Books in graphs

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Abstract

A set of q triangles sharing a common edge is called a book of size q. We write $\beta(n,m)$ for the the maximal q such that every graph $G(n,m)$ contains a book of size q. In this note

1) we compute $\beta(n,cn^2)$ for infinitely many values of c with $1/4 < c < 1/3$,

2) we show that if $m \geq (1/4-\alpha)n^2$ with $0<\alpha<17^{-3}$, and G has no book of size at least $\left(1/6-2\alpha^{1/3}\right)n$ then G contains an induced bipartite graph G_1 of order at least $\left(1-\alpha^{1/3}\right)n$ and minimal degree

$$\delta(G_1) \geq \left(\frac{1}{2}-4\alpha^{1/3}\right)n,$$

3) we apply the latter result to answer two questions of Erdős concerning the booksize of graphs $G\left(n,n^2/4-f(n)n\right)$ every edge of which is contained in a triangle, and $0<f(n)<n^{2/5-\varepsilon}$.

1 Introduction

Our notation and terminology are standard (see, e.g., [2]). Thus, $G(n,m)$ is a graph of order n and size m; for a graph G and a vertex $u \in V(G)$ we write $\Gamma(u)$ for the set of vertices adjacent to u; $d_G(u) = |\Gamma(u)|$ is the degree of u; we write $d(u)$ instead of $d_G(u)$ if the graph G is implicit. However, somewhat unusually, we set $\widehat{d}(U) = |\cap_{x\in U}\Gamma(x)|$. Unless explicitly stated, all graphs are assumed to be defined on the vertex set $[n] = \{1,2,...n\}$. Also, $k_s(G)$ is the number of s-cliques of G.

In 1962 Erdős [6] initiated the study of books in graphs. A *book* of size q consists of q triangles sharing a common edge. We write $bk(G)$ for the size of the largest book in a graph G and call it the *booksize* of G. Since 1962 books

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have attracted considerable attention both in extremal graph theory (see, e.g., [10], [5], and [4]) and in Ramsey graph theory (see, e.g., [13], [9], and [11]).

Erdős, Faudree and Rousseau defined in [5] the function

$$\beta(n,m) = \min\{bk(G) \mid G = G(n,m)\}.$$

Our aim in this paper the study of the function $\beta(n,m)$ and its variants. We shall prove a technical inequality about booksizes that we shall use to give bounds on $\beta(n,m)$ and answer two questions of Erdős.

The paper is organized as follows: in section 2 we use a counting argument of Khadžiivanov and Nikiforov [10] to prove a bound on $\beta(n,m)$ in terms of the degree sequence and other graph parameters. In particular, this result implies that $\beta\left(n, \lfloor n^2/4 \rfloor + 1\right) > n/6$, as conjectured by Erdős and proved by Edwards [3]. In addition, we determine $\beta\left(n, cn^2\right)$ for infinitely many values of c with $1/4 < c < 1/3$. In section 3 we prove that a graph $G\left(n, (1/4-\alpha)n^2\right)$ with $0 < \alpha < 17^{-3}$ either has a book of size about $n/6$ or has a large induced bipartite graph with minimal degree close to $n/2$. In the last section we make use of this structural property to answer two questions of Erdős concerning the booksize of graphs $G\left(n, n^2/4 - f(n)n\right)$, every edge of which is contained in a triangle and $0 < f(n) \leq n^{2/5-\varepsilon}$.

2 A lower bound on the booksize of a graph

In 1962 Erdős [6] conjectured that the booksize of a graph G of order n and size greater than $\lfloor n^2/4 \rfloor$ is at least $\lfloor n/6 \rfloor$, i.e., $\beta\left(n, \lfloor n^2/4 \rfloor + 1\right) \geq n/6$. This was proved by Edwards in an unpublished manuscript [3] and independently by Khadžiivanov and Nikiforov in [10].

For $r \geq 3$ and $0 \leq j < r$, we write $K_r^{(j)}$ for the graph consisting of a complete graph K_{r-1} and an additional vertex joined to precisely $r-j-1$ vertices of the K_{r-1}. We denote by $k_r^{(j)}(G)$ the number of induced subgraphs of G that are isomorphic to $K_r^{(j)}$, e. g., $k_4^{(3)}(G)$ is the number of induced subgraphs of G that are isomorphic to a triangle with an isolated vertex.

Theorem 1 *Let $G = G(n,m)$ be a graph with degree sequence $d(1), ..., d(n)$. Then,*

$$\left(6k_3(G) - \sum_{i=1}^{n} d^2(i) + nm\right) bk(G) \geq nk_3(G) + 8k_4(G) + 2k_4^{(3)}(G).$$

Proof In the proof we use some arguments from [10]. Set $\beta = bk(G)$. Clearly G contains exactly $(n-3)k_3(G)$ pairs (v,T) where $v \in V(G)$ and T is a triangle in G. Also, a K_4 subgraph of G contains exactly 4 such pairs; a $K_4^{(j)}$ subgraph contains two such pairs for $j=1$, and one such pair for $j=2$ and 3. Therefore,

$$(n-3)k_3(G) = 4k_4(G) + 2k_4^{(1)}(G) + k_4^{(2)}(G) + k_4^{(3)}(G). \tag{1}$$

We have

$$\sum_{(i,j)\in E(G)} \binom{\widehat{d}(ij)}{2} = 6k_4(G) + k_4^{(1)}(G),$$

yielding

$$\sum_{(i,j)\in E(G)} \left(\widehat{d}^2(ij) - \widehat{d}(ij)\right) = 12k_4(G) + 2k_4^{(1)}(G).$$

Since

$$\sum_{(i,j)\in E(G)} \widehat{d}(ij) = 3k_3(G), \tag{2}$$

we see that

$$\sum_{(i,j)\in E(G)} \widehat{d}^2(ij) = 12k_4(G) + 2k_4^{(1)}(G) + 3k_3(G).$$

Subtracting (1) from the last equality and rearranging the terms, we obtain

$$nk_3(G) = \sum_{(i,j)\in E(G)} \widehat{d}^2(ij) - 8k_4(G) + k_4^{(2)}(G) + k_4^{(3)}(G). \tag{3}$$

Next we shall eliminate the term $k_4^{(2)}(G)$ from (3). For every $i \in V(G)$ set $\Gamma'(i) = V(G) \setminus \Gamma(i)$. The sum $\sum_{ij\in E(G)} \widehat{d}(ij)|\Gamma'(i)\cap\Gamma'(j)|$ counts each $K_4^{(2)}$ once and each $K_4^{(3)}$ three times, so

$$\sum_{(i,j)\in E(G)} \widehat{d}(ij)|\Gamma'(i)\cap\Gamma'(j)| = k_4^{(2)}(G) + 3k_4^{(3)}(G). \tag{4}$$

Subtracting (4) from (3), we see that

$$\begin{aligned} nk_3(G) &= \sum_{(i,j)\in E(G)} \widehat{d}^2(ij) + \sum_{(i,j)\in E(G)} \widehat{d}(ij)|\Gamma'(i)\cap\Gamma'(j)| - 8k_4(G) - 2k_4^{(3)}(G) \\ &= \sum_{(i,j)\in E(G)} \widehat{d}(ij)\left(\widehat{d}(ij) + |\Gamma'(i)\cap\Gamma'(j)|\right) - 8k_4(G) - 2k_4^{(3)}(G). \quad (5) \end{aligned}$$

Noting that $\widehat{d}(ij) \leq \beta$ for every edge (i,j) and recalling (2), inequality (5) implies that

$$\begin{aligned} nk_3(G) &\leq \beta \sum_{(i,j)\in E(G)} \left(\widehat{d}(ij) + |\Gamma'(i)\cap\Gamma'(j)|\right) - 8k_4(G) - 2k_4^{(3)}(G) \\ &= \beta\left(3k_3(G) + \sum_{(i,j)\in E(G)} |\Gamma'(i)\cap\Gamma'(j)|\right) - 8k_4(G) - 2k_4^{(3)}(G) \quad (6) \end{aligned}$$

Since

$$|\Gamma'(i)\cap\Gamma'(j)| = n - d(i) - d(j) + \widehat{d}(ij).$$

we find that

$$\sum_{(i,j)\in E(G)} \left|\Gamma'(i)\cap\Gamma'(j)\right| = \sum_{(i,j)\in E(G)} \left(n - d(i) - d(j) + \widehat{d}(ij)\right)$$

$$= 3k_3(G) + nm - \sum_{i=1}^{n} d^2(i).$$

Putting this into (6) we see that

$$nk_3(G) + 8k_4(G) + 2k_4^{(3)}(G) \le 6\beta k_3(G) + \beta\left(-\sum_{i=1}^{n} d^2(i) + nm\right),$$

as claimed. □

The following corollary is due to Edwards [3].

Corollary 2 *For every graph $G = G(n,m)$ with $m > n^2/4$*

$$bk(G) \ge \frac{2m}{n} - \frac{n}{3}. \tag{7}$$

Proof With $\beta = bk(G)$, Theorem 1 implies that

$$\left(6k_3(G) - \sum_{i=1}^{n} d^2(i) + nm\right)\beta \ge nk_3(G) + 8k_4(G) + 2k_4^{(3)}(G) \ge nk_3(G),$$

and so

$$(6\beta - n)\,k_3(G) \ge \beta\left(\sum_{i=1}^{n} d^2(i) - nm\right). \tag{8}$$

Since $\sum_{i=1}^{n} d(i) = 2m$, we have

$$\sum_{i=1}^{n} d^2(i) \ge \frac{4m^2}{n} > nm; \tag{9}$$

in particular,

$$\sum_{i=1}^{n} d^2(i) - nm > 0.$$

Hence, (8) implies that $6\beta > n$. Furthermore, as $3k_3(G) \le \beta m$, we see from (8) and (9) that

$$\frac{1}{3}(6\beta - n)\,\beta m \ge \beta\left(\frac{4m^2}{n} - nm\right),$$

implying (7). □

As a consequence of Corollary 2 we easily obtain the following bound.

Corollary 3 *For every graph* $G\left(n, \lfloor n^2/4 \rfloor + 1\right)$ *we have* $bk\,(G) > n/6$. $\square$

The graph $H_{s,t}$ below, constructed by Erdős, Faudree and Rousseau in [5], shows that the bound in Corollary 2 is essentially best possible.

Example 4 *Let* $t \geq 1,\ s > 3$ *be fixed integers. Partition the vertex set* $V = [n]$ *with* $n = 3st$ *into* $3s$ *sets* V_{ij} $(i \in [3],\ j \in [r])$ *of cardinality* t. *Join two vertices* $v \in V_{ij}$ *and* $u \in V_{kl}$ *iff* $i \neq k$ *and* $j \neq l$.

By straightforward counting we see that

$$e\left(H_{s,t}\right) = 3s\left(s-1\right)t^2 = 3s\left(s-1\right)\left(\frac{n}{3s}\right)^2 = \frac{s-1}{3s}n^2,$$

and

$$bk\left(H_{s,t}\right) = \left(s-2\right)t = \frac{\left(s-2\right)n}{3s}.$$

On the other hand, from Corollary 2, we have

$$bk\left(H_{s,t}\right) \geq \frac{2e\left(H_{s,t}\right)}{n} - \frac{n}{3} = \frac{2\left(s-1\right)n}{3s} - \frac{n}{3} = \frac{\left(s-2\right)n}{3s},$$

thus, the bound in Corollary 2 is tight for n, m with $3s|n$, $s > 8$, and $m = \left(s-1\right)n^2/3s$.

A different extremal graph ([3], [10]) is defined as follows.

Example 5 *Select* 6 *disjoint sets* $A_{11}, A_{12}, A_{13}, A_{21}, A_{22}, A_{23}$ *with* $|A_{11}| = |A_{12}| = |A_{13}| = k-1$ *and* $|A_{21}| = |A_{22}| = |A_{23}| = k+1$. *Set* $V\left(G\right)$ *to be the union of all these sets. For every* $1 \leq j < k \leq 3$ *join every vertex of* A_{ij} *to every vertex of* A_{ik} *and for* $j = 1, 2, 3$ *join every vertex of* A_{1j} *to every vertex of* A_{2j}.

It is easy to check that the resulting graph has $n = 6k$ vertices, $9k^2+3 > n^2/4$ edges and its booksize is precisely $k + 1 = n/6 + 1$.

3 A stability theorem for graphs without large books

In this section we give a structural property of graphs having substantial size and whose booksize is small.

In [1] Andrásfai, Erdős and Sós proved that if G is a K_{r+1}-free graph of order n with minimal degree

$$\delta\left(G\right) > \left(1 - \frac{3}{3r-1}\right)n$$

then G is r-chromatic. We shall use this theorem to obtain a structural result related to the stability theorems of Simonovits (see, e. g., [12]).

Theorem 6 *For every α with $0 < \alpha < 10^{-5}$ and every graph $G = G(n,m)$ with*

$$m \geq \left(\frac{1}{4} - \alpha\right) n^2 \tag{10}$$

either

$$bk(G) > \left(\frac{1}{6} - 2\alpha^{1/3}\right) n \tag{11}$$

or G contains an induced bipartite graph G_1 of order at least $\left(1 - \alpha^{1/3}\right) n$ and with minimal degree

$$\delta(G_1) \geq \left(\frac{1}{2} - 4\alpha^{1/3}\right) n. \tag{12}$$

Proof If $m > n^2/4$ then Corollary 3 implies that $bk(G) > n/6$, which is stronger than (11), so we may assume that $m \leq n^2/4$. Furthermore, if $\sum_{i=1}^{n} d^2(i) > nm$ then Theorem 1 implies that

$$(6bk(G) - n)\, k_3(G) > 0,$$

and so again $bk(G) > n/6$. Therefore, we may assume

$$\sum_{i=1}^{n} d^2(i) \leq nm.$$

Clearly, from (10),

$$\frac{4m^2}{n} \geq m(n - 4\alpha n) = nm - 4\alpha nm,$$

and so,

$$\sum_{i=1}^{n} \left(d(i) - \frac{2m}{n}\right)^2 = \sum_{i=1}^{n} d^2(i) - \frac{4m^2}{n} \leq 4\alpha nm \leq \alpha n^3. \tag{13}$$

Set $\varepsilon = \alpha^{1/3}$, $M = \{u \in V(G) :\ d(u) < \frac{2m}{n} - \varepsilon n\}$ and $G_1 = G[V \backslash M]$. We claim that G_1 has the required properties. First we show that its minimal degree satisfies (12). From (13),

$$|M|\,\varepsilon^2 n^2 \leq \sum_{v \in M} \left(d(v) - \frac{2m}{n}\right)^2 < \sum_{i=1}^{n} \left(d(i) - \frac{2m}{n}\right)^2 \leq \alpha n^3.$$

Hence, $|M| < \left(\alpha/\varepsilon^2\right) n = \alpha^{1/3} n$, i.e., $v(G_1) > \left(1 - \alpha^{1/3}\right) n$. Also, for $v \in V \backslash M$, we have

$$\begin{aligned} d_{G_1}(v) &\geq d(v) - |M| > \left(\frac{2m}{n} - \varepsilon n\right) - |M| = \frac{n}{2} - 2\alpha n - \alpha^{1/3} n - |M| \\ &> \left(\frac{1}{2} - 2\alpha n - 2\alpha^{1/3}\right) n \geq \left(\frac{1}{2} - 4\alpha^{1/3}\right) n. \end{aligned} \tag{14}$$

All that remains to prove is that G_1 is bipartite. Suppose first that G_1 contains a triangle with vertices u, v, w, say. Since

$$n \geq d(u) + d(v) + d(w) - \widehat{d}(uv) - \widehat{d}(uw) - \widehat{d}(vw)$$

we find that

$$\begin{aligned} \widehat{d}(uv) + \widehat{d}(uw) + \widehat{d}(vw) &\geq d(u) + d(v) + d(w) - n \\ &\geq 3\left(\frac{1}{2} - \alpha - \sqrt[3]{\alpha}\right) n - n. \end{aligned}$$

Thus,

$$bk(G) \geq \left(\frac{1}{6} - \alpha n - \alpha^{1/3}\right) n \geq \left(\frac{1}{6} - 2\alpha^{1/3}\right) n,$$

and so (12) holds. Finally, assume that G_1 is triangle-free. Since $\alpha < 10^{-5}$,

$$\delta(G_1) \geq \left(\frac{1}{2} - 4\alpha^{1/3}\right) n > \frac{2}{5} v(G_1).$$

Hence, the case $r = 2$ of the theorem of Andrásfai, Erdős and Sós mentioned above implies that G_1 is indeed bipartite, completing the proof of Theorem 6 □

It is easily seen that if we are a little more careful in our proof of $\delta(G_1) > v(G_1)$ then the condition on α can be relaxed to $0 < \alpha < 17^{-3}$.

4 Two problems of Erdős

Erdős and Rothschild suggested the study of the booksize of graphs in which every edge is contained in a triangle. In [7] and [8] Erdős himself gave some results on such graphs. Suppose $f(n)$ is a fixed positive function of n, and let $TG(n, f)$ be the set of all graphs $G = G(n, m)$ such that every edge of G is contained in a triangle and $m > \max\left\{n^2/4 - f(n)\, n, 0\right\}$. Set

$$\gamma(n, f) = \min\left\{bk(G) \mid G \in TG(n, f)\right\}.$$

In [7], p. 91, Erdős proved that for every $c > 0$ there exists some $c_1 > 0$ such that

$$\gamma(n, c) \geq c_1 n$$

for n sufficiently large. Hence, setting

$$\underline{\lim}_{n\to\infty} \frac{\gamma(n, c)}{n} = \sigma(c),$$

we see that for every $c > 0$, $\sigma(c) > 0$. Erdős asked how large $\sigma(c)$ is. Our next theorem gives an answer that is asymptotically tight when c tends to 0.

Theorem 7 *For every function $f(n)$ with $0 < f(n) < n/4$,*

$$\gamma(n, f) > \frac{n}{12f(n) + 6}.$$

Proof From Theorem 7 we have for $\beta = bk(G)$

$$\left(6k_3(G) - \sum_{i=1}^{n} d^2(i) + nm\right)\beta \geq nk_3(G),$$

and hence,

$$(6\beta - n)\, k_3(G) \geq \beta \left(\sum_{i=1}^{n} d^2(i) - nm\right).$$

From $\sum_{i=1}^{n} d(i) = 2m$ we have $\sum_{i=1}^{n} d^2(i) \geq 4m^2/n$ and thus,

$$(6\beta - n)\, k_3(G) \geq \beta \left(\frac{4m^2}{n} - nm\right) > -4f(n)\,\beta m.$$

Clearly $3k_3 \geq m$; hence, assuming $6\beta \leq n$,

$$12f(n)\,\beta m > (n - 6\beta)\, k_3(G) \geq (n - 6\beta)\, m,$$

and the desired result follows. □

Applying Theorem 7 with $f(n) = c$, we obtain

$$\sigma(c) \geq \frac{1}{12c + 6}. \tag{15}$$

On the other hand, a slight modification of the graphs described in Example 4 gives a graph $G = G\left(n, n^2/4 - O(1)\right)$, such that every edge of G is contained in a triangle and

$$bk(G) \leq \frac{n}{6},$$

and this, together with (15), implies

$$\lim_{c \to 0} \sigma(c) = \frac{1}{6}.$$

However, for large c Theorem 7 is not precise enough. Prior to obtaining a lower bound on $\gamma(n, f)$ that is valid in a more general case of a function f, we recall the graph that Erdős outlined in [8].

Example 8 *Suppose $f(n)$ with $0 < f(n) < n/4$ tends to infinity with n; set $l_n = f(n)^{1/2}$. Define a graph G as follows: let $V(G) = [n] = A \cup B \cup C$, with $|A| = l_n^2$, $|B| = |C| = \left(n - l_n^2\right)/2$. Join every vertex of B to every vertex of C. Divide B and C into l_n roughly equal disjoint sets B_i and C_i. Join every vertex $x_{ij} \in A$ to every vertex of B_i and C_j.*

It is easily seen that $e(G) = n^2/4 - f(n)n$, every edge of G is contained in a triangle and $bk(G) = o(n)$.

In order to obtain a precise estimate of $bk(G)$ we shall describe more accurately the graph G. Suppose $f(n)$ is a function of n with $4 < f(n) < n/4$. Set $k = \left\lfloor (2f(n))^{1/2} \right\rfloor$, so that $k^2 \leq 2f(n) < (k+1)^2$. Let $n = 2kt + k^2 + s$, where $0 \leq s < 2k$. Set $V(G) = [n]$ and partition $[n]$ into $2k+2$ sets $A, B_1, ..., B_k, C_1, ..., C_k, S$ such that

$$|A| = k^2, \ \ |B_1| = ... = |B_k| = |C_1| = ... = |C_k| = t, \ \ |S| = s.$$

Join every vertex of $\cup_{i=1}^k B_i$ to every vertex $\cup_{i=1}^k C_i$; label the members of A by a_{ij} $(i, j \in [k])$, and, for every $i, j \in [k]$, join a_{ij} to all vertices of $B_i \cup C_j$. By straightforward calculations we obtain

$$\begin{aligned} e(G) &= \frac{\left(n - s - k^2\right)^2}{4} + k^2 \frac{2\left(n - s - k^2\right)}{2k} \geq \frac{\left(n - 2k - k^2\right)^2}{4} + k\left(n - 2k - k^2\right) \\ &\geq \frac{n^2}{4} - \frac{k^2 n}{2} + \frac{k^4 - 4k^2}{4} > \frac{n^2}{4} - f(n)n, \end{aligned}$$

and

$$bk(G) \leq \frac{n - s - k^2}{2k} < \frac{n}{2k} \leq \frac{n}{2\sqrt{2f(n)}}.$$

Since, obviously, $G \in TG(n, f)$, we immediately obtain the bound

$$\gamma(n, f) < \frac{n}{2\sqrt{2f(n)}}. \tag{16}$$

Our next aim is to show that, for a wide class of functions f, (16) is essentially tight.

Theorem 9 *Let $0 < c < 2/5$ and $0 < \varepsilon < 1$ be constants, and $0 < f(n) < n^c$. Then, if n is sufficiently large,*

$$\gamma(n, f) > (1 - \varepsilon) \frac{n}{2\sqrt{2f(n)}}.$$

Proof Let us start with a brief sketch of our proof. Suppose the graph G is a counterexample to our assertion. Then, from Theorem 6, G has an induced bipartite graph G_1 of order at least $n - \alpha^{1/3}n$ and large minimal degree. We show that each part of G_1 has cardinality close to $n/2$ and then consider an edge from G_1; by assumption it is contained in a triangle whose third vertex w is not in G_1. We bound the degree of w from above and then bound the number of all such vertices from below. Dropping a carefully selected number of such vertices we obtain a graph of order n_1 and size greater than $n_1^2/4$, such that n_1 is close to n. Then, by Corollary 3, this graph contains a book of size $n_1/6$, completing the proof.

Now let us give the complete proof. Set $\beta = bk(G)$ and $\alpha = f(n)/n$. Assume the assertion does not hold, i.e., there is some $\varepsilon > 0$ such that for every

F and every N there is an $n > N$ with $f(n) > F$ and a graph $G = G(n,m)$ satisfying the conditions of the theorem and with

$$\beta \leq (1-\varepsilon)\frac{1}{2}\sqrt{\frac{n}{2\alpha}}. \tag{17}$$

Then, as $\beta < n/8$, Theorem 6 implies that G has an induced bipartite graph G_1 of order at least $n - \alpha^{1/3}n$ and

$$\delta(G_1) > \left(\frac{1}{2} - 4\alpha^{1/3}\right)n = \frac{n}{2} - 4\alpha^{1/3}n. \tag{18}$$

Let $V(G_1) = B \cup C$ be a bipartition of G_1 and set $A = V(G) \backslash V(G_1)$. From (18),

$$|B| \geq \left(\frac{1}{2} - 4\alpha^{1/3}\right)n, \;\; |C| \geq \left(\frac{1}{2} - 4\alpha^{1/3}\right)n, \tag{19}$$

$$e(G_1) = e(B,C) \geq \frac{1}{2}\left(1-\alpha^{1/3}\right)n\left(\frac{1}{2} - 4\alpha^{1/3}\right)n$$
$$= \frac{n^2}{4}\left(1-\alpha^{1/3}\right)\left(1-8\alpha^{1/3}\right) > \frac{n^2}{4}\left(1-9\alpha^{1/3}\right).$$

Consider the set T of triangles containing an edge of G_1. Since every edge of G_1 is contained in a triangle and G_1 is bipartite, we see that

$$|T| \geq e(G_1) > \frac{n^2}{4}\left(1-9\alpha^{1/3}\right). \tag{20}$$

Let $D \subset A$ be the set of vertices of A that are contained in some triangle of T. We claim that for every $w \in D$, and n sufficiently large,

$$d(w) < \sqrt{\frac{n}{2\alpha}}. \tag{21}$$

Indeed, by definition, every vertex $w \in D$ is joined to some $u \in B$ and some $v \in C$. Then,

$$\beta \geq |\Gamma(u) \cap \Gamma(w)| \geq |\Gamma(u) \cap \Gamma(w) \cap C| \geq d_C(w) + d_C(u) - |C|$$
$$\geq d_C(w) + \delta(G_1) - |C|,$$

and, similarly,

$$\beta \geq |\Gamma(v) \cap \Gamma(w)| \geq |\Gamma(v) \cap \Gamma(w) \cap B| \geq d_B(w) + \delta(G_1) - |B|.$$

Hence, summing the last two inequalities and taking into account (18),

$$2\beta \geq d_B(w) + d_C(w) + 2\delta(G_1) - n + |A|$$
$$\geq d_B(w) + d_C(w) + |A| - 8\alpha^{1/3}n \geq d(w) - 8\alpha^{1/3}n.$$

To complete the proof of (21), observe that from (17), we have

$$2\beta \leq (1-\varepsilon)\sqrt{\frac{n}{2\alpha}}.$$

For every $w \in D$, let $t(w)$ be the number of triangles of T containing w. Clearly, we have

$$t(w) = \frac{1}{2}\sum_{u\in\Gamma(w)} |\Gamma(u)\cap\Gamma(w)| \leq \frac{1}{2}d(w)\beta \leq \frac{1}{4}d(w)(1-\varepsilon)\sqrt{\frac{n}{2\alpha}}.$$

This, together with (21), gives

$$t(w) < (1-\varepsilon)\frac{n}{8\alpha}. \tag{22}$$

Summing (22) for all $w \in D$, in view of (20), we obtain

$$\frac{n^2}{4}\left(1-9\alpha^{1/3}\right) < |T| = \sum_{w\in D} t(w) < |D|\frac{n(1-\varepsilon)}{8\alpha}.$$

Hence,

$$|D| > 2\alpha\frac{\left(1-9\alpha^{1/3}\right)n}{(1-\varepsilon)}.$$

Observe that, as $\alpha = f(n)/n < n^{c-1}$ and $c < 2/5$, we have $\lim_{n\to\infty}\alpha^{1/3} = 0$. Then, for n sufficiently large, we see that

$$|D| > 2(1+\varepsilon)\alpha n.$$

Select a set $D_0 \subset D$ with

$$(2+\varepsilon)\alpha n < |D_0| < (2+2\varepsilon)\alpha n. \tag{23}$$

As, from (21), for every vertex $w \in D_0$ and n sufficiently large, we have

$$d(w) < \sqrt{\frac{n}{2\alpha}},$$

then the graph $G[V\backslash D_0]$ has at least

$$e(G) - |D_0|\sqrt{\frac{n}{2\alpha}}$$

edges. We shall prove that if n is large enough then

$$\frac{n^2}{4} - \alpha n^2 - |D_0|\sqrt{\frac{n}{2\alpha}} > \frac{(n-|D_0|)^2}{4}. \tag{24}$$

Assume that (24) does not hold. Then, from (23),

$$\frac{n^2}{4} - \alpha n^2 - (2(1+\varepsilon)\alpha n)\sqrt{\frac{n}{2\alpha}} \leq \frac{n^2}{4} - \alpha n^2 - |D_0|\sqrt{\frac{n}{2\alpha}} \leq \frac{(n-|D_0|)^2}{4}$$
$$\leq \frac{(n-(2+\varepsilon)\alpha n)^2}{4}$$

and thus, after some simple algebra,

$$\frac{\varepsilon}{2} \leq (2(1+\varepsilon))\frac{1}{\sqrt{2\alpha n}} + \frac{(2+\varepsilon)^2\alpha^2}{4} < \frac{4}{\sqrt{2f(n)}} + 4n^{2c-2},$$

which is a contradiction if n is large enough. Thus, (24) holds. Then, if n is sufficiently large, Corollary 3 implies that

$$bk(G[V\backslash D_0]) > \frac{n-|D_0|}{6} > \sqrt{\frac{n}{2\alpha}}.$$

This contradiction completes our proof. □

In [7], p. 235, Erdős asked how large $\gamma(n, n^c)$ is for $0 < c < 1$. Putting $f(n) = n^{c-1}$ for $1 < c < 7/5$ and applying Theorem 9, together with (16), we obtain the following.

Corollary 10 *If $0 < c < 1$ and n is sufficiently large,*

$$\gamma(n, n^c) < \frac{1}{2\sqrt{2}}n^{1-c/2}.$$

Also, if $0 < c < 2/5$, $\varepsilon > 0$ and n is sufficiently large,

$$\gamma(n, n^c) > \frac{1-\varepsilon}{2\sqrt{2}}n^{1-c/2}.$$ □

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