot (f * g) = (x_{(1)'} \cdot f) * (x_{(2)'} \cdot g), \tag{2.6}$$

where we have used the notation $\Delta'(x) := x_{(1)'} \otimes x_{(2)'}$. Hence, we conclude that \mathcal{H}^{op} , not \mathcal{H}^* or $\widehat{\mathcal{H}}^*$, is the space of derivations on $\widehat{\mathcal{H}}$. And since in general \mathcal{H} is not cocommutative, the Leibniz rule on $\widehat{\mathcal{H}}$ does not have the same form as that on \mathcal{H} . However, if $\Delta' = \Delta$, *i.e.* \mathcal{H}^* is cocommutative, then \mathcal{H}^{op} and \mathcal{H}^* are the same, and the space of derivations is the same for $\widehat{\mathcal{H}}$ as for \mathcal{H} .

If we now look at noncommutative string theory, the algebra \mathcal{H} is the function algebra \mathcal{F} spanned by monomials in the coordinate maps x^i taking the Dp-brane into \mathbb{R}^{p+1} . The $*$ -product is introduced by using the R-matrix

$$R := e^{-\frac{i}{2}\theta^{ij}\partial_i \otimes \partial_j}, \tag{2.7}$$

where θ^{ij} is related to B_{ij} and the open string metric g_{ij} by [14]

$$\theta^{ij} := -(2\pi\alpha')^2 \left(\frac{1}{g + 2\pi\alpha' B} B \frac{1}{g - 2\pi\alpha' B} \right)^{ij}. \tag{2.8}$$

[†]This is the same well-known HA which plays a key role in the construction of the Drinfel'd double [20].

This R gives the (noncommutative) product between functions f and g as

$$f(x) * g(x) = e^{\frac{i}{2}\theta^{ij} \frac{\partial^2}{\partial \xi^i \partial \zeta^j}} f(x + \xi)g(x + \zeta) \Big|_{\xi=\zeta=0}. \quad (2.9)$$

The dually paired HA \mathcal{F}^* is the space spanned by monomials of the partial derivatives ∂_i . The coproduct is generated by $\Delta(\partial_i) = \partial_i \otimes 1 + 1 \otimes \partial_i$, and with the action on \mathcal{F} being the usual derivative, the Leibniz rule is the familiar

$$\partial_i (f(x)g(x)) = (\partial_i f(x))g(x) + f(x)(\partial_i g(x)) \quad (2.10)$$

(the \cdot signifying the action has been suppressed in the above two equations). But since this HA is cocommutative, this is also the Leibniz rule for the action of \mathcal{F}^{op} on \mathcal{F} . Hence, $\mathcal{F}^* = \mathcal{F}^{\text{op}}$, and this is the reason that one can use the familiar derivatives even when the space is noncommutative, as was done in [14].

3 The Drinfel'd Twist

The material in the preceding Section is in fact a specific example of a more general construction: Suppose \mathcal{H} is a HA such that there exists an invertible element $F \in \mathcal{H} \otimes \mathcal{H}$ which satisfies $(\epsilon \otimes \text{id})(F) = (\text{id} \otimes \epsilon)(F) = 1$ as well as the coproduct identity

$$F_{12}(\Delta \otimes \text{id})(F) = F_{23}(\text{id} \otimes \Delta)(F). \quad (3.1)$$

If this is the case, then a new HA \mathcal{H}^F , called the *Drinfel'd twist* of \mathcal{H} [20], can be defined in the following way: $\mathcal{H}^F = \mathcal{H}$ at the algebra level, and the counit and unit of \mathcal{H}^F are the same as those of \mathcal{H} . The coproduct and antipode, however, are given in terms of those on \mathcal{H} by

$$\Delta^F(f) := F\Delta(f)F^{-1}, \quad S^F(f) := \sigma^{-1}S(f)\sigma, \quad (3.2)$$

where σ is the quantity constructed from $F := F_\alpha \otimes F^\alpha$ (sum implied) via

$$\sigma := m((\text{id} \otimes S))(F) \equiv F_\alpha S(F^\alpha). \quad (3.3)$$

(The inverse can be shown to be $\sigma^{-1} = m((S \otimes \text{id})(F^{-1}))$.) For future reference, we also use the notation $\Delta^F(f) := f_{(1)F} \otimes f_{(2)F}$.

Now suppose we start with dually paired HAs \mathcal{H} and \mathcal{H}^* , and an element F in $\mathcal{H}^* \otimes \mathcal{H}^*$ satisfying (3.1) exists; then a new product, $*$, may be defined on \mathcal{H} via

$$f * g := f_{(1)}g_{(1)} \langle F^{-1}, f_{(2)} \otimes g_{(2)} \rangle. \quad (3.4)$$

We can then check associativity by first computing the triple product $(f * g) * h$:

$$\begin{aligned} (f * g) * h &= (f_{(1)}g_{(1)}) * h \langle F^{-1}, f_{(2)} \otimes g_{(2)} \rangle \\ &= f_{(1)}g_{(1)}h_{(1)} \langle F^{-1}, f_{(2)}g_{(2)} \otimes h_{(2)} \rangle \langle F^{-1}, f_{(3)} \otimes g_{(3)} \rangle \\ &= f_{(1)}g_{(1)}h_{(1)} \langle (\Delta \otimes \text{id})(F^{-1}), f_{(2)} \otimes g_{(2)} \otimes h_{(2)} \rangle \langle F^{-1}, f_{(3)} \otimes g_{(3)} \rangle \\ &= f_{(1)}g_{(1)}h_{(1)} \langle (\Delta \otimes \text{id})(F^{-1}) F_{12}^{-1}, f_{(2)} \otimes g_{(2)} \otimes h_{(2)} \rangle. \end{aligned} \quad (3.5)$$

Computing $f*(g*h)$ in a similar fashion replaces the left argument of the inner product above with $(\text{id} \otimes \Delta)(F^{-1})F_{23}^{-1}$, and if we take the inverse of (3.1), we see the two are equal, and this proves that $*$ is associative. The counit condition of F ensures that 1 is also the $*$ -multiplicative identity as well. We therefore denote by $\widehat{\mathcal{H}}$ the unital associative algebra with vector space \mathcal{H} and multiplication $*$.

One consequence of this definition of $*$ is that the Drinfel'd twist of \mathcal{H}^* is a HA of left actions on $\widehat{\mathcal{H}}$: Taking $x \in \mathcal{H}^*$ and $f, g \in \widehat{\mathcal{H}}$,

$$\begin{aligned}
x \cdot (f * g) &= x \cdot (f_{(1)}g_{(1)}) \langle F^{-1}, f_{(2)} \otimes g_{(2)} \rangle \\
&= f_{(1)}g_{(1)} \langle x, f_{(2)}g_{(2)} \rangle \langle F^{-1}, f_{(3)} \otimes g_{(3)} \rangle \\
&= f_{(1)}g_{(1)} \langle \Delta(x) \otimes F^{-1}, f_{(2)} \otimes g_{(2)} \otimes f_{(3)} \otimes g_{(3)} \rangle \\
&= f_{(1)}g_{(1)} \langle \Delta(x)F^{-1}, f_{(2)} \otimes g_{(2)} \rangle \\
&= f_{(1)}g_{(1)} \langle F^{-1}\Delta^F(x), f_{(2)} \otimes g_{(2)} \rangle \\
&= f_{(1)} * g_{(1)} \langle x_{(1)F} \otimes x_{(2)F}, f_{(2)} \otimes g_{(2)} \rangle \\
&= (x_{(1)F} \cdot f) * (x_{(2)F} \cdot g). \tag{3.6}
\end{aligned}$$

So \mathcal{H}^F has a well-defined action on $\widehat{\mathcal{H}}$, and therefore may be used as the space of derivations on $\widehat{\mathcal{H}}$.

What if \mathcal{H}^* is quasitriangular? Then we automatically have an F which satisfies (3.1), namely, $F = R_{21}^{-1}$. This follows from the coproduct properties of the R-matrix, and we recover all the results in Section 2: We see immediately that the $*$ -product given by plugging R_{21} in for F^{-1} in (3.4) is the same as (2.1). The coproduct is also the same, $\Delta^F = \tau \circ \Delta = \Delta'$. To compare the antipodes, note that σ^{-1} becomes $m(S \otimes \text{id})(R_{21})$, which is the element, usually denoted u , which generates the square of the antipode in \mathcal{H}^* : $uxu^{-1} = S^2(x)$ [24]. This immediately leads to $S^F = S^{-1} = S'$, exactly as expected, and we see that this choice of F gives $(\mathcal{H}^*)^F = \mathcal{H}^{\text{op}}$, which, as we proved, is the correct choice for the space of derivations on $\widehat{\mathcal{H}}$.

There is potentially a wider class of F s than there are of R-matrices, because the one coproduct condition on F (3.1) is less restrictive than the two coproduct conditions on R :

$$(\Delta \otimes \text{id})(R) = R_{13}R_{12}, \quad (\text{id} \otimes \Delta)(R) = R_{13}R_{23}. \tag{3.7}$$

This means that, in principle, there may be other associative $*$ -products besides the one defined using the R-matrix. However, for the noncommutative string case, the cocommutativity of \mathcal{F}^* is still a strong enough condition to force F to be identical to the R given in (2.7), so in this instance, reformulating the $*$ -product on \mathcal{F} in terms of the Drinfel'd twist does not change the fact that

it has the unique form (2.9). Therefore, using the Drinfel'd twist gives exactly the same noncommutativity to the Dp -brane-open string system as using an R-matrix does.

4 Noncommutative String Theory

We conclude this work with some brief speculations about how the Drinfel'd twist may appear in the context of a noncommutative string theory.

In [19], we conjectured that the dependence on θ^{ij} in the effective field theory would appear only through the $*$ -product, and that since this in turn was given in terms of the R-matrix R , then there would be explicit dependence on R in the action when written in terms of the commutative theory. However, this is in fact probably not the case in general, for the following reason: The $*$ -product becomes the commutative product when $\theta^{ij} = 0$, which, using the explicit form (2.7), corresponds to $R = 1 \otimes 1$. When we are dealing with a cocommutative HA where $\Delta' = \Delta$, this is an admissible R-matrix, but for the most general case where the HA may *not* be cocommutative, $1 \otimes 1$ doesn't work as an R-matrix.

However, the Drinfel'd twist construction is still applicable, because $F = 1 \otimes 1$ satisfies (3.1), and just gives the trivial case $(\mathcal{H}^*)^F = \mathcal{H}^*$. Thus, if θ is some element of a parameter space, and there exists a continuous map $\theta \mapsto F(\theta)$ which satisfies (3.1) and $F(0) = 1 \otimes 1$, we have a family of spaces $\hat{\mathcal{H}}(\theta)$ and Drinfel'd twists $\mathcal{H}^F(\theta)$ continuously connected to the undeformed cases \mathcal{H} and \mathcal{H}^* , respectively, with elements of the latter being derivations on elements of the former. This deformation does not depend on the cocommutativity, or lack thereof, of \mathcal{H}^* . Furthermore, since the coproduct of \mathcal{H}^* is dual to the multiplication on \mathcal{H} , this also implies that we do not even have to start with a commutative \mathcal{H} for this procedure to be valid.

When we look at the specific case of a Dp -brane/open string system, where we expect to be able to go continuously from the commutative case with vanishing B_{ij} to the noncommuting theory, it therefore seems reasonable to us that the θ -dependence in the effective action when expressed as an integral over the commutative space will be entirely through an F and not an R , and that the underlying structure is that of a Drinfel'd twisted HA rather than a quasitriangular one.

There is another reason to favour the Drinfel'd twist: Recall that the recent work on noncommutative string theory has been done with a *constant* B_{ij} . Since the full theory should be invariant under the gauge transformation $B_{MN} \mapsto B_{MN} + \partial_{[M}\lambda_{N]}$ for any λ_M , where $M, N = 0, \dots, 9$, taking $B_{MN} = 0$ for $M, N = p+1, \dots, 9$ and constant for $M, N = 0, \dots, p$ is a gauge fixing condition. We should therefore expect to find a Ward identity resulting from this fixing. To us, it seems very likely that this Ward identity is related to (3.1). This

explanation is attractive because, if correct, it means we do not have to impose (3.1) by hand; it comes out naturally from taking B_{ij} to be constant. And just as a Ward identity must hold to have a self-consistent theory, *i.e.* gauge invariance, so must (3.1) hold for consistency, *i.e.* $*$ is associative. For an R-matrix to be involved instead of F , we would have to come up with some way of arriving at the *two* conditions (3.7), and the single requirement that the theory have B -gauge invariance would presumably not give these.

Thus, the signs point more toward a Drinfel'd twisted rather than a quasi-triangular HA; more specifically, if we think of the 'undeformed' theory as one formulated with the HA \mathcal{H} (commutative or not), and the 'deformed' one as that on the algebra $\widehat{\mathcal{H}}$, then the former should have explicit F -dependence. This fact may therefore give some clues as to the explicit form of the undeformed low-energy effective action, even though it is presumably very complicated (unlike the deformed version, which may be very nice, *e.g.* super-Yang-Mills [14]).

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