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# AdS/CFT Correspondence with Heat Conduction

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**Abstract**

We study perturbations of the gravity dual to a perfect fluid model recently found by Janik and Perschanski [hep-th/0512162]. We solve the Einstein equations in the bulk AdS space for a metric ansatz which includes off-diagonal terms. Through holographic renormalization, we show that these terms give rise to heat conduction in the corresponding CFT on the boundary.

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# 1 Introduction

It is intriguing that strongly interacting particles produced in heavy ion collisions exhibit perfect fluid characteristics which are appropriately described by a hydrodynamic model. The hydrodynamic behavior in the context of Quantum Chromodynamics (QCD) was studied by Bjorken in the early eighties [1]. It remains a challenge to derive such behavior from first principles in QCD. On the other hand, recent experimental results at RHIC have provided experimental evidence of a hydronamic form of strongly interacting matter [2].

Much has been learned about gauge theories in the strong coupling regime through the AdS/CFT correspondence [3][4] which enables us to understand gauge theories by studying their gravity duals in AdS space with one additional dimension. Strictly speaking, this correspondence applies to the maximally symmetric $\mathcal{N} = 4$ super Yang-Mills theory. Nevertheless, its application to gauge theories with less symmetry, including QCD, has been considered. In particular, regarding heavy ion collisions, a gravity dual has been used to extract information about jet quenching [5], transport coefficients, the fireball produced at RHIC [6], as well as a bound on $\eta/s$ [7].

In an interesting recent work, Janik and Peschanski [8] discussed a solution to the Einstein equations in the bulk AdS space in the limit $\tau \to \infty$, where $\tau$ is the proper time in the longitudinal plane. By demanding regularity in the bulk, they showed that the acceptable solution corresponds through holographic renormalization [9] to a perfect fluid on the boundary of AdS.

Here we extend the results of [8] by including off-diagonal terms in the bulk metric ansatz. We solve the Einstein Equations in the bulk in the limit $\tau \to \infty$ keeping both leading and sub-leading contributions. We obtain an exact non-singular solution and show, through holographic renormalization, that it corresponds to a perturbation of the perfect fluid which includes heat conduction. We relate the temperature gradient to the form of the bulk metric.

# 2 Perfect Fluid

We are interested in understanding the behavior of a fluid described by a gauge theory in a four-dimensional space spanned by coordinates $x^\mu$, $(\mu = 0, 1, 2, 3)$. Following [1], we introduce the proper time $\tau$ and rapidity $y$ on the longitudinal plane, defined by

$$x^0 = \tau \cosh y \quad , \quad x^1 = \tau \sinh y \tag{1}$$

The transverse coordinates will be denoted by $x^\perp = (x^2, x^3)$. The gravity dual of the four-dimensional theory will be five-dimensional. Let $z$ denote the fifth dimension. We shall solve the Einstein equations in the bulk

$$R_{AB} - \frac{1}{2} g_{AB} R - 6 g_{AB} = 0 \tag{2}$$

where we chose units so that the cosmological constant is $\Lambda = -6$. Following [8], we adopt the metric ansatz

$$ds^2 = \frac{1}{z^2}\left[-e^{a(\tau,z)}d\tau^2 + \tau^2 e^{b(\tau,z)}dy^2 + e^{c(\tau,z)}dx^{\perp 2} + dz^2\right] \tag{3}$$

which is the most general bulk metric obeying boost invariance, the $y \to -y$ symmetry, plus translational and rotational invariance. It is convenient to introduce the coordinate

$$v = \frac{z}{\tau^\eta} \tag{4}$$

in terms of which the metric (3) reads

$$ds^2 = \frac{1}{v^2\tau^{2\eta}}\left[-(e^{a(\tau,v)} - \eta^2 v^2 \tau^{2(\eta-1)})d\tau^2 + \tau^2 e^{b(\tau,v)}dy^2 + e^{c(\tau,v)}dx^{\perp 2} + 2\eta v \tau^{2\eta-1}dv d\tau + \tau^{2\eta}dv^2\right] \tag{5}$$

Substituting this metric ansatz into the vacuum Einstein equations we obtain the leading behaviour in the $\tau \to \infty$ limit [8]

$$\begin{aligned} a(v) &= A(v) - 2m(v) \\ b(v) &= A(v) + 2(4\eta - 1)m(v) \\ b(v) &= A(v) + 2(1 - 2\eta)m(v) \end{aligned} \tag{6}$$

where

$$\begin{aligned} A(v) &= \frac{1}{2}\left[\ln(1 + \Delta(\eta)v^4) + \ln(1 - \Delta(\eta)v^4)\right] \\ m(v) &= \frac{1}{4\Delta(\eta)}\left[\ln(1 + \Delta(\eta)v^4) - \ln(1 - \Delta(\eta)v^4)\right] \end{aligned} \tag{7}$$

with

$$\Delta(\eta) = \sqrt{\frac{1}{3}(6\eta^2 - 4\eta + 1)} \tag{8}$$

Demanding regularity in the bulk, namely that the square of the Riemann tensor, $\mathcal{R}^2 = R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}$ be non-singular, one obtains the unique value

$$\eta = \frac{1}{3} \tag{9}$$

The above bulk metric may be related to the vacuum expectation value of the stress-energy tensor of the corresponding gauge theory on the boundary through holographic renormalization [9]. This is done as follows. The metric (3) is of the general asymptotically AdS form

$$ds^2 = \frac{g_{\mu\nu}dx^\mu dx^\nu + dz^2}{z^2} \tag{10}$$

Near the boundary at $z = 0$ we may expand

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + z^2 g^{(2)}_{\mu\nu} + z^4 g^{(4)}_{\mu\nu} + \ldots \tag{11}$$

where $g^{(0)}_{\mu\nu} = \eta_{\mu\nu}$ and $g^{(2)}_{\mu\nu} = 0$. The next coefficient, $g^{(4)}_{\mu\nu}$, is proportional to the vacuum expectation value of the stress-energy tensor. Thus,

$$\langle T_{\mu\nu} \rangle \propto g^{(4)}_{\mu\nu} \tag{12}$$

It is easily seen from the explicit form of $g^{(4)}_{\mu\nu}$, for the special value (9) of $\eta$, that the stress-energy tensor corresponds to that of a perfect fluid

$$T_{\alpha\beta} = (\rho + p) u_\alpha u_\beta + p \eta_{\alpha\beta} \tag{13}$$

obeying the equation of state $p = \frac{1}{3}\rho$ (tracelessness due to conformal invariance). Further, from the explicit form of the metric, it follows that the mass density and temperature falls off, respectively, as

$$\rho \sim \frac{1}{\tau^{4/3}} \quad , \qquad T \sim \frac{1}{\tau^{1/3}} \tag{14}$$

# 3 Perturbations about the Perfect Fluid

Next, we consider perturbations about the perfect fluid model discussed above. To this end, we add appropriate off-diagonal terms in the bulk metric (5), presumed small, that will lead to new solutions to the Einstein equations (2). To find the new solutions, we need to keep leading as well as subleading terms in the Einstein equations in the large $\tau$ limit. We therefore consider the metric ansatz

$$\begin{aligned} ds^2 &= \frac{1}{v^2 \tau^{2\eta}} \left[ -(e^{a(v)} - \eta^2 v^2 \tau^{2(\eta-1)}) d\tau^2 + \tau^2 e^{b(v)} dy^2 + e^{c(v)} dx^{\perp 2} + 2\eta v \tau^{2\eta-1} dv d\tau + \tau^{2\eta} dv^2 \right] \\ &+ 2\epsilon \tau^\lambda h_{||}(v) d\tau dy + 2\epsilon \tau^\lambda h_\perp(v) d\tau dx^\perp \end{aligned} \tag{15}$$

where the second line is the correction to the "perfect fluid" ansatz (5). We have included a cataloging (small) parameter $\epsilon$. The exponent $\lambda$ of $\tau$ and the three functions, one longitudinal $h_{||}(v)$ and two transverse $h_\perp(v) = (h_2(v), h_3(v))$, are to be determined by the Einstein equations (2).

Substituting this perturbed metric ansatz into (2), we find six additional differential equations for the $vy$, $y\tau$, the two $vx^\perp$and the two $x^\perp\tau$ components, respectively. For the parameter choice of

$$\lambda = 2\eta - 1 \tag{16}$$

we find that the $vy$ and $y\tau$ components coalesce, yielding one differential equation which uniquely determines $h_{||}(v)$

$$\begin{aligned} &\left[-36 - 144(2\eta - 1)v^4 + 216(1 - 4\eta + 2\eta^2)v^8 - 48\sigma(2\eta - 1)v^{12} - 4\sigma^2 v^{16}\right] h_{||}(v) + \\ &\quad \left[9v + 72v^5(1 - 2\eta) - 6\sigma v^9 + \sigma^2 v^{17} + 24\sigma v^{13}(-1 + 2\eta)\right] h'_{||}(v) + \\ &\quad\quad \left[9v^2 - 6\sigma v^{10} + \sigma^2 v^{18}\right] h''_{||}(v) = 0 \end{aligned} \tag{17}$$

where we defined $\sigma = 1 - 4\eta + 6\eta^2$.

The differential equation has a regular solution,

$$h_{||}(v) = \frac{1}{v^2}\left[\frac{1-\Delta v^4}{1+\Delta v^4}\right]^{(\frac{1}{2}-\eta)/\Delta}\left(1-\Delta^2 v^8\right)^{\frac{1}{2}-\frac{\eta}{\Delta}}\left[(1-\Delta v^4)^{2\eta/\Delta}-(1+\Delta v^4)^{2\eta/\Delta}\right] \tag{18}$$

It also has a singular solution which we discard. The regular solution (18) may be expanded as

$$h_{||}(v) \sim v^2 + (2\eta - 1)v^6 + \ldots \tag{19}$$

where we omitted an irrelevant overall constant factor. For the special choice (9) of $\eta$, eq. (18) reduces to

$$h_{||}(v) = \frac{v^2}{1+\frac{1}{3}v^4} \tag{20}$$

Returning to the Einstein equations, we see that our last four equations perfectly coalesce and found to yield

$$\begin{aligned}&\left(9v^2 - 6\sigma v^{10} + \sigma^2 v^{18}\right) h_\perp''(v) + \left(9v + 72v^5\eta - 6\sigma v^9 - 24\eta\sigma v^{13} + \sigma^2 v^{17}\right) h_\perp'(v)\\ &+ \left(-36 + 144v^4\eta + 72v^8\left(-1+6\eta^2\right) + 48\eta\sigma v^{12} - 4\sigma^2 v^{16}\right) h_\perp(v) = 0\end{aligned} \tag{21}$$

This differential equation admits the regular solution

$$h_\perp(v) = \frac{1}{v^2}\left(\frac{\Delta v^4 - 1}{\Delta v^4 + 1}\right)^{\eta/2\Delta}\left(1-\Delta^2 v^8\right)^{\frac{1-\eta+\Delta}{2\Delta}}\left((1-\Delta v^4)^{(\eta-1)/\Delta} - (1+\Delta v^4)^{(\eta-1)/\Delta}\right) \tag{22}$$

and a singular solution which we discard.

The regular solution (22) may be expanded as

$$h_\perp(v) \sim v^2 - \eta v^6 + \ldots \tag{23}$$

and for the special value (9) of $\eta$, it reduces to

$$h_\perp(v) = \frac{v^2}{1+\frac{1}{3}v^4} \tag{24}$$

which is identical to the form of the longitudinal function (20).

Recall that the special value (9) of $\eta$ was derived by demanding regularity of the square of the Riemann tensor, $\mathcal{R}^2 = R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}$ for the unperturbed metric ansatz (5). It takes a straightforward calculation to show that the perturbed metric (15) leads to the same conclusion, because $\mathcal{R}^2$ is not modified by the perturbation.

Finally, we find our perturbed metric (15) to take the form

$$ds^2 = \frac{1}{z^2}\left[-\frac{\left(1-\frac{e}{3}\frac{z^4}{\tau^{4/3}}\right)^2}{1+\frac{e}{3}\frac{z^4}{\tau^{4/3}}}d\tau^2 + \left(1+\frac{e}{3}\frac{z^4}{\tau^{4/3}}\right)(\tau^2 dy^2 + dx_\perp^2) + dz^2 + \frac{2\epsilon z^4}{1+\frac{e}{3}\frac{z^4}{\tau^{4/3}}}\left[\mathcal{A}_{||}dy + \mathcal{A}_\perp dx^\perp\right]\frac{d\tau}{\tau}\right] \tag{25}$$

where we included a dimensionful constant $e$ [8]. $\mathcal{A}_{||}$ and $\mathcal{A}_\perp$ are arbitrary constants characterizing the perturbation.

# 4 Hydrodynamics

Having obtained the explicit form of the bulk metric (25), we now invoke holographic renormalization [9] in order to understand the dynamics of the corresponding gauge theory on the boundary. It follows from eq. (12) and the expansions (19) and (23) of the metric perturbation that the vacuum expectation value of the gauge theory stress-energy tensor has off-diagonal components which behave as

$$\langle T_{y\tau}\rangle \sim g^{(4)}_{y\tau} \sim \frac{1}{\tau} \qquad , \qquad \langle T_{\perp\tau}\rangle \sim g^{(4)}_{\perp\tau} \sim \frac{1}{\tau} \tag{26}$$

In order to understand how our solution relates to the gauge theory fluid, let us start by working with an arbitrary stress tensor with diagonal, $\tau y$ and $\tau\perp$ components,

$$T^{\mu\nu} = \begin{pmatrix} T^{\tau\tau} & T^{\tau y} & T^{\tau 2} & T^{\tau 3} \\ T^{\tau y} & T^{yy} & 0 & 0 \\ T^{\tau 2} & 0 & T^{22} & 0 \\ T^{\tau 3} & 0 & 0 & T^{33} \end{pmatrix} . \tag{27}$$

Recalling our definition (1) of coordinates $\tau$ and $y$, the metric on the Minkowski space of the fluid reads

$$ds_4^2 = -d\tau^2 + \tau^2 dy^2 + (dx^\perp)^2 \tag{28}$$

From the local conservation law

$$\nabla_\alpha T^{\alpha\beta} = \partial_\alpha T^{\alpha\beta} + \Gamma^\alpha_{\alpha\lambda} T^{\lambda\beta} + \Gamma^\beta_{\alpha\lambda} T^{\alpha\lambda} = 0 \tag{29}$$

and using the Christoffel symbols $\Gamma^y_{y\tau} = \frac{1}{\tau} = \Gamma^y_{\tau y}$ and $\Gamma^\tau_{yy} = \tau$, we derive relations between the components of the stress tensor.

Choosing $\beta = \tau$, we obtain

$$\partial_\tau T^{\tau\tau} + \partial_y T^{\tau y} + \frac{1}{\tau} T^{\tau\tau} + \tau T^{yy} + \partial_\perp T^{\tau\perp} = 0 \tag{30}$$

Setting $\beta = y$, we deduce

$$\partial_t T^{\tau y} + \partial_y T^{yy} + \frac{3}{\tau} T^{\tau y} = 0 \tag{31}$$

and from the transverse components, $\beta = x^i$ $(i = 2, 3)$, we have

$$\partial_\tau T^{\tau i} + \frac{1}{\tau} T^{\tau i} + \partial_j T^{ij} = 0 \tag{32}$$

One more relation is a consequence of conformal invariance. Demanding tracelessness, we obtain

$$-T^{\tau\tau} + \tau^2 T^{yy} + T^{22} + T^{33} = 0 \tag{33}$$

In order to match with the expected form of the stress-energy tensor from holographic renormalization, we observe that the components of the stress-energy tensor to the order we are considering should not depend on the rapidity $y$ or the transverse coordinates $x^\perp$. We may then immediately solve for the off-diagonal components $T^{\tau y}$ and $T^{\tau\perp}$, obtaining

$$T^{\tau y} = \frac{\mathcal{C}_{||}}{\tau^3} \quad , \quad T_{\tau y} = \frac{-\mathcal{C}_{||}}{\tau} \tag{34}$$

$$T^{\tau\perp} = \frac{\mathcal{C}_\perp}{\tau} \quad , \quad T_{\tau\perp} = \frac{-\mathcal{C}_\perp}{\tau} \tag{35}$$

with the $\mathcal{C}$'s being constants. Solving for the diagonal components $T^{yy}, T^{ii}$ $(i = 2, 3)$, we obtain the stress-energy tensor in terms of $T^{\tau\tau}$ (energy density) and the three arbitrary constants $\mathcal{C}_{||}, \mathcal{C}_\perp$,

$$T^{\mu\nu} = \begin{pmatrix} T^{\tau\tau} & \frac{\mathcal{C}_{||}}{\tau^3} & \frac{\mathcal{C}_2}{\tau} & \frac{\mathcal{C}_3}{\tau} \\ \frac{\mathcal{C}_{||}}{\tau^3} & -\frac{1}{\tau}\partial_\tau T^{\tau\tau} - \frac{1}{\tau^2} T^{\tau\tau} & 0 & 0 \\ \frac{\mathcal{C}_2}{\tau} & 0 & T^{\tau\tau} + \frac{1}{2}\tau\partial_\tau T^{\tau\tau} & 0 \\ \frac{\mathcal{C}_3}{\tau} & 0 & 0 & T^{\tau\tau} + \frac{1}{2}\tau\partial_\tau T^{\tau\tau} \end{pmatrix} . \tag{36}$$

In this form, it is evident that this stress-energy tensor describes a fluid with heat conduction. Indeed, the stress-energy tensor of such a fluid in our coordinate system is easily seen to be

$$T^{\mu\nu} = \begin{pmatrix} \rho & -\frac{\kappa}{\tau^2}\partial_y T & -\kappa\partial_2 T & -\kappa\partial_3 T \\ -\frac{\kappa}{\tau^2}\partial_y T & \frac{p}{\tau^2} & 0 & 0 \\ -\kappa\partial_2 T & 0 & p & 0 \\ -\kappa\partial_3 T & 0 & 0 & p \end{pmatrix} . \tag{37}$$

The energy density and temperature behave as (14). Notice that the temperature does not enter the expression for the stress-energy tensor except through its gradient. We obtain the following behavior of the components of the stress-energy tensor,

$$\begin{aligned} T^{\tau\tau} &= \frac{\mathcal{C}_0}{\tau^{4/3}} \quad , \quad T_{\tau\tau} = \frac{\mathcal{C}_0}{\tau^{4/3}} \\ T^{yy} &= \frac{\mathcal{C}_0}{3\tau^{10/3}} \quad , \quad T_{yy} = \frac{\mathcal{C}_0}{3}\tau^{2/3} \\ T^{ii} &= \frac{\mathcal{C}_0}{3\tau^{4/3}} \quad , \quad T_{ii} = \frac{\mathcal{C}_0}{3\tau^{4/3}} \quad (i = 2, 3) \\ T^{\tau y} &= \frac{\mathcal{C}_{||}}{\tau^3} \quad , \quad T_{\tau y} = \frac{-\mathcal{C}_{||}}{\tau} \\ T^{\tau\perp} &= \frac{\mathcal{C}_\perp}{\tau} \quad , \quad T_{\tau\perp} = \frac{-\mathcal{C}_\perp}{\tau} \end{aligned} \tag{38}$$

This behavior exactly matches the expected behavior from the gravity dual. We conclude that holographic renormalization leads to a fluid with a finite temperature gradient which therefore conducts heat. This is a subleading effect in the large $\tau$ limit and was obtained as a perturbation. For the temperature of the gauge theory fluid, our conclusion is

$$T = \frac{\mathcal{T}_0}{\tau^{1/3}} - \frac{1}{\kappa\tau}\left(\mathcal{C}_{||} y + \mathcal{C}_\perp \cdot x^\perp\right) \tag{39}$$

confirming that the temperature gradient is a subleading effect. The constants $\mathcal{C}_{||}, \mathcal{C}_\perp$ single out a direction in the Minkowski space of the fluid yielding a planar heat flow. Other forms of heat conduction may be obtained by modifying the metric ansatz (e.g., by generalizing the coordinate dependence of the metric).

# 5 Conclusion

We solved the vacuum Einstein equations in $AdS_5$ for long longitudinal proper time using a metric ansatz which was based on the proposal of ref. [8]. We added off-diagonal terms to the metric and found an explicit solution by keeping both leading and subleading terms in the Einstein equations. Through holographic renormalization, we showed that the bulk metric corresponded to a gauge theory fluid in possession of a temperature gradient. The latter was a subleading effect giving rise to heat conduction in the fluid.

Further work should be done to better understand the off-diagonal perturbations of the metric. These perturbations will, interestingly, introduce bulk and shear viscosities into the gauge theory hydrodynamics. Work in this direction is in progress.

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