APPROXIMATELY BISIMILAR SYMBOLIC MODELS FOR NONLINEAR CONTROL SYSTEMS

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Abstract. Control systems are usually modeled by differential equations describing how physical phenomena can be influenced by certain control parameters or inputs. Although these models are very powerful when dealing with physical phenomena, they are less suitable to describe software and hardware interfacing the physical world. For this reason there is a growing interest in describing control systems through *symbolic models* that are abstract descriptions of the continuous dynamics, where each "symbol" corresponds to an "aggregate" of states in the continuous model. Since these symbolic models are of the same nature of the models used in computer science to describe software and hardware, they provide a unified language to study problems of control in which software and hardware interact with the physical world. Furthermore the use of symbolic models enables one to leverage techniques from supervisory control and algorithms from game theory for controller synthesis purposes. In this paper we show that every incrementally globally asymptotically stable nonlinear control system is approximately equivalent (bisimilar) to symbolic model. The approximation error is a design parameter in the construction of the symbolic model and can be rendered as small as desired. We also show that for digital control systems, and under the stronger assumption of incremental input–to–state stability, the symbolic models can be constructed through a suitable quantization of the inputs.

1. Introduction

The idea of using models at different levels of abstraction has been successfully used in the formal methods community with the purpose of mitigating the complexity of software verification. A central notion when dealing with complexity reduction, is the one of bisimulation equivalence, introduced by Milner [16] and Park [17] in the 80s'. The key idea is to find and compute an equivalence relation on the state space of the system, that respects the system dynamics. This equivalence relation induces a new system on the quotient space that shares most properties of interest with the original model. This approach leads to an alternative methodology for the analysis and control of large–scale control systems. In fact from the analysis point of view, symbolic models provide a unified framework describing continuous systems as well as, hardware and software interacting with the physical environment. Furthermore, the use of symbolic models allows one to leverage the rich literature on supervisory control [19] and algorithmic approaches to game theory [27, 5], for controller design. After the pioneering work of Alur and Dill [1] that showed existence of symbolic models for timed automata, researchers tried to identify more general classes of continuous systems admitting finite bisimulations. For example, [2] showed that o–minimal hybrid systems admit symbolic models, while discrete–time control systems have been considered in [24, 22]. In particular [24] showed that discrete–time controllable linear systems admit symbolic models. The common denominator of the work [1, 2, 24, 22] is the employed notion of bisimulation that is essentially equivalent to the classical one of Milner [16] and Park [17]. The same notion has been used in [25], [11] and [23] to reduce continuous control systems to continuous control systems.
A new twist in this research line has been recently given by the so–called *approximate bisimulation*, introduced in [10], that captures equivalence of systems in an approximate setting. By relaxing the usual notion of bisimulation to approximate bisimulation, a larger class of control systems can be expected to admit symbolic models. In fact the work in [21] shows that for every asymptotically stabilizable control system it is possible to construct a symbolic model, which is based on an approximate notion of simulation (one–sided version of

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bisimulation). However, if a controller fails to exist for the symbolic model, nothing can be concluded regarding the existence of a controller for the original model. This drawback is a direct consequence of the one–sided notion used in [21]. For this reason, an extension of the results in [21] from simulation to bisimulation is needed. The aim of this paper is precisely to provide such extension. The key idea in the results that we propose is to replace the assumption of asymptotic stabilizability of [21] with the stronger notion of asymptotic stability. *We show that every incrementally globally asymptotically stable nonlinear control system admits a symbolic model that is an approximate bisimulation, with a precision that is a–priori defined, as a design parameter.* Furthermore if the state space of the control system is bounded, which is the case in many realistic situations, the symbolic model is in fact finite. *Moreover, for incrementally input–to–state stable digital control systems, i.e. systems where control signals are piecewise–constant, a symbolic model can be obtained by quantizing the space of inputs.* As a by–product, our results also shed some light into the construction of finite abstractions in the context of quantized control systems [6, 7]. Indeed, by performing a quantization on the input space, we can guarantee that the resulting symbolic model admits a lattice structure in the state space.
Similar ideas to the ones of this paper, have been recently explored in [9] for the class of discrete–time linear control systems. A discussion on relationships between our results and the ones in [9], can be found in the last section of this paper. A preliminary version of these results appeared in [18].
This paper is organized as follows. Section 2 introduces the class of control systems considered and some stability notions upon which our results rely. Section 3 introduces the class of transition systems that we use as abstract models of control systems and the notion of approximate bisimulation. In Section 4 we show the existence of a symbolic model that is approximately bisimilar to the control system, provided that the system satisfies some stability properties. Section 5 proposes a symbolic model for the class of digital control systems and Section 6 offers an illustrative example. Finally some concluding remarks and future work are presented in Section 7.

2. Control systems and stability notions

2.1. **Notation.** The symbols $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}^+_0$ denote the natural, real, positive and nonnegative real numbers, respectively.
The identity map on a set A is denoted by 1_A. Given two sets A and B, if A is a subset of B we denote by $\imath_A : A \hookrightarrow B$ or simply by $\imath$ the natural inclusion map taking any $a \in A$ to $\imath(a) = a \in B$. Given a function $f : A \to B$ the symbol $f(A)$ denotes the image of A through f, i.e. $f(A) := \{b \in B : \exists a \in A \text{ s.t. } b = f(a)\}$. We identify a relation $R \subseteq A \times B$ with the map $R : A \to 2^B$ defined by $b \in R(a)$ if and only if $(a, b) \in R$. Given a relation $R \subseteq A \times B$, R^{-1} denotes the inverse relation of R, i.e. $R^{-1} := \{(b, a) \in B \times A : (a, b) \in R\}$. The symbol $conv(x^1, x^2, ..., x^m)$ denotes the convex hull of vectors $x^1, x^2, ..., x^m \in \mathbb{R}^n$. A bounded set of the form $conv(x^1, x^2, ..., x^m)$ is called a polytope. Given a vector $x \in \mathbb{R}^n$ we denote by x_i the i–th element of x; furthermore $\|x\|_2$ and $\|x\|$ denote the Euclidean and the infinity norm of x, respectively; we recall that $\|x\| := max\{|x_1|, |x_2|, ..., |x_n|\}$, where $|x_i|$ is the absolute value of x_i and that $\|x\| \leq \|x\|_2 \leq \sqrt{n}\|x\|$. Given a matrix M, the symbol $\|M\|$ denote the infinity norm of M; if $M \in \mathbb{R}^{n\times m}$, we recall that $\|M\| := \max_{1\leq i\leq m} \sum_{j=1}^{n} |a_{ij}|$. The symbol $\mathcal{B}_\varepsilon(x)$ denotes the closed ball centered at $x \in \mathbb{R}^n$ with radius $\varepsilon \in \mathbb{R}^+_0$, i.e. $\mathcal{B}_\varepsilon(x) = \{y \in \mathbb{R}^n : \|x - y\| \leq \varepsilon\}$. For any $A \subseteq \mathbb{R}^n$ and $\mu \in \mathbb{R}$ define $[A]_\mu := \{a \in A \mid a_i = k_i\mu,\ k_i \in \mathbb{Z}\ i = 1, ..., n\}$. By geometrical considerations on the infinity norm, for any $\mu \in \mathbb{R}^+$ and $\lambda \geq \mu/2$ the collection of sets $\{\mathcal{B}_\lambda(q)\}_{q\in[\mathbb{R}^n]_\mu}$ is a covering of $\mathbb{R}^n$, i.e. $\mathbb{R}^n \subseteq \bigcup_{q\in[\mathbb{R}^n]_\mu} \mathcal{B}_\lambda(q)$; conversely for any $\lambda < \mu/2$, $\mathbb{R}^n \nsubseteq \bigcup_{q\in[\mathbb{R}^n]_\mu} \mathcal{B}_\lambda(q)$.
Given a measurable function $f : \mathbb{R}^+_0 \to \mathbb{R}$, the (essential) supremum of f is denoted by $\|f\|_\infty$; we recall that $\|f\|_\infty := (ess)sup\{\|f(t)\|, t \geq 0\}$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be radially unbounded if $f(x) \to \infty$ as $\|x\| \to \infty$. A continuous function $\gamma : \mathbb{R}^+_0 \to \mathbb{R}^+_0$, is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$; γ is said to belong to class $\mathcal{K}_\infty$ if $\gamma \in \mathcal{K}$ and $\gamma(r) \to \infty$ as $r \to \infty$. A continuous function $\beta : \mathbb{R}^+_0 \times \mathbb{R}^+_0 \to \mathbb{R}^+_0$ is said to belong to class $\mathcal{KL}$ if for each fixed s, the map $\beta(r, s)$ belongs to class $\mathcal{K}_\infty$ with respect to r and, for each fixed r, the map $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$.
Given a metric space $(X, \mathbf{d})$, we denote by $\mathbf{d}_h$ the Hausdorff pseudo–metric induced by $\mathbf{d}$ on 2^X; we recall

that for any $X_1, X_2 \subseteq X$:

$$\mathbf{d}_h(X_1, X_2) := \max\{\vec{\mathbf{d}}_h(X_1, X_2), \vec{\mathbf{d}}_h(X_2, X_1)\},$$

where $\vec{\mathbf{d}}_h(X_1, X_2) = \sup_{x_1 \in X_1} \inf_{x_2 \in X_2} \mathbf{d}(x_1, x_2)$ is the directed Hausdorff pseudo–metric.

2.2. **Control Systems.** The class of control systems that we consider in this paper is formalized in the following definition.

Definition 2.1. A *control system* is a quadruple:

$$\Sigma = (\mathbb{R}^n, U, \mathcal{U}, f),$$

where:

- $\mathbb{R}^n$ is the state space;
- $U \subseteq \mathbb{R}^m$ is the input space;
- $\mathcal{U}$ is a subset of the set of all measurable locally essentially bounded functions of time from intervals of the form $]a, b[\subseteq \mathbb{R}$ to U with $a < 0$ and $b > 0$;
- $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ is a continuous map satisfying the following Lipschitz assumption: for every compact set $K \subset \mathbb{R}^n$, there exists a constant $L > 0$ such that $\|f(x, u) - f(y, u)\| \leq L\|x - y\|$, for all $x, y \in K$ and all $u \in U$.

An absolutely continuous curve $\mathbf{x} :]a, b[\to \mathbb{R}^n$ is said to be a *trajectory* of Σ if there exists $\mathbf{u} \in \mathcal{U}$ satisfying:

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t)),$$

for almost all $t \in]a, b[$. Although we have defined trajectories over open domains, we shall refer to trajectories $\mathbf{x} : [0, \tau] \to \mathbb{R}^n$ defined on closed domains $[0, \tau]$, $\tau \in \mathbb{R}^+$ with the understanding of the existence of a trajectory $\mathbf{x}' :]a, b[\to \mathbb{R}^n$ such that $\mathbf{x} = \mathbf{x}'|_{[0,\tau]}$. We will also write $\mathbf{x}(t, x, \mathbf{u})$ to denote the point reached at time $t \in]a, b[$ under the input $\mathbf{u}$ from initial condition x; this point is uniquely determined, since the assumptions on f ensure existence and uniqueness of trajectories.
A control system Σ is said to be *forward complete* if every trajectory is defined on an interval of the form $]a, \infty[$. Sufficient and necessary conditions for a system to be forward complete can be found in [4]. Simpler, but only sufficient, conditions for forward completeness are also available in the literature. These include linear growth or compact support of the vector field (see e.g. [15]).

2.3. **Stability notions.** The results presented in this paper will assume certain stability assumptions on the control systems. We briefly recall these notions and the results that will be used in the subsequent developments.

Definition 2.2. [3] A control system Σ is *incrementally globally asymptotically stable (*δ–GAS) if it is forward complete and there exist a $\mathcal{KL}$ function β such that for any $t \in \mathbb{R}_0^+$, any $x, y \in \mathbb{R}^n$ and any $\mathbf{u} \in \mathcal{U}$ the following condition is satisfied:

$$\|\mathbf{x}(t, x, \mathbf{u}) - \mathbf{x}(t, y, \mathbf{u})\| \leq \beta(\|x - y\|, t). \tag{2.1}$$

Definition above can be thought of as an incremental version of the classical notion of global asymptotic stability (GAS) [13]. Furthermore when f satisfies $f(0, 0) = 0$, δ–GAS implies GAS of Σ with $U = \{0\}$, by just comparing a trajectory of Σ with any $x \in \mathbb{R}^n$ and $\mathbf{u}(t) = 0$, $t \in \mathbb{R}_0^+$, with the null trajectory $\mathbf{x}(t) = 0$, $t \in \mathbb{R}_0^+$.

Definition 2.3. [3] A control system Σ is *incrementally input–to–state stable* (δ–ISS) if it is forward complete and there exist a $\mathcal{KL}$ function β and a $\mathcal{K}_\infty$ function γ such that for any $t \in \mathbb{R}_0^+$, any $x, y \in \mathbb{R}^n$ and any $\mathbf{u}, \mathbf{v} \in \mathcal{U}$ the following condition is satisfied:

$$\|\mathbf{x}(t, x, \mathbf{u}) - \mathbf{x}(t, y, \mathbf{v})\| \leq \beta(\|x - y\|, t) + \gamma(\|\mathbf{u} - \mathbf{v}\|_\infty). \tag{2.2}$$

Notice that analogously to δ–GAS, if the vector field f in the control system Σ is such that $f(0,0)=0$, δ–ISS implies input–to–state stability (ISS) [13], by just comparing an arbitrary trajectory of Σ with the null trajectory $\mathbf{x}(t)=0$, $t\in\mathbb{R}_0^+$. Furthermore by observing (2.1) and (2.2), it is not difficult to see that δ–ISS implies δ–GAS, while the converse is not true in general (see [3] for some examples).
In general, inequalities (2.1) and (2.2) are difficult to check directly. Fortunatelly δ–GAS and δ–ISS can be characterized by dissipation inequalities.

Definition 2.4. Consider a control system Σ and a smooth function $V:\mathbb{R}^n\times\mathbb{R}^n\rightarrow\mathbb{R}_0^+$. Function V is called a δ–GAS *Lyapunov function* for Σ, if there exist $\mathcal{K}_\infty$ functions α_1, α_2 and ρ such that:

(i) for any $x,y\in\mathbb{R}^n$

$$\alpha_1(\|x-y\|)\leq V(x,y)\leq\alpha_2(\|x-y\|);$$

(ii) for any $x,y\in\mathbb{R}^n$ and any $u\in U$

$$\frac{\partial V}{\partial x}f(x,u)+\frac{\partial V}{\partial y}f(y,u)<-\rho(\|x-y\|).$$

Function V is called a *δ–ISS Lyapunov function* for Σ, if there exist $\mathcal{K}_\infty$ functions α_1, α_2, ρ and σ satisfying conditions (i) and:

(iii) for any $x,y\in\mathbb{R}^n$ and any $u,v\in U$

$$\frac{\partial V}{\partial x}f(x,u)+\frac{\partial V}{\partial y}f(y,v)<-\rho(\|x-y\|)+\sigma(\|u-v\|).$$

The following result completely characterizes δ–GAS and δ–ISS in terms of existence of Lyapunov functions.

Theorem 2.5. [3] *Consider $\Sigma=(\mathbb{R}^n,U,\mathcal{U},f)$. Then:*

- *If U is compact then Σ is δ–GAS if and only if it admits a δ–GAS Lyapunov function;*
- *If U is closed, convex, contains the origin and $f(0,0)=0$, then Σ is δ–ISS if it admits a δ–ISS Lyapunov function. Moreover if U is compact, existence of a δ–ISS Lyapunov function is equivalent to δ–ISS.*

3. Approximate bisimulation

In this section we introduce a notion of approximate equivalence upon which all the results in this paper rely. We start by introducing the class of transition systems that will be used in this paper as abstract models for control systems.

Definition 3.1. A transition system is a quintuple:

$$T=(Q,L,\longrightarrow,O,H),$$

consisting of:

- A set of states Q;
- A set of labels L;
- A transition relation $\longrightarrow\subseteq Q\times L\times Q$;
- An output set O;
- An output function $H:Q\rightarrow O$.

A transition system T is said to be:

- *metric*, if the output set O is equipped with a metric $\mathbf{d}:O\times O\rightarrow\mathbb{R}_0^+$;
- *countable*, if Q and L are countable sets;
- *finite*, if Q and L are finite sets.

We will follow standard practice and denote an element $(q, l, p) \in \longrightarrow$ by $q \xrightarrow{l} p$. Transition systems capture dynamics through the transition relation. For any states $q, p \in Q$, $q \xrightarrow{l} p$ simply means that it is possible to evolve or jump from state q to state p under the action labeled by l. We will use transition systems as an abstract representation of control systems. There are several different ways in which control systems can be transformed into transition systems. We now describe one of these which has the property of capturing all the information contained in a control system Σ.
Given a control system $\Sigma = (\mathbb{R}^n, U, \mathcal{U}, f)$ define the transition system:

$$T(\Sigma) := (Q, L, \longrightarrow, O, H), \tag{3.1}$$

where:

- $Q = \mathbb{R}^n$;
- $L = \mathcal{U}$;
- $q \xrightarrow{u} p$ if there exists a trajectory $\mathbf{x} : [0, \tau] \to \mathbb{R}^n$ of Σ satisfying $\mathbf{x}(\tau, q, \mathbf{u}) = p$ for some $\tau \in \mathbb{R}^+$;
- $O = \mathbb{R}^n$;
- $H = 1_{\mathbb{R}^n}$.

Transition system $T(\Sigma)$ is metric when we regard the set $O = \mathbb{R}^n$ as being equipped with the metric $\mathbf{d}(p, q) = \|p - q\|$. Note that the state space of $T(\Sigma)$ is infinite. The aim of this paper is to study existence of countable transition systems that are approximately equivalent to $T(\Sigma)$. The notion of equivalence that we consider is the one of *bisimulation equivalence* [16, 17]. Bisimulation relations are standard mechanisms to relate the properties of transition systems [8]. Intuitively, a bisimulation relation between a pair of transition systems T_1 and T_2 is a relation between the corresponding state sets explaining how a sequence of transitions r_1 of T_1 can be transformed into a sequence of transitions r_2 of T_2 and vice versa. While typical bisimulation relations require that r_1 and r_2 are observationally indistinguishable, that is $H_1(r_1) = H_2(r_2)$, we shall relax this by requiring $H_1(r_1)$ to simply be close to $H_2(r_2)$ where closeness is measured with respect to the metric on the output set. The following notion has been introduced in [10] and in a slightly different formulation in [21].

Definition 3.2. Let $T_1 = (Q_1, L_1, \underset{1}{\longrightarrow}, O, H_1)$ and $T_2 = (Q_2, L_2, \underset{2}{\longrightarrow}, O, H_2)$ be metric transition systems with the same output set and metric $\mathbf{d}$, and let $\varepsilon \in \mathbb{R}_0^+$ be a given precision. A relation $R \subseteq Q_1 \times Q_2$ is said to be an ε–*approximate simulation* relation from T_1 to T_2, if for any $(q_1, q_2) \in R$:

(i) $\mathbf{d}(H_1(q_1), H_2(q_2)) \leq \varepsilon$;
(ii) $q_1 \underset{1}{\xrightarrow{l_1}} p_1$ implies existence of $q_2 \underset{2}{\xrightarrow{l_2}} p_2$ such that $(p_1, p_2) \in R$.

An ε–approximate simulation relation from T_1 to T_2 is an ε–*approximate bisimulation* relation (between T_1 and T_2) if R^{-1} is also an ε–approximate simulation relation from T_2 to T_1. Moreover:

- T_1 is ε–*simulated* by T_2 if there exists an ε–approximate simulation relation R from T_1 to T_2 such that $R^{-1}(Q_2) = Q_1$;
- T_1 is ε–*bisimilar* to T_2 if there exists an ε–approximate bisimulation relation R between T_1 and T_2 such that $R(Q_1) = Q_2$ and $R^{-1}(Q_2) = Q_1$.

4. Approximate bisimilar symbolic models

In the following we will work with a sub–transition system of $T(\Sigma)$ obtained by selecting those transitions from $T(\Sigma)$ that describe trajectories of duration τ for some chosen $\tau \in \mathbb{R}^+$. This can be seen as a time discretization or sampling process.

Definition 4.1. Given a control system $\Sigma = (\mathbb{R}^n, U, \mathcal{U}, f)$ and a parameter $\tau \in \mathbb{R}^+$ define the transition system:

$$T_\tau(\Sigma) := (Q_1, L_1, \underset{1}{\longrightarrow}, O_1, H_1),$$

where:

- $Q_1 = \mathbb{R}^n$;
- $L_1 = \{l_1 \in \mathcal{U} \mid \mathbf{x}(\tau, x, l_1) \text{ is defined for all } x \in \mathbb{R}^n\}$;
- $q \xrightarrow[1]{l_1} p$ if there exists a trajectory $\mathbf{x} : [0, \tau] \to \mathbb{R}^n$ of Σ satisfying $\mathbf{x}(\tau, q, l_1) = p$;
- $O_1 = \mathbb{R}^n$;
- $H_1 = 1_{\mathbb{R}^n}$.

Transition system $T_\tau(\Sigma)$ is metric when we regard $O_1 = \mathbb{R}^n$ as being equipped with the metric $\mathbf{d}(p, q) = \|p - q\|$. Note that the set of labels L_1 is composed by (only) those control signals of $\mathcal{U}$ for which a trajectory of Σ exists for any time $t \in [0, \tau]$ and for any initial condition $x \in \mathbb{R}^n$. Any measurable control input can be included in L_1 when the control system is forward complete.

In the following we show existence of a countable transition system that is approximately bisimilar to $T_\tau(\Sigma)$, provided that Σ satisfies some stability properties.

By simple considerations on the infinity norm, for any given precision $\eta \in \mathbb{R}^+$ we can approximate the state space $Q_1 = \mathbb{R}^n$ of $T_\tau(\Sigma)$ by means of the countable set $Q_2 := [\mathbb{R}^n]_\eta$ so that for any $x \in \mathbb{R}^n$ there exists $q \in [\mathbb{R}^n]_\eta$ such that $\|x - q\| \leq \eta/2$.

The approximation of the set of labels L_1 of $T_\tau(\Sigma)$ is more involved. We approximate L_1 by means of the set:

$$L_2 := \bigcup_{q \in Q_2} L_2(q), \tag{4.1}$$

where $L_2(q)$ captures the set of labels that can be applied at the state $q \in Q_2$ of the symbolic model. The definition of $L_2(q)$ is based on the notion of reachable sets. Given any state $q \in Q_1$ consider the set:

$$\mathcal{R}(\tau, q) = \left\{ p \in Q_1 : q \xrightarrow[1]{l_1} p, l_1 \in L_1 \right\}, \tag{4.2}$$

of reachable states of $T_\tau(\Sigma)$ from q. Notice that $\mathcal{R}(\tau, q)$ is well defined because of the definition of the set of labels L_1. We approximate $\mathcal{R}(\tau, q)$ by means of a countable set, as follows. Given any precision $\mu \in \mathbb{R}^+$, consider the set:

$$\mathcal{P}_\mu(\tau, q) := \{y \in [\mathbb{R}^n]_\mu : \exists z \in \mathcal{R}(\tau, q) \text{ s.t. } \|y - z\| \leq \mu/2\},$$

and define the function:

$$\psi_\mu^{\tau,q} : \mathcal{P}_\mu(\tau, q) \to L_1,$$

that associates to any $y \in \mathcal{P}_\mu(\tau, q)$ a label $l_1 = \psi_\mu^{\tau,q}(y) \in L_1$ so that $\|y - \mathbf{x}(\tau, q, l_1)\| \leq \mu/2$. Notice that the function $\psi_\mu^{\tau,q}$ is not unique. The set $L_2(q)$ appearing in (4.1) can now be defined by:

$$L_2(q) := \psi_\mu^{\tau,q}(\mathcal{P}_\mu(\tau, q)).$$

Notice that since $L_2(q)$ is the image through $\psi_\mu^{\tau,q}$ of a countable set, it is countable. Therefore L_2 as defined in (4.1) is countable, as well. Furthermore the set L_2 approximates the set L_1 in the sense that given any $q \in Q_2$, for any $l_1 \in L_1$ there exists $l_2 \in L_2(q)$ so that:

$$\|\mathbf{x}(\tau, q, l_1) - \mathbf{x}(\tau, q, l_2)\| \leq \mu. \tag{4.3}$$

We now have all the ingredients to define a symbolic model that will be used to approximate a control system. Given a control system $\Sigma = (\mathbb{R}^n, U, \mathcal{U}, f)$, any $\tau \in \mathbb{R}^+$, $\eta \in \mathbb{R}^+$ and $\mu \in \mathbb{R}^+$ define the following transition system:

$$T_{\tau,\eta,\mu}(\Sigma) := (Q_2, L_2, \underset{2}{\longrightarrow}, O_2, H_2), \tag{4.4}$$

where:

- $Q_2 = [\mathbb{R}^n]_\eta$;
- $L_2 = \bigcup_{q \in Q_2} L_2(q)$;

- $q \xrightarrow[2]{l} p$, if $l \in L_2(q)$ and $\|p - \mathbf{x}(\tau, q, l)\| \leq \eta/2$;
- $O_2 = \mathbb{R}^n$;
- $H_2 = \imath : Q_2 \hookrightarrow O_2$.

We think of $T_{\tau,\eta,\mu}(\Sigma)$ as a metric transition system where $O_2 = \mathbb{R}^n$ is equipped with the metric $\mathbf{d}(p, q) = \|p - q\|$. Parameters $\tau \in \mathbb{R}^+$, $\eta \in \mathbb{R}^+$ and $\mu \in \mathbb{R}^+$ in $T_{\tau,\eta,\mu}(\Sigma)$ can be thought of, respectively, as a sampling time, a state space and an input space quantization.
We emphasize that transition system $T_{\tau,\eta,\mu}(\Sigma)$ is countable because the sets Q_2 and L_2 are countable. Furthermore if the state space of the control system Σ is bounded, the corresponding transition system $T_{\tau,\eta,\mu}(\Sigma)$ is finite.
Note that in the definition of the transition relation $\xrightarrow[2]{}$ we require $\mathbf{x}(\tau, q, l)$ to be in the closed ball $\mathcal{B}_{\eta/2}(p)$. We can instead, require $\mathbf{x}(\tau, q, l)$ to be in $\mathcal{B}_\lambda(p)$ for any $\lambda \geq \eta/2$. However, we chose $\lambda = \eta/2$ because $\eta/2$ is the smallest value of $\lambda \in \mathbb{R}^+$ that ensures $\mathbb{R}^n \subseteq \bigcup_{q \in [\mathbb{R}^n]_\eta} \mathcal{B}_\lambda(q)$. In fact, this choice of λ reduces the number of transitions in the definition of the symbolic model (4.4).
We can now give the main result of this paper which relates δ–GAS to existence of symbolic model.

Theorem 4.2. *Consider a control system Σ and any desired precision $\varepsilon \in \mathbb{R}^+$. If Σ is δ–GAS then for any $\tau \in \mathbb{R}^+$, $\eta \in \mathbb{R}^+$ and $\mu \in \mathbb{R}^+$ satisfying the following inequality:*

$$\beta(\varepsilon, \tau) + \mu + \eta/2 \leq \varepsilon, \tag{4.5}$$

the transition system $T_\tau(\Sigma)$ is ε–bisimilar to $T_{\tau,\eta,\mu}(\Sigma)$.

Before giving the proof of this result we point out that if Σ is δ–GAS, there always exist parameters $\tau \in \mathbb{R}^+$, $\eta \in \mathbb{R}^+$ and $\mu \in \mathbb{R}^+$ satisfying condition (4.5). Indeed since β is a $\mathcal{KL}$ function, there exists a sufficiently large value of τ so that $\beta(\varepsilon, \tau) < \varepsilon$; then by choosing sufficiently small values of μ and η, condition (4.5) is fulfilled.

Proof. Consider the relation $R \subseteq Q_1 \times Q_2$ defined by $(x, q) \in R$ if and only if $||x - q|| \leq \varepsilon$. By construction $R(Q_1) = Q_2$; furthermore $Q_1 \subseteq \bigcup_{q_2 \in Q_2} \mathcal{B}_{\eta/2}(q_2)$ and therefore since by (4.5), $\eta/2 < \varepsilon$, we have that $R^{-1}(Q_2) = Q_1$. We now show that R is an ε–approximate bisimulation relation between $T_\tau(\Sigma)$ and $T_{\tau,\eta,\mu}(\Sigma)$. Consider any $(x, q) \in R$. Condition (i) in Definition 3.2 is satisfied by definition of R. Let us now show that condition (ii) in Definition 3.2 holds. Consider any $l_1 \in L_1$ and the transition $x \xrightarrow[1]{l_1} y$ in $T_\tau(\Sigma)$. Let $v = \mathbf{x}(\tau, q, l_1)$; since $\mathbb{R}^n \subseteq \bigcup_{w \in [\mathbb{R}^n]_\mu} \mathcal{B}_{\mu/2}(w)$, there exists $w \in [\mathbb{R}^n]_\mu$ such that:

$$\|v - w\| \leq \mu/2. \tag{4.6}$$

Since $v \in \mathcal{R}(\tau, q)$, it is clear that $w \in \mathcal{P}_\mu(\tau, q)$ by definition of $\mathcal{P}_\mu(\tau, q)$. Then, let $l_2 \in L_2(q)$ be given by $l_2 = \psi_\mu^{\tau,q}(w)$. By definition of $\psi_\mu^{\tau,q}$ and by setting $z = \mathbf{x}(\tau, q, l_2)$, it follows that:

$$\|w - z\| \leq \mu/2. \tag{4.7}$$

Since $Q_1 \subseteq \bigcup_{q_2 \in Q_2} \mathcal{B}_{\eta/2}(q_2)$, there exists $p \in Q_2$ such that:

$$\|z - p\| \leq \eta/2. \tag{4.8}$$

Thus, $q \xrightarrow[2]{l_2} p$ in $T_{\tau,\eta,\mu}(\Sigma)$ and since Σ is δ–GAS and by (4.6), (4.7), (4.8) and (4.5), the following chain of inequalities holds:

$$\begin{aligned}
\|y - p\| &= \|y - v + v - w + w - z + z - p\| \\
&\leq \|y - v\| + \|v - w\| + \|w - z\| + \|z - p\| \\
&\leq \beta(||x - q||, \tau) + \mu/2 + \mu/2 + \eta/2 \\
&\leq \beta(\varepsilon, \tau) + \mu + \eta/2 \leq \varepsilon.
\end{aligned}$$

Hence $(y,p) \in R$ and condition (ii) in Definition 3.2 holds. Thus R is an ε–approximate simulation relation from $T_\tau(\Sigma)$ to $T_{\tau,\eta,\mu}(\Sigma)$.
We now show that also R^{-1} is an ε–approximate simulation relation from $T_{\tau,\eta,\mu}(\Sigma)$ to $T_\tau(\Sigma)$. Consider any $l_2 \in L_2$ and the transition $q \xrightarrow[2]{l_2} p$ in $T_{\tau,\eta,\mu}(\Sigma)$. By definition of $T_{\tau,\eta,\mu}(\Sigma)$ there exists $z = \mathbf{x}(\tau, q, l_2) \in Q_1$ such that:

$$\|z - p\| \leq \eta/2. \tag{4.9}$$

Choose $l_1 = l_2 \in L_1$ and consider the transition $x \xrightarrow[1]{l_1} y$ in $T_\tau(\Sigma)$. Since Σ is δ–GAS and by conditions (4.9) and (4.5), the following chain of inequalities holds:

$$\begin{aligned} \|y-p\| &= \|y-z+z-p\| \leq \|y-z\| + \|z-p\| \\ &\leq \beta(\|x-q\|,\tau) + \eta/2 \leq \beta(\varepsilon,\tau) + \eta/2 \leq \varepsilon. \end{aligned}$$

Thus $(y,p) \in R$, which completes the proof. □

This result represents a substantial improvement over previously known classes of control systems admitting symbolic models, which included o–minimal hybrid systems [14], controllable linear systems in discrete–time [22] and stable linear systems in discrete–time [9]. However Theorem 4.2 relates $T_\tau(\Sigma)$ to the symbolic model in (4.4), whose construction is in general difficult, since it requires the computation of reachable sets. In the next section we show that for digital control systems a symbolic model can be obtained by quantizing the input space.

5. Digital control systems

In this section we specialize the results of the previous section to the case of digital control systems, i.e. control systems where control signals are piecewise–constant. In many man made systems, input signals are often physically implemented as piecewise–constant signals and this motivates our interest in this class of systems. In the following we suppose that the input space U of the considered control system $\Sigma = (\mathbb{R}^n, U, \mathcal{U}, f)$ contains the origin and that it is a hyper rectangle, i.e.

$$U := [a_1, b_1] \times [a_2, b_2] \times ... \times [a_m, b_m],$$

for some $a_i < b_i, i = 1, 2, ..., m$. Furthermore we suppose that control inputs are piecewise constant; given $\tau \in \mathbb{R}^+$, the class of inputs that we consider is:

$$\mathcal{U}_\tau := \{\mathbf{u} \in \mathcal{U} : \mathbf{u}(t) = \mathbf{u}(0), t \in [0,\tau]\}.$$

For notational simplicity, we denote by u the control input $\mathbf{u} \in \mathcal{U}_\tau$ for which $\mathbf{u}(t) = u$, $t \in [0,\tau]$.
Let us denote by $T_{\mathcal{U}_\tau}(\Sigma)$ the sub–transition system of $T_\tau(\Sigma)$ where only control inputs in $\mathcal{U}_\tau$ are considered. More formally define:

$$T_{\mathcal{U}_\tau}(\Sigma) := (Q_1, L_1, \xrightarrow[1]{}, O_1, H_1),$$

where:

- $Q_1 = \mathbb{R}^n$;
- $L_1 = U$;
- $q \xrightarrow[1]{l} p$, if there exists a trajectory $\mathbf{x}$ of Σ satisfying $\mathbf{x}(\tau, q, l) = p$;
- $O_1 = \mathbb{R}^n$;
- $H_1 = 1_{\mathbb{R}^n}$.

Transition system $T_{\mathcal{U}_\tau}(\Sigma)$ is metric when we regard $O = \mathbb{R}^n$ as being equipped with the metric $\mathbf{d}(p,q) = \|p - q\|$. Note that analogously to $T_\tau(\Sigma)$, transition system $T_{\mathcal{U}_\tau}(\Sigma)$ is not countable. Therefore we now define a suitable

countable transition system that will approximate $T_{\mathcal{U}_\tau}(\Sigma)$ with any desired precision.
Given a control system Σ, any $\tau \in \mathbb{R}^+$, $\eta \in \mathbb{R}^+$ and $\mu \in \mathbb{R}^+$, define the following transition system:

$$T_{\tau,\eta,\mu}(\Sigma) := (Q_2, L_2, \underset{2}{\longrightarrow}, O_2, H_2), \tag{5.1}$$

where:

- $Q_2 = [\mathbb{R}^n]_\eta$;
- $L_2 = [U]_\mu$;
- $q \xrightarrow[2]{l} p$, if $\|p - \mathbf{x}(\tau, q, l)\| \leq \eta/2$;
- $O_2 = \mathbb{R}^n$;
- $H_2 = \imath : Q_2 \hookrightarrow O_2$.

Analogously to transition system $T_\tau(\Sigma)$, transition system $T_{\mathcal{U}_\tau}(\Sigma)$ is countable. Notice that transition system $T_{\mathcal{U}_\tau}(\Sigma)$ is a sub–transition system of $T_\tau(\Sigma)$ and it differs from $T_\tau(\Sigma)$, (only) in the way that control inputs are approximated. In particular, the choice of labels in transition system $T_{\mathcal{U}_\tau}(\Sigma)$ does not require the knowledge of reachable set associated with Σ. This feature is essential when constructing the symbolic model. The computation of $\mathbf{x}(\tau, q, l)$ can be done either analytically or numerically; in the later case, numerical errors can be incorporated in the model, as follows. Suppose there exists a parameter $\nu \in \mathbb{R}_0^+$ so that for any state $q \in Q_2$ and control input $l \in L_2$, it is possible to evaluate $\mathbf{x}(\tau, q, l)$ by means of the numerical solution $\tilde{\mathbf{x}}(\tau, q, l)$ with precision ν, i.e.:

$$\|\mathbf{x}(\tau, q, l) - \tilde{\mathbf{x}}(\tau, q, l)\| \leq \nu.$$

Then, transition relation $\underset{2}{\longrightarrow}$ in transition system of (5.1), can be adapted to this case by saying that $q \xrightarrow[2]{l} p$, if $\|p - \tilde{\mathbf{x}}(\tau, q, l)\| \leq \eta/2 - \nu$. In fact:

$$\begin{array}{rcl} \|p - \mathbf{x}(\tau, q, l)\| & \leq & \|p - \tilde{\mathbf{x}}(\tau, q, l)\| + \|\tilde{\mathbf{x}}(\tau, q, l) - \mathbf{x}(\tau, q, l)\| \\ & \leq & \eta/2 - \nu + \nu = \eta/2, \end{array}$$

and therefore we can recover transition relation $\underset{2}{\longrightarrow}$, as defined in transition system of (5.1).
In the next section we will show the construction of such symbolic model by means of an example.
We can now give the following result that relates δ–ISS to the existence of symbolic models for digital control systems.

Theorem 5.1. *Consider a control system Σ and any desired precision $\varepsilon \in \mathbb{R}^+$. If Σ is δ–ISS then for any $\tau \in \mathbb{R}^+$, $\eta \in \mathbb{R}^+$, and $\mu \in \mathbb{R}^+$ satisfying the following inequality:*

$$\beta(\varepsilon, \tau) + \gamma(\mu) + \eta/2 \leq \varepsilon, \tag{5.2}$$

the transition system $T_{\mathcal{U}_\tau}(\Sigma)$ is ε–bisimilar to $T_{\tau,\eta,\mu}(\Sigma)$.

Before giving the proof of this result we point out that, analogously to condition (4.5) of Theorem 4.2, there always exist parameters $\tau \in \mathbb{R}^+$, $\eta \in \mathbb{R}^+$, and $\mu \in \mathbb{R}^+$ satisfying condition (5.2).

Proof. Consider the relation $R \subseteq Q_1 \times Q_2$ defined by $(x, q) \in R$ if and only if $||x - q|| \leq \varepsilon$. By construction $R(Q_1) = Q_2$; since $Q_1 \subseteq \bigcup_{q_2 \in Q_2} \mathcal{B}_{\eta/2}(q_2)$ and by (5.2), $\eta/2 < \varepsilon$, we have that $R^{-1}(Q_2) = Q_1$. We now show that R is an ε–approximate bisimulation relation between $T_{\mathcal{U}_\tau}(\Sigma)$ and $T_{\tau,\eta,\mu}(\Sigma)$. Consider any $(x, q) \in R$. Condition (i) in Definition 3.2 is satisfied by the definition of R. Let us now show that condition (ii) in Definition 3.2 holds. Consider any $l_1 \in L_1$ and the transition $x \xrightarrow[1]{l_1} y$ in $T_{\mathcal{U}_\tau}(\Sigma)$. Consider a label $l_2 \in L_2$ such that:

$$\|l_1 - l_2\| \leq \mu, \tag{5.3}$$

and set $z = \mathbf{x}(\tau, q, l_2)$. (Notice that such label $l_2 \in L_2$ exists because the assumptions on U make $L_2 = [U]_\mu$ non–empty.) For later use notice that since l_1 and l_2 are constant functions, then $\|l_1 - l_2\| = \|l_1 - l_2\|_\infty$. Since $Q_1 \subseteq \bigcup_{q_2 \in [\mathbb{R}^n]_\eta} \mathcal{B}_{\eta/2}(q_2)$, there exists $p \in Q_2$ such that:

$$\|z - p\| \leq \eta/2, \tag{5.4}$$

and therefore $q \xrightarrow[2]{l_2} p$ in $T_{\tau,\eta,\mu}(\Sigma)$. Since Σ is δ–ISS and by (5.3), (5.4) and (5.2), the following chain of inequalities holds:

$$\begin{aligned} \|y - p\| &= \|y - z + z - p\| \leq \|y - z\| + \|z - p\| \\ &\leq \beta(\|x - q\|, \tau) + \gamma(\|l_1 - l_2\|_\infty) + \eta/2 \\ &\leq \beta(\varepsilon, \tau) + \gamma(\mu) + \eta/2 \leq \varepsilon. \end{aligned}$$

Hence $(y, p) \in R$ and condition (ii) in Definition 3.2 holds. Thus R is an ε–approximate simulation relation from $T_{\mathcal{U}_\tau}(\Sigma)$ to $T_{\tau,\eta,\mu}(\Sigma)$.
We now show that also R^{-1} is an ε–approximate simulation relation from $T_{\tau,\eta,\mu}(\Sigma)$ to $T_{\mathcal{U}_\tau}(\Sigma)$. Consider any $l_2 \in L_2$ and the transition $q \xrightarrow[2]{l_2} p$ in $T_{\tau,\eta,\mu}(\Sigma)$. By definition of $T_{\tau,\eta,\mu}(\Sigma)$ there exists $z = \mathbf{x}(\tau, q, l_2) \in Q_1$ such that:

$$\|z - p\| \leq \eta/2. \tag{5.5}$$

Choose $l_1 = l_2 \in L_1$ and consider now the transition $x \xrightarrow[1]{l_1} y$ in $T_{\mathcal{U}_\tau}(\Sigma)$. Since Σ is δ–ISS and by (5.5) and (5.2), the following chain of inequalities holds:

$$\begin{aligned} \|y - p\| &= \|y - z + z - p\| \leq \|y - z\| + \|z - p\| \\ &\leq \beta(\|x - q\|, \tau) + \gamma(\|l_1 - l_2\|_\infty) + \eta/2 \\ &\leq \beta(\varepsilon, \tau) + 0 + \eta/2 \leq \varepsilon. \end{aligned}$$

Thus $(y, p) \in R$, which completes the proof. □

6. A simple example

Consider a control system:

$$\Sigma = (\mathbb{R}^2, U, \mathcal{U}, f), \tag{6.1}$$

where $U = [-0.1, 0.1] \subset \mathbb{R}$, $\mathcal{U}$ is the set of piecewise–constant functions of time taking values in U, and $f : \mathbb{R}^2 \times U \to \mathbb{R}^2$ is defined by:

$$f((x_1, x_2), u) = \begin{bmatrix} -2x_1 + x_2^2 - 7u \\ -2(1 + u^2)x_2 \end{bmatrix},$$

for any $x = (x_1, x_2) \in \mathbb{R}^2$ and any $u \in U$. We work in the compact set:

$$X = [-1, 1] \times [-1, 1].$$

The set X is invariant for the control system Σ, i.e. $\mathbf{x}(t, x, \mathbf{u}) \in X$, for any $x \in X$, any $\mathbf{u} \in \mathcal{U}$, and any time $t \in \mathbb{R}_0^+$. Indeed it is easy to see that for any state x in the boundary of X and for any control input $u \in U$, the vector field f points in X, i.e. $f(x, u) \in X$.
Given a desired precision $\varepsilon \in \mathbb{R}^+$, the goal is to find suitable parameters $\tau \in \mathbb{R}^+$, $\eta \in \mathbb{R}^+$ and $\mu \in \mathbb{R}^+$, so that transition system $T_{\tau,\eta,\mu}(\Sigma)$ as defined in (5.1) is ε–bisimilar to transition system $T_{\mathcal{U}_\tau}$. In order to find such parameters we can apply Theorem 5.1.
We start by showing that the control system Σ defined by (6.1) is δ–ISS. Consider the function $V : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_0^+$ defined by:

$$V(x, y) = \frac{1}{2}((x_1 - y_1)^2 + (x_2 - y_2)^2),$$

for any $x = (x_1, x_2), y = (y_1, y_2) \in X$. By recalling that $\|z\| \leq \|z\|_2 \leq \sqrt{n}\|z\|$ for any $z \in \mathbb{R}^n$, function V satisfies condition (i) of Definition 2.4, by choosing $\alpha_1(r) = 0.5\, r^2$ and $\alpha_2(r) = r^2$. Moreover by noting that for any $(z_1, z_2) \in \mathbb{R}^2 - \{(0,0)\}$, $z_1 z_2/(z_1^2 + z_2^2) \leq 1/2$, the following chain of equalities and inequalities holds for any $(x, y) \in X \times X$ and $x \neq y$ and for any $u, v \in U$:

$$\begin{aligned}
&\frac{\partial V}{\partial x} f(x,u) + \frac{\partial V}{\partial y} f(y,v) = -2((x_1-y_1)^2 + (x_2-y_2)^2) \\
&+(x_1-y_1)(x_2-y_2)(x_2+y_2) - 2(x_2-y_2)^2 u^2 \\
&+(-2(x_2-y_2)y_2(u+v) - 7(x_1-y_1))(u-v) \\
&\leq -2((x_1-y_1)^2 + (x_2-y_2)^2) + \\
&1/2(\max_{x_2,y_2}(x_2+y_2))((x_1-y_1)^2 + (x_2-y_2)^2) + \\
&(\max_{x,y,u,v}|-2(x_2-y_2)y_2(u+v) - 7(x_1-y_1)|)\,|u-v| \\
&= -\|x-y\|_2^2 + 14.8\,|u-v|\,.
\end{aligned}$$

Furthermore if $x = y$, inequality

$$\frac{\partial V}{\partial x} f(x,u) + \frac{\partial V}{\partial y} f(y,v) \quad \leq \quad -\|x-y\|_2^2 + 14.8\,|u-v| \tag{6.2}$$

holds as well. Thus condition (iii) of Definition 2.4 is satisfied with $\rho(r) = r^2$ and $\sigma(r) = 14.8\, r$ and therefore V is a δ–ISS Lyapunov function for Σ. Hence we conclude by Theorem 2.5 that the control system Σ is δ–ISS. In order to apply Theorem 5.1, we need to find a $\mathcal{KL}$ function β and a $\mathcal{K}_\infty$ function γ satisfying inequality (2.2). By inequality (6.2), the definition of V and the comparison lemma [13], the following inequalities hold for any $x, y \in X$, any $\mathbf{u}, \mathbf{v} \in \mathcal{U}$ and any time $t \in \mathbb{R}_0^+$:

$$\begin{aligned}
&\|\mathbf{x}(t,x,\mathbf{u}) - \mathbf{x}(t,y,\mathbf{v})\| \leq \\
&\|\mathbf{x}(t,x,\mathbf{u}) - \mathbf{x}(t,y,\mathbf{v})\|_2 = \sqrt{2V(t)} \leq \\
&\sqrt{2\left(e^{-2t}V(0) + 14.8\int_0^t e^{-2(t-\alpha)}\,|u-v|\,d\alpha\right)} \leq \\
&\sqrt{2e^{-2t}V(0)} + \sqrt{29.6\left(\int_0^\infty e^{-2\alpha} d\alpha\right)\|u-v\|_\infty} \leq \\
&\sqrt{2}e^{-t}\,\|x-y\| + \sqrt{14.8\|u-v\|_\infty}.
\end{aligned} \tag{6.3}$$

Define $\beta(r,s) := \sqrt{2}e^{-s}r$ and $\gamma(r) := \sqrt{14.8\, r}$ for any $r, s \in \mathbb{R}$. Functions β and γ are respectively a $\mathcal{KL}$ function and a $\mathcal{K}_\infty$ function and by (6.3) they satisfy inequality (2.2). We now have all the ingredients to apply Theorem 5.1. Condition (5.2) becomes:

$$\sqrt{2}e^{-\tau}\varepsilon + \sqrt{14.8\,\mu} + \eta/2 \leq \varepsilon. \tag{6.4}$$

Set the precision $\varepsilon = 0.5$ and choose $\eta = 2/3$ and $\tau = 5$; inequality (6.4) implies $\mu \leq 0.0017$ and therefore we can choose $\mu = 0.001$. The resulting transition system:

$$T_{\tau,\eta,\mu}(\Sigma) = (Q_2, L_2, \underset{2}{\longrightarrow}, O_2, H_2)$$

is defined by:

- $Q_2 = \{-\eta, 0, \eta\} \times \{-\eta, 0, \eta\}$;
- $L_2 = [U]_{0.001}$;
- $\underset{2}{\longrightarrow}$ is depicted in Figure 1;
- $O_2 = X$;
- $H_2 = \imath : Q_2 \hookrightarrow O_2$.

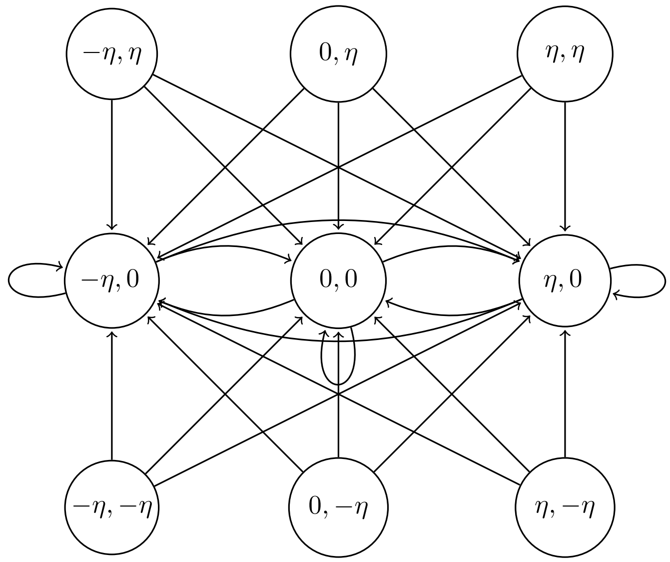

FIGURE 1. Symbolic model $T_{5,2/3,10^{-3}}(\Sigma)$ associated with the control system Σ, as defined in (6.1). An arrow from a state q to a state p means that there exists *at least* one label l in L_2 so that $\mathbf{x}(5, q, l)$ is in the closed ball $\mathcal{B}_{\eta/2}(p)$.

The transition relation $\underset{2}{\longrightarrow}$ has been obtained by analytically integrating the trajectories of Σ. Notice from Figure 1 that, outgoing transitions from states $(z_1, z_2) \in Q_2$ with $z_2 \neq 0$ end up in states $(y_1, y_2) \in Q_2$ with $y_2 = 0$ and this is a consequence of dynamics of state variable x_2 that is asymptotically stable, robustly with respect to the action of control input u. Furthermore from any state $(z_1, z_2) \in Q_2$ it is possible to reach a state $(y_1, y_2) \in Q_2$ with $z_1 \neq y_1$ and this is due to the effect of control input u in the dynamics of state variable x_1.

7. Discussion

In this work we showed existence of symbolic models that are approximately bisimilar to δ–GAS nonlinear control systems. Furthermore for δ–ISS digital control systems we showed that a symbolic model can be constructed through an input quantization.
The results in this paper follow the research line of [21] and provide important generalizations:

(i) The definition of the symbolic model in [21] relies on an (arbitrary) a–priori choice of control inputs, while the symbolic model in (4.4) captures the effect of *any* measurable control input;

(ii) The approximation notion employed in [21] is approximate simulation while the results in this paper *guarantee* approximate bisimulation.

On the basis of such generalizations, the symbolic models in (4.4) and (5.1) provide a finer description than the one proposed in [21] (see proof of Theorem 2 in [21]) and this is essential from the controller synthesis point of view. Indeed, the main drawback of the results in [21] is that if a controller fails to exist for the symbolic model in [21], nothing can be concluded regarding the existence of a controller for the original control system. Our results guarantee, instead, that given a control system and a specification, a controller exists for the original model if and only if a controller exists for the symbolic model. The key idea for obtaining such generalizations was to replace the notion of asymptotic stabilizability of [21] with the stronger notion of δ–GAS. Notice that while δ–GAS implies asymptotic stabilizability as employed in Theorem 2 of [21], the converse is not true in

general[1]. Furthermore even if a feedback control law rendering the closed–loop system δ–GAS were found, if the input space of the control system is bounded, there is no guarantee that such feedback would satisfy the input constraints.
The results in this paper share similar ideas with the ones in [9] that considers discrete–time linear control systems. When we regard discrete–time control systems as the time discretization of continuous–time control systems Theorem 5.1 extends Theorem 4 of [9] in two directions:

(i) by enlarging the class of control systems from linear to nonlinear;

(ii) by enlarging the class of input signals from piecewise–constant to measurable.

When specializing results of this paper to the class of *linear* control systems, conditions of Theorems 4.2 and 5.1 simplify. In fact given a linear control system:

$$\dot{x} = Ax + Bu,\ x \in \mathbb{R}^n,\ u \in U \subseteq \mathbb{R}^m,$$

notions of δ–GAS and δ–ISS reduce to asymptotic stability of matrix A; moreover by [20], functions β and γ appearing in inequalities (2.1) and (2.2) can be chosen as:

$$\beta(r,s) = \|e^{As}\| r; \qquad \gamma(r) = \left(\|B\| \int_0^\infty \|e^{As}\| ds \right) r. \tag{7.1}$$

Notice that functions in (7.1) are well defined, because of asymptotic stability of matrix A. The use of explicit expressions in (7.1) for β and γ simplifies indeed the search of parameters $\tau \in \mathbb{R}^+$, $\eta \in \mathbb{R}^+$ and $\mu \in \mathbb{R}^+$ satisfying conditions (4.5) and (5.2) of Theorems 4.2 and 5.1, respectively. Hence the construction of symbolic models in (4.4) and (5.1) which rely on those parameters, simplify as well. Furthermore, in contrast to the case of nonlinear control systems, the construction of the symbolic model of (4.4) can be performed for this class of systems. This can be done by using results on polytopic approximation of reachable sets for linear control systems (see e.g. [26], [12]) with compact input space. In fact by following [26], given any desired precision $\nu \in \mathbb{R}^+$, the reachable set $\mathcal{R}(\tau, q)$ of (4.2) can be approximated by a polytope $P(\tau, q)$, so that:

$$\mathbf{d}_h(P(\tau,q), \mathcal{R}(\tau,q)) \leq \nu,$$

where $\mathbf{d}_h$ is the Hausdorff pseudo–metric induced by the metric $\mathbf{d}$. The countable set $\mathcal{P}_\mu(\tau, q)$, can then be reformulated in terms of $P(\tau, q)$ rather than of $\mathcal{R}(\tau, q)$, as follows:

$$\mathcal{P}_\mu(\tau,q) := \{y \in [\mathbb{R}^n]_\mu : \exists z \in P(\tau,q) \text{ s.t. } \|y - z\| \leq \mu/2\}.$$

Symbolic model in (4.4) can be adapted to the case of linear systems by defining the set $L_2(q)$ as appearing in (4.1) by:

$$L_2(q) := \mathcal{P}_\mu(\tau, q) \tag{7.2}$$

and the transition relation $\xrightarrow[2]{}$ by:

$$q \xrightarrow[2]{l} p, \quad \text{if} \quad \|p - \mathbf{x}(\tau, q, 0) - l\| \leq \eta/2. \tag{7.3}$$

Since as already pointed out before, the set $P(\tau, q)$ can be computed, the set $L_2(q)$ can be computed as well and therefore the symbolic model in (4.4) with $L_2(q)$ given by (7.2) and $\xrightarrow[2]{}$ given by (7.3), can be constructed. Finally Theorem 4.2 can be adapted to this case by incorporating the precision ν in condition (4.5), as follows:

$$\|e^{A\tau}\|\varepsilon + \nu + \mu + \eta/2 \leq \varepsilon.$$

[1]In fact the converse is true in the case of linear control systems.

References

[1] R. Alur and D.L. Dill. A theory of timed automata. *Theoretical Computer Science*, 126(2):183–235, 1994.

[2] R. Alur, T. Henzinger, G. Lafferriere, and G.J. Pappas. Discrete abstractions of hybrid systems. *Proceedings of the IEEE*, 88(7):971–984, July 2000.

[3] D. Angeli. A Lyapunov approach to incremental stability properties. *IEEE Transactions on Automatic Control*, 47(3):410–421, 2002.

[4] D. Angeli and E.D. Sontag. Forward completeness, unboundedness observability, and their lyapunov characterizations. *Systems and Control Letters*, 38:209–217, 1999.

[5] A. Arnold, A. Vincent, and I. Walukiewicz. Games for synthesis of controllers with partial observation. *Theoretical Computer Science*, 28(1):7–34, 2003.

[6] A. Bicchi, A. Marigo, and B. Piccoli. On the rechability of quantized control systems. *IEEE Transaction on Automatic Control*, April 2002.

[7] A. Bicchi, A. Marigo, and B. Piccoli. Feedback encoding for efficient symbolic control of dynamical systems. *IEEE Transaction on Automatic Control*, 51(6):987– 1002, June 2006.

[8] E.M. Clarke, O. Grumberg, and D. Peled. *Model Checking*. MIT Press, 1999.

[9] A. Girard. Approximately bisimilar finite abstractions of stable linear systems. In A. Bemporad, A. Bicchi, and G. Buttazzo, editors, *Hybrid Systems: Computation and Control*, volume 4416 of *Lecture Notes in Computer Science*, pages 231–244. Springer Verlag, Berlin, 2007.

[10] A. Girard and G.J. Pappas. Approximate bisimulations for nonlinear dynamical systems. In *Proceedings of the 44th IEEE Conference on Decision and Control*, Seville, Spain, 2005.

[11] K.A. Grasse. Simulation and bisimulation of nonlinear control systems with admissible classes of inputs and disturbances. *SIAM Journal on Control and Optimization*, 46(2):562–584, 2007.

[12] I. Hwang, D. M. Stipanovic, and C.J. Tomlin. Polytopic approximations of reachable sets applied to linear dynamic games and to a class of nonlinear systems. In E.H. Abed, editor, *Advances in Control, Communication Networks, and Transportation Systems: In Honor of Pravin Varaiya*, Systems and Control: Foundations and Applications. Birkhuser, Boston, MA, 2005.

[13] H.K. Khalil. *Nonlinear Systems*. Prentice Hall, New Jersey, second edition, 1996.

[14] G. Lafferriere, G.J. Pappas, and S. Sastry. O-minimal hybrid systems. *Mathematics of Control, Signals and Systems*, 13(1):1–21, February 2000.

[15] E.B. Lee and L. Markus. *Foundations of Optimal Control Theory*. SIAM series in applied mathematics. Wiley, New York, Dec 1967.

[16] R. Milner. *Communication and Concurrency*. Prentice Hall, 1989.

[17] D.M.R. Park. Concurrency and automata on infinite sequences. volume 104 of *Lecture Notes in Computer Science*, pages 167–183, 1981.

[18] G. Pola, A. Girard, and P. Tabuada. Approximately bisimilar symbolic models for nonlinear control systems. In *46th IEEE Conference on Decision and Control*, New Orleans, LA, December 2007. Submitted. Available at `http://www.ee.ucla.edu/~pola/`.

[19] P.J. Ramadge and W.M. Wonham. Supervisory control of a class of discrete event systems. *SIAM Journal on Control and Optimization*, 25(1):206–230, 1987.

[20] E.D. Sontag. Input to State Stability: Basic concepts and results. In *CIME Summer Course on Nonlinear and Optimal Control Theory*, Lecture Notes in Mathematics, pages 462–488. Springer-Verlag, 2004.

[21] P. Tabuada. Approximate simulation relations and finite abstractions of quantized control systems. In A. Bemporad, A. Bicchi, and G. Buttazzo, editors, *Hybrid Systems: Computation and Control*, volume 4416 of *Lecture Notes in Computer Science*, pages 529–542. Springer Verlag, Berlin, 2007.

[22] P. Tabuada. Symbolic models for control systems. *Acta Informatica*, 43(7):477–500, February 2007. Special Issue on Hybrid Systems.

[23] P. Tabuada and G. J. Pappas. Bisimilar control affine systems. *Systems and Control Letters*, 52(1):49–58, 2004.

[24] P. Tabuada and G.J. Pappas. Linear Time Logic control of discrete-time linear systems. *IEEE Transactions on Automatic Control*, 51(12):1862–1877, 2006.

[25] A.J. van der Schaft. Equivalence of dynamical systems by bisimulation. *IEEE Transactions on Automatic Control*, 49(12):2160–2172, 2004.

[26] P. Varaiya. Reach set computation using optimal control. In *Proceedings of the KIT Workshop on Verification of Hybrid Systems*, pages 377–383, Grenoble, France, 1998.

[27] W. Zielonka. Infinite games on finitely coloured graphs with applications to automata on infinite trees. *Theoretical Computer Science*, 200:135–183, 1998.

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