Finite-time consensus for nonlinear multi-agent systems with fixed topologies

Yilun Shang[1]

Abstract

In this paper, we study finite-time state consensus problems for continuous nonlinear multi-agent systems. Building on the theory of finite-time Lyapunov stability, we propose sufficient criteria which guarantee the system to reach a consensus in finite time, provided that the underlying directed network contains a spanning tree. Novel finite-time consensus protocols are introduced as examples for applying the criteria. Simulations are also presented to illustrate our theoretical results.

Keywords: finite-time consensus; multi-agent systems; distributed control; consensus protocols.

1. Introduction

Distributed coordination for multi-agent systems has become an active research topic and attracted great attention of researchers in recent years; see e.g. [7, 10, 11, 13, 14, 15, 16]. A typical problem in the area is agreement or consensus problem, which means to design a network protocol based on the local information obtained by each agent, such that all agents finally reach an agreement on certain quantities of interest. The network protocol is an interaction rule, which ensures the whole group can achieve consensus on the shared data in a distributed manner. Consensus problems cover a very broad spectrum of applications including formation control, distributed filtering, multi-sensor data fusion, and distributed computation, to cite but a few examples. We refer the reader to the survey papers [12, 21] and references therein.

In the study of consensus problems, convergence rate is an important index to evaluate the proposed protocol. Most of the existing protocols (including those appeared in the aforementioned works) can not result in state consensus in a finite time, that is, consensus is only achieved asymptotically. Hence, finite-time consensus is more appealing and there are a number of settings where finite-time convergence is a desirable property. Recently, finite-time consensus problems have attracted the attention of some researchers. Some related works are briefly reviewed as follows. [2] introduces the signed gradient descent flows which serve as discontinuous protocols for finite-time coordination under connected undirected topologies. Two discontinuous distributed algorithms are characterized in [3] to achieve, respectively, max and min consensus in finite time over strongly connected digraphs. Several classes of continuous nonlinear protocols coming out of the typical linear protocol in [13] are considered by [19, 20]. The authors show that they are efficient finite-time agreement protocols provided the directed interconnection topology has a spanning tree. The question of having communication delays is further discussed in [17]. [8, 9] deal with finite-time consensus under a general framework for finite-time semistability of homogeneous systems, and the underlying topology is assumed to be a connected undirected graph. A continuous finite-time tracking control problem is investigated in [18] for a non-holonomic wheeled mobile robot by carefully selecting control gains.

[1]Department of Mathematics, Shanghai Jiao Tong University, Shanghai 200240, CHINA. email: shyl@sjtu.edu.cn

In this note, we aim to identify general criteria for solving finite-time consensus problems with continuous protocols under directed weighted fixed topologies. Based on the theory of finite-time Lyapunov stability [1, 5], we show that under protocols satisfying our conditions, the states of agents reach a consensus in finite time when the interaction topology has a directed spanning tree. Novel protocols are proposed as a proof of the criteria, and we corroborate their finite-time convergence property thereby.

The rest of the paper is organized as follows. In Section 2, we provide some preliminaries and formulate the finite-time convergence criteria. Section 3 contains the convergence analysis under directed fixed topologies. Some concrete examples with numerical simulations are given in Section 4 and we draw conclusion in Section 5.

2. Problem formulation

In general, information exchange between agents in a multi-agent system can be modeled by directed graphs [4, 13]. Before we proceed, we first introduce some basic concepts and notions in graph theory.

Let $\mathcal{G}(\mathcal{A}) = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}), \mathcal{A})$ be a weighted directed graph with the set of vertices $\mathcal{V}(\mathcal{G}) = \{1, 2, \cdots, n\}$ and the set of arcs $\mathcal{E}(\mathcal{G}) \subseteq \mathcal{V}(\mathcal{G}) \times \mathcal{V}(\mathcal{G})$. The vertex i in $\mathcal{G}(\mathcal{A})$ represents the ith agent, and a directed edge $(i, j) \in \mathcal{E}(\mathcal{G})$ means that agent j can directly receive information from agent i, the parent vertex. The set of neighbors of vertex i is denoted by $\mathcal{N}(\mathcal{G}, i) = \{j \in \mathcal{V}(\mathcal{G}) |\ (j, i) \in \mathcal{E}(\mathcal{G})\}$. $\mathcal{A} = (a_{ij}) \in \mathbb{R}^{n\times n}$ is called the weighted adjacency matrix of $\mathcal{G}(\mathcal{A})$ with nonnegative elements and $a_{ij} > 0$ if and only if $j \in \mathcal{N}(\mathcal{G}, i)$. The corresponding graph Laplacian $L(\mathcal{A}) = (l_{ij}) \in \mathbb{R}^{n\times n}$ can be defined as

$$l_{ij} = \begin{cases} \sum_{k=1,k\neq n}^{n} a_{ik}, & j = i \\ -a_{ij}, & j \neq i \end{cases}.$$

If $\mathcal{A}^T = \mathcal{A}$, we say $\mathcal{G}(\mathcal{A})$ is undirected. As is known, the Laplacian matrix of undirected graph is positive semidefinite.

A directed tree is a directed graph with one root vertex which has no parent vertex, every other vertex has exactly one parent, and the root can be connected to any other vertices through directed paths. A spanning tree of a directed graph $\mathcal{G}$ is a directed tree which is a spanning subgraph. A directed graph $\mathcal{G}$ is called strongly connected if there is a directed path from i to j between any two distinct vertices $i, j \in \mathcal{V}(\mathcal{G})$. An undirected graph is connected if it is strongly connected when regarded as a directed graph. A strongly connected component of a directed graph is an induced subgraph that is maximal, subject to being strongly connected. As is known, the strongly connected components of a given directed graph partition its vertex set.

Here, we consider a system consisting of n autonomous agents, indexed by $1, 2, \cdots, n$. The information interaction topology among them are described by the weighted directed graph $\mathcal{G}(\mathcal{A})$ as defined above. We further assume the diagonal entries of $\mathcal{A}$ are zeroes. The continuous-time dynamics of n agents is described as follows:

$$\dot{x}_i(t) = u_i(t), \quad i = 1, 2, \cdots, n, \tag{1}$$

where $x_i(t) \in \mathbb{R}$ is the state of the ith agent, and $u_i(t) \in \mathbb{R}$ is the state feedback, called protocol, to be designed. Denote $x(t) = (x_1(t), \cdots, x_n(t))^T$ and $1 = (1, \cdots, 1)^T$ with compatible dimensions. For a vector $z \in \mathbb{R}^n$, let $\|z\|_\infty$ denote its l^∞-norm, for a matrix $Z \in \mathbb{R}^{n\times n}$, let $\|Z\|_\infty$ denote its induced l^∞-norm, and for a number $z \in \mathbb{R}$, let $|z|$ denote its absolute value.

Given protocol $\{u_i : i = 1, 2, \cdots, n\}$, the multi-agent system is said to solve a consensus problem if for any initial states and any $i, j \in \{1, \cdots, n\}$, $|x_i(t) - x_j(t)| \to 0$ as $t \to \infty$ (c.f. [13]); and it is said to solve a finite-time consensus problem if for any initial states, there is some finite-time t^* such that $x_i(t) = x_j(t)$ for any $i, j \in \{1, \cdots, n\}$ and $t \geq t^*$ (c.f. [19]).

We now present our protocol as follows:

$$u_i = f_i\bigg(\sum_{j \in \mathcal{N}(\mathcal{G}(\mathcal{A}), i)} a_{ij}(x_j - x_i)\bigg), \tag{2}$$

where functions $f_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \cdots, n$, satisfy the following two assumptions, which will be shown as sufficient criteria for finite-time consensus:

(A1) For $i = 1, \cdots, n$, f_i is a continuous and increasing function with $f_i(z) = 0$ if and only if $z = 0$.

(A2) Given the interaction topology $\mathcal{G}(\mathcal{A})$ and initial state $x(0)$, there exist some constants $\beta > 0$ and $0 < \alpha < 1$ such that, for any $0 < |z| \leq \|L(\mathcal{A})\|_\infty \|x(0)\|_\infty$,

$$\min_{1 \leq i \leq n} \frac{f_i(z)^2}{\big(\int_0^z f_i(s)\mathrm{d}s\big)^\alpha} \geq \beta. \tag{3}$$

We give two remarks here.

Remark 1. *The continuity in Assumption (A1) is meant to guarantee the existence of solutions of differential equations (1) on $[0, \infty)$ for any initial value $x(0)$, as is indicated by Peano's Theorem (e.g. [6] pp.10).*

Remark 2. *It is easy to see that the linear protocol proposed in [13] (i.e. by setting $f_i(x) = kx$ for $k > 0$) does not satisfy Assumption (A2). In fact, consensus can never occur in a finite time for such linear protocols.*

3. Convergence analysis

In this section, the convergence property of the consensus protocol (2) for multi-agent system (1) is given. Prior to the establishment, we introduce the following two lemmas regarding the Laplacian matrix $L(\mathcal{A})$.

Lemma 1. [13, 14, 19] *Assume $\mathcal{G}(\mathcal{A})$ is a directed graph with Laplacian matrix $L(\mathcal{A})$, then we have*

(i) $L(\mathcal{A})1 = 0$ and all non-zero eigenvalues have positive real parts;

(ii) $L(\mathcal{A})$ has exactly one zero eigenvalue if and only if $\mathcal{G}(\mathcal{A})$ has a spanning tree;

(iii) If $\mathcal{G}(\mathcal{A})$ is strongly connected, then there is a positive column vector $\omega \in \mathbb{R}^n$ such that $\omega^T L(\mathcal{A}) = 0$;

(iv) Let $b = (b_1, \cdots, b_n)^T$ be a nonnegative vector and $b \neq 0$. If $\mathcal{G}(\mathcal{A})$ is undirected and connected, then $L(\mathcal{A}) + diag(b)$ is positive definite. Here, $diag(b)$ is the diagonal matrix with the (i, i) entry being b_i.

Lemma 2. [19] *Suppose $\mathcal{G}(\mathcal{A})$ is strongly connected, and ω is given as in Lemma 1. Then $\mathrm{diag}(\omega)L(\mathcal{A}) + L(\mathcal{A})^T\mathrm{diag}(\omega)$ is the graph Laplacian of the connected undirected graph $\mathcal{G}(\mathrm{diag}(\omega)\mathcal{A} + \mathcal{A}^T\mathrm{diag}(\omega))$.*

In what follows we present our main result.

Theorem 1. *If the interaction topology $\mathcal{G}(\mathcal{A})$ has a spanning tree, then the system (1) solves a finite-time consensus problem when protocol (2) is applied.*

Proof. We prove the theorem through the following three steps.

Step 1. Suppose that $\mathcal{G}(\mathcal{A})$ is strongly connected.

By Lemma 1, there exists a positive vector $\omega = (\omega_1, \cdots, \omega_n)^T \in \mathbb{R}^n$ such that $\omega^T L(\mathcal{A}) = 0$. Let $y_i = \sum_{j=1}^n a_{ij}(x_j - x_i)$ and $y = (y_1, \cdots, y_n)^T$. Therefore, $y = -L(\mathcal{A})x$, $y \perp \omega$ and $\dot{x}_i = f_i(y_i)$. Let $f = (f_1, \cdots, f_n)^T$, and then we may rewrite the system in a compact form as $\dot{x} = f(y)$. We define a Lyapunov function as:

$$V(t) = \sum_{i=1}^n \omega_i \int_0^{y_i} f_i(s)\mathrm{d}s.$$

Obviously, $V(t) \geq 0$, and $V(t) = 0$ if and only if $y(t) = 0$. Differentiating $V(t)$, we get

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} = \sum_{i=1}^n \omega_i f_i(y_i)\dot{y}_i = -f(y)^T \mathrm{diag}(\omega) L(\mathcal{A}) f(y).$$

Denote $B = \big(\mathrm{diag}(\omega)L(\mathcal{A}) + L(\mathcal{A})^T \mathrm{diag}(\omega)\big)/2$. By Lemma 2, B can be regarded as a Laplacian matrix of an connected undirected graph and hence is positive semidefinite. Suppose $V(t) \neq 0$, namely, $y \neq 0$. We obtain

$$\begin{aligned} \frac{\mathrm{d}V(t)}{\mathrm{d}t} &= -\frac{1}{2} f(y)^T \big(\mathrm{diag}(\omega)L(\mathcal{A}) + L(\mathcal{A})^T \mathrm{diag}(\omega)\big) f(y) \\ &= -\frac{f(y)^T B f(y)}{f(y)^T f(y)} \cdot \frac{f(y)^T f(y)}{V(t)^\alpha} \cdot V(t)^\alpha, \end{aligned} \tag{4}$$

where $\alpha \in (0,1)$ is defined in Assumption (A2).

Consider the first quality in the right-hand side of equality (4). Let $\mathcal{S} = \{\xi \in \mathbb{R}^n : \xi^T \xi = 1$ and the nonzero terms of $\xi_1, \cdots, \xi_n$ are not with the same sign$\}$. Then $\mathcal{S}$ is a bounded closed set. Since $\xi^T B \xi$ is a continuous function and for any $\xi \in \mathcal{S}$, $\xi^T B \xi > 0$ (involving Lemma 1 and the positive semidefiniteness of B), we have that $\min_{\xi \in \mathcal{S}} \xi^T B \xi := C_1 > 0$. Thereby

$$\frac{f(y)^T B f(y)}{f(y)^T f(y)} = \frac{f(y)^T}{\sqrt{f(y)^T f(y)}} B \frac{f(y)}{\sqrt{f(y)^T f(y)}} \geq C_1.$$

Note that

$$0 < |y_i(t)| \leq \|y(t)\|_\infty = \| - L(\mathcal{A})x(t)\|_\infty \leq \|L(\mathcal{A})\|_\infty \|x(t)\|_\infty \leq \|L(\mathcal{A})\|_\infty \|x(0)\|_\infty,$$

where the last inequality follows from the fact that $\|x(t)\|_\infty$ is non-increasing. By exploiting C_r-inequality and (3) in Assumption (A2), we get

$$\frac{f(y)^T f(y)}{V(t)^\alpha} = \frac{\sum_{i=1}^n f_i(y_i)^2}{\big(\sum_{i=1}^n \omega_i \int_0^{y_i} f_i(s)\mathrm{d}s\big)^\alpha} \geq \frac{\sum_{i=1}^n f_i(y_i)^2}{\sum_{i=1}^n \omega_i^\alpha \big(\int_0^{y_i} f_i(s)\mathrm{d}s\big)^\alpha} \geq C_2 \beta,$$

where $C_2 = 1/\max_{1 \leq i \leq n} \omega_i^\alpha > 0$. Combing these with Equation (4) yields

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} \leq -C_1 C_2 \beta V(t)^\alpha.$$

Consider the differential equation

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = -C_1 C_2 \beta v(t)^\alpha$$

with initial value $v(0) = V(0)$, and its unique solution is shown to be given by

$$v(t) = \begin{cases} \big(-C_1C_2\beta(1-\alpha)t + V(0)^{1-\alpha}\big)^{\frac{1}{1-\alpha}}, & t < t^* \\ 0, & t \geq t^* \end{cases}$$

where $t^* = V(0)^{1-\alpha}/C_1C_2\beta(1-\alpha)$. By Comparison Principle of differential equations (e.g. [6] pp.26), we have $V(t) \leq v(t)$. Consequently, $V(t)$ and $y(t)$ approach zero in finite time t^*. Since $y = -L(\mathcal{A})x$, $y = 0$ implies that $x \in \mathrm{span}\{1\} = \{c1 : c \in \mathbb{R}\}$ and $\dot{x}(t) = 0$ by using Lemma 1 and Assumption (A1). Hence the system solves a finite-time consensus problem.

Step 2. Suppose that $\mathcal{G}(\mathcal{A})$ has a spanning tree with root vertex i, and the subgraph induced by the remaining vertices is strongly connected. Moreover, we suppose there exists no directed path connecting those vertices to i.

Without loss of generality, assume that the root vertex i is vertex n. From the protocol (2), we see the state x_n is time-invariant, and $a_{n1} = \cdots = a_{nn} = 0$. Let $b_i = a_{in}$ for $1 \leq i \leq n-1$, $\widetilde{b} = (b_1, \cdots, b_{n-1})^T$ and $\widetilde{\mathcal{A}} = (a_{ij})_{1\leq i,j\leq n-1} \in \mathbb{R}^{(n-1)\times(n-1)}$. By our assumption, $\widetilde{b} \neq 0$. Denote $z_i = x_i - x_n$ for $i = 1, \cdots, n-1$, and $z = (z_1, \cdots, z_{n-1})^T$. Then for $i = 1, \cdots, n-1$,

$$\dot{z}_i = \dot{x}_i = f_i\bigg(\sum_{j=1}^{n-1} a_{ij}(z_j - z_i) - b_i z_i\bigg).$$

Let $y_i = \sum_{j=1}^{n-1} a_{ij}(z_j - z_i) - b_i z_i$ for $i = 1, \cdots, n-1$, and $\widetilde{y} = (y_1, \cdots, y_{n-1})^T$. Then we obtain

$$\dot{y}_i = \sum_{j=1}^{n-1} a_{ij}\big(f_j(y_j) - f_i(y_i)\big) - b_i f_i(y_i).$$

Since the subgraph $\mathcal{G}(\widetilde{\mathcal{A}})$ induced by $\{1, \cdots, n-1\}$ is strongly connected, by Lemma 1, there exists $\widetilde{\omega} = (\omega_1, \cdots, \omega_{n-1})^T$ such that $\widetilde{\omega}^T L(\widetilde{\mathcal{A}}) = 0$. Define a Lyapunov function $\widetilde{V}(t) = \sum_{i=1}^{n-1} \omega_i \int_0^{y_i} f_i(s)\mathrm{d}s$. Then

$$\frac{\mathrm{d}\widetilde{V}(t)}{\mathrm{d}t} = -f(\widetilde{y})^T \mathrm{diag}(\widetilde{\omega})\big(L(\widetilde{\mathcal{A}}) + \mathrm{diag}(\widetilde{b})\big) f(\widetilde{y}) = -f(\widetilde{y})^T \widetilde{B} f(\widetilde{y}),$$

where $\widetilde{B} = \big(\big(\mathrm{diag}(\widetilde{\omega})L(\widetilde{\mathcal{A}}) + L(\widetilde{\mathcal{A}})^T \mathrm{diag}(\widetilde{\omega})\big)/2\big) + \mathrm{diag}(\widetilde{\omega})\mathrm{diag}(\widetilde{b})$. From Lemma 1 and 2, $\widetilde{B}$ is positive definite. Denote the smallest eigenvalue of it as $\lambda_1(\widetilde{B}) > 0$. Suppose $\widetilde{V}(t) \neq 0$, by utilizing Rayleigh-Ritz Theorem, we obtain

$$\frac{\mathrm{d}\widetilde{V}(t)}{\mathrm{d}t} = -\frac{f(\widetilde{y})^T \widetilde{B} f(\widetilde{y})}{f(\widetilde{y})^T f(\widetilde{y})} \cdot \frac{f(\widetilde{y})^T f(\widetilde{y})}{\widetilde{V}(t)^\alpha} \cdot \widetilde{V}(t)^\alpha \leq -\lambda_1(\widetilde{B}) C_2 \beta \widetilde{V}(t)^\alpha.$$

Thereby, arguing as in Step 1 we get that $\widetilde{V}(t)$ and $\widetilde{y}(t)$ will reach zero in finite time $\widetilde{t}^* = V(0)^{1-\alpha}/\lambda_1(\widetilde{B})C_2\beta(1-\alpha)$. By Lemma 1, $L(\widetilde{\mathcal{A}}) + \mathrm{diag}(\widetilde{b})$ is positive definite and thus non-degenerate. Note that $\widetilde{y} = -\big(L(\widetilde{\mathcal{A}}) + \mathrm{diag}(\widetilde{b})\big)z$, and then $\widetilde{y} = 0$ yields $z = 0$. Consequently, we obtain $x = x_n 1$ and the system solves a finite-time consensus problem with the group decision value x_n.

Step 3. Suppose that $\mathcal{G}(\mathcal{A})$ has a spanning tree.

This general case can be proved by induction exactly as in [20]. We sketch the proof here for completeness. We introduce another directed graph, denoted by $\mathcal{G}^c(\mathcal{A})$, consisting of all strongly connected components $u_1, \cdots, u_k$ of $\mathcal{G}(\mathcal{A})$, such that $(u_i, u_j) \in \mathcal{E}(\mathcal{G}^c)$ if and only if there exist $i' \in \mathcal{V}(u_i)$ and $j' \in \mathcal{V}(u_j)$ satisfying $(i', j') \in \mathcal{E}(\mathcal{G})$.

The dynamics of agents corresponding to the vertex set of the root of $\mathcal{G}^c(\mathcal{A})$ is not affected by others and the local interconnection topology among them is strongly connected. Hence by Step 1, the states of them will reach consensus in a finite time. Denote the finial state by x_0. The induction step can proceed along every path from root to leaves in $\mathcal{G}^c(\mathcal{A})$ by employing Step 2 repeatedly. Since there is a finite number of agents, the system solves a finite-time consensus problem with finial state x_0. $\square$

4. Examples

In this section, to illustrate our theoretical results derived in the above section, we will provide two concrete examples. Both are seen to solve finite-time consensus problems.

Example 1. In protocol (2) for $i = 1, \cdots, n$, take

$$f_i(z) = a_i \mathrm{sign}(z)|z|^{c_i} + b_i z, \quad z \in \mathbb{R} \tag{5}$$

where $a_i > 0$, $b_i \geq 0$, $0 < c_i < 1$, and $\mathrm{sign}(\cdot)$ is the sign function defined as

$$\mathrm{sign}(z) = \begin{cases} 1, & z > 0 \\ 0, & z = 0 \\ -1, & z < 0 \end{cases}.$$

The above protocol is a generalization of some protocols introduced in [19, 20]. In the sequel, we will show that it meets our criteria (A1) and (A2).

Claim 1. *Suppose the interaction topology $\mathcal{G}(\mathcal{A})$ has a spanning tree, then the system (1) solves a finite-time consensus problem when protocol (5) is applied.*

Proof. In view of Theorem 1, we need to verify the assumptions (A1) and (A2) for (5).

It is easy to see Assumption (A1) is satisfied. To prove (A2), let $c = \max_{1\leq i\leq n} c_i$, $\alpha = 2c/(1+c)$ and

$$\beta = \min_{1\leq i\leq n} \left\{ \frac{a_i^2 \cdot \min\left\{ \left(\|L(\mathcal{A})\|_\infty \|x(0)\|_\infty\right)^{2c_i - \frac{2c(1+c_i)}{1+c}}, \left(\|L(\mathcal{A})\|_\infty \|x(0)\|_\infty\right)^{2c_i - \frac{4c}{1+c}} \right\}}{2 \cdot \max\left\{ \left(\frac{a_i}{1+c_i}\right)^{\frac{2c}{1+c}}, \left(\frac{b_i}{2}\right)^{\frac{2c}{1+c}} \right\}} \right\}.$$

Note that $\mathrm{d}|z|^{k+1}/\mathrm{d}t = (k+1)\mathrm{sign}(z)|z|^k$ for $k > 0$. Hence, we have

$$\frac{f_i(z)^2}{\left(\int_0^z f_i(s)\mathrm{d}s\right)^\alpha} = \frac{\left(a_i \mathrm{sign}(z)|z|^{c_i} + b_i z\right)^2}{\left(\frac{a_i}{1+c_i}|z|^{1+c_i} + \frac{b_i}{2}|z|^2\right)^\alpha} \geq \frac{a_i^2 |z|^{2c_i}}{\left(\frac{a_i}{1+c_i}\right)^{\frac{2c}{1+c}} |z|^{\frac{2c(1+c_i)}{1+c}} + \left(\frac{b_i}{2}\right)^{\frac{2c}{1+c}} |z|^{\frac{4c}{1+c}}} \geq \beta,$$

where the last inequality follows from the fact $2c_i - \frac{4c}{1+c} \leq 2c_i - \frac{2c(1+c_i)}{1+c} \leq 0$. $\square$

Example 2. In protocol (2) for $i = 1, \cdots, n$, take

$$f_i(z) = \begin{cases} -a_i \mathrm{sign}(z)|z|^{c_i} \ln|z|, & 0 < |z| \leq e^{-1} \\ a_i \mathrm{sign}(z)|z|^{c_i}, & |z| > e^{-1} \\ 0, & z = 0 \end{cases} \tag{6}$$

where $a_i > 0$, $0 < c_i < 2/3$.

We will show that (6) is also a finite-time consensus protocol.

Claim 2. *Suppose the interaction topology $\mathcal{G}(\mathcal{A})$ has a spanning tree, then the system (1) solves a finite-time consensus problem when protocol (6) is applied.*

Proof. By straightforward calculation, it is easy to see that Assumption (A1) holds. Note that $a_i\mathrm{sign}(z)|z|^{c_i} \leq f_i(z) \leq a_i\mathrm{sign}(z)|z|^{c_i/2}$, when $|z| \leq e^{-1}$. Let $c = \max_{1\leq i\leq n} c_i$, $\alpha = 4c/(2+c)$,

$$\beta_1 = \min_{1\leq i\leq n} \frac{a_i^2\big(\|L(\mathcal{A})\|_\infty\|x(0)\|_\infty\big)^{2c_i - \frac{4c(1+c_i)}{2+c}}}{2\big(\frac{a_i}{1+c_i}\big)^{\frac{4c}{2+c}}}, \quad \beta_2 = \min_{1\leq i\leq n} \frac{a_i^2\big(\|L(\mathcal{A})\|_\infty\|x(0)\|_\infty\big)^{2c_i - \frac{2c(2+c_i)}{2+c}}}{2\big(\frac{2a_i}{2+c_i}\big)^{\frac{4c}{2+c}}}$$

and $\beta = \min\{\beta_1, \beta_2\}$. We may obtain (3) with a similar reasoning as in Claim 1. $\square$

Remark 3. *It is noteworthy that both examples above are not Lipschitz continuous at some points. Since solutions reach span$\{1\}$ in finite time, there is no uniqueness of solutions in backwards time. Therefore, the Lipschitz condition must be violated (e.g. [6] pp.8).*

To illustrate, we show simulation results involving four agents using protocols (5) and (6) respectively over directed network topology $\mathcal{G}$ as shown in Fig. 1. Note that $\mathcal{G}$ in this case has a spanning tree, implying that the conditions of Claim 1 and Claim 2 are satisfied. For simplicity, we assume that $a_{ij} = 1$ if $(j,i) \in \mathcal{E}(\mathcal{G})$, and $a_{ij} = 0$ otherwise. Take initial value $x(0) = (2, -1, 3, -2)^T$. Consider the following two cases of (5) and (6) respectively:

(i) For $i = 1, \cdots, n$, take $f_i(z) = \mathrm{sign}(z)|z|^{3/4} + z$, $z \in \mathbb{R}$.

(ii) For $i = 1, \cdots, n$, take

$$f_i(z) = \begin{cases} -\mathrm{sign}(z)|z|^{1/2}\ln|z|, & 0 < |z| \leq e^{-1} \\ \mathrm{sign}(z)|z|^{1/2}, & |z| > e^{-1} \\ 0, & z = 0 \end{cases}.$$

The simulation results are shown in Fig. 2 and Fig. 3, respectively.

5. Conclusion

Finite-time consensus problems for continuous nonlinear multi-agent systems are investigated in this paper. We propose general sufficient criteria, under which the system achieves a consensus, provided that the underlying interaction topology has a directed spanning tree. We introduce new finite-time distributed protocols as examples for using the criteria. Simulation results are given to demonstrate the effectiveness of our theoretical results. Since we only study the case when interconnection topologies are fixed, how to consider the switching topology is our future research.

References

[1] S. P. Bhat, D. S. Bernstein, Finite-time stability of continuous autonomous systems. *SIAM J. Control Optim.*, 38(2000) pp.751–766

[2] J. Cortés, Finite-time convergent gradient flows with applications to network consensus. *Automatica*, 42(2006) pp.1993–2000

[3] J. Cortés, Distributed algorithms for reaching consensus on general functions. *Automatica*, 44(2008) pp.726–737

[4] C. Godsil, G. Royle, *Algebraic Graph Theory*. Springer-Verlag, New York, 2001

[5] V. T. Haimo, Finite time controllers. *SIAM J. Control Optim.*, 24(1986) pp.760–770

[6] P. Hartman, *Ordinary Differential Equations*. Birkhäuser, Boston, 1982

[7] Y. Hatano, M. Mesbahi, Agreement over random networks. *IEEE Trans. Autom. Control*, 50(2005) pp.1867–1872

[8] Q. Hui, W. M. Haddad, S. P. Bhat, Semistability theory for differential inclusions with applications to consensus problems in dynamical networks with switching topology. *Proc. American Control Conference*, 2008 pp.3981–3986

[9] Q. Hui, W. M. Haddad, S. P. Bhat, Finite-time semistability and consensus for nonlinear dynamical networks. *IEEE Trans. on Autom. Control*, 53(2008) pp.1887–1900

[10] A. Jadbabaie, J. Lin, A. S. Morse, Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Trans. Autom. Control*, 48(2003) pp.988–1001

[11] R. Olfati-Saber, Flocking for multi-agent dynamic systems: Algorithms and theory. *IEEE Trans. on Autom. Control*, 51(2006) pp.401–420

[12] R. Olfati-Saber, J. A. Fax, R. M. Murray, Consensus and cooperation in networked multi-agent systems. *Proceedings of the IEEE*, 95(2007) pp.215–233

[13] R. Olfati-Saber, R. M. Murray, Consensus problems in networks of agents with switching topology and time-delays. *IEEE Trans. Autom. Control*, 49(2004) pp.1520–1533

[14] W. Ren, R. W. Beard, Consensus seeking in multi-agent systems under dynamically changing interaction topologies. *IEEE Trans. on Autom. Control*, 50(2005) pp.655–661

[15] W. Ren, R. W. Beard, E. M. Atkins, Information consensus in multivehicle cooperative control. *IEEE Control Systems Magazine*, 27(2007) pp.71–82

[16] Y. Shang, On consensus of nonlinear multi-agent systems under directed interaction topologies. *Proc. International Conference on Cyber-Enabled Distributed Computing and Knowledge Discovery*, 2009, accepted.

[17] L. Wang, Z. Chen, Z. Liu, Z. Yuan, Finite time agreement protocol design of multi-agent systems with communication delays. *Asian Journal of Control*, 11(2009) pp. 281–286

[18] Z. Wang, S. Li, S. Fei, Finite-time tracking control of a nonholonomic mobile robot. *Asian Journal of Control*, 11(2009) pp. 344–357

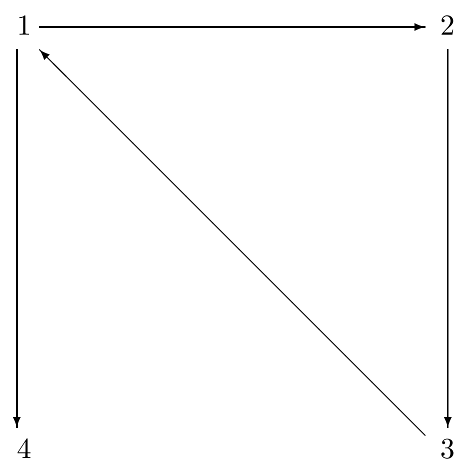
[19] L. Wang, F. Xiao, Finite-time consensus problems for networks of dynamic agents. arXiv:math/0701724v1

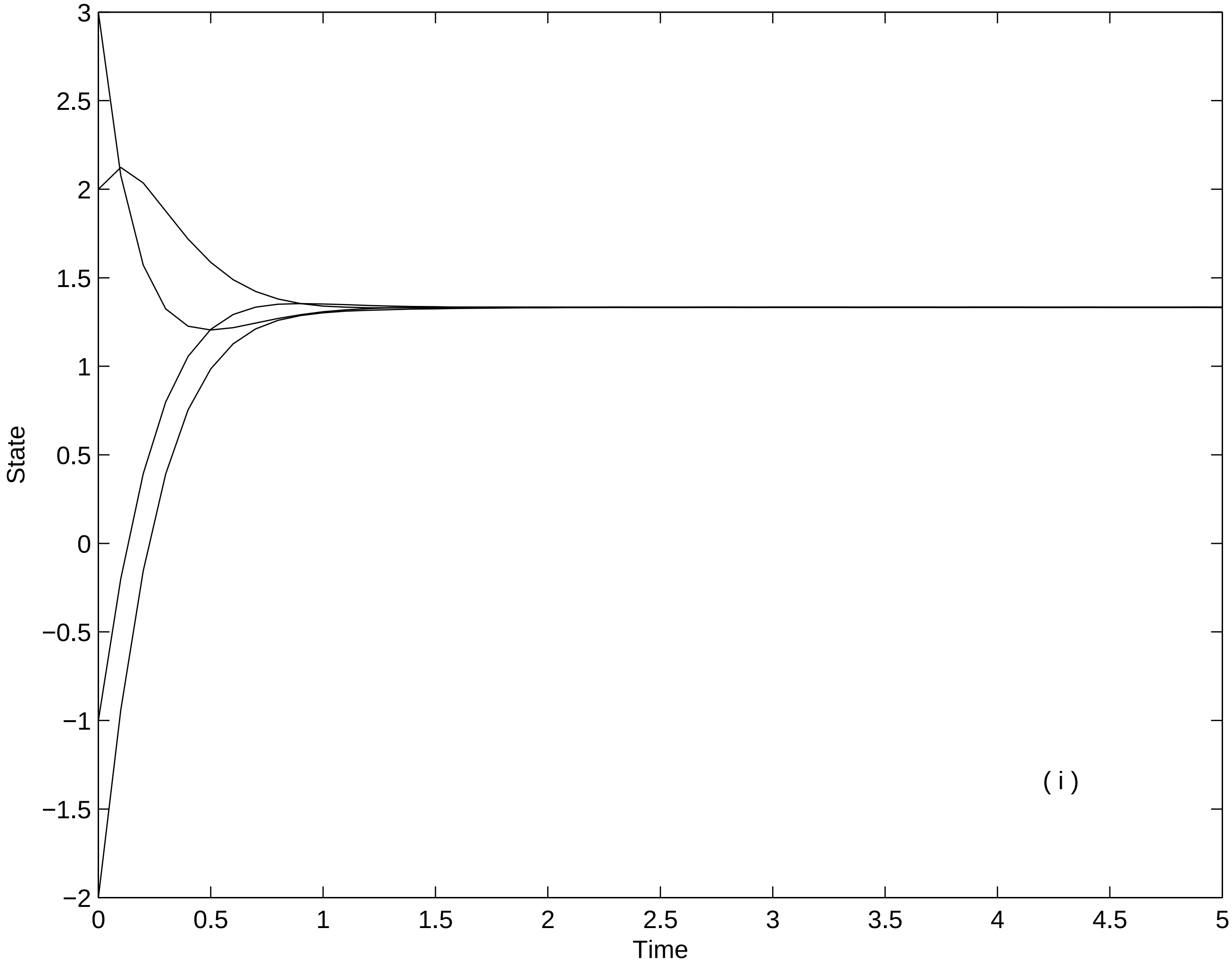
[20] F. Xiao, L. Wang, Y. Jia, Fast information sharing in networks of autonomous agents. *Proc. American Control Conference*, 2008 pp.4388–4393

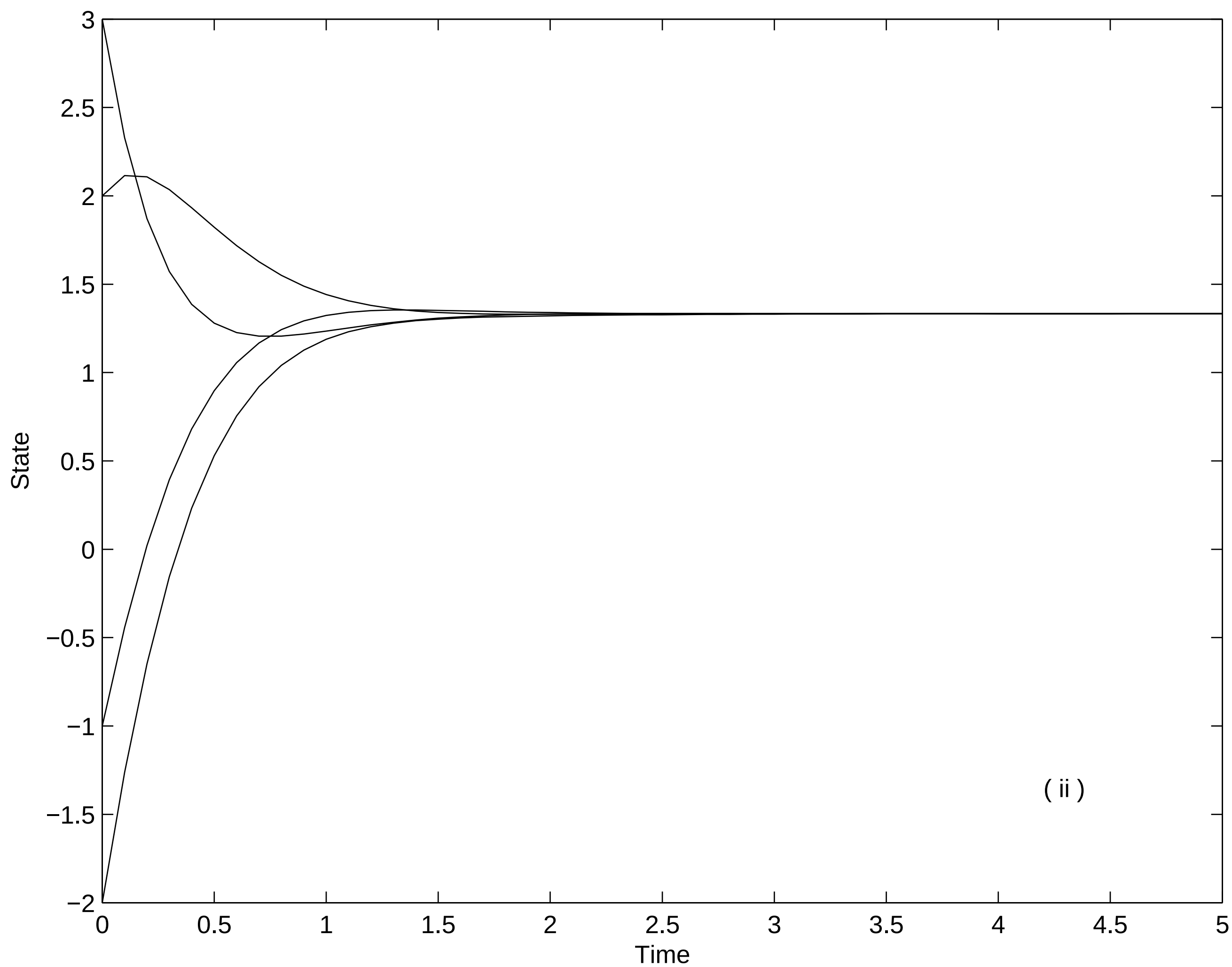
[21] Y. Zheng, L. Shu, Z. Lin, Recent research and progress on consensus problem. *System Engineering Theory and Practice*, 28(2008) pp.27–34

Figure captions

Fig. 1 Directed network $\mathcal{G}$ of four vertices. $\mathcal{G}$ has $0-1$ weights.
Fig. 2 Evolution of states over $\mathcal{G}$ with protocol (2) and (i).
Fig. 3 Evolution of states over $\mathcal{G}$ with protocol (2) and (ii).




3
2.5
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
State
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
Time
(i)



State
Time
(ii)