A Sequent Calculus for Skeptical Default Logic

P. A. Bonatti and N. Olivetti

Dip. di Informatica - Università di Torino
Corso Svizzera 185, I-10149 Torino, Italy
E-mail: {bonatti,olivetti}@di.unito.it

Abstract. In this paper, we contribute to the proof-theory of Reiter's Default Logic by introducing a sequent calculus for skeptical reasoning. The main features of this calculus are simplicity and regularity, and the fact that proofs can be surprisingly concise and, in many cases, involve only a small part of the default theory.

1 Introduction

Non-monotonic logics play a fundamental role in knowledge representation and commonsense reasoning, as well as in the theory of programming languages.[1] The semantic and algorithmic aspects of non-monotonic reasoning have been extensively investigated (e.g. see [22, 26, 13, 17, 18, 9, 29, 33, 25] and [30, 27, 3, 4, 7, 35, 1, 2, 31, 36]). On the other hand, the proof-theoretic aspects are not yet completely understood.

The fundamental papers by Gabbay [14] , Makinson [24] and Kraus, Lehmann and Magidor [19] , focus their attention on general properties of non-monotonic inference, rather than on specific formalisms. In particular, they do not axiomatize any form of non-monotonic assumption making. A similar consideration holds for the papers by Bochman [5] and Nait Abdallah [28]. The only complete axiomatizations are Levesque's Hilbert-style system for skeptical reasoning in a generalized autoepistemic logic [20],[2] Olivetti's sequent calculus for minimal entailment [32] , and Bonatti's sequent calculi for credulous reasoning in default logic and normal autoepistemic logic [6] . A novel feature of [6] is the use of an *axiomatic rejection method* (cf. [21, 37, 38, 10, 11, 40, 41, 39, 8]) for checking the consistency of the defaults' justifications, and for deriving negative autoepistemic literals (disbeliefs).

In the present paper, we proceed along the line of research initiated in [6]. We introduce a sequent calculus for skeptical default reasoning, with two main goals in mind. First, we are aiming at a terse, abstract characterization of default inference, without committing to any specific proof strategy. The resulting calculus combines the constructive nature of algorithmic approaches with the declarative nature of axiomatic systems. Among the possible applications of such a theoretical tool, we mention the following.

[1] We mention the semantics of negation in logic programming, inheritance in object oriented languages, and the structured operational semantics of process algebras.

[2] From Levesque's system, Jiang [16] derived a resolution principle for clausal autoepistemic theories.

– It constitutes an intermediate step toward a general framework, where different proof strategies and heuristics can be explored and compared.
– It is a useful tool for investigating the power and the efficiency of nonmonotonic reasoning (e.g., it can be used to prove non-elementary speed-up results [12]).
– It facilitates the understanding of nonmonotonic logics, and constitutes a promising didactic tool.

Secondly, we want our calculus to yield concise proofs, where the defaults which are irrelevant to the conclusion play no essential role, by analogy with the following examples.

Example 1. Consider an arbitrary default theory T containing (among other components) the default

$$\delta = \frac{:A}{\neg A}.$$

We claim that T skeptically entails $\neg A$. To see this, note that δ cannot be applied, because its consequent denies the premise. Thus, in order to block δ, each extension of T must contain $\neg A$, which means that $\neg A$ must be skeptically derivable. Note that the other components of T (propositional formulae and defaults different from δ) play no role in the above argument.

Example 2. Consider an arbitrary default theory T containing (among other components) the following sentence and default rule:

$$\frac{:B}{\neg A}, \quad \neg A \vee \neg B \to C\,.$$

Any such T entails skeptically C. Indeed, if the default is applicable, then its conclusion and the implication $\neg A \vee \neg B \to C$ suffice to prove C; alternatively, if the default is not applicable, then the negation of its justification (i.e. $\neg B$) must be derivable (we do not care how), and through $\neg A \vee \neg B \to C$ we may conclude C, also in this case. None of T's sentences and defaults, except the above ones, play any role in this argument.

Note that ignoring part of the given theory is not a trivial task in nonmonotonic reasoning. For instance, it seems impossible to achieve a similar behavior in credulous default logic. Difficulties are strictly related to the following property.

Proposition 1. *For all sentences C and default theories T, such that T credulously entails C, there exists a default δ such that $T \cup \{\delta\}$ does not entail credulously C.*

In other words, the answer to a credulous reasoning problem cannot be a function of a strict subset of the given theory T. All the current approaches to skeptical reasoning are based on credulous reasoning, in the sense that they enumerate all the extensions of the given theory—with the exception of [27], where autoepistemic theories are translated into classical propositional theories (usually much larger than the given theory) which can be queried through classical theorem proving. The enumeration-based approaches are inefficient in two respects; first, the number of extensions can be exponential in the size of the default theory; secondly, by the above proposition, all the rules of the given theory need to be considered. The calculus introduced in this paper does not have the above limitations, and by this very fact it is fundamentally different from all the previous enumeration-based approaches.

The paper is organized as follows. In the next section we recall some basic definitions and properties concerning default logic and the rejection method introduced in [8]. In Section 3 we introduce the skeptical default sequent calculus and demonstrate its main properties. The paper is concluded by a brief comparison between the skeptical calculus and the credulous calculus of [6].

2 Preliminaries

2.1 Propositional Default Logic

Here, only some basic notions are recalled; for more details see [34, 22]. Let $\mathcal{L}$ be a standard propositional language. A *default* is an inference rule of the form:

$$\frac{\alpha : \beta_1, \cdots, \beta_n}{\gamma} \, ,$$

sometimes denoted by $\alpha : \beta_1, \cdots, \beta_n \,/\, \gamma$, where $\alpha, \beta_1, \cdots, \beta_n, \gamma \in \mathcal{L}$. Roughly speaking, the intuitive meaning of the above default is: if α can be derived and each β_i is consistent with the rest of the theory, then conclude γ. For all defaults δ having the above structure, the precondition α, which is called *prerequisite*, will be denoted by $p(\delta)$; the set of sentences $\{\beta_1, \cdots, \beta_n\}$, called *justification*, will be denoted by $j(\delta)$; and γ, which is the *conclusion* of the default, will be denoted by $c(\delta)$. We will employ $\neg j(\delta)$ as an abbreviation for $\{\neg\beta_1, \cdots, \neg\beta_n\}$. A *default theory* is a pair $\langle W, D \rangle$, where W is a set of sentences and D is a set of defaults. We shall often identify a default theory $\langle W, D \rangle$ with the set $W \cup D$. A theory[3] E is a *default extension* (or simply an extension) of $\langle W, D \rangle$ if and only if $E = \bigcup_{i<\omega} E_i$, where

$$\begin{aligned} E_0 &= W \, , \\ E_{i+1} &= \mathrm{Th}(E_i) \cup \{\, c(\delta) \mid \delta \in D,\ E_i \vdash p(\delta),\ E \cap \neg j(\delta) = \emptyset \,\} \, . \end{aligned}$$

A default theory Γ *skeptically entails* a sentence ψ if ψ belongs to all the extensions of Γ. Γ *credulously entails* a sentence ψ if ψ belongs to at least one extension of Γ.[4]

Example 3. (Nixon's Diamond) The statements

[3] By theory, we mean a set of sentences, closed under classical entailment.

[4] Sometimes "credulous" and "skeptical" are replaced by "brave" and "cautious", in the literature.

Nixon is a quaker (Q);
Nixon is a republican (R);
If Nixon is a republican then, if possible, assume that Nixon is not a pacifist ($\neg P$);
If Nixon is a quaker then, if possible, assume that Nixon is a pacifist (P);

can be represented by the default theory $\langle W, D \rangle$ where $W = \{Q, R\}$ and $D = \{(R : \neg P / \neg P), (Q : P/P)\}$. The reader may easily verify that this theory has two extensions: $E' = \mathrm{Th}(\{Q, R, \neg P\})$ and $E'' = \mathrm{Th}(\{Q, R, P\})$. Intuitively, in E', the first default is applied; its consequent blocks the second default. Symmetrically, in E'', the second default is applied, and blocks the first one. Clearly, Q and R are skeptically entailed by $\langle W, D \rangle$, while P and $\neg P$ are entailed only credulously.

Definition 2. A default δ is **active** in a set of sentences E if and only if $E \vdash p(\delta)$ and $E \cap \neg j(\delta) = \emptyset$.

Intuitively, a default is active in E if its preconditions are satisfied in the context defined by E.[5]

Example 4. In the above example, the default $R : \neg P / \neg P$ is active in E', while $Q : P/P$ is active in E''.

Lemma 3. [6] *Assume that δ is not active in a theory E. Then E is an extension of $\langle W, D \rangle$ iff E is an extension of $\langle W, D \cup \{\delta\} \rangle$.*

2.2 Anti-Sequent Calculus

An *anti-sequent* is a pair of sets of sentences $\langle \Gamma, \Sigma \rangle$, denoted by $\Gamma \not\vdash \Sigma$. As usual, Γ, α is an abbreviation for $\Gamma \cup \{\alpha\}$. The intended meaning of $\Gamma \not\vdash \Sigma$ is: there exists a model of Γ where all the sentences of Σ are false. If M is such a model, then we say that the anti-sequent is *true* and that M is an *anti-model* for it.

An anti-sequent $\Gamma \not\vdash \Sigma$ is an axiom of the anti-sequent calculus if, and only if, Γ and Σ are disjoint sets of propositional variables. The rules of the calculus are listed in Fig. 1.

Note that most rules coincide with their classical counterparts. The only difference is that the classical rules with two premisses are split into pairs of rules with one premise (e.g. $\not\vdash \bullet\wedge$ and $\not\vdash \wedge\bullet$). Intuitively, this means that what is exhaustive search in classical sequent calculus, becomes nondeterminism in anti-sequent calculus. The above proof-system is sound and complete.

[5] The notion of active default is essentially similar to the notion of *generating default* which, however, is defined for extensions only. When E is an arbitrary set of sentences, the set of active defaults does not necessarily generate E; hence the change of terminology.

$$\frac{\Gamma \nvdash \Sigma, \alpha}{\Gamma, \neg\alpha \nvdash \Sigma}\ (\neg \nvdash) \qquad \frac{\Gamma, \alpha \nvdash \Sigma}{\Gamma \nvdash \Sigma, \neg\alpha}\ (\nvdash \neg)$$

$$\frac{\Gamma, \alpha, \beta \nvdash \Sigma}{\Gamma, \alpha \wedge \beta \nvdash \Sigma}\ (\wedge \nvdash) \qquad \frac{\Gamma \nvdash \Sigma, \alpha}{\Gamma \nvdash \Sigma, \alpha \wedge \beta}\ (\nvdash \bullet\wedge) \qquad \frac{\Gamma \nvdash \Sigma, \beta}{\Gamma \nvdash \Sigma, \alpha \wedge \beta}\ (\nvdash \wedge\bullet)$$

$$\frac{\Gamma, \alpha \nvdash \Sigma}{\Gamma, \alpha \vee \beta \nvdash \Sigma}\ (\bullet\vee \nvdash) \qquad \frac{\Gamma, \beta \nvdash \Sigma}{\Gamma, \alpha \vee \beta \nvdash \Sigma}\ (\vee\bullet \nvdash) \qquad \frac{\Gamma \nvdash \Sigma, \alpha, \beta}{\Gamma \nvdash \Sigma, \alpha \vee \beta}\ (\nvdash \vee)$$

$$\frac{\Gamma \nvdash \Sigma, \alpha}{\Gamma, \alpha \to \beta \nvdash \Sigma}\ (\bullet \to \nvdash) \qquad \frac{\Gamma, \beta \nvdash \Sigma}{\Gamma, \alpha \to \beta \nvdash \Sigma}\ (\to \bullet \nvdash) \qquad \frac{\Gamma, \alpha \nvdash \Sigma, \beta}{\Gamma \nvdash \Sigma, \alpha \to \beta}\ (\nvdash \to)$$

Fig. 1. Rules of anti-sequent calculus

Theorem 4 [8]. *An anti-sequent $\Gamma \nvdash \Sigma$ is true if and only if it is provable.*

The anti-sequent calculus preserves many properties of the standard sequent calculus. For example, the above rules are perfectly symmetric, and satisfy the subformula property; for each connective there exist rules for introducing it in the left-hand side and in the right-hand side of anti-sequents; moreover, as we already pointed out, the new rules are strikingly similar to their classical counterparts. The rule below is a counterpart of the classical Cut rule, and generalizes Łukasiewicz's detachment rule.

$$\frac{\Gamma \nvdash \Sigma \qquad \Gamma, \alpha \vdash \Sigma}{\Gamma \nvdash \Sigma, \alpha}\ \text{Cut 2}\,.$$

This rule is manifestly sound. By Theorem 4, we have that the calculus without Cut 2 is complete, so we get the admissibility of Cut 2 for free. In the anti-sequent calculus, proofs correspond to counterexamples. Note that a derivation in the anti-sequent calculus is linear, and hence contains exactly one axiom, that constitutes a partial anti-model for the conclusion of the proof. In some other rejection methods there is no clear correspondence between proofs and counterexamples (see [8] for further details).

3 The Skeptical Calculus for Default Logic

3.1 Residual Rules

As an intermediate step towards a skeptical calculus for default logic, in this section we develop a calculus for propositional logic extended with ordinary monotonic inference rules of the form α/β. We call such a rule *residue* because, for our purpose, it is what is left of a default rule after deleting all justifications. Let $\mathcal{L}$ denote the propositional language, we define $\mathcal{L}^{res}$, the language of residues, as follows:

$$\mathcal{L}^{res} = \mathcal{L} \cup \{\alpha/\beta \mid \alpha, \beta \in \mathcal{L}\}.$$

Given a subset S of $\mathcal{L}^{res}$, we are interested in the deductive closure of S under classical provability *and* residual rules.

Definition 5. Let $S \subseteq \mathcal{L}^{res}$; the deductive closure of S, denoted by $\mathrm{Th}^{res}(S)$ is the least set $S' \subseteq \mathcal{L}$ which satisfies the following conditions:

a) $S \cap \mathcal{L} \subseteq S'$;
b) if $S' \vdash \alpha$, then $\alpha \in S'$;
c) if $\alpha \in S'$, and $\alpha/\gamma \in S$, then $\gamma \in S'$.

We now show that given S, $\mathrm{Th}^{res}(S)$ exists and can be generated inductively.

Proposition 6. *Given S, let*

$$S_0 = S \cap \mathcal{L}\,;$$
$$S_{i+1} = \mathrm{Th}(S_i) \cup \{\beta \mid \alpha/\beta \in S \ \wedge \ S_i \vdash \alpha\}\,.$$

Then $\mathrm{Th}^{res}(S) = \bigcup_i S_i$.

In Fig. 2 we give a sequent and anti-sequent calculus for residual rules, that is for Th^{res}. Sequents and antisequents have respectively the form $\Gamma \vdash \Delta$ and $\Gamma \not\vdash \Delta$, where Γ is a finite subset of $\mathcal{L}^{res}$ and Δ is a finite subset of $\mathcal{L}$.

In the next lemma we state some easy properties of the closure operator Th^{res} which are needed in the next theorem.

Proposition 7. *Let $S \subseteq \mathcal{L}^{res}$, then*

1. $\mathrm{Th}(S) \subseteq \mathrm{Th}^{res}(S)$;
2. *if* $S \subseteq S'$, *then* $\mathrm{Th}^{res}(S) \subseteq \mathrm{Th}^{res}(S')$;
3. $\mathrm{Th}^{res}(S \cup \{\alpha/\beta\}) \subseteq \mathrm{Th}^{res}(S \cup \{\beta\})$;
4. *if* $\alpha \in \mathrm{Th}^{res}(S)$, *then* $\mathrm{Th}^{res}(S \cup \{\alpha/\beta\}) = \mathrm{Th}^{res}(S \cup \{\beta\})$;
5. *if* $\alpha \notin \mathrm{Th}^{res}(S)$, *then* $\mathrm{Th}^{res}(S \cup \{\alpha/\beta\}) = \mathrm{Th}^{res}(S)$.

Theorem 8. *The standard sequent calculus and the anti-sequent calculus extended with* (Re1)-(Re4) *are complete w.r.t. residual rules. That is*

(i) $\Gamma \vdash \Delta$ *is derivable iff* $\bigvee \Delta \in \mathrm{Th}^{res}(\Gamma)$;
(ii) $\Gamma \not\vdash \Delta$ *is derivable iff* $\bigvee \Delta \notin \mathrm{Th}^{res}(\Gamma)$.

$$\textbf{(Re1)}\ \frac{\Gamma \vdash \Delta}{\Gamma,\, \alpha/\gamma \vdash \Delta} \qquad \textbf{(Re2)}\ \frac{\Gamma \vdash \alpha \qquad \Gamma,\, \gamma \vdash \Delta}{\Gamma,\, \alpha/\gamma \vdash \Delta}$$

$$\textbf{(Re3)}\ \frac{\Gamma \nvdash \Delta \qquad \Gamma \nvdash \alpha}{\Gamma,\, \alpha/\gamma \nvdash \Delta} \qquad \textbf{(Re4)}\ \frac{\Gamma,\, \gamma \nvdash \Delta}{\Gamma,\, \alpha/\gamma \nvdash \Delta}$$

Plus the rules of classical sequent calculus and anti-sequent calculus, restricted to $\mathcal{L}$.

Fig. 2. Classical calculi extended with residues

3.2 The Skeptical Sequent Calculus

In order to simplify the presentation, we first introduce a basic version of the skeptical calculus; a generalized version (which leads to more efficient deductions) will be introduced in the next section.

The skeptical sequent calculus exploits *constraints* of the form $\mathbf{M}\alpha$ or $\mathbf{L}\alpha$, where $\alpha \in \mathcal{L}$. Intuitively, $\mathbf{M}$ and $\mathbf{L}$ are analogous to a possibility modality and to a necessity modality, respectively. We say that a set of sentences E *satisfies* a constraint $\mathbf{M}\alpha$ if $E \nvdash \neg\alpha$; we say that E satisfies $\mathbf{L}\alpha$ if $E \vdash \alpha$.

A *skeptical default sequent* is a 3-tuple $\langle \Sigma, \Gamma, \Delta \rangle$, denoted by $\Sigma; \Gamma \vdash \Delta$, where Σ is a set of constraints, Γ is a propositional default theory, and Δ is a set of propositional sentences. The intended meaning of the above sequent is: $\bigvee \Delta$ belongs to all the extensions of Γ that satisfy the constraints Σ. When this is the case, we say that the sequent is *true*.

The *skeptical sequent calculus* for default logic comprises the axioms and rules illustrated in Fig. 3. Intuitively, (Sk1) explores the alternative cases where the justifications $\beta_1 \dots \beta_n$ are/are not consistent; in the first case (first premise) the default is equivalent to the residue α/γ (and is replaced with the latter); the other premisses correspond to all the possible ways of contradicting the justifications; clearly, in these cases, the default cannot be applied and can therefore be removed. When the set of constraints cannot be possibly satisfied, the skeptical sequent is vacuously true; the rules (Sk2) and (Sk3) detect this condition. Finally, (Sk4) captures the property that default logic extends classical logic.

Theorem 9. *The skeptical calculus is sound and complete, that is, a skeptical sequent is derivable if, and only if, it is true.*

Example 5. Consider the default theory $\Gamma = \{ : B/\neg A,\ : A/\neg B,\ \neg A \vee \neg B \rightarrow C \}$. This theory skeptically entails C. Fig. 4 illustrates a proof of C (the classical part of the derivations is omitted).

$$(\mathbf{Sk1})\quad \frac{\mathbf{M}\beta_1,\ldots,\mathbf{M}\beta_n, \Sigma\,;\,\Gamma,\,\alpha/\gamma \vdash \Delta \qquad \mathbf{L}\neg\beta_1, \Sigma\,;\,\Gamma \vdash \Delta \quad \ldots \quad \mathbf{L}\neg\beta_n, \Sigma\,;\,\Gamma \vdash \Delta}{\Sigma\,;\,\Gamma,\,\frac{\alpha\,:\,\beta_1\ldots\beta_n}{\gamma} \vdash \Delta}$$

$$(\mathbf{Sk2})\,\frac{\Gamma \vdash \neg\alpha}{\mathbf{M}\alpha,\,\Sigma\,;\,\Gamma \vdash \Delta}\ (\Gamma \subseteq \mathcal{L}^{res}) \qquad (\mathbf{Sk3})\,\frac{\Gamma \not\vdash \alpha}{\mathbf{L}\alpha,\,\Sigma\,;\,\Gamma \vdash \Delta}\ (\Gamma \subseteq \mathcal{L}^{res})$$

$$(\mathbf{Sk4})\quad \frac{\Gamma \vdash \Delta}{\Sigma\,;\,\Gamma \vdash \Delta}\ (\Gamma \subseteq \mathcal{L}^{res})$$

Plus the rules for residues, restricted to $\mathcal{L}^{res}$.

Fig. 3. Skeptical sequent calculus

4 Enhanced Calculus

According to the skeptical calculus introduced in the previous section, the rules for residues (i.e. the monotonic part of the calculus) cannot be applied until all *proper* defaults—that is, defaults with nonempty justification—have been eliminated. Intuitively, we are forced to verify, for each possible subset of the defaults, whether it generates an extension or not. This causes proof trees to be exponentially large in the size of the default theory; more precisely, each proof tree has at least 2^n nodes, where n is the number of defaults occurring in the root. However, in general, it is not necessary to consider every default, in order to derive a skeptical conclusion (cf. examples 1 and 2). In this section, we show that a sound generalization of the skeptical rules can be used to reduce dramatically the proof size.

The generalized rules are illustrated in Fig. 5 (they are meant to replace (Sk2)-(Sk4)). The basic idea behind (Sk2') and (Sk4') is that each extension of Γ that satisfies Σ, contains both the propositional sentences of Γ and all the sentences α such that $\mathbf{L}\alpha \in \Sigma$; moreover, these sentences are closed under classical entailment and the residues occurring in Γ. Therefore, the sentences in this closure can be used to prove the conclusion of a skeptical sequent, as in (Sk4'), or to prove that no extension of Γ can possibly satisfy a constraint $\mathbf{M}\alpha$, as in (Sk2'). In order to understand (Sk3'), note that any extension of Γ is the closure of the propositional sentences of Γ, plus the consequents of some of its defaults. The set $\Gamma^{res+cons}$ is an upper approximation of these sentences; any sentence which does not follow from $\Gamma^{res+cons}$ cannot belong to any extension of Γ; this observation is used in (Sk3') to conclude that no extension of Γ satisfies

$$\dfrac{\begin{array}{c}\vdots\;\Pi_1\\ \mathbf{M}B;\,\neg A,\,\frac{:A}{\neg B},\,\neg A\vee\neg B\rightarrow C\vdash C\end{array}\qquad\begin{array}{c}\vdots\;\Pi_2\\ \mathbf{L}\neg B;\,\frac{:A}{\neg B},\,\neg A\vee\neg B\rightarrow C\vdash C\end{array}}{;\,\frac{:B}{\neg A},\,\frac{:A}{\neg B},\,\neg A\vee\neg B\rightarrow C\vdash C}\,(\mathrm{Sk1})$$

where Π_1 is

$$\dfrac{\dfrac{\neg A,\,\neg B,\,\neg A\vee\neg B\rightarrow C\vdash\neg A}{\mathbf{M}A,\mathbf{M}B;\,\neg A,\,\neg B,\,\neg A\vee\neg B\rightarrow C\vdash C}\,(\mathrm{Sk2})\qquad\dfrac{\begin{array}{c}\vdots\\ \neg A,\,\neg A\vee\neg B\rightarrow C\vdash C\end{array}}{\mathbf{L}\neg A,\mathbf{M}B;\,\neg A,\,\neg A\vee\neg B\rightarrow C\vdash C}\,(\mathrm{Sk4})}{\mathbf{M}B;\,\neg A,\,\frac{:A}{\neg B},\,\neg A\vee\neg B\rightarrow C\vdash C}\,(\mathrm{Sk1})$$

and Π_2 is

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \neg B,\,\neg A\vee\neg B\rightarrow C\vdash C\end{array}}{\mathbf{M}A,\mathbf{L}\neg B;\,\neg B,\,\neg A\vee\neg B\rightarrow C\vdash C}\,(\mathrm{Sk4})\qquad\dfrac{\dfrac{\dfrac{A,C\not\vdash}{C\not\vdash\neg A}\,(\not\vdash\neg)}{\neg A\vee\neg B\rightarrow C\not\vdash\neg A}\,(\rightarrow\bullet\not\vdash)}{\mathbf{L}\neg A,\mathbf{L}\neg B;\,\neg A\vee\neg B\rightarrow C\vdash C}\,(\mathrm{Sk3})}{\mathbf{L}\neg B;\,\frac{:A}{\neg B},\,\neg A\vee\neg B\rightarrow C\vdash C}\,(\mathrm{Sk1})$$

Fig. 4. An example

$\mathbf{L}\alpha$, and hence the conclusion of (Sk3') is vacuously true.[6]

Theorem 10. *The rules* (Sk2')-(Sk4') *are sound.*

Remark. The rules (Sk2)-(Sk4) are special cases of (Sk2')-(Sk4'), therefore, the enhanced calculus is complete, by Theorem 9.

The next examples show the effectiveness of the generalized rules in reducing the length of the proofs.

[6] In the extended version of the paper we introduce also a modification of (Sk1), called (Sk1'), where each premise $\mathbf{L}\neg\beta_i,\Sigma;\Gamma\vdash\Delta$ is replaced with $\mathbf{M}\beta_1\ldots\mathbf{M}\beta_{i-1},\mathbf{L}\neg\beta_i,\Sigma;\Gamma\vdash\Delta$ (the order of the β_is is irrelevant, it can be any permutation of the justifications of the default). This modification affects neither soundness nor completeness. The advantage of (Sk1') is that *a justification β may occur in many defaults without affecting the size of the proofs* (up to a constant). Intuitively, after some assumption on β_i has been done, (that is, either $\mathbf{M}\beta_i\in\Sigma$ or $\mathbf{L}\neg\beta_i\in\Sigma$), the corresponding constraint can be immediately used to simplify (possibly eliminate) the selected default.

$$(\mathbf{Sk2'})\ \frac{\Sigma', \Gamma' \vdash \neg\alpha}{\mathbf{M}\alpha,\ \Sigma\,;\ \Gamma \vdash \Delta} \qquad (\mathbf{Sk3'})\ \frac{\Gamma^{res+cons} \nvdash \alpha}{\mathbf{L}\alpha,\ \Sigma\,;\ \Gamma \vdash \Delta} \qquad (\mathbf{Sk4'})\ \frac{\Sigma', \Gamma' \vdash \Delta}{\Sigma\,;\ \Gamma \vdash \Delta}$$

where $\Sigma' \subseteq \{\,\alpha \mid \mathbf{L}\alpha \in \Sigma\,\}$, $\Gamma' \subseteq \Gamma^{res} \stackrel{\text{def}}{=} \Gamma \cap \mathcal{L}^{res}$, and $\Gamma^{res+cons} = \Gamma^{res} \cup \{\,c(\delta) \mid \delta \in \Gamma,\ j(\delta) \neq \emptyset\,\}$.

Fig. 5. Enhanced rules

$$\dfrac{\dfrac{\dfrac{\vdots}{\neg A,\ \neg A \vee \neg B \to C \vdash C}}{\mathbf{M}B;\ \neg A,\ \frac{:A}{\neg B},\ \neg A \vee \neg B \to C \vdash C}\,(\text{Sk4}') \qquad \dfrac{\dfrac{\vdots}{\neg B,\ \neg A \vee \neg B \to C \vdash C}}{\mathbf{L}\neg B;\ \frac{:A}{\neg B},\ \neg A \vee \neg B \to C \vdash C}\,(\text{Sk4}')}{;\ \frac{:B}{\neg A}, \frac{:A}{\neg B},\ \neg A \vee \neg B \to C \vdash C}\,(\text{Sk1})$$

Fig. 6. An example

Example 6. Consider the default theory $\Gamma = \{\,:B/\neg A,\ :A/\neg B,\ \neg A \vee \neg B \to C\,\}$ of Ex. 5. With the enhanced calculus, the proof of C can be greatly simplified, as shown by Fig. 6 (cf. Fig. 4). Note that $:A/\neg B$ plays no essential role in the proof, and that it might be replaced by any default theory Γ, as in Fig. 7. The latter proof, is actually a formalization of the argument presented in Ex. 2; the leftmost branch shows that if $:A/\neg B$ is applied, then C can be derived from its conclusion; the other branch shows that if $:A/\neg B$ cannot be applied, then $\neg B$ must be derivable, which suffices to obtain C. An important feature of this proof is that it has constant length for all Γ, no matter how many defaults are

$$\dfrac{\dfrac{\dfrac{\vdots}{\neg A, \neg A \vee \neg B \to C \vdash C}}{\mathbf{M}B;\ \neg A, \Gamma,\ \neg A \vee \neg B \to C \vdash C}\,(\text{Sk4}') \qquad \dfrac{\dfrac{\vdots}{\neg B, \neg A \vee \neg B \to C \vdash C}}{\mathbf{L}\neg B;\ \Gamma,\ \neg A \vee \neg B \to C \vdash C}\,(\text{Sk4}')}{;\ \frac{:B}{\neg A}, \Gamma, \neg A \vee \neg B \to C \vdash C}\,(\text{Sk1})$$

Fig. 7. A generic proof

in Γ.

Example 7. Consider an arbitrary default theory T containing the default $\delta = (: A/\neg A)$, as in Ex. 1. The informal proof of $\neg A$ presented there, can be formalized as shown in Fig. 8(a). The leftmost branch shows that the default cannot be applied; the rightmost branch derives $\neg A$ from the assumption that the default is blocked ($\neg A$, in the left-hand side of the upper right sequent, is obtained from the constraint $\mathbf{L}\neg A$ immediately below). Note that all the defaults of Γ are indeed ignored.

If we further assume that A does not occur in $T \setminus \{\delta\}$, then δ cannot be blocked and T has no extensions. Consequently, any contradiction $\bot$ is skeptically derivable. Fig. 8(b) contains the schema of the formal proof (the completeness of the anti-sequent calculus guarantees that the right branch of the proof can be successfully completed).

$$\dfrac{\dfrac{\neg A \vdash \neg A}{\mathbf{M}A;\, \Gamma,\, \neg A \vdash \neg A}(\mathrm{Sk}2') \quad \dfrac{\neg A \vdash \neg A}{\mathbf{L}\neg A;\, \Gamma \vdash \neg A}(\mathrm{Sk}4')}{;\, \Gamma,\, \dfrac{:A}{\neg A} \vdash \neg A}(\mathrm{Sk}1)$$

(a)

$$\dfrac{\dfrac{\neg A \vdash \neg A}{\mathbf{M}A;\, \Gamma,\, \neg A \vdash \bot}(\mathrm{Sk}2') \quad \dfrac{\begin{array}{c}\vdots\\ \Gamma^{res+cons} \nvdash \neg A\end{array}}{\mathbf{L}\neg A;\, \Gamma \vdash \bot}(\mathrm{Sk}3')}{;\, \Gamma,\, \dfrac{:A}{\neg A} \vdash \bot}(\mathrm{Sk}1)$$

(b)

Fig. 8. An example

Two interesting rules, which together capture the notion of *cumulativity* (cf. [14, 24, 19]) are shown in Fig. 9. It is well known that (Cut) is sound for skeptical default inference, while (WM) is not [24]. Although (Cut) is not needed for the completeness of the skeptical calculus, it can be useful for capturing certain proof strategies. These aspects will be tackled in an extended version of the paper.

$$\dfrac{\Sigma;\, \Gamma \vdash \alpha \quad \Sigma;\, \Gamma,\, \alpha \vdash \Delta}{\Sigma;\, \Gamma \vdash \Delta}(\mathrm{Cut}) \qquad \dfrac{\Sigma;\, \Gamma \vdash \alpha \quad \Sigma;\, \Gamma \vdash \Delta}{\Sigma;\, \Gamma,\, \alpha \vdash \Delta}(\mathrm{WM})$$

Fig. 9. Cumulativity rules (Cut and Weak Monotony)

5 A comparison with the credulous calculus

In this section we reformulate the credulous calculus proposed in [6] in order to make an informal comparison with the skeptical one presented in this paper.

The calculus presented in [6] makes use of sequents with a different structure and does not use the constraints. In Fig. 10 we give the rules of the credulous calculus, rephrased to match the structure of the skeptical sequents.

$$(\mathbf{Cr1})\quad \frac{\Gamma \vdash \Delta}{;\, \Gamma \vdash \Delta}\ (\Gamma \subseteq \mathcal{L})$$

$$(\mathbf{Cr2})\ \frac{\Gamma \vdash \alpha \quad \Sigma\,;\, \Gamma \vdash \Delta}{\mathbf{L}\alpha,\, \Sigma\,;\, \Gamma \vdash \Delta}\ (\Gamma \subseteq \mathcal{L}) \qquad (\mathbf{Cr3})\ \frac{\Gamma \not\vdash \neg\alpha \quad \Sigma\,;\, \Gamma \vdash \Delta}{\mathbf{M}\alpha,\, \Sigma\,;\, \Gamma \vdash \Delta}\ (\Gamma \subseteq \mathcal{L})$$

$$(\mathbf{Cr4})\ \frac{\mathbf{M}\neg\alpha,\, \Sigma\,;\, \Gamma \vdash \Delta}{\Sigma\,;\, \Gamma,\, (\alpha : \beta_1 \ldots \beta_n/\gamma) \vdash \Delta} \qquad (\mathbf{Cr5})\ \frac{\mathbf{L}\neg\beta_i,\, \Sigma\,;\, \Gamma \vdash \Delta}{\Sigma\,;\, \Gamma,\, (\alpha : \beta_1 \ldots \beta_n/\gamma) \vdash \Delta}$$

$$(\mathbf{Cr6})\quad \frac{\Gamma \cap \mathcal{L} \vdash \alpha \quad \mathbf{M}\beta_1, \ldots \mathbf{M}\beta_n,\, \Sigma\,;\, \Gamma, \gamma \vdash \Delta}{\Sigma\,;\, \Gamma,\, (\alpha : \beta_1 \ldots \beta_n/\gamma) \vdash \Delta}$$

Plus the rules for classical sequents and anti-sequents restricted to $\mathcal{L}$.

Fig. 10. Credulous sequent calculus

The most prominent difference with respect to the skeptical calculus is that residual rules are not needed. A similar approach might be taken in the skeptical framework, at the price of a certain loss of elegance. The major reason for adopting residues, however, is flexibility, as explained in Ex. 8 below.

To improve the understanding of the relations between the two calculi, we note that the constraints in the skeptical case are simply assumptions, whereas in the credulous case they must be satisfied. This explains a certain duality between the rules of the credulous calculus and the rules of the skeptical one. Rule $(Cr1)$ is dual of $(Sk4)$, rules $(Cr2)$ and $(Cr3)$ are duals of $(Sk2)$ and $(Sk3)$. Rules $(Cr4)$, $(Cr5)$, $(Cr6)$ corresponds to $(Sk1)$. A default can be introduced if it is unapplicable (either by rule $(Cr4)$: its prerequisite cannot be proved, or by $(Cr5)$: one of its justification is inconsistent), or it is applicable (rule $(Cr6)$). While, in the skeptical case, we must prove the conclusion of the sequent in both cases, in the credulous case, we choose one of the alternatives, and we keep it.

Example 8. $\Gamma = \{ : A/\neg B,\ \neg B : C/D,\ : B/\neg A \}$. We have that Γ credulously entails D, but not skeptically. Here below is a derivation of $;\, \Gamma \vdash D$, in the credulous calculus:

$$\cfrac{\vdash True \qquad \cfrac{\neg B \vdash \neg B \qquad \cfrac{\cfrac{\neg B \vdash \neg B \qquad \cfrac{\neg B, D \not\vdash \neg A \qquad \cfrac{\neg B, D \not\vdash \neg C \qquad \cfrac{\neg B, D \vdash D}{;\, \neg B, D \vdash D}\,Cr1}{\mathbf{M}C;\, \neg B, D \vdash D}\,Cr3}{\mathbf{M}A, \mathbf{M}C;\, \neg B, D \vdash D}\,Cr3}{\mathbf{L}\neg B, \mathbf{M}A, \mathbf{M}C;\, \neg B,\, D,\, : B/\neg A \vdash D}\,Cr2}{\mathbf{M}A, \mathbf{M}C;\, \neg B,\, D,\, \frac{: B}{\neg A} \vdash D}\,Cr5}{\mathbf{M}A;\, \neg B,\, \frac{\neg B : C}{D},\, \frac{: B}{\neg A} \vdash D}\,Cr6}{;\, \frac{: A}{\neg B},\, \frac{\neg B : C}{D},\, \frac{: B}{\neg A} \vdash D}\,Cr6$$

Note that in $(Cr6)$, the prerequisite must be proved by the propositional part of the theory. In this way, the provability of the prerequisite is checked immediately, and residual rules are not needed. This restriction, however, forces an ordering in the introduction (or elimination, if we inspect a derivation backwards) of defaults. For example, in the shown derivation (inspecting it backwards) we cannot "eliminate" $\neg B :\ C/D$ before $: A/\neg B$, since we need the consequent of the latter default to prove the prerequisite of the former. This lack of flexibility may prevent some interesting proof-strategies from being represented in the above credulous calculus.

Since this calculus is just a restatement of the one presented in [6], we have soundness and completeness wrt. credulous reasoning.

Theorem 11. *$\Sigma; \Gamma \vdash \Delta$ is derivable in the credulous calculus iff there is an extension E of Γ, which satisfies all constraints in Σ, and such that $\bigvee \Delta \in E$.*

We conclude by noting that the above calculus is not the only possible formulation of credulous default inference; it is possible to give an alternative formulation which makes use of residual rules and is *perfectly symmetric*, or dual, to the skeptical calculus. A presentation of this latter formulation and a discussion of the relationships between the dual calculi will be deferred to a full paper.

6 Further Work

We are currently adapting the skeptical calculus to other non-monotonic formalisms, such as Autoepistemic Logic, Circumscription and cumulative variants of Default Logic. Moreover, we are exploring proof-strategies which proceed in a goal-directed fashion as long as possible (the cut rule of Fig. 9 plays an important role, in this respect). A strictly related topic is the development of refinements of the calculus, suitable for Logic Programming languages.

An interesting direction for further research concerns first-order default logic. Tiomkin [39] introduced an anti-sequent calculus, complete w.r.t. finite models, that can be used in place of the anti-sequent calculus adopted here. One interesting problem is the identification of a class of default theories for which the resulting proof-system is complete. On the other hand, one may explore variants of Default Logic based on semi-decidable notions of consistency, stronger than the classical one.

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