arXiv:math/0501303v2 [math.ST] 4 Apr 2005

REFINEMENT INEQUALITIES AMONG SYMMETRIC DIVERGENCE MEASURES

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Abstract. There are three classical divergence measures in the literature on information theory and statistics, namely, Jeffryes-Kullback-Leiber's *J-divergence*, Sibson-Burbea-Rao's *Jensen-Shannon divegernce* and Taneja's *arithemtic - geometric mean divergence*. These bear an interesting relationship among each other and are based on logarithmic expressions. The divergence measures like *Hellinger discrimination*, *symmetric* χ^2-*divergence*, and *triangular discrimination* are not based on logarithmic expressions. These six divergence measures are symmetric with respect to probability distributions. In this paper some interesting inequalities among these symmetric divergence measures are studied. Refinement over these inequalities is also given. Some inequalities due to Dragomir et al. [6] are also improved.

1. Introduction

Let

$$\Gamma_n = \left\{ P = (p_1, p_2, ..., p_n) \middle| p_i > 0, \sum_{i=1}^{n} p_i = 1 \right\}, \; n \geqslant 2,$$

be the set of all complete finite discrete probability distributions. For all $P, Q \in \Gamma_n$, the following measures are well known in the literature on information theory and statistics:

- **Hellinger Discrimination**

$$h(P||Q) = 1 - B(P||Q) = \frac{1}{2}\sum_{i=1}^{n}(\sqrt{p_i} - \sqrt{q_i})^2, \tag{1.1}$$

where

$$B(P||Q) = \sqrt{p_i q_i}, \tag{1.2}$$

is the well-known Bhattacharyya [1] *coefficient*.

- **Triangular Discrimination**

$$\Delta(P||Q) = 2\left[1 - W(P||Q)\right] = \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{p_i + q_i}, \tag{1.3}$$

where

$$W(P||Q) = \sum_{i=1}^{n} \frac{2 p_i q_i}{p_i + q_i}, \tag{1.4}$$

2000 *Mathematics Subject Classification.* 94A17; 26D15.

Key words and phrases. J-divergence; Jensen-Shannon divergence; Arithmetic-geometric mean divergence; Triangular discrimination; Symmetric chi-square divergence; Hellinger discrimination; Csiszár's f-divergence; Information inequalities.

is the well-known *harmonic mean divergence.*

- **Symmetric Chi-square Divergence**

$$\Psi(P||Q) = \chi^2(P||Q) + \chi^2(Q||P) = \sum_{i=1}^{n} \frac{(p_i - q_i)^2(p_i + q_i)}{p_i q_i}, \tag{1.5}$$

where

$$\chi^2(P||Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{q_i} = \sum_{i=1}^{n} \frac{p_i^2}{q_i} - 1, \tag{1.6}$$

is the well-known χ^2-*divergence* (Pearson [10])

- **J-Divergence**

$$J(P||Q) = \sum_{i=1}^{n} (p_i - q_i) \ln(\frac{p_i}{q_i}). \tag{1.7}$$

- **Jensen-Shannon Divergence**

$$I(P||Q) = \frac{1}{2}\left[\sum_{i=1}^{n} p_i \ln\left(\frac{2p_i}{p_i + q_i}\right) + \sum_{i=1}^{n} q_i \ln\left(\frac{2q_i}{p_i + q_i}\right)\right]. \tag{1.8}$$

- **Arithmetic-Geometric Mean Divergence**

$$T(P||Q) = \sum_{i=1}^{n} \left(\frac{p_i + q_i}{2}\right) \ln\left(\frac{p_i + q_i}{2\sqrt{p_i q_i}}\right). \tag{1.9}$$

After simplification, we can write

$$J(P||Q) = 4\left[I(P||Q) + T(P||Q)\right]. \tag{1.10}$$

The measures $I(P||Q)$, $J(P||Q)$ and $T(P||Q)$ can be written as

$$J(P||Q) = K(P||Q) + K(Q||P), \tag{1.11}$$

$$I(P||Q) = \frac{1}{2}\left[K\left(P||\frac{P+Q}{2}\right) + K\left(Q||\frac{P+Q}{2}\right)\right], \tag{1.12}$$

and

$$T(P||Q) = \frac{1}{2}\left[K\left(\frac{P+Q}{2}||P\right) + K\left(\frac{P+Q}{2}||Q\right)\right], \tag{1.13}$$

where

$$K(P||Q) = \sum_{i=1}^{n} p_i \log\left(\frac{p_i}{q_i}\right), \tag{1.14}$$

is the well known Kullback-Leibler [9] *relative information.*

We call the measures given in (1.1), (1.3), (1.5), (1.7), (1.9) and (1.10) as *symmetric divergence measures*, since they are symmetric with respect to the probability distributions P and Q. The measure (1.1) is due to Hellinger [7]. The measure (1.5) is due to Dragomir et al. [6], and recently has been studied by Taneja [15]. The measure (1.7) is due to

Jeffreys [8], and later Kullback-Leibler [9] studied it extensively. Some times it is called as Jeffreys-Kullback-Leibler's *J-divergence*. The measure (1.8) is due to Sibson [11], and later Burbea and Rao [2, 3] studied it extensively. Initially, it was called as *information radius*, but now a days it is famous as *Jensen-Shannon divegence*. The measure (1.9) is due to Taneja [15], and is known by *arithmetic-geometric mean divergence*. For one parametric generalizations of the measures given above refer to Taneja [17, 18]. A general study of information and divergence measures and their generalizations can be seen in Taneja [12, 13, 14].

In this paper our aim is to obtain an inequality its improvement in terms of above symmetric divergence measures. This we shall do by the application of some properties of Csiszár's $f-$*divergence*.

2. Csiszár's $f-$Divergence

Given a function $f:[0,\infty)\to\mathbb{R}$, the *f-divergence* measure introduced by Csiszár's [4] is given by

$$C_f(P||Q)=\sum_{i=1}^{n} q_i f\left(\frac{p_i}{q_i}\right), \tag{2.1}$$

for all $P,Q\in\Gamma_n$.

The following theorem is well known in the literature.

Theorem 2.1. *(Csiszár's* [4, 5]*). If the function f is convex and normalized, i.e., $f(1)=0$, then the $f-$divergence, $C_f(P||Q)$ is nonnegative and convex in the pair of probability distribution $(P,Q)\in\Gamma_n\times\Gamma_n$.*

Recently, Taneja [16, 18] established the following property of the measure (2.1).

Theorem 2.2. *(Taneja* [15]*). Let $f_1,f_2:I\subset\mathbb{R}_+\to\mathbb{R}$ two generating mappings are normalized, i.e., $f_1(1)=f_2(1)=0$ and satisfy the assumptions:*

(i) f_1 and f_2 are twice differentiable on (a,b);

(ii) there exists the real constants m,M such that $m<M$ and

$$m\leqslant\frac{f_1''(x)}{f_2''(x)}\leqslant M,\ f_2''(x)>0,\ \forall x\in(a,b), \tag{2.2}$$

then we have the inequalities:

$$m\ C_{f_2}(P||Q)\leqslant C_{f_1}(P||Q)\leqslant M\ C_{f_2}(P||Q). \tag{2.3}$$

Proof. Let us consider the functions $\eta_{m.s}(\cdot)$ and $\eta_{M.s}(\cdot)$ given by

$$\eta_m(x)=f_1(x)-m\ f_2(x), \tag{2.4}$$

and

$$\eta_M(x)=M\ f_2(x)-f_1(x), \tag{2.5}$$

respectively, where m and M are as given by (2.2).

Since $f_1(x)$ and $f_2(x)$ are normalized, i.e., $f_1(1)=f_2(1)=0$, then $\eta_m(\cdot)$ and $\eta_M(\cdot)$ are also normalized, i.e., $\eta_m(1)=0$ and $\eta_M(1)=0$. Also, the functions $f_1(x)$ and $f_2(x)$ are twice differentiable. Then in view of (2.2), we have

$$\eta_m''(x)=f_1''(x)-m\ f_2''(x)=f_2''(x)\left(\frac{f_1''(x)}{f_2''(x)}-m\right)\geqslant 0, \tag{2.6}$$

and

$$\eta_M''(x) = M\ f_2''(x) - f_1''(x) = f_2''(x)\left(M - \frac{f_1''(x)}{f_2''(x)}\right) \geqslant 0, \tag{2.7}$$

for all $x \in (r, R)$.

In view of (2.6) and (2.7), we can say that the functions $\eta_m(\cdot)$ and $\eta_M(\cdot)$ given by (2.4) and (2.5) respectively, are convex on (r, R).

According to Theorem 2.1, we have

$$C_{\eta_m}(P||Q) = C_{f_1 - m f_2}(P||Q) = C_{f_1}(P||Q) - m\ C_{f_2}(P||Q) \geqslant 0, \tag{2.8}$$

and

$$C_{\eta_M}(P||Q) = C_{M f_2 - f_1}(P||Q) = M\ C_{f_2}(P||Q) - C_{f_1}(P||Q) \geqslant 0. \tag{2.9}$$

Combining (2.8) and (2.9) we have the proof of (2.3). □

Now, based on Theorem 2.1, we shall give below the *convexity* and *nonnegativity* of the *symmetric divergence measures* given in Section 1.

Example 2.1. *(Hellinger discrimination). Let us consider*

$$f_h(x) = \frac{1}{2}(\sqrt{x} - 1)^2,\ x \in (0, \infty), \tag{2.10}$$

in (2.1), then we have $C_f(P||Q) = h(P||Q)$, where $h(P||Q)$ is as given by (1.1).

Moreover,

$$f_h'(x) = \frac{\sqrt{x} - 1}{2\sqrt{x}},$$

and

$$f_h''(x) = \frac{1}{4x\sqrt{x}}. \tag{2.11}$$

Thus we have $f_h''(x) > 0$ for all $x > 0$, and hence, $f_h(x)$ is strictly convex for all $x > 0$. Also, we have $f_h(1) = 0$. In view of this we can say that the Hellinger discrimination given by (1.1) is nonnegative and convex in the pair of probability distributions $(P, Q) \in \Gamma_n \times \Gamma_n$.

Example 2.2. *(Triangular discrimination). Let us consider*

$$f_\Delta(x) = \frac{(x - 1)^2}{x + 1},\ x \in (0, \infty), \tag{2.12}$$

in (2.1), then we have $C_f(P||Q) = \Delta(P||Q)$, where $\Delta(P||Q)$ is as given by (1.3).

Moreover,

$$f_\Delta'(x) = \frac{(x - 1)(x + 3)}{(x + 1)^2},$$

and

$$f_\Delta''(x) = \frac{8}{(x + 1)^3}. \tag{2.13}$$

Thus we have $f_\Delta''(x) > 0$ for all $x > 0$, and hence, $f_\Delta(x)$ is strictly convex for all $x > 0$. Also, we have $f_\Delta(1) = 0$. In view of this we can say that the triangular discrimination given by (1.3) is nonnegative and convex in the pair of probability distributions $(P, Q) \in \Gamma_n \times \Gamma_n$.

Example 2.3. *(Symmetric chi-square divergence). Let us consider*

$$f_\Psi(x) = \frac{(x-1)^2(x+1)}{x},\ x \in (0,\infty), \tag{2.14}$$

in (2.1), then we have $C_f(P||Q) = \Psi(P||Q)$, where $\Psi(P||Q)$ is as given by (1.5).
Moreover,

$$f'_\Psi(x) = \frac{(x-1)(2x^2+x+1)}{x^2},$$

and

$$f''_\Psi(x) = \frac{2(x^3+1)}{x^3}. \tag{2.15}$$

Thus we have $f''_\Psi(x) > 0$ for all $x > 0$, and hence, $f_\Psi(x)$ is strictly convex for all $x > 0$. Also, we have $f_\Psi(1) = 0$. In view of this we can say that the symmetric chi-square divergence given by (1.5) is nonnegative and convex in the pair of probability distributions $(P,Q) \in \Gamma_n \times \Gamma_n$.

Example 2.4. *(J-divergence). Let us consider*

$$f_J(x) = (x-1)\ln x,\ x \in (0,\infty), \tag{2.16}$$

in (2.1), then we have $C_f(P||Q) = J(P||Q)$, where $J(P||Q)$ is as given by (1.7).
Moreover,

$$f'_J(x) = 1 - x^{-1} + \ln x,$$

and

$$f''_J(x) = \frac{x+1}{x^2}. \tag{2.17}$$

Thus we have $f''_J(x) > 0$ for all $x > 0$, and hence, $f_J(x)$ is strictly convex for all $x > 0$. Also, we have $f_J(1) = 0$. In view of this we can say that the J-divergence given by (1.7) is nonnegative and convex in the pair of probability distributions $(P,Q) \in \Gamma_n \times \Gamma_n$.

Example 2.5. *(JS-divergence). Let us consider*

$$f_I(x) = \frac{x}{2}\ln x + \frac{x+1}{2}\ln\left(\frac{2}{x+1}\right),\ x \in (0,\infty), \tag{2.18}$$

in (2.1), then we have $C_f(P||Q) = I(P||Q)$, where $I(P||Q)$ is as given by (1.8).
Moreover,

$$f'_I(x) = \frac{1}{2}\ln\left(\frac{2x}{x+1}\right),$$

and

$$f''_I(x) = \frac{1}{2x(x+1)}. \tag{2.19}$$

Thus we have $f''_I(x) > 0$ for all $x > 0$, and hence, $f_I(x)$ is strictly convex for all $x > 0$. Also, we have $f_I(1) = 0$. In view of this we can say that the JS-divergence given by (1.8) is nonnegative and convex in the pair of probability distributions $(P,Q) \in \Gamma_n \times \Gamma_n$.

Example 2.6. *(AG-Divergence). Let us consider*

$$f_T(x) = \left(\frac{x+1}{2}\right)\ln\left(\frac{x+1}{2\sqrt{x}}\right),\ x \in (0,\infty), \tag{2.20}$$

in (2.1), then we have $C_f(P||Q) = T(P||Q)$, where $T(P||Q)$ is as given by (1.9).

Moreover,

$$f_T'(x) = \frac{1}{4}\left[1 - x^{-1} + 2\ln\left(\frac{x+1}{2\sqrt{x}}\right)\right],$$

and

$$f_T''(x) = \frac{x^2+1}{4x^2(x+1)}. \tag{2.21}$$

Thus we have $f_T''(x) > 0$ for all $x > 0$, and hence, $f_T(x)$ is strictly convex for all $x > 0$. Also, we have $f_T(1) = 0$. In view of this we can say that the AG-divergence given by (1.9) is nonnegative and convex in the pair of probability distributions $(P,Q) \in \Gamma_n \times \Gamma_n$.

3. Inequalities Among the Measures

In this section we shall apply the Theorem 2.2 to obtain inequalities among the measures given in Section 1. We have considered only the symmetric measures given in (1.1), (1.3), (1.5), (1.7)-(1.9).

Theorem 3.1. *The following inequalities among the divergence measures hold:*

$$\frac{1}{4}\Delta(P||Q) \leqslant I(P||Q) \leqslant h(P||Q) \leqslant \frac{1}{8}J(P||Q) \leqslant T(P||Q) \leqslant \frac{1}{16}\Psi(P||Q). \tag{3.1}$$

The proof of the above theorem is based on the following propositions, where we have proved each part separately.

Proposition 3.1. *The following inequality hold:*

$$\frac{1}{4}\Delta(P||Q) \leqslant I(P||Q). \tag{3.2}$$

Proof. Let us consider

$$g_{I\Delta}(x) = \frac{f_I''(x)}{f_\Delta''(x)} = \frac{(x+1)^2}{16x}, \quad x \in (0,\infty), \tag{3.3}$$

where $f_I''(x)$ and $f_\Delta''(x)$ are as given by (2.19) and (2.13) respectively.

From (3.3), we have

$$g_{I\Delta}'(x) = \frac{(x-1)(x+1)}{16x^2}\begin{cases} \geqslant 0, & x \geqslant 1 \\ \leqslant 0, & x \leqslant 1 \end{cases}. \tag{3.4}$$

In view of (3.4), we conclude that the function $g_{I\Delta}(x)$ is decreasing in $x \in (0,1)$ and increasing in $x \in (1,\infty)$, and hence

$$m = \sup_{x\in(0,\infty)} g_{I\Delta}(x) = g_{I\Delta}(1) = \frac{1}{4}. \tag{3.5}$$

Applying the inequalities (2.3) for the measures $\Delta(P||Q)$ and $I(P||Q)$ along with (3.5) we get the required result. □

Proposition 3.2. *The following inequality hold:*

$$I(P||Q) \leqslant h(P||Q). \tag{3.6}$$

Proof. Let us consider

$$g_{Ih}(x) = \frac{f_I''(x)}{f_h''(x)} = \frac{2\sqrt{x}}{x+1}, \quad x \in (0,\infty), \tag{3.7}$$

where $f_I''(x)$ and $f_h''(x)$ are as given by (2.19) and (2.11) respectively.

From (3.7), we have

$$g_{Ih}'(x) = -\frac{x-1}{\sqrt{x}(x+1)^2} \begin{cases} \geqslant 0, & x \leqslant 1 \\ \leqslant 0, & x \geqslant 1 \end{cases}. \tag{3.8}$$

In view of (3.8), we conclude that the function $g_{Ih}(x)$ is increasing in $x \in (0,1)$ and decreasing in $x \in (1,\infty)$, and hence

$$M = \sup_{x\in(0,\infty)} g_{Ih}(x) = g_{Ih}(1) = 1. \tag{3.9}$$

Applying the inequalities (2.3) for the measures $I(P||Q)$ and $h(P||Q)$ along with (3.9) we get the required result. □

Proposition 3.3. *The following inequality hold:*

$$h(P||Q) \leqslant \frac{1}{8} J(P||Q). \tag{3.10}$$

Proof. Let us consider

$$g_{Jh}(x) = \frac{f_J''(x)}{f_h''(x)} = \frac{4(x+1)}{\sqrt{x}}, \quad x \in (0,\infty), \tag{3.11}$$

where $f_J''(x)$ and $f_h''(x)$ are as given by (2.17) and (2.11) respectively.

From (3.11) we have

$$g_{Jh}'(x) = \frac{2(x-1)}{x\sqrt{x}} \begin{cases} \geqslant 0, & x \geqslant 1 \\ \leqslant 0, & x \leqslant 1 \end{cases}. \tag{3.12}$$

In view of (3.12), we conclude that the function $g_{Jh}(x)$ is decreasing in $x \in (0,1)$ and increasing in $x \in (1,\infty)$, and hence

$$m = \inf_{x\in(0,\infty)} g_{Jh}(x) = g_{Jh}(1) = 8. \tag{3.13}$$

Applying the inequalities (2.3) for the measures $h(P||Q)$ and $J(P||Q)$ along with (3.13) we get the required result. □

Proposition 3.4. *The following inequality hold:*

$$\frac{1}{8} J(P||Q) \leqslant T(P||Q). \tag{3.14}$$

Proof. Let us consider

$$g_{JT}(x) = \frac{f_J''(x)}{f_T''(x)} = \frac{4(x+1)^2}{x^2+1}, \quad x \in (0,\infty), \tag{3.15}$$

where $f_J''(x)$ and $f_T''(x)$ are as given by (2.17) and (2.21) respectively.

From (3.15) we have

$$g'_{JT}(x) = -\frac{8(x-1)(x+1)}{(x^2+1)^2} \begin{cases} \geqslant 0, & x \leqslant 1 \\ \leqslant 0, & x \geqslant 1 \end{cases}. \tag{3.16}$$

In view of (3.16) we conclude that the function $g_{JT}(x)$ is increasing in $x \in (0,1)$ and decreasing in $x \in (1,\infty)$, and hence

$$M = \sup_{x \in (0,\infty)} g_{JT}(x) = g_{JT}(1) = 8. \tag{3.17}$$

Applying the inequality (2.3) for the measures $J(P||Q)$ and $T(P||Q)$ along with (3.17) we get the required result. □

Proposition 3.5. *The following inequality hold:*

$$T(P||Q) \leqslant \frac{1}{16}\Psi(P||Q). \tag{3.18}$$

Proof. Let us consider

$$g_{T\Psi}(x) = \frac{f''_T(x)}{f''_\Psi(x)} = \frac{x(x^2+1)}{8(x+1)(x^3+1)}, \quad x \in (0,\infty), \tag{3.19}$$

where $f''_T(x)$ and $f''_\Psi(x)$ are as given by (2.21) and (2.15) respectively.

From (3.19) we have

$$g'_{T\Psi}(x) = -\frac{(x-1)(x^4+4x^2+1)}{8(x+1)^3(x^2-x+1)^2} \begin{cases} \geqslant 0, & x \leqslant 1 \\ \leqslant 0, & x \geqslant 1 \end{cases}. \tag{3.20}$$

In view of (3.20) we conclude that the function $g_{T\Psi}(x)$ is increasing in $x \in (0,1)$ and decreasing in $x \in (1,\infty)$, and hence

$$M = \sup_{x \in (0,\infty)} g_{T\Psi}(x) = g_{T\Psi}(1) = \frac{1}{16}. \tag{3.21}$$

Applying the inequality (2.3) for the measures $T(P||Q)$ and $\Psi(P||Q)$ along with (3.21) we get the required result. □

The proof of the inequalities given in (3.1) follows by combining the results given in (3.2), (3.6), (3.10), 5.14) and (3.18) respectively.

Dragomir et al. [6] proved the following two inequalities involving the measures (1.3), (1.5) and (1.7):

$$0 \leqslant \frac{1}{2}J(P||Q) - \Delta(P||Q) \leqslant \frac{1}{12}D^*(P||Q), \tag{3.22}$$

and

$$0 \leqslant \frac{1}{2}\Psi(P||Q) - J(P||Q) \leqslant \frac{1}{6}D^*(P||Q), \tag{3.23}$$

where

$$D^*(P||Q) = \sum_{i=1}^{n} \frac{(p_i-q_i)^4}{\sqrt{(p_iq_i)^3}}. \tag{3.24}$$

In the following section we shall improve the inequalities given in (3.1). An improvement over the inequalities (3.22) and (3.23) along with their unification is also presented.

4. Difference of Divergence Measures

Let us consider the following *nonnegative* differences:

$$D_{\Psi T}(P||Q) = \frac{1}{16}\Psi(P||Q) - T(P||Q), \tag{4.1}$$

$$D_{\Psi J}(P||Q) = \frac{1}{16}\Psi(P||Q) - \frac{1}{8}J(P||Q), \tag{4.2}$$

$$D_{\Psi h}(P||Q) = \frac{1}{16}\Psi(P||Q) - h(P||Q), \tag{4.3}$$

$$D_{\Psi I}(P||Q) = \frac{1}{16}\Psi(P||Q) - I(P||Q), \tag{4.4}$$

$$D_{\Psi \Delta}(P||Q) = \frac{1}{16}\Psi(P||Q) - \frac{1}{4}\Delta(P||Q), \tag{4.5}$$

$$D_{TJ}(P||Q) = T(P||Q) - \frac{1}{8}J(P||Q), \tag{4.6}$$

$$D_{Th}(P||Q) = T(P||Q) - h(P||Q), \tag{4.7}$$

$$D_{TI}(P||Q) = T(P||Q) - I(P||Q), \tag{4.8}$$

$$D_{T\Delta}(P||Q) = T(P||Q) - \frac{1}{4}\Delta(P||Q), \tag{4.9}$$

$$D_{Jh}(P||Q) = \frac{1}{8}J(P||Q) - h(P||Q), \tag{4.10}$$

$$D_{JI}(P||Q) = \frac{1}{8}J(P||Q) - I(P||Q), \tag{4.11}$$

$$D_{J\Delta}(P||Q) = \frac{1}{8}J(P||Q) - \frac{1}{4}\Delta(P||Q), \tag{4.12}$$

$$D_{hI}(P||Q) = h(P||Q) - I(P||Q), \tag{4.13}$$

$$D_{h\Delta}(P||Q) = h(P||Q) - \frac{1}{4}\Delta(P||Q), \tag{4.14}$$

and

$$D_{I\Delta}(P||Q) = I(P||Q) - \frac{1}{4}\Delta(P||Q). \tag{4.15}$$

In the examples below we shall show the convexity of the above measures (4.1)-(4.15). In view of Theorem 2.1 and Examples 2.1-2.6, it is sufficient to show the nonnegativity of the second order derivative of generating function in each case.

Example 4.1. *We can write*

$$D_{\Psi T}(P||Q) = \frac{1}{16}\Psi(P||Q) - T(P||Q) = \sum_{i=1}^{n} q_i f_{\Psi T}\left(\frac{p_i}{q_i}\right),$$

where

$$f_{\Psi T}(x) = \frac{1}{16} f_{\Psi}(x) - f_T(x), \ x > 0.$$

Moreover, we have

$$f''_{\Psi T}(x)=\frac{1}{16}f''_{\Psi}(x)-f''_{T}(x)=\frac{x^3+1}{8x^3}-\frac{x^2+1}{4x^2(x+1)}=\frac{(x-1)^2(x^2+x+1)}{8x^3(x+1)}\geqslant 0,\ \forall x>0, \tag{4.16}$$

where $f''_{\Psi}(x)$ *and* $f''_{T}(x)$ *are as given by (2.15) and (2.21) respectively.*

Example 4.2. *We can write*

$$D_{\Psi J}(P||Q)=\frac{1}{16}\Psi(P||Q)-\frac{1}{8}J(P||Q)=\sum_{i=1}^{n}q_i f_{\Psi J}\left(\frac{p_i}{q_i}\right),$$

where

$$f_{\Psi J}(x)=\frac{1}{16}f_{\Psi}(x)-\frac{1}{8}f_{J}(x),\ x>0.$$

Moreover, we have

$$f''_{\Psi J}(x)=\frac{1}{16}f''_{\Psi}(x)-\frac{1}{8}f''_{J}(x)=\frac{1}{8}\left(\frac{x^3+1}{x^3}-\frac{x+1}{x^2}\right)=\frac{(x-1)^2(x+1)}{8x^3}\geqslant 0,\ \forall x>0, \tag{4.17}$$

where $f''_{\Psi}(x)$ *and* $f''_{J}(x)$ *are as given by (2.15) and (2.17) respectively.*

Example 4.3. *We can write*

$$D_{\Psi h}(P||Q)=\frac{1}{16}\Psi(P||Q)-h(P||Q)=\sum_{i=1}^{n}q_i f_{\Psi h}\left(\frac{p_i}{q_i}\right),$$

where

$$f_{\Psi h}(x)=\frac{1}{16}f_{\Psi}(x)-f_{h}(x),\ x>0.$$

Moreover, we have

$$f''_{\Psi h}(x)=\frac{1}{16}f''_{\Psi}(x)-f''_{h}(x)=\frac{1}{4}\left(\frac{x^3+1}{2x^3}-\frac{1}{x\sqrt{x}}\right)=\frac{\left(x\sqrt{x}-1\right)^2}{8x^3}\geqslant 0,\ \forall x>0, \tag{4.18}$$

where $f''_{\Psi}(x)$ *and* $f''_{h}(x)$ *are as given by (2.15) and (2.11) respectively.*

Example 4.4. *We can write*

$$D_{\Psi I}(P||Q)=\frac{1}{16}\Psi(P||Q)-I(P||Q)=\sum_{i=1}^{n}q_i f_{\Psi I}\left(\frac{p_i}{q_i}\right),$$

where

$$f_{\Psi I}(x)=\frac{1}{16}f_{\Psi}(x)-f_{I}(x),\ x>0.$$

Moreover, we have

(4.19)
$$f''_{\Psi I}(x) = \frac{1}{16} f''_{\Psi}(x) - f''_{I}(x) = \frac{1}{2x}\left(\frac{x^3+1}{4x^2} - \frac{1}{x+1}\right) = \frac{(x-1)^2(x^2+3x+1)}{8x^3(x+1)} \geqslant 0, \ \forall x > 0,$$

where $f''_{\Psi}(x)$ *and* $f''_{I}(x)$ *are as given by (2.15) and (2.19) respectively.*

Example 4.5. *We can write*

$$D_{\Psi\Delta}(P||Q) = \frac{1}{16}\Psi(P||Q) - \frac{1}{4}\Delta(P||Q) = \sum_{i=1}^{n} q_i f_{\Psi\Delta}\left(\frac{p_i}{q_i}\right),$$

where

$$f_{\Psi\Delta}(x) = \frac{1}{4}\left(\frac{1}{4} f_{\Psi}(x) - f_{\Delta}(x)\right), \ x > 0.$$

Moreover, we have

(4.20)
$$f''_{\Psi\Delta}(x) = \frac{1}{4}\left(\frac{1}{4} f''_{\Psi}(x) - f''_{\Delta}(x)\right) = \frac{x^3+1}{8x^3} - \frac{2}{(x+1)^3} = \frac{(x-1)^2(x^4+5x^3+12x^2+5x+1)}{8x^3(x+1)^3} \geqslant 0, \ \forall x > 0,$$

where $f''_{\Psi}(x)$ *and* $f''_{\Delta}(x)$ *are as given by (2.15) and (2.13) respectively.*

Example 4.6. *We can write*

$$D_{TJ}(P||Q) = T(P||Q) - \frac{1}{8}J(P||Q) = \sum_{i=1}^{n} q_i f_{TJ}\left(\frac{p_i}{q_i}\right),$$

where

$$f_{TJ}(x) = f_T(x) - \frac{1}{8} f_J(x), \ x > 0.$$

Moreover, we have

(4.21)
$$f''_{TJ}(x) = f''_{T}(x) - \frac{1}{8} f''_{J}(x) = \frac{x^2+1}{4x^2(x+1)} - \frac{x+1}{8x^2} = \frac{(x-1)^2}{8x^2(x+1)} \geqslant 0, \ \forall x > 0,$$

where $f''_{T}(x)$ *and* $f''_{J}(x)$ *are as given by (2.21) and (2.17) respectively.*

Example 4.7. *We can write*

$$D_{Th}(P||Q) = T(P||Q) - h(P||Q) = \sum_{i=1}^{n} q_i f_{Th}\left(\frac{p_i}{q_i}\right),$$

where

$$f_{Th}(x) = f_T(x) - f_h(x), \ x > 0.$$

Moreover, we have

$$f''_{Th}(x) = f''_T(x) - f''_h(x) = \frac{1}{4}\left(\frac{x^2+1}{x^2(x+1)} - \frac{1}{x\sqrt{x}}\right) = \frac{\left(\sqrt{x}-1\right)^2\left(x+\sqrt{x}+1\right)}{4x^2(x+1)} \geqslant 0, \ \forall x > 0, \tag{4.22}$$

where $f''_T(x)$ and $f''_h(x)$ are as given by (2.21) and (2.11) respectively.

Example 4.8. *We can write*

$$D_{TI}(P||Q) = T(P||Q) - I(P||Q) = \sum_{i=1}^{n} q_i f_{TI}\left(\frac{p_i}{q_i}\right),$$

where

$$f_{TI}(x) = f_T(x) - f_I(x), \ x > 0.$$

Moreover, we have

$$f''_{TI}(x) = f''_T(x) - f''_I(x) = \frac{x^2+1}{4x^2(x+1)} - \frac{1}{2x(x+1)} = \frac{(x-1)^2}{4x^2(x+1)} \geqslant 0, \ \forall x > 0, \tag{4.23}$$

where $f''_T(x)$ and $f''_I(x)$ are as given by (2.21) and (2.19) respectively.

Example 4.9. *We can write*

$$D_{T\Delta}(P||Q) = T(P||Q) - \frac{1}{4}\Delta(P||Q) = \sum_{i=1}^{n} q_i f_{T\Delta}\left(\frac{p_i}{q_i}\right),$$

where

$$f_{T\Delta}(x) = f_T(x) - \frac{1}{4} f_\Delta(x), \ x > 0.$$

Moreover, we have

$$f''_{T\Delta}(x) = f''_T(x) - \frac{1}{4} f''_\Delta(x) = \frac{x^2+1}{4x^2(x+1)} - \frac{8}{(x+1)^3} = \frac{(x-1)^2(x^2+4x+1)}{4x^2(x+1)^3} \geqslant 0, \ \forall x > 0, \tag{4.24}$$

where $f''_T(x)$ and $f''_\Delta(x)$ are as given by (2.21) and (2.13) respectively.

Example 4.10. *We can write*

$$D_{Jh}(P||Q) = \frac{1}{8}J(P||Q) - h(P||Q) = \sum_{i=1}^{n} q_i f_{Jh}\left(\frac{p_i}{q_i}\right),$$

where

$$f_{Jh}(x) = \frac{1}{8} f_J(x) - f_h(x), \ x > 0.$$

Moreover, we have

$$f''_{Jh}(x) = \frac{1}{8} f''_J(x) - f''_h(x) = \frac{x+1}{8x^2} - \frac{1}{4x\sqrt{x}} = \frac{\left(\sqrt{x}-1\right)^2}{8x^2} \geqslant 0, \ \forall x > 0, \tag{4.25}$$

where $f_J''(x)$ and $f_h''(x)$ are as given by (2.17) and (2.11) respectively.

Example 4.11. *We can write*

$$D_{JI}(P||Q) = \frac{1}{8}J(P||Q) - I(P||Q) = \sum_{i=1}^{n} q_i f_{JI}\left(\frac{p_i}{q_i}\right),$$

where

$$f_{JI}(x) = \frac{1}{8}f_J(x) - f_I(x), \; x > 0.$$

Moreover, we have

$$\begin{aligned} f_{JI}''(x) &= \frac{1}{8}f_J''(x) - f_I''(x) \\ &= \frac{x+1}{8x^2} - \frac{1}{2x(x+1)} = \frac{(x-1)^2}{8x^2(x+1)} \geqslant 0, \; \forall x > 0, \end{aligned} \tag{4.26}$$

where $f_J''(x)$ and $f_I''(x)$ are as given by (2.17) and (2.19) respectively.

Example 4.12. *We can write*

$$D_{J\Delta}(P||Q) = \frac{1}{8}J(P||Q) - \frac{1}{4}\Delta(P||Q) = \sum_{i=1}^{n} q_i f_{J\Delta}\left(\frac{p_i}{q_i}\right),$$

where

$$f_{J\Delta}(x) = \frac{1}{8}f_J(x) - \frac{1}{4}f_\Delta(x), \; x > 0.$$

Moreover, we have

$$\begin{aligned} f_{J\Delta}''(x) &= \frac{1}{8}f_J''(x) - \frac{1}{4}f_\Delta''(x) = \frac{x+1}{8x^2} - \frac{2}{(x+1)^3} \\ &= \frac{(x-1)^2(x^2+6x+1)}{8x^2(x+1)^3} \geqslant 0, \; \forall x > 0, \end{aligned} \tag{4.27}$$

where $f_J''(x)$ and $f_\Delta''(x)$ are as given by (2.17) and (2.13) respectively.

Example 4.13. *We can write*

$$D_{hI}(P||Q) = h(P||Q) - I(P||Q) = \sum_{i=1}^{n} q_i f_{hI}\left(\frac{p_i}{q_i}\right),$$

where

$$f_{hI}(x) = f_h(x) - f_I(x), \; x > 0.$$

Moreover, we have

$$\begin{aligned} f_{hI}''(x) &= f_h''(x) - f_I''(x) \\ &= \frac{1}{4x\sqrt{x}} - \frac{1}{2x(x+1)} = \frac{(\sqrt{x}-1)^2}{4x^{3/2}(x+1)} \geqslant 0, \; \forall x > 0, \end{aligned} \tag{4.28}$$

where $f_h''(x)$ and $f_I''(x)$ are as given by (2.11) and (2.19) respectively.

Example 4.14. *We can write*

$$D_{h\Delta}(P||Q) = h(P||Q) - \frac{1}{4}\Delta(P||Q) = \sum_{i=1}^{n} q_i f_{h\Delta}\left(\frac{p_i}{q_i}\right),$$

where

$$f_{h\Delta}(x) = f_h(x) - \frac{1}{4} f_\Delta(x),\ x > 0.$$

Moreover, we have

$$(4.29)\qquad f''_{h\Delta}(x) = f''_h(x) - \frac{1}{4} f''_\Delta(x) = \frac{1}{4x\sqrt{x}} - \frac{2}{(x+1)^3}$$
$$= \frac{(\sqrt{x}-1)^2\left[(\sqrt{x}+1)^2(x+1)+4x\right]}{4x^{3/2}(x+1)^3} \geqslant 0,\ \forall x > 0,$$

where $f''_h(x)$ and $f''_\Delta(x)$ are as given by (2.11) and (2.13) respectively.

Example 4.15. *We can write*

$$D_{I\Delta}(P||Q) = I(P||Q) - \frac{1}{4}\Delta(P||Q) = \sum_{i=1}^{n} q_i f_{I\Delta}\left(\frac{p_i}{q_i}\right),$$

where

$$f_{I\Delta}(x) = f_I(x) - \frac{1}{4} f_\Delta(x),\ x > 0.$$

Moreover, we have

$$(4.30)\qquad f''_{I\Delta}(x) = f''_I(x) - \frac{1}{4} f''_\Delta(x)$$
$$= \frac{1}{2x(x+1)} - \frac{2}{(x+1)^3} = \frac{(x-1)^2}{2x(x+1)^3} \geqslant 0,\ \forall x > 0,$$

where $f''_I(x)$ and $f''_\Delta(x)$ are as given by (2.19) and (2.13) respectively.

Thus in view of Theorem 2.1 and Examples 4.1-4.15, we can say that the *divergence measures* given in (4.1)-(4.15) are all *nonnegative* and *convex* in the pair of probability distributions $(P, Q) \in \Gamma_n \times \Gamma_n$.

5. Refinement Inequalities

In view of (3.1), the following inequalities are obvious:

$$(5.1)\qquad D_{\Psi T}(P||Q) \leqslant D_{\Psi J}(P||Q) \leqslant D_{\Psi h}(P||Q) \leqslant D_{\Psi I}(P||Q) \leqslant D_{\Psi\Delta}(P||Q),$$

$$(5.2)\qquad D_{TJ}(P||Q) \leqslant D_{Th}(P||Q) \leqslant D_{TI}(P||Q) \leqslant D_{T\Delta}(P||Q),$$

$$(5.3)\qquad D_{Jh}(P||Q) \leqslant D_{JI}(P||Q) \leqslant D_{J\Delta}(P||Q)$$

and

$$(5.4)\qquad D_{hI}(P||Q) \leqslant D_{h\Delta}(P||Q).$$

In view of the relation (1.10), we have the following equality:

$$(5.5)\qquad D_{JI}(P||Q) = \frac{1}{2} D_{TI}(P||Q) = D_{TJ}(P||Q).$$

In this section our aim is to establish refinement inequalities improving the one given in (3.1). This refinement is given in the following theorem.

Theorem 5.1. *The following inequalities hold:*

$$D_{I\Delta}(P||Q) \leqslant \frac{2}{3}D_{h\Delta}(P||Q) \leqslant 2D_{hI}(P||Q) \leqslant D_{TJ}(P||Q), \tag{5.6}$$

$$D_{I\Delta}(P||Q) \leqslant \frac{2}{3}D_{h\Delta}(P||Q) \leqslant \frac{1}{2}D_{J\Delta}(P||Q) \leqslant \frac{1}{3}D_{T\Delta}(P||Q) \leqslant D_{TJ}(P||Q), \tag{5.7}$$

and

$$\begin{aligned} D_{TJ}(P||Q) &\leqslant \frac{2}{3}D_{Th}(P||Q) \leqslant 2D_{Jh}(P||Q) \leqslant \frac{1}{6}D_{\Psi\Delta}(P||Q) \\ &\leqslant \frac{1}{5}D_{\Psi I}(P||Q) \leqslant \frac{2}{9}D_{\Psi h}(P||Q) \leqslant \frac{1}{4}D_{\Psi J}(P||Q) \leqslant \frac{1}{3}D_{\Psi T}(P||Q), \end{aligned} \tag{5.8}$$

The proofs of the inequalities (5.6)-(5.8) are based on the following propositions.

Proposition 5.1. *We have*

$$D_{I\Delta}(P||Q) \leqslant \frac{2}{3}D_{h\Delta}(P||Q). \tag{5.9}$$

Proof. Let us consider

$$\begin{aligned} g_{I\Delta\_h\Delta}(x) = \frac{f''_{I\Delta}(x)}{f''_{h\Delta}(x)} &= \frac{2\sqrt{x}\,(x-1)^2}{(x+1)^3 - 8\left(\sqrt{x}\right)^3},\ x \neq 1 \\ &= \frac{2\sqrt{x}\left(\sqrt{x}+1\right)^2}{\left(\sqrt{x}+1\right)^2(x+1)+4x} \end{aligned}$$

for all $x \in (0,\infty)$, where $f''_{I\Delta}(x)$ and $f''_{h\Delta}(x)$ are as given by (4.30) and (4.29) respectively.

Calculating the first order derivative of the function $g_{I\Delta\_h\Delta}(x)$ with respect to x, one gets

$$\begin{aligned} g'_{I\Delta\_h\Delta}(x) &= -\frac{\left(\sqrt{x}+1\right)\left(x^{5/2} - 2x^{3/2} + 3x^2 + 2x - 3\sqrt{x} - 1\right)}{\sqrt{x}\left[x^2 + 6x + 2\sqrt{x}\,(x+1) + 1\right]^2} \\ &= -\frac{(x-1)(x+1)\left(x+4\sqrt{x}+1\right)}{\sqrt{x}\left[x^2 + 6x + 2\sqrt{x}\,(x+1) + 1\right]^2} \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases}. \end{aligned} \tag{5.10}$$

In view of (5.10) we conclude that the function $g_{I\Delta\_h\Delta}(x)$ is increasing in $x \in (0,1)$ and decreasing in $x \in (1,\infty)$, and hence

$$M = \sup_{x\in(0,\infty)} g_{I\Delta\_h\Delta}(x) = g_{I\Delta\_h\Delta}(1) = \frac{2}{3}. \tag{5.11}$$

By the application of (2.3) with (5.11) we get (5.9). □

Proposition 5.2. *We have*

$$D_{h\Delta}(P||Q) \leqslant 3D_{hI}(P||Q). \tag{5.12}$$

Proof. Let us consider

$$g_{h\Delta\_hI}(x) = \frac{f''_{h\Delta}(x)}{f''_{hI}(x)} = \frac{(x+1)\left(\sqrt{x}+1\right)^2 + 4x}{(x+1)^2},\ x \in (0,\infty),$$

where $f''_{h\Delta}(x)$ and $f''_{hI}(x)$ are as given by (4.29) and (4.28) respectively.

Calculating the first order derivative of the function $g_{h\Delta\_hI}(x)$ with respect to x, one gets

$$g'_{h\Delta\_hI}(x) = -\frac{4x^{3/2}+x^2-4\sqrt{x}-1}{\sqrt{x}\,(x+1)^3} = -\frac{(x-1)\,(x+4\sqrt{x}+1)}{\sqrt{x}\,(x+1)^3}\begin{cases} >0, & x<1 \\ <0, & x>1 \end{cases}. \tag{5.13}$$

In view of (5.13) we conclude that the function $g_{h\Delta\_hI}(x)$ is increasing in $x \in (0,1)$ and decreasing in $x \in (1,\infty)$, and hence

$$M = \sup_{x\in(0,\infty)} g_{h\Delta\_hI}(x) = g_{h\Delta\_hI}(1) = 3. \tag{5.14}$$

By the application of (2.3) with (5.14) we get (5.12). □

Remark 5.1. *In view of Propositions 5.1 and 5.2, and the inequality (3.1) we conclude that*

$$I(P||Q) \leqslant \frac{2}{3}h(P||Q) + \frac{1}{12}\Delta(P||Q) \leqslant h(P||Q). \tag{5.15}$$

Proposition 5.3. *We have*

$$D_{hI}(P||Q) \leqslant \frac{1}{2}D_{TJ}(P||Q). \tag{5.16}$$

Proof. Let us consider

$$g_{hI\_TJ}(x) = \frac{f''_{hI}(x)}{f''_{TJ}(x)} = \frac{2\sqrt{x}}{(\sqrt{x}+1)^2},\ x \in (0,\infty),$$

where $f''_{hI}(x)$ and $f''_{TJ}(x)$ are as given by (4.28) and (4.21) respectively.

Calculating the first order derivative of the function $g_{hI\_TJ}(x)$ with respect to x, one gets

$$g'_{hI\_TJ}(x) = -\frac{\sqrt{x}-1}{\sqrt{x}\,(\sqrt{x}+1)^3}\begin{cases} >0, & x<1 \\ <0, & x>1 \end{cases}. \tag{5.17}$$

In view of (5.17), we conclude that the function $g_{hI\_TJ}(x)$ is increasing in $x \in (0,1)$ and decreasing in $x \in (1,\infty)$, and hence

$$M = \sup_{x\in(0,\infty)} g_{hI\_TJ}(x) = g_{hI\_TJ}(1) = \frac{1}{2}. \tag{5.18}$$

By the application of (2.3) with (5.18) we get (5.16). □

Remark 5.2. *In view of Propositions 5.3 and the inequality (3.1) we conclude the following inequality*

$$h(P||Q) \leqslant \frac{1}{16}J(P||Q) + \frac{1}{2}I(P||Q) \leqslant \frac{1}{8}J(P||Q). \tag{5.19}$$

Combining the inequalities (5.9), (5.12) and (5.16) we get (5.6).

Proposition 5.4. *We have*

$$D_{h\Delta}(P||Q) \leqslant \frac{3}{4} D_{J\Delta}(P||Q). \tag{5.20}$$

Proof. Let us consider

$$\begin{aligned} g_{h\Delta\_J\Delta}(x) = \frac{f''_{h\Delta}(x)}{f''_{J\Delta}(x)} &= \frac{2\sqrt{x}\left[(x+1)^3 - 8x^{3/2}\right]}{(x-1)^2(x^2+6x+1)},\ x \neq 1 \\ &= \frac{2\sqrt{x}\left[\left(\sqrt{x}+1\right)^2(x+1)+4x\right]}{\left(\sqrt{x}+1\right)^2(x^2+6x+1)}, \end{aligned}$$

for all $x \in (0,\infty)$, where $f''_{h\Delta}(x)$ and $f''_{J\Delta}(x)$ are as given by (4.29) and (4.27) respectively.

Calculating the first order derivative of the function $g_{h\Delta\_J\Delta}(x)$ with respect to x, one gets

$$\begin{aligned} g'_{h\Delta\_J\Delta}(x) &= -\frac{1}{\sqrt{x}\left(\sqrt{x}+1\right)^3(x^2+6x+1)^2}\Big[3x^4 - 4x^3 - 18x^2 - 12x - 1 \\ &\qquad + \sqrt{x}\left(x^4 + 12x^3 + 18x^2 + 4x - 3\right)\Big] \\ &= -\frac{\left(\sqrt{x}-1\right)(x+1)^2\left(x^2+4x\sqrt{x}+14x+4\sqrt{x}+1\right)}{\sqrt{x}\left(\sqrt{x}+1\right)^3(x^2+6x+1)^2} \begin{cases} >0, & x<1 \\ <0, & x>1 \end{cases}. \end{aligned} \tag{5.21}$$

In view of (5.21) we conclude that the function $g_{h\Delta\_J\Delta}(x)$ is increasing in $x \in (0,1)$ and decreasing in $x \in (1,\infty)$, and hence

$$M = \sup_{x\in(0,\infty)} g_{h\Delta\_J\Delta}(x) = g_{h\Delta\_J\Delta}(1) = \frac{3}{4}. \tag{5.22}$$

By the application of (2.3) with (5.22) we get (5.20). □

Remark 5.3. *In view of Proposition 5.4 and the inequality (3.1) we conclude the following inequality*

$$h(P||Q) \leqslant \frac{3}{32} J(P||Q) + \frac{1}{16}\Delta(P||Q) \leqslant \frac{1}{8} J(P||Q). \tag{5.23}$$

Proposition 5.5. *We have*

$$D_{J\Delta}(P||Q) \leqslant \frac{2}{3} D_{T\Delta}(P||Q). \tag{5.24}$$

Proof. Let us consider

$$g_{J\Delta\_T\Delta}(x) = \frac{f''_{J\Delta}(x)}{f''_{T\Delta}(x)} = \frac{x^2+6x+1}{2(x^2+4x+1)},\ x \in (0,\infty),$$

where $f''_{J\Delta}(x)$ and $f''_{T\Delta}(x)$ are as given by (4.27) and (4.24) respectively.

Calculating the first order derivative of the function $g_{J\Delta\_T\Delta}(x)$ with respect to x, one gets

$$g'_{J\Delta\_T\Delta}(x) = -\frac{(x-1)(x+1)}{(x^2+4x+1)^2} \begin{cases} >0, & x<1 \\ <0, & x>1 \end{cases}. \tag{5.25}$$

In view of (5.25) we conclude that the function $g_{J\Delta\_T\Delta}(x)$ is increasing in $x \in (0,1)$ and decreasing in $x \in (1,\infty)$, and hence

$$M = \sup_{x\in(0,\infty)} g_{J\Delta\_T\Delta}(x) = g_{J\Delta\_T\Delta}(1) = \frac{2}{3}. \tag{5.26}$$

By the application of (2.3) with (5.26) we get (5.24). □

Proposition 5.6. *We have*

$$D_{T\Delta}(P||Q) \leqslant 3D_{TJ}(P||Q). \tag{5.27}$$

Proof. Let us consider

$$g_{T\Delta\_TJ}(x) = \frac{f''_{T\Delta}(x)}{f''_{TJ}(x)} = \frac{2(x^2+4x+1)}{(x+1)^2},\ x \in (0,\infty),$$

where $f''_{T\Delta}(x)$ and $f''_{TJ}(x)$ are as given by (4.24) and (4.21) respectively.

Calculating the first order derivative of the function $g_{T\Delta\_TJ}(x)$ with respect to x, one gets

$$g'_{T\Delta\_TJ}(x) = -\frac{4(x-1)}{(x+1)^3}\begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases}. \tag{5.28}$$

In view of (5.28) we conclude that the function $g_{T\Delta\_TJ}(x)$ is increasing in $x \in (0,1)$ and decreasing in $x \in (1,\infty)$, and hence

$$M = \sup_{x\in(0,\infty)} g_{T\Delta\_TJ}(x) = g_{T\Delta\_TJ}(1) = 3. \tag{5.29}$$

By the application of (2.3) with (5.29) we get (5.27). □

Remark 5.4. *In view of Propositions 5.5 and 5.6, and the inequality (3.1) we conclude the following inequality*

$$\frac{1}{8}J(P||Q) \leqslant \frac{2}{3}T(P||Q) + \frac{1}{12}\Delta(P||Q) \leqslant T(P||Q). \tag{5.30}$$

Combining the inequalities (5.9), (5.20), (5.24) and (5.27), we get (5.7).

Proposition 5.7. *We have*

$$D_{TJ}(P||Q) \leqslant \frac{2}{3}D_{Th}(P||Q). \tag{5.31}$$

Proof. Let us consider

$$\begin{aligned} g_{TJ\_Th}(x) = \frac{f''_{TJ}(x)}{f''_{Th}(x)} &= \frac{(x-1)^2}{2\left[x^2+1-2\sqrt{x}\,(x+1)\right]},\ x \neq 1 \\ &= \frac{\left(\sqrt{x}+1\right)^2}{2\left(x+\sqrt{x}+1\right)}, \end{aligned}$$

for all $x \in (0,\infty)$, where $f''_{TJ}(x)$ and $f''_{Th}(x)$ are as given by (4.21) and (4.22) respectively.

Calculating the first order derivative of the function $g_{TJ\_Th}(x)$ with respect to x, one gets

$$g'_{TJ\_Th}(x) = -\frac{\left(\sqrt{x}-1\right)\left(\sqrt{x}+1\right)}{4\sqrt{x}\left(x+\sqrt{x}+1\right)}\begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases}. \tag{5.32}$$

In view of (5.32) we conclude that the function $g_{TJ\_Th}(x)$ is increasing in $x \in (0,1)$ and decreasing in $x \in (1,\infty)$, and hence

$$M = \sup_{x\in(0,\infty)} g_{TJ\_Th}(x) = g_{TJ\_Th}(1) = \frac{2}{3}. \tag{5.33}$$

By the application of (2.3) with (5.33) we get (5.31). □

Proposition 5.8. *We have*

$$D_{Th}(P||Q) \leqslant 3D_{Jh}(P||Q). \tag{5.34}$$

Proof. Let us consider

$$\begin{aligned} g_{Th\_Jh}(x) = \frac{f''_{Th}(x)}{f''_{Jh}(x)} &= \frac{2\left[x^2+1-\sqrt{x}\,(x+1)\right]}{(x+1)\left(\sqrt{x}-1\right)^2},\ x \neq 1 \\ &= \frac{2(x+\sqrt{x}+1)}{x+1}, \end{aligned}$$

for all $x \in (0,\infty)$, where $f''_{Th}(x)$ and $f''_{Jh}(x)$ are as given by (4.22) and (4.25) respectively.

Calculating the first order derivative of the function $g_{Th\_Jh}(x)$ with respect to x, one gets

$$g'_{Th\_Jh}(x) = -\frac{x-1}{\sqrt{x}\,(x+1)^2} \begin{cases} < 0, & x < 1 \\ < 0, & x > 1 \end{cases}. \tag{5.35}$$

In view of (5.35) we conclude that the function $g_{Th\_Th}(x)$ is increasing in $x \in (0,1)$ and decreasing in $x \in (1,\infty)$, and hence

$$M = \sup_{x\in(0,\infty)} g_{Th\_Jh}(x) = g_{Th\_Jh}(1) = 3. \tag{5.36}$$

By the application of (2.3) with (5.36) we get (5.34). □

Remark 5.5. *In view of Propositions 5.7 and 5.8, and the inequality (3.1) we conclude the following inequality*

$$h(P||Q) \leqslant \frac{T(P||Q)+2h(P||Q)}{3} \leqslant \frac{1}{8}J(P||Q). \tag{5.37}$$

Proposition 5.9. *We have*

$$D_{Jh}(P||Q) \leqslant \frac{1}{12}D_{\Psi\Delta}(P||Q). \tag{5.38}$$

Proof. Let us consider

$$\begin{aligned} g_{Jh\_\Psi\Delta}(x) = \frac{f''_{Jh}(x)}{f''_{\Psi\Delta}(x)} &= \frac{x\left(\sqrt{x}-1\right)^2(x+1)^3}{(x-1)^2\left(x^4+5x^3+12x^2+5x+1\right)},\ x \neq 1 \\ &= \frac{x(x+1)^3}{\left(\sqrt{x}+1\right)^2\left(x^4+5x^3+12x^2+5x+1\right)}. \end{aligned}$$

for all $x \in (0,\infty)$, where $f''_{Jh}(x)$ and $f''_{\Psi\Delta}(x)$ are as given by (4.25) and (4.20) respectively.

Calculating the first order derivative of the function $g_{Jh\_\Psi\Delta}(x)$ with respect to x, one gets

$$g'_{Jh\_\Psi\Delta}(x) = -\frac{(\sqrt{x}-1)(x+1)^2}{(\sqrt{x}+1)^3(x^4+5x^3+12x^2+5x+1)^2}\times \times\left[x^5+5x^4+6x^2(\sqrt{x}-1)^2+5x+1 +\sqrt{x}\left(x^4+3x^3+4x^2+3x+1\right)\right]. \tag{5.39}$$

From (5.39), one gets

$$g'_{Jh\_\Psi\Delta}(x)\begin{cases}>0, & x<1\\ <0, & x>1\end{cases}. \tag{5.40}$$

In view of (5.40) we conclude that the function $g_{Jh\_\Psi\Delta}(x)$ is increasing in $x\in(0,1)$ and decreasing in $x\in(1,\infty)$, and hence

$$M=\sup_{x\in(0,\infty)} g_{Jh\_\Psi\Delta}(x) = g_{Jh\_\Psi\Delta}(1)=\frac{1}{12}. \tag{5.41}$$

By the application of (2.3) with (5.41) we get (5.38). □

Remark 5.6. *In view of Propositions 5.9, and the inequality (3.1) we conclude the following inequality*

$$\frac{3}{2}J(P||Q)+\frac{1}{4}\Delta(P||Q)\leqslant\frac{1}{16}\Psi(P||Q)+12h(P||Q). \tag{5.42}$$

Proposition 5.10. *We have*

$$D_{\Psi\Delta}(P||Q)\leqslant\frac{6}{5}D_{\Psi I}(P||Q). \tag{5.43}$$

Proof. Let us consider

$$g_{\Psi\Delta\_\Psi I}(x)=\frac{f''_{\Psi\Delta}(x)}{f''_{\Psi I}(x)}=\frac{x^4+5x^3+2x^2+5x+1}{(x+1)^2(x^2+3x+1)},\ x\in(0,\infty),$$

where $f''_{\Psi\Delta}(x)$ and $f''_{\Psi I}(x)$ are as given by (4.20) and (4.19) respectively.

Calculating the first order derivative of the function $g_{\Psi\Delta\_\Psi I}(x)$ with respect to x, one gets

$$g'_{\Psi\Delta\_\Psi I}(x)=-\frac{4x(x-1)(2x+1)(x+2)}{(x+1)^3(x^2+3x+1)^2}\begin{cases}>0, & x<1\\ <0, & x>1\end{cases}. \tag{5.44}$$

In view of (5.44) we conclude that the function $g_{\Psi\Delta\_\Psi I}(x)$ is increasing in $x\in(0,1)$ and decreasing in $x\in(1,\infty)$, and hence

$$M=\sup_{x\in(0,\infty)} g_{\Psi\Delta\_\Psi I}(x)=g_{\Psi\Delta\_\Psi I}(1)=\frac{6}{5}. \tag{5.45}$$

By the application of (2.3) with (5.45) we get (5.43). □

Remark 5.7. *In view of Propositions 5.10, and the inequality (3.1) we conclude the following inequality*

$$I(P||Q)\leqslant\frac{1}{6}\left[\frac{1}{16}\Psi(P||Q)+\frac{5}{4}\Delta(P||Q)\right]\leqslant\frac{1}{16}\Psi(P||Q). \tag{5.46}$$

Proposition 5.11. *We have*

$$D_{\Psi I}(P||Q) \leqslant \frac{10}{9} D_{\Psi h}(P||Q). \tag{5.47}$$

Proof. Let us consider

$$g_{\Psi I\_\Psi h}(x) = \frac{f''_{\Psi I}(x)}{f''_{\Psi h}(x)} = \frac{(x-1)^2(x^2+3x+1)}{(x+1)\left(x\sqrt{x}-1\right)^2},\ x \neq 1$$
$$= \frac{\left(\sqrt{x}+1\right)^2\left(x^2+3x+1\right)}{(x+1)\left(x+\sqrt{x}+1\right)^2}.$$

for all $x \in (0,\infty)$, where $f''_{\Psi I}(x)$ and $f''_{\Psi h}(x)$ are as given by (4.19) and (4.18) respectively.

Calculating the first order derivative of the function $g_{\Psi I\_\Psi h}(x)$ with respect to x, one gets

$$g'_{\Psi I\_\Psi h}(x) = -\frac{(x-1)\left(3x+\sqrt{x}+3\right)}{\left(x+\sqrt{x}+1\right)^3(x+1)^2} \begin{cases} > 0, & x < 1 \\ > 0, & x > 1 \end{cases}. \tag{5.48}$$

In view of (5.48) we conclude that the function $g_{\Psi I\_\Psi h}(x)$ is increasing in $x \in (0,1)$ and decreasing in $x \in (1,\infty)$, and hence

$$M = \sup_{x\in(0,\infty)} g_{\Psi I\_\Psi h}(x) = g_{\Psi I\_\Psi h}(1) = \frac{10}{9}. \tag{5.49}$$

By the application of (2.3) with (5.49) we get (5.47). □

Remark 5.8. *In view of Propositions 5.11, and the inequality (3.1) we conclude the following inequality*

$$h(P||Q) \leqslant \frac{1}{10}\left[\frac{1}{16}\Psi(P||Q) + 9I(P||Q)\right] \leqslant \frac{1}{16}\Psi(P||Q). \tag{5.50}$$

Proposition 5.12. *We have*

$$D_{\Psi h}(P||Q) \leqslant \frac{9}{8} D_{\Psi J}(P||Q). \tag{5.51}$$

Proof. Let us consider

$$g_{\Psi h\_\Psi J}(x) = \frac{f''_{\Psi h}(x)}{f''_{\Psi J}(x)} = \frac{\left(x+\sqrt{x}+1\right)^2}{(\sqrt{x}+1)^2(x+1)},\ x \in (0,\infty),$$

where $f''_{\Psi h}(x)$ and $f''_{\Psi J}(x)$ are as given by (4.18) and (4.17) respectively.

Calculating the first order derivative of the function $g_{\Psi h\_\Psi J}(x)$ with respect to x, one gets

$$g'_{\Psi h\_\Psi J}(x) = -\frac{(\sqrt{x}-1)\left(x+\sqrt{x}+1\right)}{(\sqrt{x}+1)^3(x+1)^2} \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases} \tag{5.52}$$

In view of (5.52) we conclude that the function $g_{\Psi h\_\Psi J}(x)$ is monotonically increasing in $x \in (0,1)$ and decreasing in $x \in (1,\infty)$, and hence

$$M = \sup_{x\in(0,\infty)} g_{\Psi h\_\Psi J}(x) = g_{\Psi h\_\Psi J}(1) = \frac{9}{8}. \tag{5.53}$$

By the application of (2.3) with (5.53) we get (5.51). □

Remark 5.9. *In view of Propositions 5.12, and the inequalities (3.1) we conclude the following inequality*

$$\frac{1}{8}J(P||Q) \leqslant \frac{1}{9}\left[\frac{1}{16}\Psi(P||Q) + 8h(P||Q)\right] \leqslant \frac{1}{16}\Psi(P||Q). \tag{5.54}$$

Proposition 5.13. *We have*

$$D_{\Psi J}(P||Q) \leqslant \frac{4}{3}D_{\Psi T}(P||Q). \tag{5.55}$$

Proof. Let us consider

$$g_{\Psi J\_\Psi T}(x) = \frac{f''_{\Psi J}(x)}{f''_{\Psi T}(x)} = \frac{(x+1)^2}{x^2+x+1},\ x \in (0,\infty),$$

where $f''_{\Psi J}(x)$ and $f''_{\Psi T}(x)$ are as given by (4.17) and (4.16) respectively.

Calculating the first order derivative of the function $g_{\Psi J\_\Psi T}(x)$ with respect to x, one gets

$$g'_{\Psi J\_\Psi T}(x) = -\frac{(x-1)(x+1)}{(x^2+x+1)^2}\begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases}. \tag{5.56}$$

In view of (5.56) we conclude that the function $g_{\Psi J\_\Psi T}(x)$ is increasing in $x \in (0,1)$ and decreasing in $x \in (1,\infty)$, and hence

$$M = \sup_{x\in(0,\infty)} g_{\Psi J\_\Psi T}(x) = g_{\Psi J\_\Psi T}(1) = \frac{4}{3}. \tag{5.57}$$

By the application of (2.3) with (5.57) we get (5.55). □

Remark 5.10. *In view of Propositions 5.13, and the inequality (3.1) we conclude the following inequality*

$$T(P||Q) \leqslant \frac{1}{32}\left[\frac{1}{2}\Psi(P||Q) + 3J(P||Q)\right] \leqslant \frac{1}{16}\Psi(P||Q). \tag{5.58}$$

Combining (5.31), (5.34), (5.38), (5.43), (5.47), (5.51) and (5.55) we get (5.8). Thus the combination of the Propositions 5.1-5.13 completes the proof of the Theorem 5.1.

6. Final Comments

(i) In view of inequalities (5.15), (5.19), (5.24), (5.31), (5.38) and (5.58), we have the following improvement over the inequality (3.1):

$$\begin{aligned} \frac{1}{4}\Delta(P||Q) &\leqslant I(P||Q) \leqslant \frac{2}{3}h(P||Q) + \frac{1}{12}\Delta(P||Q) \leqslant h(P||Q) \\ &\leqslant \frac{1}{16}J(P||Q) + \frac{1}{2}I(P||Q) \leqslant \frac{1}{3}T(P||Q) + \frac{2}{3}h(P||Q) \\ &\leqslant \frac{1}{8}J(P||Q) \leqslant \frac{2}{3}T(P||Q) + \frac{1}{12}\Delta(P||Q) \leqslant T(P||Q) \\ &\leqslant \frac{1}{32}\left[\frac{1}{2}\Psi(P||Q) + 3J(P||Q)\right] \leqslant \frac{1}{16}\Psi(P||Q). \end{aligned} \tag{6.1}$$

(ii) For simplicity, if we write, the divergence measures given in (4.1)-(4.15) by $D_1 - D_{15}$ respectively, then the Theorem 5.1 resumes in the following inequalities:

(a) $D_{15} \leqslant \frac{2}{3}D_{14} \leqslant 2D_{13} \leqslant D_6$;
(b) $D_{15} \leqslant \frac{2}{3}D_{14} \leqslant \frac{1}{2}D_{12} \leqslant \frac{1}{3}D_9 \leqslant D_6$;
(c) $D_6 \leqslant \frac{2}{3}D_7 \leqslant 2D_{10} \leqslant \frac{1}{6}D_5 \leqslant \frac{1}{5}D_4 \leqslant \frac{2}{9}D_3 \leqslant \frac{1}{4}D_2 \leqslant \frac{1}{3}D_1$.

(iii) Following the similar lines of the propositions given in section 5, we can easily prove the following inequality,

$$D_{\Psi T}(P||Q) \leqslant \frac{1}{64}D^*(P||Q). \tag{6.2}$$

where $D^*(P||Q)$ is as given by (3.24).

The inequality (6.2) together with Theorem 5.1 gives us the following improvement over the inequalities (3.22) and (3.23):

$$D_{J\Delta}(P||Q) \leqslant \frac{1}{2}D_{\Psi J}(P||Q) \leqslant \frac{2}{3}D_{\Psi T}(P||Q) \leqslant \frac{1}{96}D^*(P||Q). \tag{6.3}$$

or equivalently,

$$D_{12} \leqslant \frac{1}{2}D_2 \leqslant \frac{2}{3}D_1 \leqslant \frac{1}{96}D^*.$$

From the inequality (6.3) and item (ii)(b)-(c), we observe that there are many *divergence measures* in between $D_{J\Delta}(P||Q)$ and $D_{\Psi J}(P||Q)$. Thus the inequality (6.3) improves the results due to Dragomir et al. [6].

(iv) The inequalities (5.42) and (5.54) can be written as

$$\begin{aligned} \frac{1}{8}J(P||Q) &\leqslant \frac{1}{12}\left[\frac{1}{16}\Psi(P||Q) + 12h(P||Q) - \frac{1}{4}\Delta(P||Q)\right] \\ &\leqslant \frac{1}{9}\left[\frac{1}{16}\Psi(P||Q) + 8h(P||Q)\right] \leqslant \frac{1}{16}\Psi(P||Q). \end{aligned} \tag{6.4}$$

The middle inequalities of (6.4) follow in view of (5.7) and (5.8).

(v) The inequalities (5.50) and (5.54) can be written as

$$\begin{aligned} h(P||Q) &\leqslant \frac{1}{10}\left[\frac{1}{16}\Psi(P||Q) + 9I(P||Q)\right] \\ &\leqslant \frac{1}{9}\left[\frac{1}{16}\Psi(P||Q) + 8h(P||Q)\right] \leqslant \frac{1}{16}\Psi(P||Q). \end{aligned} \tag{6.5}$$

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