THE ACTIONS OF $Out(F_k)$ ON THE BOUNDARY OF OUTER SPACE AND ON THE SPACE OF CURRENTS: MINIMAL SETS AND EQUIVARIANT INCOMPATIBILITY

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Abstract. We prove that for $k \geq 5$ there does not exist a continuous map $\partial CV(F_k) \to \mathbb{P}Curr(F_k)$ that is either $Out(F_k)$-equivariant or $Out(F_k)$-anti-equivariant. Here $\partial CV(F_k)$ is the "length-function" boundary of Culler-Vogtmann's Outer space $CV(F_k)$, and $\mathbb{P}Curr(F_k)$ is the space of projectivized geodesic currents for F_k.

We also prove that, if $k \geq 3$, for the action of $Out(F_k)$ on $\mathbb{P}Curr(F_k)$ and for the diagonal action of $Out(F_k)$ on the product space $\partial CV(F_k) \times \mathbb{P}Curr(F_k)$ there exist unique non-empty minimal closed $Out(F_k)$-invariant sets.

Our results imply that for $k \geq 3$ any continuous $Out(F_k)$-equivariant embedding of $CV(F_k)$ into $\mathbb{P}Curr(F_k)$ (such as the Patterson-Sullivan embedding) produces a new compactification of Outer space, different from the usual "length-function" compactification $\overline{CV(F_k)} = CV(F_k) \cup \partial CV(F_k)$.

1. Introduction

For a free group F of finite rank $k \geq 2$ Culler-Vogtmann's *Outer space* [13] $CV(F)$ is a fundamental object for studying the group $Out(F)$ and the properties of individual automorphisms of F. Outer space is a close cousin of the Teichmüller space $\mathcal{T}(S_g)$ of a hyperbolic surface S_g. The analogy between the action of the mapping class group on Teichmüller space and the action of $Out(F)$ on $CV(F)$ is an important source of mathematical and philosophical inspiration in the study of $Out(F)$. However, it is well-understood that Outer space $CV(F)$ is more complicated than Teichmüller space. The elements of $CV(F)$ are minimal free and discrete actions of F on $\mathbb{R}$-trees with quotient metric graphs of volume one. Any such F-action on a tree T defines a translation length function $||\cdot||_T : F \to \mathbb{R}_{\geq 0}$ and thus a point in $\mathbb{R}^F$. This gives an embedding $CV(F) \subset \mathbb{P}\mathbb{R}^F$, and its closure $\overline{CV(F)}$ is the "length function" compactification of $CV(F)$, see §3 below. It is known that the elements of $\overline{CV(F)}$ are precisely the projective classes of *very small* minimal actions of F on $\mathbb{R}$-trees [6, 12].

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The recent work [10, 18, 19, 20, 21] shows that there is another fundamental structure related to $Out(F)$, namely the space $Curr(F)$ of *geodesic currents* on F, that is, the space of positive Radon F-invariant and flip-invariant measures on the space $\partial^2 F$ of pairs of distinct points of ∂F. Here the "flip map" is $\sigma : \partial^2 F \to \partial^2 F$, $\sigma : (\xi, \zeta) \mapsto (\zeta, \xi)$. The corresponding projectivized space with respect to scalar multiples is denoted $\mathbb{P}Curr(F)$. There is a natural action of $Out(F)$ on $Curr(F)$ that quotients through to the action of $Out(F)$ on $\mathbb{P}Curr(F)$. The advantage of the projectivized space $\mathbb{P}Curr(F)$ is that it is compact.

In the study of Teichmüller space geodesic currents are natural and important objects, as elucidated, in particular, in the work of Bonahon [3, 4]. Thus the points of the Thurston boundary $\partial\mathcal{T}(S_g)$ of Teichmüller space are geodesic laminations on S_g equipped with transverse measures. When pulled back to the universal cover of S_g, these transverse measures lift to $\pi_1(S_g)$-invariant measures on $\partial^2\mathbb{H}^2 = \partial^2\pi_1(S_g)$, that is, geodesic currents. These transverse measures are important, in particular, because they are used to define the metric on an $\mathbb{R}$-tree dual to a geodesic lamination.

It turns out that for a free group F geodesic currents and $\mathbb{R}$-trees are naturally "transversal objects". There is a canonical $Out(F)$-equivariant *intersection form* (see [19, 20, 26])

$$I : cv(F) \times Curr(F) \to \mathbb{R}$$

where $cv(F)$ is the space of all (i.e. with arbitrary covolume) $\mathbb{R}$-trees equipped with a free and discrete minimal action of F. This intersection form has some important properties in common with Bonahon's notion of an intersection number between geodesic currents on a hyperbolic surface. In particular, for every $\mathbb{R}$-tree $T \in cv(F)$ and every non-trivial $g \in F$ we have

$$I(T, \mu_g) = ||g||_T,$$

where μ_g is the "counting" or "rational" current corresponding to the conjugacy class $[g]$ (see Definition 2.6 below for the precise definition of μ_g), and $||g||_T$ denotes the translation length on T of the element $g \in F$.

The study of $Curr(F)$ has already led to some new results about the geometry and dynamics of free group automorphisms. For example, it is proved [19] that for every $\phi \in Aut(F)$ and every free basis A of F the *conjugacy distortion spectrum* $I_A(\phi)$ *of* ϕ is a rational interval $I_A(\phi) = [\lambda_1, \lambda_2] \cap \mathbb{Q}$ where by definition

$$I_A(\phi) := \{\frac{||\phi(g)||_A}{||g||_A} : g \in F, g \neq 1\}.$$

Moreover, the extremal distortions λ_1, λ_2 actually belong to $I_A(\phi)$, that is they are realized as distortions of some non-trivial conjugacy classes. The geodesic currents approach also provided a theoretical explanation for the experimental results regarding the behavior of Whitehead's algorithm [17, 21, 23]. The work of Coulbois, Hilion and Lustig [8, 9, 10] explored the idea of an $\mathbb{R}$-tree dual to a measured geodesic lamination in the context of free groups and some new and unexpected behavior was uncovered. Other results regarding currents and free group automorphisms can be found in [17, 18, 19, 20, 21, 22, 18, 24, 27].

Since both $\overline{CV(F)}$ and $\mathbb{P}Curr(F)$ are fundamental and intimately connected compact spaces endowed with natural $Out(F)$-actions, it is interesting to understand how the dynamical properties of these actions are related to each other. There is an additional reason to be interested in such questions, which is motivated by what happens for hyperbolic surfaces. Bonahon proved that if S_g is a compact oriented hyperbolic surface and $\mathcal{T}(S_g)$ is the Teichmüller space of S_g, then the Liouville map $L : \mathcal{T}(S_g) \to \mathbb{P}Curr(\pi_1(S_g))$ is a topological embedding equivariant with respect to the action of the mapping class group $Mod(S_g)$ of S_g. Moreover, it turns out that the map L extends to a homeomorphism (which is necessarily $Mod(S_g)$-equivariant) from the Thurston compactification $\overline{\mathcal{T}(S_g)}$ to the closure of the image of L. It is natural to ask if there is an analogue of this result for free groups, where the Thurston compactification of Teichmüller space is replaced by the length function compactification $\overline{CV(F)}$ of Outer space. It turns out that the answer to this question is negative. The reason for this is very general and is given by the following theorem that we establish in this paper:

Theorem A. *Let F be a free group of finite rank $k \geq 5$. Then there exists no continuous $Out(F)$-equivariant map $\partial CV(F) \to \mathbb{P}Curr(F)$. Similarly, there is no continuous $Out(F)$-anti-equivariant map $\partial CV(F) \to \mathbb{P}Curr(F)$.*

There are several equivalent descriptions of the Liouville map $L : \mathcal{T}(S_g) \to \mathbb{P}Curr(\pi_1(S_g))$ mentioned above. One of such descriptions involves characterizing the Liouville current as the Patterson-Sullivan current corresponding to the hyperbolic structure. It turns out that this characterization generalizes to the case of free groups. In [24] Kapovich and Nagnibeda proved that the Patterson-Sullivan map $\mathcal{P} : CV(F) \to \mathbb{P}Curr(F)$ is an $Out(F)$-equivariant topological embedding. Theorem A implies that, provided F has rank at least five, $\mathcal{P}$ does not extend to a continuous map from $\overline{CV(F)}$.

While proving Theorem A we obtain some new information about the dynamics of the action of $Out(F)$ on $\mathbb{P}Curr(F)$. In [27] Reiner Martin introduces the following subset of $\mathbb{P}Curr(F)$: the set $\mathcal{M}^{\mathbb{P}Curr} \subseteq \mathbb{P}Curr(F)$ is defined as the closure in $\mathbb{P}Curr(F)$ of the set of projectivized rational currents $[\mu_g]$ corresponding to all the primitive elements g of F. Although suggested by the name, Reiner Martin does not prove or conjecture that $\mathcal{M}^{\mathbb{P}Curr}$ is the unique smallest non-empty closed $Out(F)$-invariant subset of $\mathbb{P}Curr(F)$. We shall prove here that this is indeed the case, provided F has rank bigger or equal to three:

Theorem B. *Let F be a free group of finite rank $k \geq 3$. Then $\mathcal{M}^{\mathbb{P}Curr} \subseteq \mathbb{P}Curr(F)$ is the unique smallest non-empty closed $Out(F)$-invariant subset of $\mathbb{P}Curr(F)$.*

Since $\mathcal{M}^{\mathbb{P}Curr}$ is infinite dimensional [27], when F has rank $k \geq 3$, and $\partial CV(F)$ is finite dimensional [29, 30, 14], Theorem B implies:

Corollary 1.1. *Let F be a free group of finite rank $k \geq 3$, there is no $Out(F)$-equivariant (or $Out(F)$-anti-equivariant) topological embedding from $\partial CV(F)$ to $\mathbb{P}Curr(F)$.*

Thus we see that in this case the closure of the image of any $Out(F)$-equivariant topological embedding $CV(F) \to \mathbb{P}Curr(F)$ gives a new compactification of Outer space $CV(F)$, different from the length function compactification $\overline{CV(F)}$. In particular, this applies to the Patterson-Sullivan embedding $\mathcal{P} : CV(F) \to \mathbb{P}Curr(F)$. Together Theorem A and Theorem B imply:

Corollary 1.2. *Let F be a finitely generated free group of rank $k \geq 2$ and let $\mathcal{P} : CV(F) \to \mathbb{P}Curr(F)$ be the Patterson-Sullivan embedding.*

If $k \geq 3$ then $\mathcal{P}$ does not extend to a homeomorphism from $\overline{CV(F)}$ to the closure of the image of $\mathcal{P}$, and $\mathcal{P}$ does not induce a topological embedding $\partial CV(F) \to \mathbb{P}Curr(F)$.

Moreover, if $k \geq 5$, then $\mathcal{P}$ does not extend to a continuous map from $\overline{CV(F)}$ to $\mathbb{P}Curr(F)$.

All of the above results reflect the fact that the dynamics of the $Out(F)$ action on $\mathbb{P}Curr(F)$ is rather different from the dynamics of the $Out(F)$ action on $\overline{CV(F)}$ and on $\partial CV(F)$. In particular, one can expect to find new information about the dynamical properties of free group automorphisms by considering the action of $Out(F)$ on the space of geodesic currents.

We will describe briefly the idea of the proof of Theorem A. Suppose that $\tau : \partial CV(F) \to \mathbb{P}Curr(F)$ is an $Out(F)$-equivariant continuous map. First we compare the dynamics of the action of simple Dehn twists on $\partial CV(F)$ and $\mathbb{P}Curr(F)$. In both cases this dynamics turns out to be of a "parabolic" nature. Using this fact we show that τ must take the length function $[T_D] \in \partial CV(F)$, corresponding to a simple Dehn twist D, to the rational current $[\mu_b] \in \mathbb{P}Curr(F)$ where $b \in F$ is the "twistor" of D (see Section 5 for the definitions related to simple Dehn twists). Next we use the results of Cohen-Lustig [12] about the dynamics of Dehn multi-twists on $\partial CV(F)$. We exhibit two specific points $[T_{D_1}], [T_{D_2}] \in \partial CV(F)$, corresponding to simple Dehn twists D_1 and D_2 with the twistors b_1 and b_2, and a multi-twist ϕ of F with the following properties. On the trees side, we have $\lim_{n\to\infty} \phi^n[T_{D_1}] = \lim_{n\to\infty} \phi^n[T_{D_2}]$. On the other hand, on the currents side, ϕ, b_1, b_2 are chosen so that $\lim_{n\to\infty} \phi^n[\mu_{b_1}] \neq \lim_{n\to\infty} \phi^n[\mu_{b_2}]$. Since, by the first step, we must have $\tau([T_{D_1}]) = [\mu_{b_1}]$ and $\tau([T_{D_2}]) = [\mu_{b_1}]$, we get a contradiction with the assumption that τ is continuous and $Out(F)$-equivariant.

The ideas involved in the proof of Theorem A also yield:

Theorem C. *Let be a free group of finite rank $k \geq 3$.*

Then there exists a unique minimal non-empty closed $Out(F)$-invariant subset $\mathcal{M}^2$ of $\partial CV(F) \times \mathbb{P}Curr(F)$. Moreover, this set $\mathcal{M}^2$ is equal to the closure of all points of $\partial CV(F) \times \mathbb{P}Curr(F)$ of the form $([T_D], [\mu_b])$ where D is a simple Dehn twist of F and b is the twistor of D.

It turns out that in the context of Theorem A there are no continuous equivariant or anti-equivariant maps going in the other direction:

Theorem D. *Let F be a free group of finite rank $k \geq 3$. Then there does not exist a continuous $Out(F)$-equivariant (or $Out(F)$-anti-equivariant) map $\mathcal{M}^{\mathbb{P}Curr} \to \partial CV(F)$. Hence there does not exist a continuous $Out(F)$-equivariant (or $Out(F)$-anti-equivariant) map $\mathbb{P}Curr(F) \to \partial CV(F)$.*

The proof of Theorem D turns out to be somewhat easier than that of Theorem A. First, we show, again by exploring the parabolic dynamics, that if $\tau : \mathcal{M}^{\mathbb{P}Curr} \to \partial CV(F)$ is a map as in Theorem D then for every simple Dehn twist D of F with twistor b we have $\tau([\mu_b]) = [T_D]$. Then, since $k \geq 3$, it is easy two produce two simple Dehn twists D and D' with the same twistor b such that $[T_D] \neq [T_{D'}]$, yielding a contradiction.

2. Basic definitions

We will give only a quick review of the basic concepts related to geodesic currents on free groups. We refer the reader to [19, 20] for a detailed treatment of this topic.

Convention 2.1. For the remainder of the paper, unless specified otherwise, let F be a finitely generated free group of rank $k \geq 2$. We will denote by ∂F the hyperbolic boundary of F in the sense of the theory of word-hyperbolic groups. Since F is free, ∂F can also be viewed as the space of ends of F with the standard ends-space topology.

Thus ∂F is a topological space homeomorphic to the Cantor set. We will also denote

$$\partial^2 F := \{(\zeta, \xi) : \zeta, \xi \in \partial F \text{ and } \zeta \neq \xi\}.$$

Denote by $\sigma : \partial^2 F \to \partial^2 F$ the *flip* map $\sigma : (\zeta, \xi) \mapsto (\xi, \zeta)$ for $(\zeta, \xi) \in \partial^2 F$.

Definition 2.2 (Geodesic Currents)**.** Let F be a free group of finite rank $k \geq 2$. A *geodesic current* on F is a positive Radon measure on $\partial^2 F$ that is F-invariant and σ-invariant. We denote the space of all geodesic currents on F by $Curr(F)$.

The space $Curr(F)$ comes equipped with a weak topology that in this case can be characterized as follows: for $\nu_n, \nu \in Curr(F)$ we have $\lim_{n\to\infty} \nu_n = \nu$ if and only if for every two disjoint closed-open sets $S, S' \subseteq \partial F$ we have $\lim_{n\to\infty} \nu_n(S \times S') = \nu(S \times S')$.

Note that the above definition and notations are consistent with [8, 9, 10] but are a little different from those used in [20]. Namely, in [20] $Curr(F)$ denotes all F-invariant positive Radon measure on $\partial^2 F$. The subspace of those such measures that are also σ-invariant is denoted in [20] by $Curr_s(F)$.

Definition 2.3 (Projectivized Geodesic Currents)**.** For two non-zero geodesic currents $\nu_1, \nu_2 \in Curr(F)$ we say that ν_1 is *projectively equivalent* to ν_2, denoted $\nu_1 \sim \nu_2$, if there exists a non-zero scalar $r \in \mathbb{R}$ such that $\nu_2 = r\nu_1$. We denote

$$\mathbb{P}Curr(F) := \{\nu \in Curr(F) : \nu \neq 0\}/\sim$$

and call it the *space of projectivized geodesic currents on* F. Elements of $\mathbb{P}Curr(F)$ (that is, scalar equivalence classes of elements of $Curr(F)$) are called *projectivized geodesic currents.* The space $\mathbb{P}Curr(F)$ is endowed with the quotient topology. We will denote the $\sim$-equivalence class of a non-zero geodesic current ν by $[\nu]$.

Definition 2.4 ($Out(F)$-action)**.** Let $\phi \in Aut(F)$ be an automorphism. It is well known that ϕ extends to a homeomorphism of ∂F and hence induced a homeomorphism $\hat{\phi}$ of $\partial^2 F$.

For any $\nu \in Curr(F)$ define a measure $\phi\nu$ on $\partial^2 F$ as a pull-back:

$$(\phi\nu)(S) := \nu\big((\hat{\phi})^{-1}(S)\big)$$

for $S \subseteq \partial^2 F$. It can be shown [20] that $\phi\nu \in Curr(F)$ is a geodesic current on F and, moreover, $\phi\nu$ only depends on ν and on the outer automorphism class $[\phi] \in Out(F)$. The map $Aut(F) \times Curr(F) \to Curr(F)$, $(\phi, \nu) \mapsto \phi\nu$ defines a continuous left action of $Aut(F)$ on $Curr(F)$ by linear transformations that factors through to a left continuous action of $Out(F)$ on $Curr(F)$. Moreover, this action commutes with scalar multiplication and hence defines a continuous left action of $Out(F)$ on $\mathbb{P}Curr(F)$.

Definition 2.5 (Coordinates on $Curr(F)$)**.** Let $A = \{a_1, \dots, a_k\}$ be a free basis of F and let $Cay(F, A)$ be the Cayley graph of F with respect to A. Thus $Cay(F, A)$ is a $2k$-regular tree.

Let γ be a non-empty geodesic segment in $Cay(F, A)$ that begins and ends at vertices of $Cay(F, A)$. The path γ comes equipped with a *label* v which is a freely reduced word over A.

We denote by $Cyl_A(\gamma)$ the set of all $(\xi, \zeta) \in \partial^2 F$ such that the directed geodesic from ξ toζ in $Cay(F, A)$ contains the segment γ. Thus $Cyl_A(\gamma)$ is a closed-open subset of $\partial^2 F$.

Let $\nu \in Curr(F)$ be arbitrary and let $v \in F$ be a non-trivial freely reduced word. Since ν is F-invariant, if γ is a segment in $Cay(F, A)$ with label v, then $\nu(Cyl_A(\gamma))$ depends only on ν and v but not on the choice of a lift γ of v. Moreover, since ν is σ-invariant, we have $\nu(Cyl_A(\gamma)) = \nu(Cyl_A(\gamma^{-1}))$.

We denote

$$(v; \nu)_A := \frac{1}{2}(\nu(Cyl_A(\gamma)) + \nu(Cyl_A(\gamma^{-1}))) = \nu(Cyl_A(\gamma)),$$

where γ is any lift of v to $Cay(F, A)$.

Definition 2.6 (Rational currents)**.** Let $g \in F$ be a non-trivial element that is not a proper power. Then there exist two distinct points $g_+, g_- \in \partial F$ such that $\lim_{n\to\infty} g^n = g_+$ and $\lim_{n\to\infty} g^{-n} = g_-$. We define

$$\mu_g := \sum_{h\in[g]} \left(\delta_{(h_-,h_+)} + \delta_{(h_+,h_-)}\right),$$

where $[g]$ is the conjugacy class of g in F, and $\delta_{(\zeta,\xi)}$ is the Dirac measure defined by the point $(\zeta,\xi) \in \partial^2 F$.

If $g \in F$ is a non-trivial element that is a proper power, we can uniquely write $g = f^m$ where $m > 1$ and f is not a proper power. We set $\mu_g := m\mu_f$.

Thus for every non-trivial $g \in F$ we have $\mu_g \in Curr(F)$ is a geodesic current on F. We say that $\nu \in Curr(F)$ is a *rational current* if ν has the form $\nu = s\mu_g$ for some $s > 0$ and $g \in F$.

Proposition 2.7. [27, 19, 20] *The set of rational currents is dense in $Curr(F)$.*

Remark 2.8 (Rational currents and cyclic words)**.** Let A be a free basis of F. It is often convenient to represent conjugacy classes in F by "cyclic words". A *cyclic word* w over A is a non-trivial cyclically reduced word in A written on a circle clockwise without a specified base-point. If v is a freely reduced word, a vertex on a cyclic word w is an *occurrence* of v in w if we can read v in w from this vertex going forward clockwise and without leaving the labelled circle. The number of occurrences of $v^{\pm 1}$ in w is denoted $(v;w)_A$. Thus by definition $(v;w)_A = (v^{-1};w)_A$. We denote the number of vertices in a cyclic word w in the basis A by $||w||_A$; this coincides with the word length of w in $A^{\pm 1}$, since w is (cyclically) reduced.

Clearly there is a bijective correspondence between cyclic words over A and non-trivial conjugacy classes in F.

A simple but important observation [19, 20] says that if a cyclic word w represents a conjugacy class $[g]$ in F then for every freely reduced word v over A we have

$$(v;\mu_g)_A := (v;w)_A.$$

Definition 2.9 (Length)**.** Let A be a free basis of F and let $\nu \in Curr(F)$ be a current. Denote

$$||\nu||_A := \sum_{a\in A} (a;\nu)_A.$$

We call $||\nu||_A$ the *length* of ν with respect to A.

Remark 2.8 implies that for any $a \in A$ the bracket $(a;\mu_g)_A$ equals to the number of occurrences of a or a^{-1} in the cyclically reduced word in $A^{\pm 1}$ that represents g. For every non-trivial $g \in F$ we have $||\mu_g||_A = ||g||_A$, the cyclically reduced length of g with respect to A. Note also that for any current μ the length $||\mu||_A$ is precisely equal to the intersection form $I(T_A,\mu)$, where T_A denotes the Cayley tree associated to the basis A, with all edges of length 1.

We will need the following basic facts [20], where for any element $g \in F$ we denote by $|g|_A$ the length of the reduced word in the basis A that represents g.

Lemma 2.10. *Let A be a free basis of F and let $\nu \in Curr(F)$. Then for every freely reduced word $v \in F$ we have*

1. $(v;\nu)_A \ge 0$;
2. $(v;\nu)_A = \sum_{a\in A^{\pm 1}:|va|_A=|v|_A+1}(va;\nu)_A$;
3. $(v;\nu)_A = \sum_{a\in A^{\pm 1}:|av|_A=|v|_A+1}(av;\nu)_A$;
4. *For every $m \ge 1$*
$$||\nu||_A = \frac{1}{2}\sum_{u\in F:|u|_A=m}(u;\nu)_A.$$
5. *Let $\nu_n, \nu \in Curr(F)$. Then $\lim_{n\to\infty}\nu_n = \nu$ in $Curr(F)$ if and only if for every $v \in F$, $v \ne 1$ we have*
$$\lim_{n\to\infty}(v;\nu_n)_A = (v;\nu)_A.$$
6. *Let $\nu_n, \nu \in Curr(F)$ be nonzero currents. Then we have*
$$\lim_{n\to\infty}[\nu_n] = [\nu] \quad in \quad \mathbb{P}Curr(F)$$
if and only if
$$\lim_{n\to\infty}\frac{\nu_n}{||\nu_n||_A} = \frac{\nu}{||\nu||_A} \quad in \quad Curr(F).$$

Corollary 2.11. *Let A be a free basis of F and let $a \in A$. For nonzero currents $\nu_n \in Curr(F)$ we have*
$$\lim_{n\to\infty}[\nu_n] = [\mu_a]$$
if and only if
$$\lim_{n\to\infty}\frac{(a;\nu_n)_A}{||\nu_n||_A} = 1.$$

Proof. The "only if" direction is obvious. Suppose that $\lim_{n\to\infty}\frac{(a;\nu_n)_A}{||\nu_n||_A} = 1$. We claim that $\lim_{n\to\infty}\frac{\nu_n}{||\nu_n||_A} = \mu_a$ in $Curr(F)$. Note that
$$||\frac{\nu_n}{||\nu_n||_A}||_A = 1.$$

Definition 2.9 implies that $\lim_{n\to\infty}\frac{(x;\nu_n)_A}{||\nu_n||_A} = 0$ for every $x \in A, x \ne a$. If $x \in A$, $x \ne a$ then by parts (2) and (3) of Lemma 2.10 for every freely reduced word u involving $x^{\pm 1}$ we have $(u;\nu_n)_A \le (x;\nu_n)_A$. Therefore for such u we have $\lim_{n\to\infty}\frac{(u;\nu_n)_A}{||\nu_n||_A} = 0$. Hence, by part (4) of Lemma 2.10 for every $m \ge 1$ we have $\lim_{n\to\infty}\frac{(a^m;\nu_n)_A}{||\nu_n||_A} = 1$. Thus we have verified that for every nontrivial $v \in F$ we have $\lim_{n\to\infty}\frac{(v;\nu_n)_A}{||\nu_n||_A} = (v;\mu_a)$. Hence

$\lim_{n\to\infty} \frac{\nu_n}{||\nu_n||_A} = \mu_a$ in $Curr(F)$ and $\lim_{n\to\infty}[\nu_n] = [\mu_a]$ in $\mathbb{P}Curr(F)$, as required.

□

Corollary 2.11 and Remark 2.8 easily imply:

Corollary 2.12. *Let A be a free basis of F and let $a \in A$. Then for every $g \in F$ we have*

$$\lim_{n\to\infty} [\mu_{a^n g}] = \lim_{n\to\infty} [\mu_{a^{-n} g}] = [\mu_a]$$

in $\mathbb{P}Curr(F)$.

3. Conventions regarding Outer space

We refer the reader to [1, 12, 13, 16, 31] for detailed information regarding Outer space and its boundary. We recall here briefly some basic notions and facts.

Let F be a free group of finite rank $k \geq 2$. We denote by $cv(F)$ the *non-projectivized Outer space* of F, which consists of all $\mathbb{R}$-trees T equipped with an action of F by isometries which is minimal, free and discrete. Such an action is characterized (up to F-equivariant isometries) by the associated translation length function $||\cdot||_T : F \to \mathbb{R}$, and for the purpose of this paper it is convenient to identify the point $T \in cv(F)$ with the corresponding length function $||\cdot||_T \in \mathbb{R}^F$, i.e. $cv(F) \subset \mathbb{R}^F$. Note that for every $T \in cv(F)$ the quotient by the F-action is a finite connected metric marked graph without valence 1 vertices: this gives an alternative characterization of the points in $cv(F)$. A typical example is the Cayley tree T_A associated to a basis A of F, which has as quotient a rose with $\#A$ leaves, usually assumed to be of unit length.

We denote by $CV(F)$ the subset of $cv(F)$ corresponding to those actions where the quotient metric graphs have volume one. Alternatively, we can also view $CV(F)$ as the projectivized quotient $\mathbb{P}cv(F) := cv(F)/\sim$, where $\sim$ corresponds to scalar multiplication. Namely, for every $T \in cv(F)$ there exists a unique rescaled tree λT with the quotient of volume one. The space $CV(F)$ is called the *Outer space* of F, and the projective class of a tree T is denoted by $[T]$.

There is a *left* action $Aut(F) \times cv(F) \to cv(F)$ given by $||w||_{\phi T} = ||\phi^{-1}(w)||_T$, for all $w \in F$ and all $T \in cv(F)$. This action leaves $CV(F)$ invariant. Moreover, the subgroup $Inn(F)$ of inner automorphisms of F acts trivially on $cv(F)$ and the actions of $Aut(F)$ on $cv(F)$ and $CV(F)$ factor through to left actions of $Out(F)$ on $cv(F)$ and $CV(F)$ accordingly.

The space $cv(F) \subset \mathbb{R}^F$ is endowed with the weak topology of pointwise convergence on finite subsets of F. The action of $Out(F)$ on $cv(F)$ is an action by homeomorphisms and $CV(F) \subseteq cv(F)$ is a closed $Out(F)$-invariant subset.

Outer space $CV(F) = \mathbb{P}cv(F) \subset \mathbb{P}\mathbb{R}^F$ can be compactified with respect to weak convergence of projective classes of length functions, and the result is denoted $\overline{CV(F)}$ and is called the *length function compactification* of $CV(F)$. The difference $\overline{CV(F)} - CV(F)$ is denoted $\partial CV(F)$ and is called the *length-function boundary* or sometimes the *Thurston boundary* of $CV(F)$. It turns out [6, 12] that $\overline{CV(F)}$ consists precisely of all *very small* minimal actions of F on $\mathbb{R}$-trees.

4. Minimal sets

Throughout this section assume that the rank k of F satisfies $k \geq 3$. As Guirardel showed in [16], there exists a unique minimal non-empty closed $Out(F)$-invariant subset $\mathcal{M}^{CV} \subseteq \partial CV(F)$. This set obviously has the property that for every $x \in \mathcal{M}^{CV}$ the orbit $Out(F)x$ is dense in $\mathcal{M}^{CV}$. Moreover, Guirardel describes in detail some of the points that belong to $\mathcal{M}^{CV}$. We shall only need the following basic fact [16]:

Proposition 4.1. *There exists a point $[T] \in \partial CV(F)$ corresponding to a free action of F on an $\mathbb{R}$-tree T such that $[T] \in \mathcal{M}^{CV}$. Moreover, since the $Out(F)$-orbit of $[T]$ is dense in $\mathcal{M}^{CV}$, the set of points of $\mathcal{M}^{CV}$ corresponding to free actions of F is dense in $\mathcal{M}^{CV}$.*

Specifically, one can choose $[T]$ in Proposition 4.1 to be the attracting fixed point in $\partial CV(F)$ of any outer automorphism α of F which is irreducible with irreducible powers and does not have periodic conjugacy classes. Levitt and Lustig [25] proved that such automorphisms have "North-South" dynamics on $\partial CV(F)$ (and indeed on the entire space $\overline{CV(F)}$) and hence, clearly, their attracting fixed points must belong to $\mathcal{M}^{CV}$. It is also known that these attracting fixed points correspond to free actions.

Similarly but not quite analogously, Reiner Martin [27] introduces the following notion in the context of currents:

Definition 4.2 (Minimal set in $\mathbb{P}Curr(F)$)**.** Set $\mathcal{M}^{\mathbb{P}Curr} \subseteq \mathbb{P}Curr(F)$ to be the closure in $\mathbb{P}Curr(F)$ of the set

$$\{[\mu_g] : g \in F \text{ is a primitive element}\}.$$

Here by a primitive element we mean an element that belongs to some free basis of F. Recall that, if $\phi \in Aut(F)$ and $g \in F$ is non-trivial, then $\phi[\mu_g] = [\mu_{\phi(g)}]$. Note that for any primitive element g the set $\mathcal{M}^{\mathbb{P}Curr}$ is the closure of the $Out(F)$-orbit of $[\mu_g]$. Thus $\mathcal{M}^{\mathbb{P}Curr}$ is a closed $Out(F)$-invariant subset of $\mathbb{P}Curr(F)$. We will need the following basic result of Reiner Martin [27]. For completeness we include a proof.

Proposition 4.3. *Let F be a free group of finite rank $k \geq 3$. Then the space $\mathcal{M}^{\mathbb{P}Curr}$ is infinite dimensional.*

Proof. We will denote $F = F(a, b, c, \dots)$ and $F' := F(a, b)$. There is a natural topological embedding (see [20] for proof)

$$\iota : \mathbb{P}Curr(F') \to \mathbb{P}Curr(F)$$

with the property that for every $g \in F'$ we have $\iota([\mu'_g]) = \mu_g$ where $\mu'_g \in Curr(F')$ and $\mu_g \in Curr(F)$ are the rational currents corresponding to g considered as the element of F' and F accordingly.

We claim that $\mathcal{M}^{\mathbb{P}Curr}$ contains a copy $\iota(\mathbb{P}Curr(F'))$ of $\mathbb{P}Curr(F')$. Since rational currents are dense in $\mathbb{P}Curr(F')$, to establish the claim it suffices to show that for every $g \in F'$ we have $\mu_g \in \mathcal{M}^{\mathbb{P}Curr}$. Let $u = u(a, b) \in F'$ be an arbitrary non-trivial freely reduced word. Then for every $n \geq 1$ the element $g_n := cu^n$ is primitive in F. Thus $[\mu_{g_n}] \in \mathcal{M}^{\mathbb{P}Curr}$. It is easy to see that $\lim_{n\to\infty}[\mu_{g_n}] = [\mu_u] \in \mathbb{P}Curr(F)$. Thus indeed $\mathcal{M}^{\mathbb{P}Curr}$ contains the image of the topological embedding ι. Since $\mathbb{P}Curr(F')$ is infinite dimensional [27, 19, 20] it now follows that so is $\mathcal{M}^{\mathbb{P}Curr}$.

□

5. Simple Dehn twists and their action on $\mathbb{P}Curr(F)$

Definition 5.1. Let F be a free group of finite rank $k \geq 2$ with a free basis $A = \{a_1, \dots, a_k\}$. Denote $a = a_1, b = a_2$. Let $D : F \to F$ be the automorphism defined as $D(a) = ab$, $D(a_i) = a_i$ for $2 \leq i \leq k$. We will call D a *simple Dehn twist* with respect to A and say that b is the *twistor* of D.

Definition 5.2 (Splitting corresponding to a Dehn twist)**.** Let D be a simple Dehn twist of F as in Definition 5.1.

Denote $b_1 := aba^{-1} \in F$. Note that the elements $b_1, b, a_3, \dots, a_k$ freely generate a subgroup of F that we denote by F_0.

Consider the following HNN-extension decomposition of F:

$$F = \langle F_0, a | aba^{-1} = b_1 \rangle.$$

Let $T_D \in cv(F)$ be the Bass-Serre tree corresponding to this HNN-extension. We call T_D *the tree corresponding to the simple Dehn twist* D.

Definition 5.3. [Critical set] Let D be the simple Dehn twist with respect to a free basis A with twistor b, as in Definition 5.1. Denote

$$Y := \{[\nu] : 0 \neq \nu \in Curr(F), \frac{1}{2}(a; \nu)_A = (ab; \nu)_A = (ab^{-1}; \nu)_A\} \subseteq \mathbb{P}Curr(F).$$

We call Y the *critical set* of D in $\mathbb{P}Curr(F)$.

Convention 5.4. Until the end of this section, unless specified otherwise, we assume that D is a simple Dehn twist of F with twistor b as in Definition 5.1.

We will need the following fact [12]:

Proposition 5.5. *Let* $[T] \in \overline{CV(F)}$ *be such that* $||b||_T > 0$. *Then*

$$\lim_{n\to\infty} D^n[T] = \lim_{n\to\infty} D^{-n}[T] = [T_D].$$

Recall that in our coordinate notation we have $(a_i;\nu)_A = (a_i^{-1};\nu)_A$ for $i=1,\dots,k$ and for every $\nu\in Curr(F)$.

Lemma 5.6. *Let $\nu\in Curr(F)$ be a current. We have:*

1. $(a_i;D\nu)_A = (a_i;\nu)_A$ *for* $i\ne 2$.
2. $(b;D\nu)_A = (b;\nu)_A + (a;\nu)_A - 2(ab^{-1};\nu)_A$.
3. $(ab^{-1};D\nu)_A = (ab^{-2};\nu)_A + (ab^{-1}a^{-1};\nu)_A$ *and hence* $(ab^{-1};D\nu)_A \le (ab^{-1},\nu)_A$.
4. $||D\nu||_A - ||\nu||_A = (b;D\nu)_A - (b;\nu)_A = (a;\nu)_A - 2(ab^{-1};\nu)_A$. *Consequently,* $||D\nu||_A > ||\nu||_A$ *if and only if* $(a;\nu)_A > 2(ab^{-1};\nu)_A$.

Proof. Parts (1), (2) and (3) are straightforward for cyclic words and hence for rational currents. Therefore they follow for arbitrary currents by continuity since rational currents are dense in $Curr(F)$. Clearly, part (4) follows from parts (1) and (2).

□

Lemma 5.7. *Let $\nu\in Curr(F)$ and suppose that $||D\nu||_A > ||\nu||_A$. Then $||D^2\nu||_A > ||D\nu||_A$ and $||D^2\nu||_A - ||D\nu||_A \ge ||D\nu||_A - ||\nu||_A$.*

Proof. By assumption and by Lemma 5.6 (4) we have $||D\nu||_A - ||\nu||_A = (a;\nu)_A - 2(ab^{-1};\nu)_A > 0$.

By Lemma 5.6 (1) and (3) we have $(a;D\nu)_A = (a;\nu)_A$ and $(ab^{-1};D\nu)_A \le (ab^{-1};\nu)_A$.

Hence

$$||D^2\nu||_A - ||D\nu||_A = (a;D\nu)_A - 2(ab^{-1};D\nu)_A \ge$$
$$(a;\nu)_A - 2(ab^{-1};\nu)_A = ||D\nu||_A - ||\nu||_A > 0$$

and the statement of the lemma follows.

□

Corollary 5.8. *Let $\nu\in Curr(F)$ and suppose that $(a;\nu)_A > 2(ab^{-1};\nu)_A$. Then we have:*

$$\lim_{n\to\infty} ||D^n\nu||_A = \infty, \quad \lim_{n\to\infty}\frac{1}{||D^n\nu||_A}D^n\nu = \mu_b \quad \textit{and}$$
$$\lim_{n\to\infty}[D^n\nu] = [\mu_b] \quad \textit{in} \quad \mathbb{P}Curr(F).$$

Proof. Lemma 5.6 (4) and Lemma 5.7 imply that $||D\nu||_A > ||\nu||_A$ and that for $n\ge 1$ we have

$$||D^n\nu||_A \ge ||\nu||_A + n(||D\nu||_A - ||\nu||_A).$$

Hence $\lim_{n\to\infty}||D^n\nu|| = \infty$. For every $a_i\ne b$ we have $(a_i,D^n\nu)_A = (a_i,\nu)_A$. Hence for $a_i\ne b$ one obtains

$$\lim_{n\to\infty}\frac{(a_i,D^n\nu)_A}{||D^n\nu||_A} = 0.$$

From the definition of $||D^n\nu||_A$ it follows

$$\lim_{n\to\infty} \frac{(b, D^n\nu)_A}{||D^n\nu||_A} = 1.$$

Therefore by Corollary 2.11 one has $\lim_{n\to\infty}[D^n\nu] = [\mu_b]$ in $\mathbb{P}Curr(F)$, as claimed.

□

Corollary 5.9. *Let $\nu \in Curr(F)$ be a non-zero current such that $[\nu] \notin Y$. Then in $\mathbb{P}Curr(F)$ one has either $\lim_{n\to\infty}[D^n\nu] = [\mu_b]$ or $\lim_{n\to\infty}[D^{-n}\nu] = [\mu_b]$.*

Proof. We claim that at least one of the following holds: $(a;\nu)_A > 2(ab;\nu)_A$ or $(a;\nu)_A > 2(ab^{-1};\nu)_A$. Suppose not. Then $\frac{1}{2}(a;\nu)_A \le (ab;\nu)_A$ and $\frac{1}{2}(a;\nu)_A \le (ab^{-1};\nu)_A$. Since $[\nu] \notin Y$, at least one of these inequalities must be strict and hence $(a;\nu)_A < (ab;\nu)_A + (ab^{-1};\nu)_A$. On the other hand, Lemma 2.10 (4) implies that $(a;\nu)_A \ge (ab;\nu)_A + (ab^{-1};\nu)_A$, yielding a contradiction.

Thus we have indeed either $(a;\nu)_A > 2(ab;\nu)_A$ or $(a;\nu)_A > 2(ab^{-1};\nu)_A$. If the former holds, then Corollary 5.8 applied to D^{-1} gives $\lim_{n\to\infty}[D^{-n}\nu] = [\mu_b]$. If the latter inequality holds then Corollary 5.8 applied to D we have $\lim_{n\to\infty}[D^n\nu] = [\mu_b]$.

□

We can now prove that $\mathcal{M}^{\mathbb{P}Curr}$ is indeed the smallest non-empty closed $Out(F)$-invariant subset of F assuming that F has rank at least three.

Theorem 5.10 (Minimality of $\mathcal{M}^{\mathbb{P}Curr}$)**.** *Let F be a free group of rank $k \ge 3$. Let $S \subseteq \mathbb{P}Curr(F)$ be a non-empty closed $Out(F)$-invariant subset. Then one has:*

$$\mathcal{M}^{\mathbb{P}Curr} \subseteq S.$$

Proof. Fix a free basis $A = \{a_1, a_2, a_3, \dots, a_k\}$ of A and denote $a = a_1$, $b = a_2$, $c = a_3$.

Since S is non-empty, there exists a non-zero current $\nu \in Curr(F)$ such that $[\nu] \in S$.

There exists i such that $(a_i;\nu)_A > 0$ and without loss of generality we may assume that $i = 1$, so that $(a;\nu)_A > 0$.

Let D be the simple Dehn twist with twistor b defined as $D(a) = ab$ and $D(a_i) = a_i$ for $i \ge 2$. Let $Y \subseteq \mathbb{P}Curr(F)$ be the critical set of D as in Definition 5.3.

Suppose first that $[\nu] \notin Y$. Then by Corollary 5.9 either $\lim_{n\to\infty}[D^n\nu] = [\mu_b]$ or $\lim_{n\to\infty}[D^{-n}\nu] = [\mu_b]$. In either case, since S is closed and $Out(F)$-invariant, it follows that $[\mu_b] \in S$ and hence $\mathcal{M}^{\mathbb{P}Curr} \subseteq S$.

Suppose now that $[\nu] \in Y$, that is

$$\frac{1}{2}(a;\nu)_A = (ab^{-1};\nu)_A = (ab;\nu)_A$$

By part (4) of Lemma 2.10 this implies that $(ac;\nu)_A = (ac^{-1};\nu)_A = 0$.

Consider now the simple Dehn twist D' of F defined as $D'(a) = ac$, $D'(a_i) = a_i$ for $2 \leq i \leq k$. Thus c is the twistor of D'. Since $(ac^{-1};\nu)_A = 0$ and $(a;\nu)_A > 0$, we have $(a;\nu)_A - 2(ac^{-1};\nu)_A > 0$. Thus by Corollary 5.8 applied to D' we have $\lim_{n\to\infty}[(D')^n\nu] = [\mu_c]$. Again, since S is closed and $Out(F)$-invariant it follows that $[\mu_c] \in S$ and hence $\mathcal{M}^{\mathbb{P}Curr} \subseteq S$.

□

Corollary 5.11. *Let F be a free group of finite rank $k \geq 3$. Then there does not exist an $Out(F)$-equivariant topological embedding of $\partial CV(F)$ to $\mathbb{P}Curr(F)$.*

Similarly, there exists no $Out(F)$-anti-equivariant topological embedding of $\partial CV(F)$ to $\mathbb{P}Curr(F)$.

Proof. Suppose $j : \partial CV(F) \to \mathbb{P}Curr(F)$ is an $Out(F)$-equivariant (or $Out(F)$-anti-equivariant) topological embedding. Since $\partial CV(F)$ is compact, this implies that the image of j is a closed $Out(F)$-invariant subset of $\mathbb{P}Curr(F)$. Hence by Theorem 5.10 the image of j contains the set $\mathcal{M}^{\mathbb{P}Curr}$. By Proposition 4.3 the set $\mathcal{M}^{\mathbb{P}Curr}$ is infinite dimensional. On the other hand, $\partial CV(F)$ is finite dimensional [29, 30, 14], yielding a contradiction with the fact that j is a homeomorphism from $\partial CV(F)$ onto its image $j(\partial CV(F))$.

□

In particular, Corollary 5.11 implies that if $\tau : CV(F) \to \mathbb{P}Curr(F)$ is an $Out(F)$-equivariant topological embedding then τ does not extend to a homeomorphism from the length-function compactification $\overline{CV(F)} = CV(F) \cup \partial CV(F)$ to the closure of the image of τ.

6. Rigid points

Convention 6.1. For the remainder of this section we assume that F is a free group of rank $k \geq 3$ and that $\tau : \partial CV(F) \to \mathbb{P}Curr_S(F)$ is a continuous map that is either $Out(F)$-equivariant or $Out(F)$-anti-equivariant.

Proposition 6.2. *For any simple Dehn twist $D \in Aut(F)$ with twistor $b \in F$ we have $\tau([T_D]) = [\mu_b]$.*

Proof. Let A be the basis of F used to define D as in Definition 5.1. Let $Y \subseteq \mathbb{P}Curr(F)$ be the critical set of D, as defined in Definition 5.3. Thus

$$Y = \{[\nu] : \nu \in Curr(F), \nu \neq 0, \frac{1}{2}(a;\nu)_A = (ab;\nu)_A = (ab^{-1};\nu)_A\}.$$

Note that Y is a closed non-empty subset of $\mathbb{P}Curr(F)$ and that Y contains the set

$$Y_0 := \{[\nu] \in \mathbb{P}Curr(F) : \nu \in Curr(F), \nu \neq 0, (a;\nu)_A = 0\}.$$

Moreover, $[\mu_a] \notin Y$ and hence $\mathcal{M}^{\mathbb{P}Curr} \not\subseteq Y$.

We claim that there exists $[T] \in \mathcal{M}^{CV}$ such that $||b||_T > 0$ and $\tau([T]) \notin Y$. Indeed, suppose not, so that for every $[T] \in \mathcal{M}^{CV}$ with $||b||_T > 0$ we have $\tau([T]) \in Y$. By Proposition 4.1 the set of length functions corresponding to free actions is dense in $\mathcal{M}^{CV}$ and for each of them b has non-zero translation length. Hence $\tau(\mathcal{M}^{CV}) \subseteq Y$ since Y is closed. The space $\mathcal{M}^{CV}$ is compact and $Out(F)$-invariant. Hence $\tau(\mathcal{M}^{CV})$ is a non-empty compact (and thus closed) subset of $\mathbb{P}Curr(F)$ that is $Out(F)$-invariant. Therefore by Theorem 5.10 $\mathcal{M}^{\mathbb{P}Curr} \subseteq \tau(\mathcal{M}^{CV}) \subseteq Y$, yielding a contradiction with the earlier conclusion that $\mathcal{M}^{\mathbb{P}Curr} \not\subseteq Y$.

Thus indeed there exists $x = [T] \in \mathcal{M}^{CV}$ such that $||b||_T > 0$ and $\tau(x) \notin Y$.

Therefore, by Corollary 5.9 we have

$$\text{either } \lim_{n\to\infty} D^n \tau(x) = [\mu_b] \text{ or } \lim_{n\to\infty} D^{-n}\tau(x) = [\mu_b].$$

Suppose the former holds (the other case is symmetric) and that we have $\lim_{n\to\infty} D^n\tau(x) = [\mu_b]$.

If τ is $Out(F)$-equivariant, then by Proposition 5.5 we have $[T_D] = \lim_{n\to\infty} D^n x$, and hence the continuity of τ implies

$$\tau([T_D]) = \lim_{n\to\infty} \tau(D^n x) = \lim_{n\to\infty} D^n\tau(x) = [\mu_b].$$

If τ is $Out(F)$-anti-equivariant, then, again by Proposition 5.5 we have $[T_D] = \lim_{n\to\infty} D^{-n}x$ and hence

$$\tau([T_D]) = \lim_{n\to\infty} \tau(D^{-n}x) = \lim_{n\to\infty} D^n\tau(x) = [\mu_b].$$

Thus $\tau([T_D]) = [\mu_b]$, as required.

□

7. Non-existence of equivariant maps

Convention 7.1. Throughout this section let F be a free group of finite rank $k \geq 5$ with a free basis $A = \{a_1, \dots, a_k\}$. We denote $a = a_1$, $b = a_2$, $c = a_3$, $d = a_4$ and $e = a_5$.

Let ϕ denote the automorphism of F defined as $\phi(a) = ab$, $\phi(e) = ed$ and $\phi(a_i) = a_i$ for $i \neq 1, 5$.

Definition 7.2 (The length function $\Delta(\rho, \theta)$.)**.** Denote $b_1 = aba^{-1}$, $d_1 = ede^{-1}$ and put $H := \langle b, b_1, c, d, d_1, a_6, \dots, a_k\rangle \leq F$. (Note that H is freely generated by these elements).

Consider the following HNN-extension splitting of F with stable letters a, e and the base H:

$$F = \langle H, a, e | aba^{-1} = b_1, ede^{-1} = d_1\rangle.$$

Let $\mathbb{A}$ be the graph of groups corresponding to this splitting. Let $\rho > 0$, $\theta > 0$ be positive numbers.

We denote by $\Delta(\rho,\theta)\in cv(F)$ the hyperbolic length function on F coming from $\mathbb{A}$ where in $\mathbb{A}$ the loop-edge labelled a is given length ρ and the loop-edge labelled e is given length θ.

The following result is due to Cohen-Lustig (see [12], Theorem 13.2).

Proposition 7.3. *Let $[T]\in\partial CV(F)$ be such that $||b||_T>0$ and $||d||_T>0$. Then*

$$\lim_{n\to\infty}\phi^n([T])=\lim_{n\to\infty}\phi^{-n}([T])=[\Delta(||b||_T,||d||_T)]$$

in $\partial CV(F)$.

Theorem 7.4. *Let F be a free group of rank $k\ge 5$. Then there does not exist a continuous $Out(F)$-equivariant map $\tau:\partial CV(F)\to\mathbb{P}Curr(F)$. Similarly, there exists no continuous $Out(F)$-anti-equivariant map $\partial CV(F)\to\mathbb{P}Curr(F)$.*

Proof. Suppose $\tau:\partial CV(F)\to\mathbb{P}Curr(F)$ is a continuous map that is $Out(F)$-equivariant.

Consider the free bases $A'=\{a',b',c',d',e',a_6,\dots,a_k\}$ and $A''=\{a'',b'',c'',d'',e'',a_6,\dots,a_k\}$ of F defined via

$$a=a',e=e',b=b'a'c',d=a'b'd',c=a'b'$$

and

$$a=a'',e=e'',b=b''a''c'',d=a''b''d'',c=e''b''.$$

Note that we have $b'=a^{-1}c$ and $b''=e^{-1}c$.

Let D' and D'' be automorphisms of F defined as follows. We have $D'(a')=(a'b')$ and D' fixes all other elements of A'. Similarly $D''(a'')=a''b''$ and D'' fixes all other elements of A''. Let $T_{D'}$ and $T_{D''}$ be the length functions on F defined as in Definition 5.2.

Since $b=b'a'c'$ and $d=a'b'd'$, we have $||b||_{T_{D'}}=1$ and $||d||_{T_{D'}}=1$. Similarly, $b=b''a''c''$ and $d=a''b''d''$, so that $||b||_{T_{D''}}=1$ and $||d||_{T_{D''}}=1$. Therefore by Proposition 7.3 we have

$$\lim_{n\to\infty}\phi^n[T_{D'}]=\lim_{n\to\infty}\phi^n[T_{D''}]=[\Delta(1,1)]\in\partial CV(F).$$

On the other hand, by Proposition 6.2 we have $\tau([T_{D'}])=[\mu_{b'}]=[\mu_{a^{-1}c}]$ and $\tau([T_{D''}])=[\mu_{b''}]=[\mu_{e^{-1}c}]$.

By definition of ϕ for $n\ge 1$ we have $\phi^n(a^{-1}c)=b^{-n}a^{-1}c$ and hence $\phi^n[\mu_{b'}]=\phi^n[\mu_{a^{-1}c}]=[\mu_{b^{-n}a^{-1}c}]$. Similarly, for $n\ge 1$ we have $\phi^n(e^{-1}c)=d^{-n}e^{-1}c$ and hence $\phi^n[\mu_{b''}]=\phi^n[\mu_{e^{-1}c}]=[\mu_{d^{-n}e^{-1}c}]$. Therefore, by Corollary 2.12, we see that

$$\lim_{n\to\infty}\phi^n[\mu_{b'}]=[\mu_b]\in\mathbb{P}Curr(F)$$

and

$$\lim_{n\to\infty}\phi^n[\mu_{b''}]=[\mu_d]\in\mathbb{P}Curr(F).$$

Since $[\mu_b] \neq [\mu_d]$, we have

$$\lim_{n\to\infty} \phi^n[T_{D'}] = \lim_{n\to\infty} \phi^n[T_{D''}] \text{ in } \partial CV(F) \text{ but}$$
$$\lim_{n\to\infty} \phi^n\tau[T_{D'}] \neq \lim_{n\to\infty} \phi^n\tau[T_{D''}] \text{ in } \mathbb{P}Curr(F),$$

yielding a contradiction with the assumption that τ is continuous and $Out(F)$-equivariant.

The same proof shows that there does not exist a continuous $Out(F)$-anti-equivariant map $\tau_0 : \partial CV(F) \to \mathbb{P}Curr(F)$. The only changes needed to be made in the argument are the following. Using Proposition 6.2 we still conclude that $\tau_0([T_{D'}]) = [\mu_{b'}] = [\mu_{a^{-1}c}]$ and $\tau_0([T_{D''}]) = [\mu_{b''}] = [\mu_{e^{-1}c}]$. As before, we have

$$\lim_{n\to\infty} \phi^n[T_{D'}] = \lim_{n\to\infty} \phi^n[T_{D''}] \text{ in } \partial CV(F).$$

By anti-equivariance, $\tau_0(\phi^n x) = \phi^{-n}\tau_0(x)$ for every $x \in \partial CV(F)$. For $n \geq 1$ we have $\phi^{-n}(a^{-1}c) = b^n a^{-1}c$, $\phi^{-n}(e^{-1}c) = d^n e^{-1}c$ and so $\phi^{-n}([\mu_{a^{-1}c}]) = [\mu_{b^n a^{-1}c}]$ and $\phi^{-n}([\mu_{e^{-1}c}]) = [\mu_{d^n e^{-1}c}]$. Therefore, again by Corollary 2.12, we have

$$\lim_{n\to\infty} \phi^{-n}([\mu_{a^{-1}c}]) = [\mu_b] \text{ and } \lim_{n\to\infty} \phi^{-n}([\mu_{e^{-1}c}]) = [\mu_d],$$

and, since $[\mu_b] \neq [\mu_d]$ we get a contradiction with the continuity of τ_0 as before.

□

Remark 7.5. The proof of Theorem 7.4 actually implies that there exists neither an $Out(F)$-equivariant nor an $Out(F)$-anti-equivariant continuous map from $\mathcal{M}^{CV}$ to $\mathbb{P}Curr(F)$. Indeed, the same holds with $\mathcal{M}^{CV}$ replaced by any non-empty closed $Out(F)$-invariant (or anti-invariant) subspace of $\partial CV(F)$.

8. The minimal set in $\partial CV(F) \times \mathbb{P}Curr(F)$

Definition 8.1. Let F be a free group of finite rank $k \geq 3$. Let $\mathcal{M}^2$ be the closure in $\partial CV(F) \times \mathbb{P}Curr(F)$ of all points of the form $([T_D], [\mu_g])$, where D is a simple Dehn twist of F with respect to any base, and $g \in F$ is the twistor of D. Recall that g is always a primitive element, according to the conventions in this paper, see Definition 5.1.

Note that by definition $\mathcal{M}^2$ is an $Out(F)$-invariant subset of $\partial CV(F) \times \mathbb{P}Curr(F)$. In fact, if D is any simple Dehn twist of F with twistor b, then $\mathcal{M}^2$ is the closure of the $Out(F)$-orbit of the point $([T_D], [\mu_b])$.

Theorem 8.2. *Let be a free group of finite rank $k \geq 3$. Then $\mathcal{M}^2$ is the unique minimal non-empty closed $Out(F)$-invariant subset of $\partial CV(F) \times \mathbb{P}Curr(F)$.*

Proof. Let $S \subseteq \partial CV(F) \times \mathbb{P}Curr(F)$ be a non-empty closed $Out(F)$-invariant subset of $\partial CV(F) \times \mathbb{P}Curr(F)$. We need to show that $\mathcal{M}^2 \subseteq S$.

It is clear that the coordinate projections of $\mathcal{M}^2$ to the spaces $\partial CV(F)$ and $\mathbb{P}Curr(F)$ are compact (and hence closed) $Out(F)$-invariant sets and therefore they contain $\mathcal{M}^{CV}$ and $\mathcal{M}^{\mathbb{P}Curr}$ accordingly.

Choose a simple Dehn twist D of F with a twistor b. Let $Y \subseteq \mathbb{P}Curr(F)$ be the critical set of D, defined in Definition 5.3.

We claim that there exists a point $([T], [\nu]) \in S$ such that $||b||_T > 0$ and $\nu \notin Y$. Suppose not. Then for every point $([T], [\nu]) \in S$ either $||T||_b = 0$ or $\nu \in Y$. Choose $([T], [\nu]) \in S$ such that T corresponds to a free action of F. (Such a point exists since the first coordinate projection of S contains $\mathcal{M}^{CV}$). Then $\phi[T]$ corresponds to a free action for every $\phi \in Out(F)$. For every $\phi \in Out(F)$ we have $\phi([T], [\nu]) = (\phi[T], \phi[\nu]) \in S$ and thus $\phi\nu \in Y$. Therefore the closure C of the orbit $Out(F)([T], [\nu])$ is a subset of $\partial CV(F) \times Y$. The second coordinate projection of C is a closed $Out(F)$-invariant subset of $\mathbb{P}Curr(F)$ that is contained in Y. This is a contradiction to Theorem 5.10, since Y does not contain $\mathcal{M}^{\mathbb{P}Curr}$.

Thus the claim is verified and there exists a point $([T], [\nu]) \in S$ such that $||b||_T > 0$ and $\nu \notin Y$. By definition of Y one has either $\frac{1}{2}(a; \nu)_A > (ab^{-1}; \nu)_A$ or $\frac{1}{2}(a; \nu)_A > (ab; \nu)_A$. Suppose the former (the other case is symmetric). Then by Proposition 5.5 and Corollary 5.8 we have

$$\lim_{n\to\infty} D^n([T], [\nu]) = ([T_D], [\mu_b]).$$

Since S is closed and $Out(F)$-equivariant, it follows that $([T_D], [\mu_b]) \in S$. Hence the closure of the $Out(F)$-orbit of $([T_D], [\mu_b])$, that is the set $\mathcal{M}^2$, is also contained in S, as claimed.

□

9. Maps from currents to the boundary of Outer space

Lemma 9.1. *Let F be a free group of finite rank $k \geq 3$ and let D be a simple Dehn twists of F with twistor b. Let $Y \subseteq \mathbb{P}Curr(F)$ be the critical set of D. Then $\mathcal{M}^{\mathbb{P}Curr} - Y$ is dense in $\mathcal{M}^{\mathbb{P}Curr}$.*

Proof. Recall that rational currents corresponding to primitive elements are dense in $\mathcal{M}^{\mathbb{P}Curr}$, by the definition of the latter. Hence it suffices to show that if g is a primitive element with $[\mu_g] \in Y$ then $[\mu_g]$ can be approximated by elements of $\mathcal{M}^{\mathbb{P}Curr} - Y$.

Let $A = \{a, b, c, \dots\}$ be the free basis of F that appears in the definition of the simple Dehn twist D. Suppose g is a primitive element such that $[\mu_g] \in Y$. By definition of Y it follows that, when expressed as a reduced word in $A^{\pm 1}$, the element g involves an even number of occurrences of $a^{\pm 1}$. After replacing g by its conjugate if necessary, we may assume that g is represented by a cyclically reduced word in A.

Choose a free basis B of F containing g. By looking at the abelianization of F we see that there must exist an element $f \in B$, $f \neq g^{\pm 1}$ such that f, when expressed in A, involves an odd number of occurrences of $a^{\pm 1}$. Thus $(a;\mu_f)_A$ is odd and hence by definition of Y we have $[\mu_f] \notin Y$. Moreover, for every integer $n \geq 1$ we have that $(a;\mu_{g^n f})_A$ is odd and hence $[\mu_{g^n f}] \notin Y$. On the other hand $g^n f$ is primitive for every integer n and

$$\lim_{n\to\infty} [\mu_{g^n f}] = [\mu_g].$$

Since $[\mu_{g^n f}] \in \mathcal{M}^{\mathbb{P}Curr} - Y$, the lemma is proved.

□

Proposition 9.2. *Let F be a free group of finite rank $k \geq 3$. Let $\tau : \mathcal{M}^{\mathbb{P}Curr} \to \partial CV(F)$ be a continuous map that is either $Out(F)$-equivariant or $Out(F)$-anti-equivariant.*

Then for every simple Dehn twist D of F with twistor b we have $\tau([\mu_b]) = [T_D]$.

Proof. Let $Y \subseteq \mathbb{P}Curr(F)$ be the critical set of D.

We claim that there exists a point $[\mu] \in \mathcal{M}^{\mathbb{P}Curr} - Y$ such that for $[T] = \tau([\mu])$ we have $||b||_T > 0$. Suppose not. Then, since by Lemma 9.1 the set $\mathcal{M}^{\mathbb{P}Curr} - Y$ is dense in $\mathcal{M}^{\mathbb{P}Curr}$ and since τ is continuous, it follows that the image of the map τ is contained in the set $\{[T] \in \partial CV(F) : ||b||_T = 0\}$. Since τ is either $Out(F)$-equivariant or $Out(F)$-anti-equivariant, it follows that the set $\{[T] \in \partial CV(F) : ||b||_T = 0\}$ contains a closed $Out(F)$-invariant subset of $\partial CV(F)$ and hence it contains the set $\mathcal{M}^{CV}$. This contradicts the fact that $\mathcal{M}^{CV}$ has elements corresponding to free actions.

Thus indeed, there exists a point $[\mu] \in \mathcal{M}^{\mathbb{P}Curr} - Y$ such that for $[T] = \tau([\mu])$ we have $||b||_T > 0$. Then $\lim_{n\to\infty} D^n([T]) = \lim_{n\to\infty} D^{-n}([T]) = [T_D]$ and either $\lim_{n\to\infty} D^n([\mu]) = [\mu_b]$ or $\lim_{n\to\infty} D^{-n}([\mu]) = [\mu_b]$. By assumptions on τ it follows that $\tau([\mu_b]) = [T_D]$, as claimed.

□

Theorem 9.3. *Let F be a free group of finite rank $k \geq 3$. Then there does not exist a continuous map $\tau : \mathcal{M}^{\mathbb{P}Curr} \to \partial CV(F)$ that is either $Out(F)$-equivariant or $Out(F)$-anti-equivariant.*

Proof. Suppose $\tau : \mathcal{M}^{\mathbb{P}Curr} \to \partial CV(F)$ is a continuous map that is either $Out(F)$-equivariant or $Out(F)$-anti-equivariant.

Let $A = \{a_1, \dots, a_k\}$ be a free basis of F. Denote $a_1 = a$, $a_2 = b$ and $a_3 = c$. Let D be the simple Dehn twist with twistor b defined as $D(a) = ab$ and $D(a_i) = a_i$ for $i \geq 2$.

Consider the free basis $A' = \{a'_1, \dots, a'_k\}$ where $a'_3 = ca$ and $a'_i = a_i$ for $i \neq 3$. Let D' be the simple Dehn twist defined as $D'(a'_1) = a'_1 a'_2 = ab$ and $D'(a'_i) = a'_i$ for $i \geq 2$. Then D' has twistor $a'_2 = b$. Since both D and D' are simple Dehn twists with the same twistor b, Proposition 9.2 implies that $\tau([\mu_b]) = [T_D] = [T_{D'}]$. However, for $c' := a'_3 = ca$ we have $||c'||_{T_D} = 1$ and $||c'||_{T_{D'}} = 0$. Hence $[T_D] \neq [T_{D'}]$, yielding a contradiction.

□

Since $\mathcal{M}^{\mathbb{P}Curr}$ is a closed $Out(F)$-invariant subset of $\mathbb{P}Curr(F)$, Theorem 9.3 immediately implies:

Corollary 9.4. *Let F be a free group of finite rank $k \geq 3$. Then there does not exist a continuous map $\tau : \mathbb{P}Curr(F) \to \partial CV(F)$ that is either $Out(F)$-equivariant or $Out(F)$-anti-equivariant.*

10. Outlook

The following question seems natural, and a positive answer would be useful for several applications, some indicated below. In this section we assume that F is a finitely generated free group of rank $k \geq 3$.

Question 10.1. Let $z(F)$ be the closure of the (non-projectivized) Outer space $cv(F)$ in the ambient vector space $\mathbb{R}^F$, given by the canonical embedding $cv(F) \subset \mathbb{R}^F$ via the translation length function as explained in §3.

Does the intersection form $I : cv(F) \times Curr(F) \to \mathbb{R}$ admit a continuous extension $\overline{I} : z(F) \times Curr(F) \to \mathbb{R}$?

Note that if such $\overline{I}$ exists, it must be $Out(F)$-invariant since the intersection form I is $Out(F)$-invariant. Similarly, $\overline{I}$ will have to be $\mathbb{R}$-homogeneous with respect to the first argument and linear with respect to the second argument.

The second author has produced a preliminary preprint [26] which addresses this question and sketches the proof of a positive answer. Working out some of the details still needs to be done.

If a continuous extension $\overline{I}$ does exist, then the notion of having zero intersection number makes sense for the elements of $\overline{CV(F)}$ and $\mathbb{P}Curr(F)$. Namely, for $[T] \in \overline{CV(F)}$ and $[\nu] \in \mathbb{P}Curr(F)$ we say that $([T], [\nu]) \in I_0$ if $\overline{I}(T, \nu) = 0$. It is easy to see that I_0 has to be closed and $Out(F)$-invariant.

Question 10.2. Let F be free of finite rank $k \geq 3$. Recall that $\mathcal{M}^2$ is the smallest non-empty closed $Out(F)$-invariant subset of $\partial CV(F) \times \mathbb{P}Curr(F)$. Is it true that

$$\mathcal{M}^2 = I_0 \cap \left(\mathcal{M}^{CV} \times \mathcal{M}^{\mathbb{P}Curr}\right) ?$$

We do not know whether the answer to the inclusion "$\supset$" should expected to be positive. For the inclusion "$\subset$ " we will now give an argument, based on the assumption that Question 10.1 has a positive answer:

We already know that for a simple Dehn twist D of F corresponding to A with twistor $b \in A$ we have $([T_D], [\mu_b]) \in \mathcal{M}^2$. Let $T \in cv(F)$ be the action of F on its Cayley graph with respect to A. It is known and easy to see directly that $\frac{1}{n} D^n T \to_{n\to\infty} T_D$ in $z(F)$. Also, we have $D^n \mu_a = \mu_{D^n a} = \mu_{ab^n}$,

$||D^n\mu_a||_A = ||ab^n||_A = n+1$ and $\lim_{n\to\infty} \frac{1}{n+1} D^n\mu_a = \mu_b$. Therefore

$$I(\frac{1}{n}D^nT, \frac{1}{n+1}D^n\mu_a) = \frac{1}{n(n+1)} I(D^nT, D^n\mu_a) = \frac{1}{n(n+1)} I(T, \mu_a) \to_{n\to\infty} 0.$$

Hence the presumed continuity of $\overline{I}$ implies that $([T_D], [\mu_b]) \in I_0$. By equivariance of $\overline{I}$ we get $\phi([T_D], [\mu_b]) \in I_0$ for every $\phi \in Out(F)$. Since the $Out(F)$-orbit of $([T_D], [\mu_b])$ is dense in $\mathcal{M}^2$, it follows that $\mathcal{M}^2 \subseteq I_0$ and therefore

$$\mathcal{M}^2 \subseteq I_0 \cap \left(\mathcal{M}^{CV} \times \mathcal{M}^{\mathbb{P}Curr}\right).$$

In this context we would also point to the work in progress of the second author with Coulbois and Hilion [11] where it is shown that the minimal set $\mathcal{M}^2$ is contained in a subset $\mathcal{L}^2$ of $\mathcal{M}^{CV} \times \mathcal{M}^{\mathbb{P}Curr}$ that is described in terms of algebraic laminations: A pair (T, μ) defines a point of $\mathcal{L}^2$ if the *dual algebraic lamination* $L^2(T)$ associated to T in [9] contains the *support* $L^2(\mu)$ associated to μ in [10]. Recent results lead the second author to the belief that $\mathcal{L}^2$ agrees with $I_0 \cap \left(\mathcal{M}^{CV} \times \mathcal{M}^{\mathbb{P}Curr}\right)$.

Question 10.3. Kapovich and Nagnibeda [24] proved that the Patterson-Sullivan map $\mathcal{P} : CV(F) \to \mathbb{P}Curr(F)$ is a continuous $Out(F)$-equivariant topological embedding. Hence the closure $\overline{image(\mathcal{P})}$ of the image of $\mathcal{P}$ is a closed $Out(F)$-invariant set. Hence by Theorem 5.10 $\overline{image(\mathcal{P})}$ contains the minimal set $\mathcal{M}^{\mathbb{P}Curr}$.

Is it true that

$$\overline{image(\mathcal{P})} = image(\mathcal{P}) \cup \mathcal{M}^{\mathbb{P}Curr}\,?$$

We would like to finish this section with a question of more speculative character:

Question 10.4. For any point $[T] \in CV(F)$, is the set $\mathcal{P}^2$ of accumulation points of the $Out(F)$-orbit of the pair $([T], [\mathcal{P}(T)])$ strictly smaller than $I_0 \cap \left(\mathcal{M}^{CV} \times \mathcal{M}^{\mathbb{P}Curr}\right)$ (or than $\mathcal{L}^2$) ? Is it perhaps equal to $\mathcal{M}^2$?

Note that for any current $0 \neq \mu \in Curr(F)$ and for any $[T] \in CV(F)$ the set of accumulation points of the $Out(F)$-orbit of the pair $([T], [\mu])$ is contained in $\partial CV \times \mathcal{P}Curr(F)$, and, since it is closed, $Out(F)$-invariant and non-empty, it must contain $\mathcal{M}^2$. We do not know whether it depends on the choice of T and μ, and we do also not know whether it can ever be strictly bigger than $\mathcal{M}^2$.

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