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TORIFICATION AND FACTORIZATION OF BIRATIONAL MAPS

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ABSTRACT. Building on work of the fourth author in [68], we prove the weak factorization conjecture for birational maps in characteristic zero: a birational map between complete nonsingular varieties over an algebraically closed field K of characteristic zero is a composite of blowings up and blowings down with smooth centers.

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CONTENTS

0. INTRODUCTION

We work over an algebraically closed field K of characteristic 0. We denote the multiplicative group of K by K^*.

0.1. **Statement of the main result.** The purpose of this paper is to give a proof for the following weak factorization conjecture of birational maps:

Theorem 0.1.1 (Weak Factorization). [1] *Let $\phi : X_1 \dashrightarrow X_2$ be a birational map between proper nonsingular algebraic varieties X_1 and X_2 over an algebraically closed field K of characteristic zero, and let $U \subset X_1$ be the maximal open set where ϕ is an isomorphism. Then ϕ can be factorized into a sequence of blowings up and blowings down with smooth centers disjoint from U, namely, there exists a sequence of birational maps between proper nonsingular algebraic varieties*

$$X_1 = V_0 \overset{\varphi_1}{\dashrightarrow} V_1 \overset{\varphi_2}{\dashrightarrow} \cdots \overset{\varphi_i}{\dashrightarrow} V_i \overset{\varphi_{i+1}}{\dashrightarrow} V_{i+1} \overset{\varphi_{i+2}}{\dashrightarrow} \cdots \overset{\varphi_{l-1}}{\dashrightarrow} V_{l-1} \overset{\varphi_l}{\dashrightarrow} V_l = X_2$$

where

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[1] Another proof of this theorem was independently given by the fourth author in [69].

1. *$\phi = \varphi_l \circ \varphi_{l-1} \circ \cdots \varphi_2 \circ \varphi_1$,*
2. *φ_i are isomorphisms on U, and*
3. *either $\varphi_i : V_i \dashrightarrow V_{i+1}$ or $\varphi_i^{-1} : V_{i+1} \dashrightarrow V_i$ is a morphism obtained by blowing up a smooth center disjoint from U.*

Furthermore, there is an index i_0 such that for all $i \leq i_0$ the map $V_i \dashrightarrow X_1$ is a projective morphism, and for all $i \geq i_0$ the map $V_i \dashrightarrow X_2$ is a projective morphism. In particular, if X_1 and X_2 are projective then all the V_i are projective.

Remark 0.1.2. In a later version of this paper, we plan to deal with the the following cases where we face some more generalized settings:

(i) the base field K is not algebraically closed;
(ii) $\phi : X_1 \dashrightarrow X_2$ is a birational map between proper algebraic spaces X_1 and X_2;
(iii) $\phi : X_1 \dashrightarrow X_2$ is a bimeromorphic map between compact complex manifolds X_1 and X_2.
(iv) X_1 and X_2 are varieties with a group action and the birational map $X_1 \dashrightarrow X_2$ is equivariant. Can we then take all the intermediate blowings up and down equivariant?

Remark 0.1.3. The assumption on the characteristic of the base field K to be zero is essential in several places of our proof where we use canonical (embedded) resolution of singularities ([23], [66], [7]) and Luna's Étale Slice Theorem.

0.2. **Strong factorization.** If we insist in the assertion above that $\phi_1^{-1}, \ldots, \phi_{i_0}^{-1}$ and $\phi_{i_0+1}, \ldots, \phi_l$ be regular maps for some i_0, we obtain the following strong factorization conjecture.

Conjecture 0.2.1 (Strong Factorization)**.** *Let the situation be as in Theorem 1. Then there exists a diagram*

$$\begin{array}{ccccc} & & Y & & \\ & \psi_1 \swarrow & & \searrow \psi_2 & \\ X_1 & & \stackrel{\phi}{\dashrightarrow} & & X_2 \end{array}$$

where the morphisms ψ_1 and ψ_2 are composites of blowings up of smooth centers disjoint from U.

0.3. **Early origins of the problem.** The history of the factorization problem of birational maps could be traced back to the Italian school of algebraic geometers, who already knew that the operation of blowing up points on surfaces is a fundamental source of richness for surface geometry: the importance of the strong factorization theorem in dimension 2 (see [70]) cannot be overestimated in the analysis of the birational geometry of algebraic surfaces. We can only guess that Zariski, possibly even members of the Italian school, contemplated the problem in higher dimension early on, but refrained from stating it before results on resolution of singularities were available. The question of strong factorization was explicitly stated by Hironaka as "Question (F′)" in [23], Chapter 0, §6, and the question of weak factorization in [49]. The problem remained largely open in higher dimensions despite the efforts and interesting results of many (see e.g. Crauder [11], Kulikov [38], Schaps [59]). These were summarized by Pinkham [52], where the weak factorization conjecture is explicitly stated.

0.4. **The toric case.** For toric birational maps, the equivariant versions of the weak and strong factorization conjectures was posed in [49] and came to be known as Oda's weak and strong conjectures. Not only is the toric version a special case of the general factorization conjectures, most examples demonstrating the difficulties in higher dimensions are also toric (see Hironaka [22], Sally [57], Shannon [61]). Thus it presented a substantial challenge and combinatorial difficulty. In dimension 3, Danilov's proof [16] was later supplemented by Ewald [18]. Oda's weak conjecture was solved in arbitrary dimension by J. Włodarczyk in [67], and Oda's strong conjecture was proven by R. Morelli in [43] (see also [44], [3]). An important combinatorial notion which Morelli introduced into this study is that of a *cobordism* between fans. The algebro-geometric realization of Morelli's combinatorial cobordism is the notion of a *birational cobordism* introduced in [68] .

0.5. **The local version.** There is a local version of the factorization conjecture, formulated and proved in dimension 2 by Abhyankar ([5], Theorem 3). Christensen [9] posed the problem in general and solved it for some special cases in dimension 3. Here the varieties X_1 and X_2 are replaced by appropriate birational local rings dominated by a fixed valuation, and blowings up are replaced by monoidal transforms subordinate to the valuation. This local conjecture was recently solved by S. D. Cutkosky in its strongest form in a series of papers [12, 13, 14]. In a sense, Cutkosky's result says that there are no local obstructions to solving the global strong factorization conjecture.

0.6. **Birational cobordisms.** Our method is based upon the theory of birational cobordisms [68]. As mentioned above, this theory was inspired by the combinatorial notion of polyhedral cobordisms of R. Morelli [43], which was used in his proof of strong factorization for birational toric maps.

Given a birational map $\phi : X_1 \dashrightarrow X_2$, a *birational cobordism* $B_\phi(X_1, X_2)$ is a variety of dimension $\dim(X_1) + 1$ with an action of the multiplicative group K^*. It is analogous to the usual cobordism $B(M_1, M_2)$ between differentiable manifolds M_1 and M_2 given by a Morse function f. In the differential setting one can construct an action of the additive real group $\mathbb{R}$, where the "time" $t \in \mathbb{R}$ acts as a diffeomorphism induced by integrating the vector field $grad(f)$; hence the multiplicative group $(\mathbb{R}_{>0}, \times) = \exp(\mathbb{R}, +)$ acts as well. The critical points of f are precisely the fixed points of the action of the multiplicative group, and the homotopy type of fibers of f changes when we pass through these critical points. Analogously, in the algebraic setting "passing through" the fixed points of the K^*-action induces a birational transformation. Looking at the action on the tangent space at each fixed point, we obtain a locally toric description of the transformation. This already gives the main result of [68] : a factorization of ϕ into *locally toric birational maps* among varieties with locally toric structures. Such birational transformations can also be interpreted using the work of Thaddeus and others ([64, 65, 24, 8]), which describes the change of Geometric Invariant Theory quotients associated to a change of linearization.

0.7. **Locally toric versus toroidal structures.** Considering the fact that strong factorization has been proven for *toroidal* birational maps ([67], [43], [3]), one might naïvely think that this would already provide a proof for Theorem 1.

However, in the locally toric structure obtained from a cobordism, the embedded tori chosen may vary from point to point, while a toroidal structure (see below) requires the embedded tori to be induced from one fixed open set. Thus there is still a gap between the notion of locally toric birational maps and that of toroidal birational maps.

0.8. **Torification.** In order to bridge over this gap, we follow ideas introduced in [1], and blow up the birational cobordism $B_\phi(X_1, X_2)$ along *torific ideals*. This operation induces a toroidal structure in a neighborhood of a connected component F of the fixed point set, on which the action of K^* is a *toroidal action* (we say that the blowup *torifies* the action of K^*). Now the birational transformation passing through F is toroidal, and hence we can apply the factorization theorem for toroidal birational maps. As these toroidal structures may have singularities, we use the canonical resolution of singularities at the final stage to complete the proof of Theorem 1.

At the moment, we can only analyze the effect of blowing up the torific ideals in a neighborhood of one connected component of the fixed point set at a time and fails to carry it out simultaneously for the entire fixed point set in the birational cobordism $B_\phi(X_1, X_2)$. This remains as an obstruction for us to obtain a proof for the strong factorization conjecture. We hope to remove this obstruction in the near future.

0.9. **Relation with the minimal model program.** It is worthwhile to note the relation of the factorization problem to the development of the Mori program. Hironaka [21] used the cone of effective curves to study the properties of birational morphisms. This direction was further developed and given a decisive impact by Mori [45], who, motivated by Kleiman's criterion for ampleness [36], introduced the notion of extremal rays and systematically used it in an attempt to construct minimal models in higher dimension, called *the minimal model program*. Danilov [16] introduced the notion of *canonical and terminal singularities* in conjunction with the toric factorization problem. This was developed by Reid into a general theory of these singularities [53, 54], which appear in an essential way in the minimal model program. The minimal model program is so far proven up to dimension 3 ([46], see also [29, 30, 31, 37, 62]), and for toric varieties in arbitrary dimension (See [55]). In the steps of the minimal model program one is only allowed to contract a divisor into a variety with terminal singularities, or to perform a flip, modifying some codimension 2 loci. This allows a factorization of a given birational morphism into such "elementary operations". An algorithm to factorize birational maps among uniruled varieties, known as *Sarkisov's program,* has been developed and carried out in dimension 3 (see [58, 56, 10], and see [39] for the toric case). Still, we do not know of a way to solve the classical factorization problem using such a factorization.

0.10. **Relation with the toroidalization problem.** In [2], Theorem 2.1, it is proven that given a morphism of projective varieties $X \to B$, there are modifications $m_X : X' \to X$ and $m_B : B' \to B$, with a lifting $X' \to B'$ which has a toroidal structure. The *toroidalization problem* (see [2], [3], [34]) is that of obtaining such m_X and m_B which are composites of blowings up with smooth centers (maybe even with centers supported only over the locus where $X \to B$ is not toroidal).

The proof in [2] relies on the work of De Jong [27] and methods of [1]. The authors of the present paper have tried to use these methods to approach the factorization conjectures, so far without success; one notion we do use in this paper is the torific ideal of [1]. It would be interesting if one can turn this approach on its head and prove a result on toroidalization using factorization.

0.11. **Relation with the proof in** [69]**.** Another proof of the weak factorization theorem was given independently by the fourth author in [69]. The main difference of the two approaches is that in the current paper we are using objects such as torifying ideals defined locally on each quasielementary piece of a cobordism. The blowup of a torifying ideal gives the quasielementary cobordism a toroidal structure. These toroidal modifications are then pieced together using canonical resolution of singularities. In [69] the objects of study, such as painted semifans, are defined globally on a cobordism. The structure these semifans define on the cobordism is something between our notion of locally toric and toroidal embedding.

0.12. **Outline of the paper.** In section 1 we discuss locally toric and toroidal structures, and reduce the proof of Theorem 1 to the case where ϕ is a projective morphism.

Suppose now we have a projective morphism $\phi : X_1 \to X_2$. In section 2 we apply the theory of birational cobordisms to obtain a factorization into locally toric maps. In case X_1 and X_2 are projective, we use a geometric invariant theory analysis, inspired by Thaddeus's work, which shows that the intermediate varieties can be chosen to be projective. In all cases our cobordism B is relatively projective over X_2.

In section 3 we construct, for each connected component of the fixed-point set $F \subset B$, an ideal sheaf I_F whose blowing up *torifies* the action of K^* on B in a neighborhood of F. In other words, there is a neighborhood B_F of F such that K^* acts toroidally on the variety obtained by blowing up B_F along I_F.

In section 4 we prove the weak factorization theorem by piecing together the toroidal birational transforms induced by the neighborhoods B_F of a cobordism B near fixed point components F as in section 3. This is done using several applications of canonical resolution of singularities, Luna's Fundamental Lemma and Étale Slice Theorem.

In section 5 we discuss some problems related to strong factorization.

1. Preliminaries

Definition 1.0.1. Suppose a reductive group G acts on an algebraic variety X. We denote by X/G the geometric quotient representing the space of orbits, and by $X//G$ the space of equivalence classes of orbits, where two orbits are equivalent if their closures intersect.

1.1. **Toric varieties.** Let $N \cong \mathbb{Z}^n$ be a lattice and $\sigma \subset N_{\mathbb{R}}$ a strictly convex rational polyhedral cone. We denote the dual lattice by M and the dual cone by $\sigma^\vee \subset M_{\mathbb{R}}$. The *affine toric variety* $X = X(N, \sigma)$ is

$$X = \operatorname{Spec} K[M \cap \sigma^\vee].$$

For an $m \in M \cap \sigma^\vee$ we denote its image in the semigroup algebra $K[M \cap \sigma^\vee]$ by z^m.

More generally, the toric variety associated to a fan Σ in $N_{\mathbb{R}}$ is denoted by $X(N, \Sigma)$.

If $X_1 = X(N, \Sigma_1)$ and $X_2 = X(N, \Sigma_2)$ are two toric varieties, the embeddings of the torus $T = \operatorname{Spec} K[M]$ in both of them defines a birational toric map $X_1 \dashrightarrow X_2$. This map is a morphism if and only if every cone in Δ_1 is contained in a cone of Δ_2

Suppose that K^* acts on an affine toric variety $X = X(N, \sigma)$ as a 1-parameter subgroup of the torus T, corresponding to a lattice point $a \in N$. If $t \in K^*$ and $m \in M$, the action on the monomial z^m is given by

$$t^*(z^m) = t^{(a,m)} \cdot z^m,$$

where $(\cdot, \cdot)$ is the natural pairing on $N \times M$. The K^*-invariant monomials correspond to lattice points $M \cap a^\perp$, hence

$$X//K^* \cong \operatorname{Spec} K[M \cap \sigma^\vee \cap a^\perp],$$

which is again an affine toric variety defined by the lattice $\pi(N)$ and cone $\pi(\sigma)$ where $\pi : N_{\mathbb{R}} \to N_{\mathbb{R}}/\mathbb{R}\cdot a$ is the projection. This quotient is a geometric quotient precisely when $\pi : \sigma \to \pi(\sigma)$ is an isomorphism.

1.2. **Locally toric and toroidal structures.** There is some confusion in the literature between the notion of *toroidal embeddings* and toroidal maps ([33], [2]) and that of *toroidal varieties* (see [15]), which we prefer to call *locally toric varieties* (and morphisms). A crucial issue in this paper is the distinction between the two notions.

Definition 1.2.1.
1. A variety W is *locally toric* if for every closed point $p \in W$ there exists an étale neighborhood $\iota_p : V_p \to W$ of p and an étale morphism $\eta_p : V_p \to X_p$ to a toric variety X_p. We call V_p (together with the maps ι_p, η_p) a toric chart at p.
2. An open embedding $U \subset W$ is *toroidal* if W is locally toric and for every $p \in W$ there exists a toric chart V_p satisfying $\iota_p^{-1}(U) = \eta_p^{-1}(T)$, where $T \subset X_p$ is the torus. We call such charts toroidal.

Definition 1.2.2 ([23],[26])**.** Let $\psi : W_1 \dashrightarrow W_2$ be a rational map defined on a dense open subset U. Denote by Γ_ψ the closure of the graph of ψ_U in $W_1 \times W_2$. We say that ψ is *proper* if the projections $\Gamma_\psi \to W_1$ and $\Gamma_\psi \to W_2$ are both proper.

Definition 1.2.3.
1. A proper birational map $\psi : W_1 \dashrightarrow W_2$ between two locally toric varieties W_1 and W_2 is said to be *locally toric* if for any point $(p, q) \in \Gamma_\psi$ we can find toric charts V_p at p and V_q at q, a lifting of ψ to $\varphi : V_p \dashrightarrow V_q$ and a birational toric map $\nu : X_p \dashrightarrow X_q$ such that the following diagram commutes
$$\begin{array}{ccccc} X_p & \stackrel{\eta_p}{\leftarrow} & V_p & \stackrel{\iota_p}{\rightarrow} & W_1 \\ \nu \downarrow & & \varphi \downarrow & & \downarrow \psi \\ X_q & \stackrel{\eta_q}{\leftarrow} & V_q & \stackrel{\iota_q}{\rightarrow} & W_2 \end{array}$$
2. If, moreover, $U_1 \subset W_1$ and $U_2 \subset W_2$ are toroidal embeddings, the birational map $\psi : W_1 \dashrightarrow W_2$ is said to be *toroidal* if the charts V_p and V_q above can be chosen toroidal.

Remark 1.2.4. Note that a toroidal birational map $(U_1 \subset W_1) \dashrightarrow (U_2 \subset W_2)$ induces an isomorphism between the open sets U_1 and U_2.

Definition 1.2.5. Let W be a locally toric (resp. toroidal) variety with a K^* action. We say that the action is locally toric (resp. toroidal) if at any $p \in W$ there exists a toric (resp. toroidal) chart V_p with a K^* action on V_p such that

(i) the étale morphisms ι_p and η_p are K^*-equivariant, where K^* acts on the toric variety X_p as a subgroup of the torus $T \subset X_p$;
(ii) let O_p be the orbit of p, let $O' = \iota_p^{-1}(O_p)$, and $O'' = \eta_p(O')$. Then O', O'' are closed, and ι_p, η_p are injective when restricted to O'.

Definition 1.2.6. Let W_1 and W_2 be two locally toric (resp. toroidal) varieties with locally toric (resp. toroidal) K^*-action, and let $\psi : W_1 \dashrightarrow W_2$ be a proper birational map. Then ψ is said to be *locally toric K^*-equivariant* (resp. *toroidal K^*-equivariant*) if we can find toric (resp. toroidal) charts V_p and V_q for any $(p, q) \in \Gamma_\psi$ defining locally toric (resp.toroidal) K^* action at p and q as in Definition 1.2.5 and a locally toric (resp. toroidal) structure on the map ψ as in Definition 1.2.3.

Definition 1.2.7 (cf. [48], p. 198)**.** Let V and X be affine varieties with K^*-actions, and let $\eta : V \to X$ be a K^*-equivariant étale morphism. Then η is said to be *strongly étale* if

(i) the quotient map $V//K^* \to X//K^*$ is étale, and
(ii) the natural map
$$V \to X \times_{X//K^*} V//K^*$$
is an isomorphism.

It follows from Luna's Fundamental Lemma that for a toric chart defining the structure of a locally toric K^*-action in Definition 1.2.5 the maps ι_p and η_p are strongly étale. Hence the locally toric structure descends to the quotient.

Lemma 1.2.8. *Let a locally toric K^*-action on a locally toric variety W be given, and assume that W/K^* exists. Then W/K^* is again locally toric.*

If, moreover, $U_W \subset W$ is a toroidal embedding and the action is toroidal, then $U_W/K^* \subset W/K^*$ is a toroidal embedding.

Proof. Let V_p be an affine toric chart at $p \in W$, where ι_p and η_p are K^*-equivariant strongly étale morphisms. Then
$$\iota_p/K^* : V_p/K^* \to W/K^*,$$
$$\eta_p/K^* : V_p/K^* \to X_p/K^*$$
are again étale, and X_p/K^* is an affine toric variety. This gives us a toric chart at the image of p in W/K^*.

If $U_W \subset W$ is a toroidal embedding then U_W, coresponding to the torus in a chart, is clearly K^*-invariant. Now if V_p as above is a toroidal chart at $p \in W$ then $\eta_p^{-1}(T) = \iota_p^{-1}(U_W)$ implies that
$$(\eta_p/K^*)^{-1}(T/K^*) = (\iota_p/K^*)^{-1}(U_W/K^*).$$
Since T/K^* is the torus in the toric variety X_p/K^*, this toric chart is actually toroidal. □

Lemma 1.2.9. *Let $\psi : W_1 \dashrightarrow W_2$ be a locally toric K^* equivariant (resp. toroidal K^* equivariant) birational map. Assume that W_1/K^* and W_2/K^* exist. Then the birational map $\psi/K^* : W_1/K^* \dashrightarrow W_2/K^*$ is locally toric (resp. toroidal).*

Proof. Let ψ be given in toric charts (which also define locally toric K^*action on either variety) at $p \in W_1$ and $q \in W_2$ by the commutative diagram
$$\begin{array}{ccccc} X_p & \overset{\eta_p}{\leftarrow} & V_p & \overset{\iota_p}{\rightarrow} & W_1 \\ \nu \downarrow & & \varphi \downarrow & & \downarrow \psi \\ X_q & \overset{\eta_q}{\leftarrow} & V_q & \overset{\iota_q}{\rightarrow} & W_2 \end{array}$$
Since all these maps are K^*-equivariant and the horizontal morphisms are strongly étale, we get a commutative diagram of quotients with horizontal morphisms étale
$$\begin{array}{ccccc} X_p/K^* & \overset{\eta_p/K^*}{\leftarrow} & V_p/K^* & \overset{\iota_p/K^*}{\rightarrow} & W_1/K^* \\ \downarrow & & \downarrow & & \downarrow \\ X_q/K^* & \overset{\eta_q/K^*}{\leftarrow} & V_q/K^* & \overset{\iota_q/K^*}{\rightarrow} & W_2/K^* \end{array}$$
providing a locally toric structure on the map $\psi/K^* : W_1/K^* \dashrightarrow W_2/K^*$.

If everything is toroidal, we get a corresponding fiber square of open subsets

$$\begin{array}{ccccc} T/K^* & \stackrel{\eta_p/K^*}{\longleftarrow} & U_{V_p}/K^* & \stackrel{\iota_p/K^*}{\longrightarrow} & U_{W_1}/K^* \\ \downarrow & & \downarrow & & \downarrow \\ T/K^* & \stackrel{\eta_q/K^*}{\longleftarrow} & U_{V_q}/K^* & \stackrel{\iota_q/K^*}{\longrightarrow} & U_{W_2}/K^* \end{array}$$

hence the locally toric structure on the quotient morphism is in fact toroidal. □

Remark 1.2.10. Note that the previous lemmas remain true when we replace the geometric quotients / by // and require that all charts in the proofs satisfy $\eta_p(V_p)//K^* \to W//K^*$ and $\iota_p(V_p)//K^* \to X_p//K^*$ are inclusions. This condition is satisfied, for example, if all charts can be chosen such that if an orbit in W lies in the image of V_p then its closure in W also lies in the image, and the same for X_p.

Lemma 1.2.11. *Let W be a nonsingular variety with a K^*-action. Then W has a locally toric structure such that the action is locally toric.*

Proof. Let $p \in W$ be a closed point, G_p the stabilizer of p in K^*, and N_p the normal vector space to the orbit of p at p. We choose an affine open K^*-invariant neighborhood U of p such that the orbit of p is closed in U [63].

By Luna's Étale Slice Theorem there exists a nonsingular locally closed G_p-invariant subvariety $S \subset W$ with $p \in S$ such that, shrinking U if necessary, we have a diagram of fiber squares

$$\begin{array}{ccccccc} K^* \times_{G_p} N_p & \stackrel{\eta_p}{\longleftarrow} & K^* \times_{G_p} S & \stackrel{\iota_p}{\twoheadrightarrow} & U & \subset & W \\ \downarrow & & \downarrow & & \downarrow & & \\ N_p/G_p & \longleftarrow & S/G_p & \twoheadrightarrow & U/K^* & & \end{array}$$

where K^* acts on the first factors of $K^* \times_{G_p} N_p$ and $K^* \times_{G_p} S$, the upper horizontal maps are strongly étale, hence the lower horizontal maps are étale as well. Now to prove the first claim, it suffices to notice that $K^* \times_{G_p} N_p$ is a toric variety on which K^* acts as a subgoup of the torus. □

We call the charts constructed in the lemma *Luna's toric charts.*

Remark 1.2.12. If $p \in W$ is a fixed point then $S = K^* \times_{G_p} S$ is a K^*-invariant open neighborhood of p and ι_p is an embedding. If the variety W is proper then every point has a limit fixed point, and it follows that such Zariski local toric charts cover W. We will not use this fact.

Remark 1.2.13. We remark that if $\eta : V \to X$ is an étale morphism and I a sheaf of ideals on X then we get an étale morphism from the blowup of V along $\eta^{-1}(I)$ to the blowup of X along I. If the morphism ϕ is strongly étale and the ideal I is K^*-equivariant, the morphism of the blowups is also strongly étale.

1.3. **Relation with toroidal embeddings of** [33]**.** The results in [33] on toroidal embeddings assume that the embedding is without self-intersection. This means that the irreducible components of the toroidal divisor $W \setminus U_W$ are normal. If the divisors are not normal, we can take a sequence of blowups with smooth centers which preserves the toroidal structure to make the complement of U_W a divisor with simple normal crossings.

It is shown in [33] that proper birational *allowable* morphisms to a given toroidal embedding X are in one-to-one correspondence with subdivisions of the associated polyhedral complex Δ_X. A toroidal morphism $W_1 \to W_2$ is said to be allowable if the charts defining the toroidal structure on the morphism as in Definition 1.2.3 can be chosen so that $\iota_p(V_p) = \psi^{-1}\iota_q(V_q)$. It follows from the results in [2] that every proper birational toroidal morphism $Y \to X$ is allowable: first, the toroidal morphism induces a morphism of polyhedral complexes $\Delta_Y \to \Delta_X$ which is automatically a subdivision; if $Y' \to X$ is the allowable morphism corresponding to this subdivision, the morphism $Y \to X$ factors through $Y \to Y'$; now the latter is an equidimensional proper birational morphism between normal varieties, hence it is an isomorphism.

1.4. **Strong factorization for toroidal birational maps.** Morelli's proof of the strong factorization theorem for birational toric maps can be extended to the case of birational toroidal maps. This is proved in [3] for toroidal morphisms. The general case of a proper birational toroidal map $W_1 \dashrightarrow W_2$ follows from this when we take the normalization of the graph of the birational map in the product of the two toroidal embeddings. Then it naturally inherits a toroidal structure, it dominates the original two toroidal embeddings by toroidal birational morphisms, and admits a toroidal resolution of singularities. Now we apply [3] to the first projection to get $W \to W_1$ a sequence of smooth toroidal blowings up and $W \to W_2$ a toroidal morphism. Another application of [3] to the toroidal morphism $W \to W_2$ finishes the proof.

We remark that Morelli's theorem for toric birational maps works over any field, and its extension to the toroidal case in [3] works over any perfect field.

1.5. **Canonical resolution of singularities and canonical principalization.** In the following, we will use canonical versions of Hironaka's theorems on resolution of singularities and principalization of an ideal, proved in [7].

A canonical (embedded) resolution of singularities $\widetilde{W} \to W$ consists of a composite of blowings up with smooth centers, and it is canonical in the sense that if $\theta : W \to W$ is a (local) automorphism then the ideals blown up are invariant under pullback by θ and hence θ can be lifted to an automorphism $\widetilde{\theta} : \widetilde{W} \to \widetilde{W}$. Since the ideals blown up in this process depend only on the formal germs of W, the canonical resolution behaves well with respect to étale morphisms: if $V \to W$ is étale, we get an étale morphism of canonical resolutions $\widetilde{V} \to \widetilde{W}$.

By *canonical principalization of an ideal sheaf* in a nonsingular variety we mean the canonical embedded resolution of singularities of the subscheme defined by the ideal sheaf; i.e., a composite of blowings up with smooth centers such that the total transform of the ideal is a strict divisor of normal crossings. Embedded resolution of singularities of a non-reduced subscheme is discussed in Section 11 of [7].

1.6. **Reduction to projective morphisms.** We start with a birational map

$$\phi : X_1 \dashrightarrow X_2$$

between proper nonsingular algebraic varieties X_1 and X_2 defined over K and restricting to an isomorphism on an open set U.

Lemma 1.6.1. *There is a commutative diagram*

$$\begin{array}{ccc} X_1' & \stackrel{\phi'}{\to} & X_2' \\ g_1 \downarrow & & \downarrow g_2 \\ X_1 & \stackrel{\phi}{\dashrightarrow} & X_2 \end{array}$$

such that g_1 and g_2 are composites of blowings up with smooth centers disjoint from U, and ϕ' is a projective birational morphism.

Proof. By Hironaka's theorem on elimination of indeterminacies, there is a morphism $g_2 : X_2' \to X_2$ which is a composite of blowings up with smooth centers disjoint from U, such that the birational map $h := \phi^{-1} \circ g_2 : X_2' \to X_1$ is a morphism:

$$\begin{array}{ccc} & & X_2' \\ & h \swarrow & \downarrow g_2 \\ X_1 & \stackrel{\phi}{\dashrightarrow} & X_2 \end{array}.$$

By the same theorem, there is a morphism $g_1 : X_1' \to X_1$ which is a composite of blowings up with smooth centers disjoint from U, such that $\phi' := h^{-1} \circ g_1 : X_1' \to X_2'$ is a morphism. Since the composite $h \circ \phi' = g_1$ is projective, it follows that ϕ' is projective. □

Thus we may replace $X_1 \dashrightarrow X_2$ by $X_1' \to X_2'$ and assume from now on that ϕ is a projective morphism.

2. Birational Cobordisms

2.1. Definitions.

Definition 2.1.1 ([68])**.** Let $\phi : X_1 \dashrightarrow X_2$ be a birational map between two algebraic varieties X_1 and X_2 over K, isomorphic on an open set U. An algebraic variety B is called a *birational cobordism* for ϕ and denoted by $B_\phi(X_1, X_2)$ if it satisfies the following conditions.

1. The multiplicative group K^* acts effectively on $B = B_\phi(X_1, X_2)$.
2. The sets
$$\begin{array}{rcl} B_- & := & \{x \in B : \ \lim_{t\to 0}\ t(x) \text{ does not exist in } B\} \\ \text{and } B_+ & := & \{x \in B : \ \lim_{t\to \infty} t(x) \text{ does not exist in } B\} \end{array}$$
are nonempty Zariski open subsets of B.
3. There are isomorphisms
$$B_-/K^* \xrightarrow{\sim} X_1 \quad \text{and} \quad B_+/K^* \xrightarrow{\sim} X_2.$$
4. Considering the rational map $\psi : B_- \dashrightarrow B_+$ induced by the inclusions $B_- \cap B_+ \subset B_-$ and $B_- \cap B_+ \subset B_+$, the following diagram commutes:
$$\begin{array}{ccc} B_- & \stackrel{\psi}{\dashrightarrow} & B_+ \\ \downarrow & & \downarrow \\ X_1 & \stackrel{\phi}{\dashrightarrow} & X_2 \end{array}.$$

We say that B *respects the open set* U if U is contained in the image of $B_- \cap B_+/K^*$.

Definition 2.1.2 ([68])**.** Let $B = B_\phi(X_1, X_2)$ be a birational cobordism, and let $F \subset B^{K^*}$ be a connected component of the fixed-point set. We define

$$\begin{array}{rcl} F^+ & = & \{x \in B |\ \lim_{t\to 0}\ t(x) \in F\} \\ F^- & = & \{x \in B |\ \lim_{t\to \infty} t(x) \in F\} \\ F^\pm & = & F^+ \cup F^- \\ F^* & = & F^\pm \smallsetminus F \end{array}$$

Definition 2.1.3 ([68])**.** A *nonsingular* birational cobordism $B = B_\phi(X_1, X_2)$ is said to be *elementary* if the fixed-point set B^{K^*} is connected.

Definition 2.1.4. A nonsingular birational cobordism $B = B_\phi(X_1, X_2)$ is said to be *quasi-elementary* if for any two connected components $F_1 \neq F_2 \subset B^{K^*}$ we have $F_1^{\pm} \cap F_2^{\pm} = \emptyset$.

Definition 2.1.5 ([68]). Let $B = B_\phi(X_1, X_2)$ be a birational cobordism. We define a relation $\prec$ among connected components of B^{K^*} as follows: let $F_1 \neq F_2 \subset B^{K^*}$ be two connected components, and set $F_1 \prec F_2$ if $F_1^+ \cap F_2^- \neq \emptyset$.

Definition 2.1.6 (cf. [43],[68]). We say that a birational cobordism $B = B_\phi(X_1, X_2)$ is *collapsible* if the relation $\prec$ is a (strict) pre-order, namely, it can be extended to a strict order.

2.2. **The main example.** We now recall a fundamental example of an elementary birational cobordism in the toric setting, discussed in [68]:

Example 2.2.1. Let $B = \mathbb{A}^n = \operatorname{Spec} K[z_1, \cdots, z_n]$ and let $t \in K^*$ act by

$$t(z_1, \ldots, z_i, \ldots, z_n) = (t^{\alpha_1} z_1, \ldots, t^{\alpha_i} z_i, \ldots, t^{\alpha_n} z_n).$$

We regard $\mathbb{A}^n$ as a toric variety defined by a lattice $N \cong \mathbb{Z}^n$ and a regular cone $\sigma \in N_\mathbb{R}$ generated by the standard basis

$$\sigma = \langle v_1, \ldots, v_n \rangle.$$

The dual cone $\sigma^\vee$ is generated by the dual basis $v_1^*, \ldots, v_n^*$, and we identify $z^{v_i^*} = z_i$. The K^*-action then corresponds to a one-parameter subgroup

$$a = (\alpha_1, \ldots, \alpha_n) \in N.$$

We have the obvious description of the sets B_+ and B_-:

$$\begin{aligned} B_+ &= \{(z_1, \cdots, z_n); z_i \neq 0 \text{ for some } i \text{ with } \alpha_i = (v_i^*, a) > 0\}, \\ B_- &= \{(z_1, \cdots, z_n); z_i \neq 0 \text{ for some } i \text{ with } \alpha_i = (v_i^*, a) < 0\}. \end{aligned}$$

We define the upper boundary and lower boundary of σ to be

$$\begin{aligned} \partial_+\sigma &= \{x \in \sigma; x + \epsilon \cdot (-a) \notin \sigma \text{ for } \epsilon > 0\}, \\ \partial_-\sigma &= \{x \in \sigma; x + \epsilon \cdot a \notin \sigma \text{ for } \epsilon > 0\}). \end{aligned}$$

Then we obtain the description of B_+ and B_- as the toric varieties corresponding to the fans $\partial_+\sigma$ and $\partial_-\sigma$ in $N_\mathbb{R}$.

Let $\pi : N_\mathbb{R} \longrightarrow N_\mathbb{R}/\mathbb{R} \cdot a$ be the projection. Then $B//K^*$ is again an affine toric variety defined by the lattice $\pi(N)$ and cone $\pi(\sigma)$. Similarly, one can check that the geometric quotients B_-/K^* and B_+/K^* are toric varieties defined by fans $\pi(\partial_+\sigma)$ and $\pi(\partial_-\sigma)$. Since both $\pi(\partial_+\sigma)$ and $\pi(\partial_-\sigma)$ are subdivisions of $\pi(\sigma)$, we get a diagram of birational toric maps

$$\begin{array}{ccc} B_-/K^* & \overset{\varphi}{\dashrightarrow} & B_+/K^* \\ & \searrow \quad \swarrow & \\ & B//K^* & \end{array}$$

More generally, one can prove that if Σ is a fan in $N_\mathbb{R}$ with lower boundary $\partial_-\Sigma$ and upper boundary $\partial_+\Sigma$ relative to an element $a \in N$, then the toric variety corresponding to Σ, with the K^*-action given by the one-parameter subgroup $a \in N$, is a birational cobordism between the two toric varieties corresponding to $\pi(\partial_-\Sigma)$ and $\pi(\partial_+\Sigma)$ as fans in $N_\mathbb{R}/\mathbb{R} \cdot a$.

For the details, we refer the reader to [43], [68] and [3]. □

2.3. **Construction of a cobordism.** It was shown in [68] that cobordisms exist for any birational map $X_1 \dashrightarrow X_2$. We give a very special case of this construction.

Theorem 2.3.1. *Let $\phi : X_1 \to X_2$ be a projective birational morphism between proper nonsingular algebraic varieties, which is an isomorphism on an open set U. Then there is a proper nonsingular algebraic variety $\overline{B}$ with a K^* action, satisfying the following properties:*

1. *There exist closed embeddings $\iota_1 : X_1 \hookrightarrow \overline{B}^{K^*}$ and $\iota_2 : X_2 \hookrightarrow \overline{B}^{K^*}$ with disjoint images.*
2. *The open subvariety $B = \overline{B} \setminus (\iota_1(X_1) \cup \iota_2(X_2))$ is a birational cobordism between X_1 and X_2 respecting the open set U.*
3. *There is a vector bundle $E \to X_2$ with a K^* action, and a closed K^*-equivariant embedding $\overline{B} \subset \mathbb{P}(E)$.*

Proof. Let $J \subset \mathcal{O}_{X_2}$ be an ideal sheaf which does not vanish on U and such that $\phi : X_1 \to X_2$ is the blowup morphism of X_2 along J. Let I_0 be the ideal of the point $0 \in \mathbb{P}^1$. Consider $W_0 = X_2 \times \mathbb{P}^1$ and let $p : W_0 \to X_2$ and $q : W_0 \to \mathbb{P}^1$ be the projections. Let $I = (p^{-1}J + q^{-1}I_0)\mathcal{O}_{W_0}$. Let W be the blowup of W_0 along I.

We claim that X_1 and X_2 lie in the nonsingular locus of W. For $X_2 \cong X_2 \times \{\infty\} \subset X_2 \times \mathbb{P}^1$ this is clear. Since X_1 is nonsingular, embedded in W as the strict transform of $X_2 \times \{0\} \subset X_2 \times \mathbb{P}^1$, to prove that X_1 lies in the nonsingular locus, it suffices to prove that X is a Cartier divisor in W. We look at local coordinates. Let $A = \Gamma(V, \mathcal{O}_V)$ for some affine open subset $V \subset X_2$, and let $y_1, \ldots, y_m$ be a set of generators of J on V. Then on the affine open subset $V \times \mathbb{A}^1 \subset X_2 \times \mathbb{P}^1$ with coordinate ring $A[x]$, the ideal I is generated by $y_1, \ldots, y_m, x$. The charts of the blowup containing the strict transform of $\{x = 0\}$ are of the form

$$\operatorname{Spec} A[x]\left[\frac{y_1}{y_i}, \ldots, \frac{y_m}{y_i}, \frac{x}{y_i}\right] = \operatorname{Spec} A\left[\frac{y_1}{y_i}, \ldots, \frac{y_m}{y_i}, \frac{x}{y_i}\right].$$

The strict transform of $\{x = 0\}$ is defined by $\frac{x}{y_i}$, hence it is Cartier.

Let $\overline{B} \to W$ be a canonical resolution of singularities. Then conditions 1 and 2 are clearly satisfied. Condition 3 follows from the fact that the projection $X_2 \times \mathbb{P}^1 \to X_2$ composed with the blowup of the ideal I and canonical resolution of singularities is a K^*-equivariant projective morphism. □

We refer the reader to [68] for more details.

We call a variety $\overline{B}$ as in the theorem a *compactified, relatively projective cobordism.*

2.4. **Collapsibility and Projectivity.** Let $B = B_\phi(X_1, X_2)$ be a birational cobordism. We seek a criterion for collapsibility of B.

Let $\mathcal{C}$ be the set of connected components of $B_\phi(X_1, X_2)^{K^*}$, and let $\chi : \mathcal{C} \to \mathbb{Z}$ be a function. We say that χ is strictly increasing if $F \prec F' \Rightarrow \chi(F) < \chi(F')$. The following lemma is obvious:

Lemma 2.4.1. *Assume there exists a strictly increasing function χ. Then $\prec$ is a strict pre-order, and B is collapsible. Conversely, suppose B is collapsible. Then there exists a strictly increasing function χ.* □

Remark 2.4.2. It is evident that every strictly increasing function can be replaced by one which induces a strict total order. However, it will be convenient for us to consider arbitrary strictly increasing functions.

Let χ be a strictly increasing function, and let $a_0 < a_1 \cdots < a_k \in \mathbb{Z}$ be the values of χ.

Definition 2.4.3. We denote

1. $F_{a_i} = \cup\{F|\chi(F) = a_i\}$.
2. $F^+_{a_i} = \cup\{F^+|\chi(F) = a_i\}$.
3. $F^-_{a_i} = \cup\{F^-|\chi(F) = a_i\}$.
4. $F^\pm_{a_i} = \cup\{F^\pm|\chi(F) = a_i\}$.
5. $F^*_{a_i} = \cup\{F^*|\chi(F) = a_i\}$.
6. $B_{a_i} = B \smallsetminus (\bigcup\{F^-|\chi(F) < a_i\} \quad \cup \quad \bigcup\{F^+|\chi(F) > a_i\})$.

The following is an immediate extension of Proposition 1 of [68] .

Proposition 2.4.4. 1. *B_{a_i} is a quasi-elementary cobordism.*
2. *For $i = 0, \ldots, k-1$ we have $(B_{a_i})_+ = (B_{a_{i+1}})_-$.*

The following is an analogue of Lemma 1 of [68] in the case of the cobordisms we have constructed.

Proposition 2.4.5. *Let $E \to X_2$ be a locally free sheaf with a K^* action, and let $\overline{B} \subset \mathbb{P}(E)$ be a compactified, relatively projective cobordism embedded K^*-equivariantly. Then there exists a strictly increasing function χ for the cobordism $B = \overline{B} \smallsetminus (X_1 \cup X_2)$. In particular, the cobordism is collapsible.*

Proof. Since K^* acts trivially on X_2, there exists a direct sum decomposition

$$E = \bigoplus_{b \in \mathbb{Z}} E_b$$

where E_b is the subbundle on which the action of K^* is given by the character $t \mapsto t^b$. (Proof: If $K^d \times V \to V$ is a trivialization of E over an invariant affine open subset $V \subset X_2$ then the action of K^* on E in this trivialization is given by a group homomorphism $K^* \to \Gamma(V, GL_d(K))$. Since K^* is reductive, we can diagonalize all these matrices simultaneously.) Denote by $b_0, \ldots, b_k$ the characters which figure in this representation, and let $d_i = \operatorname{rank} E_i$. Note that there are disjoint embeddings $\mathbb{P}(E_{b_i}) \subset \mathbb{P}(E)$.

Let $p \in B$ be a fixed point lying in the fiber $\mathbb{P}(E_q)$ over $q \in X_2$. We choose a basis $(x_{b_0,1}, \ldots, x_{b_0,d_0}, \ldots x_{b_k,1}, \ldots, x_{b_k,d_k})$ of E_q where $x_{b_i,j} \in E_{b_i}$ and use the following lemma:

Lemma 2.4.6. *Suppose $p \in \mathbb{P}(E_q)^{K^*}$ is a fixed point with homogeneous coordinates*

$$(p_{b_0,1}, \ldots, p_{b_0,d_0}, \ldots, p_{b_k,1}, \ldots, p_{b_k,d_k}).$$

Then there is an i_p such that $p_{b_i,j} = 0$ whenever $i \neq i_p$. In particular, $p \in \mathbb{P}(E_{b_{i_p}}) \subset \mathbb{P}(E)$. □

If $F \subset B^{K^*}$ is a connected component of the fixed point set, then it follows from the lemma that $F \subset \mathbb{P}(E_{b_i})$ for some i. We define

$$\chi(F) = b_i.$$

To check that χ is strictly increasing, consider a point $p \in B$ such that $\lim_{t\to 0} t \cdot p \in F_1$ and $\lim_{t\to\infty} t \cdot p \in F_2$ for some fixed point components F_1 and F_2. Let the coordinates of p in the fiber over $q \in X_2$ be $(p_{b_0,1}, \ldots, p_{b_0,d_0}, \ldots, p_{b_k,1}, \ldots, p_{b_k,d_k})$. Now

$$\lim_{t\to 0} t \cdot p \in \mathbb{P}(E_{b_{min}}),$$

$$\lim_{t\to\infty} t \cdot p \in \mathbb{P}(E_{b_{max}}),$$

where

$$b_{min} = \min\{b_i : p_{b_i,j} \neq 0 \text{ for some } j\},$$
$$b_{max} = \max\{b_i : p_{b_i,j} \neq 0 \text{ for some } j\}.$$

Thus, if F_1 and F_2 are distinct, the orbit of p is not constant and

$$\chi(F_1) = b_{min} < b_{max} = \chi(F_2).$$

□

2.5. **Geometric invariant theory and projectivity.** In this section we use geometric invariant theory, and the ideas of M. Thaddeus and others (see e.g. [64, 65, 24]), in order to obtain a result about relative projectivity of quotients.

We continue with the notation of the last section. Consider the vector bundle E and its decomposition according to the character. Let $\{b_j\}$ be the characters of the action of K^* on E, and $\{a_i\}$ the subset of those b_j that are images of χ. If we use the Veronese embedding $\overline{B} \subset \mathbb{P}(\mathrm{Sym}^2(E))$ and replace E by $\mathrm{Sym}^2(E)$, we may assume that a_i are even, in particular $a_{i+1} > a_i + 1$ (this is a technical condition which comes up handy in what follows).

Denote by $\rho_0(t)$ the action of $t \in K^*$ on E. For any $r \in \mathbb{Z}$ consider the "twisted" action $\rho_r(t) = t^{-r} \cdot \rho_0(t)$. Note that the induced action on $\mathbb{P}(E)$ does not depend on the "twist" r. Considering the decomposition $E = \bigoplus E_{b_j}$, we see that $\rho_r(t)$ acts on E_{b_j} by multiplication by $t^{b_j - r}$.

We can apply geometric invariant theory in its relative form (see, e.g., [51], [25]) to the action $\rho_r(t)$ of K^*. Recall that a point $p \in \mathbb{P}(E)$ is said to be semistable with respect to ρ_r, written $p \in (\mathbb{P}(E), \rho_r)^{ss}$, if there is a positive n and a local section $s \in \mathrm{Sym}^n(E)$ such that $s(p) \neq 0$, and such that s is invariant under the action induced by ρ_r on $\mathrm{Sym}^n(E)$. The main result of geometric invariant theory implies that

$$\underset{X_2 \times \mathbb{P}^1}{\mathcal{P}roj} \bigoplus_{n \geq 0}^{\infty} (\mathrm{Sym}^n(E))^{\rho_r} = (\mathbb{P}(E), \rho_r)^{ss} // K^*.$$

We denote by $(\overline{B}, \rho_r)^{ss} = \overline{B} \cap (\mathbb{P}(E), \rho_r)^{ss}$.

The numerical criterion of semistability (see [48]) immediately implies the following:

Lemma 2.5.1. *For $0 < i < k$ we have*

1. $(\overline{B}, \rho_{a_i})^{ss} = B_{a_i}$;
2. $(\overline{B}, \rho_{a_i+1})^{ss} = (B_{a_i})_+$;
3. $(\overline{B}, \rho_{a_i-1})^{ss} = (B_{a_i})_-$;

Thus we obtain:

Proposition 2.5.2. *The morphisms $(B_{a_i})_+/K^* \to X_2$, $(B_{a_i})_-/K^* \to X_2$ and $B_{a_i}//K^* \to X_2$ are projective.*

2.6. **The main result of** [68]**.** Let $B \subset \overline{B}$ be a collapsible nonsingular birational cobordism. Then we can write B as a union of quasi-elementary cobordisms $B = \cup_i B_{a_i}$, with $(B_{a_i})_+ = (B_{a_{i+1}})_-$. By Lemma 1.2.11 each B_{a_i} has a locally toric structure such that the action of K^* is locally toric.

Lemma 2.6.1. *For each a_i and for each $p \in B_{a_i}$ there exists a toric chart V_p at p such that if an orbit in B_{a_i} (resp. X_p) lies in the image of V_p then its closure in B_{a_i} (resp. X_p) also lies in the image.*

Proof. Note that the connected components of F_{a_i} are closed in B_{a_i} and their images in $B_{a_i}//K^*$ are also closed. For a point $p \in B_{a_i} \smallsetminus F^{\pm}_{a_i}$ we take a Luna's chart V_p such that $\iota_p(V_p)$ lies in the open set $B_{a_i} \smallsetminus F^{\pm}_{a_i}$, hence no point in the image has a limit in B_{a_i}. A point $p \in F^{\pm}_{a_i}$ has a unique limit $p' \in F \subset B^{K^*}_{a_i}$. We take a Luna's chart at p' such that the image of $\iota_p(V_p)$ in $B_{a_i}//K^*$ is disjoint from the images of the fixed point components other than F. Such charts satisfy the requirement of the lemma for ι_p. The requirement for η_p is automatically satisfied by the construction of Luna's charts. □

Lemma 2.6.2. *For each a_i we have a diagram of locally toric maps*

$$\begin{array}{ccc} (B_{a_i})_-/K^* & \overset{\varphi_i}{\dashrightarrow} & (B_{a_i})_+/K^* \\ & \searrow \quad \swarrow & \\ & B_{a_i}//K^* & \end{array}$$

which in toric charts is described by the main example in Section 2.2.

Proof. We cover B_{a_i} with charts as in the previous lemma, and apply Lemma 1.2.9 to the locally toric K^*-equivariant birational morphisms $(B_{a_i})_\pm \hookrightarrow B_{a_i}$ (see Remark 1.2.10). □

Composing the birational transformations obtained from each B_{a_i} we get a slight refinement of the main result in [68] .

Theorem 2.6.3. *Let $\phi : X_1 \dashrightarrow X_2$ be a birational map between proper nonsingular algebraic varieties X_1 and X_2 over an algebraically closed field K of characteristic zero, and let $U \subset X_1$ be the maximal open set where ϕ is an isomorphism. Then there exists a sequence of birational maps between proper nonsingular algebraic varieties*

$$X_1 = W_0 \overset{\varphi_1}{\dashrightarrow} W_1 \overset{\varphi_2}{\dashrightarrow} \cdots \overset{\varphi_i}{\dashrightarrow} W_i \overset{\varphi_{i+1}}{\dashrightarrow} W_{i+1} \overset{\varphi_{i+2}}{\dashrightarrow} \cdots \overset{\varphi_{l-1}}{\dashrightarrow} W_{l-1} \overset{\varphi_l}{\dashrightarrow} W_l = X_2$$

where

1. $\phi = \varphi_l \circ \varphi_{l-1} \circ \cdots \varphi_2 \circ \varphi_1$,
2. *φ_i are isomorphisms on U, and*
3. *For each i, the map φ_i is locally toric, and étale locally equivalent to the map φ described in 2.2.*

Furthermore, there is an index i_0 such that for all $i \leq i_0$ the map $W_i \dashrightarrow X_1$ is a projective morphism, and for all $i \geq i_0$ the map $W_i \dashrightarrow X_2$ is a projective morphism. In particular, if X_1 and X_2 are projective then all the W_i are projective.

Remark 2.6.4. For the projectivity claim 2, we take the first i_0 terms in the factorization to come from Hironaka's elimination of indeterminacies in Lemma 1.6.1, which is projective over X_1, whereas the last terms come from $\overline{B}$, which is projective over X_2, and the geometric invariant theory considerations as in Proposition 2.5.2.

3. Torification

We wish to replace the locally toric factorization of Theorem 2.6.3 by a toroidal factorization. This amounts to replacing B with a locally toric K^*-action by some B' with a toroidal K^*-action. We call such a procedure *torification.* The basic idea, which goes back at least to Hironaka, is that if one blows up an ideal, the exceptional divisors provide the resulting variety with useful extra structure. The ideal we construct, called a *torific ideal*, is closely related to the torific ideal of [1].

3.1. **Construction of a torific ideal.**

Definition 3.1.1. Let V be an algebraic variety with a K^*-action, $p \in V$ a closed point, $G_p \subset K^*$ the stabilizer of p. Fix an integer α. Then we define

$$J_{\alpha,p} \subset \mathcal{O}_{V,p}$$

to be the ideal generated by the semi-invariant functions $f \in \mathcal{O}_{V,p}$ of character G_p-character α, that is, for $t \in G_p$ and a generator $f \in J_{\alpha,p}$ we have

$$t^*(f) = t^\alpha f.$$

Lemma 3.1.2. *Let V be a nonsingular variety with a K^*-action, and consider*

$$\mathcal{O}_{V,p} \hookrightarrow \hat{\mathcal{O}}_{V,p}.$$

We lift the action of G_p to $\hat{\mathcal{O}}_{V,p}$ and define

$$\hat{J}_{\alpha,p} \subset \hat{\mathcal{O}}_{V,p}$$

to be the ideal generated by semi-invariants $f \in \hat{\mathcal{O}}_{V,p}$ of character G_p-character α. Then $\hat{J}_{\alpha,p}$ is generated by the image of $J_{\alpha,p}$.

Proof. Using Luna's Étale Slice Theorem we choose G_p-semi-invariant coordinates $z_1, \dots, z_n$ at p. Then $\hat{\mathcal{O}}_{V,p} \cong K[\![z_1, \dots, z_n]\!]$, and the claim is obvious. □

For the rest of this section, we let let B be a *nonsingular quasi-elementary* cobordism, $B = B_{a_i}$ for some i according to our previous notation, and let $F_0 = B^{K^*}$. Recall the notation F_0^* above. This is a constructible set in B.

Proposition 3.1.3. *There exists a unique coherent K^*-equivariant ideal sheaf I_α on B, such that for all $p \in B \smallsetminus F_0^*$ we have $(I_\alpha)_p = J_{\alpha,p}$.*

Definition 3.1.4. The sheaf I_α is called *the α-torific ideal sheaf* of the action of K^* on B.

Remark 3.1.5. Notice that the collection of ideals $J_{\alpha,p}$ for $p \in B$ do not define a coherent sheaf of ideals in general. As an example, let $B = \mathbb{A}^2$, and let the action of K^* be

$$t \cdot (x, y) = (tx, t^{-1}y).$$

Then at $p = (0, 0)$, the stabilizer is $G_p = K^*$, and $J_{1,p} = (x)$. Any other point $q \in B \smallsetminus \{p\}$ has trivial stabilizer, hence $J_{1,q}$ is trivial. These germs do not form a coherent ideal sheaf on B. In this case, the ideal sheaf generated by x is the sheaf I_1 of the proposition.

Proof of 3.1.3. First we prove the uniqueness of I_α. Clearly the stalks of I_α are uniquely defined at all points $p \in B \smallsetminus F_0^*$. Now if $p \in F_0^*$ then p has a unique limit fixed point $p' \in F_0$. Since I_α is uniquely determined at p', hence also near p' by coherence, it follows from K^*-equivariance that I_α is uniquely defined at p.

To prove the existence of I_α we cover B with toric charts as in Lemma 2.6.1. With such charts, it follows that F_0^* defined in B, V_p and X_p correspond to each other. By Lemma 3.1.2 we are reduced to the case where B is toric, and the proposition follows from the lemma below. □

Lemma 3.1.6. *Let $B = X(N, \sigma)$ be a nonsingular affine toric variety on which K^* acts as a subgroup of the torus. Let $p \in B$ be a point lying in the unique closed stratum of B, and let G_p be the stabilizer of p in K^*. Then I_α is the ideal generated by all monomials on which G_p acts by character α.*

Proof. By abuse of notation we will identify a lattice point $m \in M$ with the corresponding monomial z^m. We will also use the same letter α to denote a character of a subgroup of G_p.

For any $q \in B$ the ideal $J_{\alpha,q}$ is generated by all monomials regular at q on which G_q acts by character α. (To see this, write a function $f \in \mathcal{O}_{B,q}$ as a power series in the monomials. Then f is semi-invariant of character α if and only if each term is semi-invariant of character α.) In particular, the claim of the lemma is true at p.

Let $q \in B \setminus F_0^*$ be a point different from p, and let τ be the smallest face of σ such that q lies in the affine open toric subvariety $X(N, \tau)$. The monomials invertible at q correspond to the lattice points in $M \cap \tau^\perp$.

If $m_p \in M \cap \sigma^\vee$ is a monomial regular at p on which G_p acts by character α then clearly m_p is also regular at q and $G_q \subset G_p$ acts on it by character α.

Conversely, let $m_q \in M \cap \tau^\vee$ be a monomial regular at q, on which G_q acts by character α. We have to show that m_q lies in the ideal generated by monomials regular at p and of G_p-character α. In other words, we have to find a decomposition

$$m_q = m_p + m'$$

where $m_p \in M \cap \sigma^\vee$ has G_p-character α, and $m' \in M \cap \tau^\vee$.

Note that even though G_q acts on m_q by character α, it is not true in general that G_p acts on m_q by character α. However, by Lemma 3.1.7 below we can find an $m'_1 \in M \cap \tau^\perp$ such that G_p acts on $m_q - m'_1$ by character α. Now we look for $m' \in M \cap \tau^\vee$ of the form $m' = m'_1 + m'_2$ such that $m_p = m_q - m' \in M \cap \sigma^\vee$ and G_p acts trivially on m'_2. Then clearly G_p acts on m_p by character α.

Let $B = \operatorname{Spec} K[z_1, \ldots, z_m, z_{m+1}^{\pm 1}, \ldots, z_n^{\pm 1}]$, where K^* acts on z_i by character a_i. Let q have coordinates $(q_1, \ldots, q_n)$. If q is fixed by K^* then for all i such that $q_i \neq 0$ we have $a_i = 0$. If q is not fixed, we may assume that there exist both positive and negative a_i for i such that $q_i \neq 0$. If B^{K^*} is nonempty, this is precisely the condition for $q \notin F_0^*$. If B^{K^*} is empty then $a_i \neq 0$ for some $i \geq m$, and we may replace z_i by z_i^{-1} if necessary.

In all cases we can choose integers $c_i < 0$ such that

$$\sum_{q_i \neq 0} c_i a_i = 0.$$

The monomial

$$z^{m'_2} = \prod_{q_i \neq 0} z_i^{c_i}$$

is invertible at q and fixed by K^*, hence also fixed by G_p. We may take a multiple of m'_2 if necessary such that

$$m = m_q - m'_1 - m'_2 \in M \cap \sigma^\vee.$$

This completes the proof of Proposition 3.1.3. □

Lemma 3.1.7. *Let $B = X(N, \sigma)$ be a nonsingular affine toric variety on which K^* acts as a subgroup of the torus. Let $p \in B$ be a closed point lying in the unique closed stratum* *of B, and let G_p be the stabilizer of p in K^*. If G_p is finite then there exists a monomial, invertible on B, on which K^* acts by character $\#G_p$.*

Proof. Consider the group of K^*-characters defined by $M \cap \sigma^\perp$, and let $\alpha \in \mathbb{Z}$ be the positive generator of this group. Now G_p is the subgroup of K^* acting trivially on the invertible monomials $M \cap \sigma^\perp$. Hence G_p consists of all α'th roots of unity, and $\#G_p = \alpha$. □

3.2. **The torifying property of the torific ideal.** Let B be a quasi-elementary nonsingular cobordism. Choose $a_1, \dots, a_m \in \mathbb{Z}$ a finite set of integers containing all characters of the G_p-action on the tangent space of B at p for all $p \in B$. Let

$$I = I_{a_1} \cdots I_{a_m}$$

be the product of the a_i-torific ideals, and let $B^{tor} \to B$ be the blowup of B along I. Since I is K^*-equivariant, we can lift the action of K^* to B^{tor}. Denote by $D \subset B^{tor}$ the total transform of the support of I, and $U_{B^{tor}} = B^{tor} \setminus D$.

Proposition 3.2.1. *The embedding $U_{B^{tor}} \subset B^{tor}$ is toroidal and K^* acts toroidally on this embedding.*

Definition 3.2.2. We call I a *torific* ideal and $B^{tor} \to B$ a *torific* blowup.

Proof. We take toric charts as in Lemma 2.6.1. Then

$$\iota_p^*(I) = \eta_p^*(I^{X_p}),$$

where I^{X_p} is a torific ideal in X_p. The blowups of these ideals provide a locally toric structure on B^{tor}, such that the total transforms of the ideals correspond. Thus we are reduced to the case where B is a toric variety.

Now if $B = X(N, \sigma)$ is a nonsingular toric variety then $T \subset B^{tor}$ is again toric (with K^* acting as a subgroup of the torus), and the divisor D consists of a subset of the toric divisors. We have to show that we can "remove" the toric divisors not in D so that the embedding and the K^*-action remain toroidal. For this, it suffices to show that if $Y \subset B^{tor}$ is any affine open toric subvariety intersecting a toric divisor E not in D, we can write

$$Y \cong Y' \times \mathbb{A}^1$$
$$E \cong Y' \times \{0\}$$

such that the K^*-action on Y is a product of a K^*-action on Y' with the trivial K^*-action on $\mathbb{A}^1$.

It follows from the Lemma 3.2.3 below that $U_{B^{tor}} \subset B^{tor}$ is a toroidal embedding, and for p a point lying in the closed stratum of B, the stabilizer G_p acts toroidally on this toroidal embedding. Now Lemma 3.2.5 shows that then also K^* acts toroidally on this embedding. □

Lemma 3.2.3. *Let $B^{tor} = X(N, \Sigma)$ be the toric variety obtained from a nonsingular toric variety $B = X(N, \sigma)$ by blowing up a torific ideal I. Let $p \in B$ be a point in the closed stratum, and G_p the stabilizer of p in K^*. Then for each irreducible toric divisor $E \subset B^{tor}$ either*

(i) *The divisor E lies in the total transform of I; or*

(ii) *For any affine open toric subvariety $Y \subset B^{tor}$ intersecting E, we can write*

$$Y \cong Y' \times \mathbb{A}^1,$$
$$E \cong Y' \times \{0\},$$

such that the G_p-action on Y is a product of a G_p-action on Y' with the trivial G_p-action on $\mathbb{A}^1$.

Proof. Let $\sigma = \langle v_1, \dots, v_m \rangle$, $\sigma^\vee = \langle v_1^*, \dots, v_m^*, \pm v_{m+1}^*, \dots, \pm v_n^* \rangle$, and let G_p act on z_i by character a_i. The only toric divisors that do not lie in the support of I are among the strict transforms of the divisors $\{z_i = z^{v_i^*} = 0\} \subset B$.

Let E be the strict transform of $\{z_1 = 0\}$. Then the ideal I_{a_1} contains z_1. If I_{a_1} is principal, we are in case (i). Otherwise, we choose monomial generators for I_{a_1} corresponding to lattice points $v_1^*, m_1, \dots, m_l$ in $M \cap \sigma^\vee$. We may assume that m_i do not contain v_1^*, i.e, all m_i lie in the face $v_1^\perp \cap \sigma^\vee = \langle v_2^*, \dots, \pm v_n^* \rangle$ of $\sigma^\vee$.

Let τ be a maximal cone in the subdivision of σ corresponding to blowing up I_{a_1} and normalizing. Now E intersects $X(N, \tau)$ if and only if v_1 lies in τ, and this means that $\tau^\vee$ has one of its facets lying in the plane $v_1^\perp$. Such τ corresponds to a chart of the blowup where we invert one of the m_i, say m_1. Hence $\tau^\vee$ is generated by

$$v_1^* - m_1, m_2 - m_1, \dots, m_l - m_1, v_2^*, \dots, \pm v_n^*.$$

Since the coefficient of v_1^* in $v_1^* - m_1$ is one, and the other generators lie in $v_1^\perp$, we can write

$$M \cong \mathbb{Z} \cdot (v_1^* - m_1) \oplus M \cap v_1^\perp$$
$$\tau^\vee \cong \langle v_1^* - m_1 \rangle \times \langle m_2 - m_1, \dots, m_l - m_1, v_2^*, \dots, \pm v_n^* \rangle,$$

which gives $Y = X(N, \tau)$ a product structure. Since all monomials are G_p-semi-invariants, the action of G_p on Y is a product of actions on the two terms. Moreover, the strict transform E of $\{z_1 = 0\}$ is defined by $z^{v_1^* - m_1}$ and G_p acts trivially on this monomial.

It remains to show that if we blow up the ideals I_β for $\beta \neq a_1$, the subdivided cones in τ have the same product structure. One can define the ideal $I_{\beta,\tau}$ the same way as $I_\beta = I_{\beta,\sigma}$. The lemma below shows that $I_{\beta,\tau}$ is generated by I_β (via the inclusion $\sigma^\vee \subset \tau^\vee$). Hence we may blow up $I_{\beta,\tau}$ instead of I_β. Since G_p acts on $v_1^* - m_1$ by character 0, the ideals $I_{\beta,\tau}$ are generated by monomials in the second term of the product. So, blowing up $I_{\beta,\tau}$ preserves the product structure. □

Lemma 3.2.4. *Let τ be a maximal cone in the subdivision of σ corresponding to blowing up I_α for some character α and normalizing. Then $I_{\beta,\tau}$ is generated by I_β.*

Proof. Since τ is a maximal cone in the subdivision, it follows that the stabilizer G_q of a point q lying in the closed stratum of $X(N, \tau)$ is the same as G_p. We have

$$\tau^\vee = \langle m_2 - m_1, \dots, m_l - m_1, v_1^*, \dots, \pm v_n^* \rangle$$

for some generators m_i of I_α. Thus an element $m \in M \cap \tau^\vee$ can be written as a linear combination

$$m = b_1(m_2 - m_1) + \ldots + b_l(m_l - m_1) + c_1 v_1^* + \ldots + c_n v_n^*$$

for some integers $b_i, c_j \geq 0$ for $i = 1, \dots, l$, $j = 1, \dots, m$. If m happens to be a generator of $I_{\beta,\tau}$, i.e., G_p acts on m by character β, then also G_p acts on $m' = c_1 v_1^* + \ldots + c_n v_n^*$ by character β, and m' is a generator of I_β. □

Lemma 3.2.5. *Let $Y = X(N,\sigma)$ be an affine toric variety on which K^* acts as a subgroup of the torus, and let E be an irreducible toric divisor in Y. Suppose we have*

$$Y \cong Y' \times \mathbb{A}^1,$$
$$E \cong Y' \times \{0\},$$

such that the G_p-action on Y is a product of a G_p-action on Y' with the trivial G_p-action on $\mathbb{A}^1$, where $p = (p', 0)$ is a point in the closed stratum of Y. Then we can write

$$Y \cong Y'' \times \mathbb{A}^1,$$
$$E \cong Y'' \times \{0\},$$

such that the K^*-action on Y is a product of a K^*-action on Y'' with the trivial K^*-action on $\mathbb{A}^1$.

Proof. We have correspoding splittings of the lattice M and the cone $\sigma^\vee$:

$$M \cong M' \oplus \mathbb{Z},$$
$$\sigma^\vee \cong \sigma' \times \langle v^* \rangle.$$

where the monomial z^{v^*} defines the divisor E. Let K^* act on v^* by character b. Since G_p acts trivially on v^*, we get that $\#G_p$ divides b. Now by Lemma 3.1.7 there exists an $m \in M \cap \sigma^\perp$ such that K^* acts trivially on $v^* - m$. Since z^m is invertible on Y, the monomial z^{v^*-m} defines the divisor E, and we can write

$$\sigma^\vee \cong \sigma' \times \langle v^* - m \rangle$$

prooving the lemma. □

Example 3.2.6. Consider $B = \mathbb{A}^3 = \operatorname{Spec} K[z_1, z_2, z_3]$, where $t \in K^*$ acts as

$$t \cdot (z_1, z_2, z_3) = (t^2 z_1, t z_2, t^{-1} z_3).$$

We have the following generators of the torific ideals I_α:

$$I_2 = \{z_1, z_2^2\}$$
$$I_1 = \{z_2, z_1 z_3\}$$
$$I_{-1} = \{z_3\}.$$

If we regard $B = X(N,\sigma)$ as the toric variety corresponding to the regular cone

$$\sigma = \langle v_1, v_2, v_3 \rangle \subset N_{\mathbb{R}},$$

then B^{tor} is described by the fan covered by the following three maximal cones

$$\begin{aligned} \sigma_1 &= \langle v_1, v_3, 2v_1 + v_2 \rangle \\ \sigma_2 &= \langle v_1 + v_2, v_2 + v_3, v_2 \rangle \\ \sigma_3 &= \langle v_3, 2v_1 + v_2, v_1 + v_2, v_2 + v_3 \rangle. \end{aligned}$$

The dual cone $\sigma_2^\vee$ has the product description

$$\begin{aligned} \sigma_2^\vee &= \langle v_2^* - (v_1^* + v_3^*), v_1^*, v_3^* \rangle \\ &= \langle v_2^* - (v_1^* + v_3^*) \rangle \times \langle v_1^*, v_3^* \rangle. \end{aligned}$$

Thus from the description of the original toric structure $X(N,\sigma_2) = \operatorname{Spec} k[z_2/z_1 z_3, z_1, z_3]$, and even if we remove the divisor $\{z_2/z_1 z_3 = 0\}$, it still has the toroidal embedding structure

$$X(N,\sigma_2) \smallsetminus (\{z_1 = 0\} \cup \{z_3 = 0\}) \subset X(N,\sigma_2).$$

As z_2/z_1z_3 is invariant, the action of K^* is toroidal. For example, at $0 \in X(N, \sigma_2)$ we have a toric chart

$$\begin{aligned} \mathbb{G}_m \times \mathbb{A}^2 &\to \mathbb{A}^1 \times \mathbb{A}^2 \cong X(N, \sigma_2) \\ (x_1, x_2, x_3) &\mapsto (x_1 - 1, x_2, x_3). \end{aligned}$$

Globally, the divisors corresponding to the new 1-dimensional rays

$$D_{\langle 2v_1+v_2\rangle}, D_{\langle v_2+v_3\rangle}, D_{\langle v_1+v_2\rangle},$$

together with $D_{\langle v_3\rangle}$ coming from I_{-1}, are obtained through the blowup of the torific ideals. Considering

$$U_{B^{tor}} = B^{tor} \smallsetminus (D_{\langle v_3\rangle} \cup D_{\langle 2v_1+v_2\rangle} \cup D_{\langle v_2+v_3\rangle} \cup D_{\langle v_1+v_2\rangle})$$

we obtain a toroidal structure $U_{B^{tor}} \subset B^{tor}$ compatible with the K^*-action. □

4. A proof for the weak factorization theorem

4.1. **Lifting the toroidal structures.** In Theorem 2.6.3 we have constructed a factorization of the given birational map ϕ into locally toric birational maps

$$\phi : X_1 = W_1 \stackrel{\varphi_1}{\dashrightarrow} W_2 \stackrel{\varphi_2}{\dashrightarrow} \cdots W_j \stackrel{\varphi_j}{\dashrightarrow} W_{j+1} \cdots W_{l-1} \stackrel{\varphi_{l-1}}{\dashrightarrow} W_l = X_2$$

where $W_j = (B_{a_j})_-/K^*$ and $W_{j+1} = (B_{a_j})_+/K^*$, and the birational map $\varphi_j : W_j \dashrightarrow W_{j+1}$ is induced by the elementary cobordism B_{a_j} associated to the value a_j of a strictly increasing function $\varphi : \mathcal{C} \to \mathbb{Z}$.

Let $B^{tor}_{a_j} \to B_{a_j}$ be a torific blowup. Write $W^{tor}_{j-} = B^{tor}_{a_j-}/K^*$ and $W^{tor}_{j+} = B^{tor}_{a_j+}/K^*$. We have natural birational morphisms

$$f_{j-} : W^{tor}_{j-} \to W_j \text{ and } f_{j+} : W^{tor}_{j+} \to W_{j+1}.$$

Lemma 4.1.1. *The morphisms f_{j-} and f_{j+} can be obtained by blowing up ideals I_{j-} and I_{j+} whose zero sets are disjoint from the open set $U \in X_1$ on which $\phi : X_1 \dashrightarrow X_2$ is an isomorphism. In particular, U is canonically an open set in $W^{tor}_{j\pm}$ and the map $W^{tor}_{j-} \dashrightarrow W^{tor}_{j+}$ is an isomorphism on U.*

Proof. We discuss f_{j-}, since the case of f_{j+} is identical. First we claim that the morphism f_{j-} is projective.

We take a toric chart V_p as in the proof of Proposition 3.1.3 so that the torific ideals on B_{a_j} and X_p agree when pulled back to V_p. Hence blowing up these ideals provides a toric chart for $B^{tor}_{a_j}$, and we are reduced to the case where B_{a_j} is toric.

If $B_{a_j} = X(N, \sigma)$ is an affine toric variety then blowing up a torific ideal corresponds to a projective subdivision of the cone σ. This subdivision restricts to a projective subdivision of $\partial_-\sigma$, and hence also of $\pi(\partial_-\sigma)$. Now it follows that the morphism f_{j-} corresponding to this subdivision of $\pi(\partial_-\sigma)$ is projective.

Since f_{j-} is projective, it is isomorphic to a blowup of an ideal sheaf. Next we describe this ideal sheaf explicitly. Take an f_{j-}-ample divisor A^{tor}_{j-} on W^{tor}_{j-}. Since W_j has only quotient singularities (the cones in $\pi(\partial_-\sigma)$ are simplicial), the divisor $A_{j-} := {f_{j-}}_*(A^{tor}_{j-})$ is $\mathbb{Q}$-Cartier. Therefore, by replacing A^{tor}_{j-} with a sufficiently divisible multiple, we may assume from the beginning that both A^{tor}_{j-} and A_{j-} are Cartier divisors. Then we have

$$A^{tor}_{j-} = f^*_{j-}(A_{j-}) - E^{tor}_{j-}$$

for some effective Cartier divisor E^{tor}_{j-} consisting of exceptional divisors for f_{j-} so that $\mathcal{O}_{W^{tor}_{j-}}(-E^{tor}_{j-})$ is f_{j-}-ample and that f_{j-} is the blowup of the ideal

$$I_{j-} := f_{j-*}\mathcal{O}_{W^{tor}_{j-}}(-E^{tor}_{j-}).$$

Note that the torific ideals are trivial at the points $p \in B \smallsetminus F_0^*$ whose stabilizers are trivial, hence on the locus $B_+ \cap B_-$. Therefore, over $U = (B_+ \cap B_-)/K^*$ the birational morphism $f_{j-} : W^{tor}_{j-} \to W_j$ is an isomorphism, and hence I_{j-} is trivial over U. ☐

By Lemma 2.6.2 we have the toroidal structures

$$U^{tor}_{j-} \subset W^{tor}_{j-} \text{ and } U^{tor}_{j+} \subset W^{tor}_{j+}$$

induced from $U_j \subset B^{tor}_{a_j}$, and the birational map

$$W^{tor}_{j-} \overset{\varphi^{tor}_j}{\dashrightarrow} W^{tor}_{j+}$$

is toroidal with respect to these toroidal structures.

Let $W^{res}_j \to W_j$ be a canonical resolution of singularities [7].

Proposition 4.1.2. *Let $I^{res}_{j-} = I_{j-}\mathcal{O}_{W^{res}_j}$ and $I^{res}_{j+} = I_{j+}\mathcal{O}_{W^{res}_{j+1}}$. Let $W^{can}_{j-} \to W^{res}_j$ be a canonical principalization of I^{res}_{j-}, and similarly let $W^{can}_{j+} \to W^{res}_{j+1}$ be a canonical principalization of I^{res}_{j+}.*

Denote by $h_{j-} : W^{can}_{j-} \to W^{tor}_{j-}$ and $h_{j+} : W^{can}_{j+} \to W^{tor}_{j+}$ the induced morphisms. Let $U^{can}_{j-} = h^{-1}_{j-}(U^{tor}_{j-})$ and $U^{can}_{j+} = h^{-1}_{j+}(U^{tor}_{j+})$.

$$\begin{array}{ccccccc} W^{can}_{j-} & \overset{h_{j-}}{\to} & W^{tor}_{j-} & \overset{\varphi^{tor}_j}{\dashrightarrow} & W^{tor}_{j+} & \overset{h_{j+}}{\leftarrow} & W^{can}_{j+} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ W^{res}_j & \to & W_j & \overset{\varphi_j}{\dashrightarrow} & W_{j+1} & \leftarrow & W^{res}_{j+1} \end{array}$$

Then we have

(i) *$U^{can}_{j-} \subset W^{can}_{j-}$ and $U^{can}_{j+} \subset W^{can}_{j+}$ are toroidal embeddings, and*
(ii) *the morphisms h_{j-} and h_{j+} are toroidal with respect to these toroidal structures.*

Proof. We discuss h_{j-}, since the case of h_{j+} is identical. Observe that canonical resolutions can be "pulled back" by étale morphisms and that all the relevant ideals above have the corresponding ideals on the toric charts so that their pull-backs by $(i_p)_\pm/K^*$ and $(\eta_p)_\pm/K^*$ coincide. Therefore, in order to show the assertion, we can use the locally toric charts as in the proof of Proposition 3.1.3.

Let $B_{a_j} = \operatorname{Spec} K[z_1, \ldots, z_m, z^{\pm 1}_{m+1}, \ldots, z^{\pm 1}_n]$, and let σ be the cone in $N_\mathbb{R}$ describing its toric structure. Let $a \in N$ be the one-parameter subgroup corresponding to the action of K^* and let $\pi : N_\mathbb{R} \to N_\mathbb{R}/\mathbb{R}\cdot a$ be the projection.

By the Main Example of Section 2.2, $W_j = (B_{a_j})_-/K^*$ corresponds to the fan $\pi(\partial_-\sigma)$. We have three subdivisions of $\pi(\partial_-\sigma)$: first $\Sigma^{tor} \to \pi(\partial_-\sigma)$ induced from that of σ by the torification (blowing up of the torific ideal), second $\Sigma^{res} \to \pi(\partial_-\sigma)$ obtained from the process of taking the canonical resolution of singularities of W_j, and third $\Sigma^{can} \to \Sigma^{res}$ obtained from the canonical principalization of the ideal I^{res}_{j-}. (Remark that the canonical resolution of singularities and the canonical principalization blow up nonsingular toric subvarieties, hence the blowups are toric.)

$$\begin{array}{ccc} \Sigma^{can} & \longrightarrow & \Sigma^{tor} \\ \downarrow & & \downarrow \\ \Sigma^{res} & \longrightarrow & \pi(\partial_{-}\sigma) \end{array}$$

Recall that the toroidal structure on $W_{j-}^{tor} = X(N, \Sigma^{tor})$ is obtained by removing some of the toric divisors from $X(N, \Sigma^{tor})$. We have to show that we can also remove the inverse images of these divisors from $X(N, \Sigma^{can})$. For this it suffice to show the following:

> Let v_1 be a generator of σ, corresponding to the divisor $E \subset X(N, \sigma)$ whose strict transform is removed from the torification of $X(N, \sigma)$. Let $\tau \in \Sigma^{tor}$ be any cone containing $\pi(v_1)$. Recall that there is a factorization $\tau = \tau' \times \langle v_1 \rangle$ induced from the product structure described in the proof of Lemma 3.2.3, which descends also to the product structure on the quotient by K^*. Then for any (new) ray $w \in \Sigma^{can}$, we have
> $$w \subset \tau \Longrightarrow w \in \tau'.$$

In fact, the above implies that both Σ^{can} and Σ^{tor} have a product structure near v_i and the map between them preserves this structure.

Using the notation in the proof of Proposition 3.2.1, τ has its dual generated by

$$< v_1^* - m_1, m_2 - m_1, \cdots, m_l - m_1, v_2^*, \cdots, \pm v_n^* > \cap a^{\perp},$$

where $(v_1^* - m_1, \cdot) = 0$ defines the face τ' of τ. Therefore, it is sufficient to show

$$(v_1^* - m_1, w) = \mathrm{ord}_{E_w}(z_1 \cdot z^{-m_1}) = 0,$$

where $\mathrm{ord}_{E_w}(r)$ denotes the order of the rational function r on the divisor E_w (exceptional for h_{j-}) corresponding to the ray w.

Now consider the family of automorphisms θ_c, parameterized by $c \in K$, of $K[z_1, \cdots, z_n^{\pm 1}]$ defined by

$$\theta_c(z_1) = z_1 + c \cdot z^{m_1}, \theta_c(z_i) = z_i \text{ for } i = 2, \ldots, n.$$

Observe that these automorphisms preserve the torific ideals because z_1 and m_1 have the same characters. It follows that θ_c induces a family of automorphisms of W_{j-}^{tor} mapping the exceptional divisor E_{j-}^{tor} to itself. Since the irreducible components of the exceptional divisor are discrete, they are preserved by these automorphisms, and therefore the automorphisms also preserve I_{j-}.

By the canonicity of the resolutions the automorphisms θ_c induce automorphisms of W_{j-}^{can}, which preserve the divisor E_w exceptional for h_{j-} by the same argument as above.

Then

$$\begin{aligned} \mathrm{ord}_{E_w}(z_1 \cdot z^{-m_1}) &= \mathrm{ord}_{E_w}(\theta_c(z_1 \cdot z^{-m_1})) \\ &= \mathrm{ord}_{E_w}((z_1 + c \cdot z^{m_1}) \cdot z^{-m_1}) \\ &= \mathrm{ord}_{E_w}(z_1 \cdot z^{-m_1} + c). \end{aligned}$$

Since the above equality holds for any $c \in K$, we conclude

$$\mathrm{ord}_{E_w}(z_1 \cdot z^{-m_1}) = 0.$$

□

4.2. **Conclusion of the proof of Theorem 1.** Since $X_1 = W_0$ and $X_2 = W_l$ are nonsingular, we have $W_0^{res} = W_0$ and $W_l^{res} = W_l$. For each $j = 0, \dots, l-1$ we have obtained a diagram

$$\begin{array}{ccccccc}
 & & W_{j-}^{can} & \overset{\varphi_j^{can}}{\dashrightarrow} & W_{j+}^{can} & & \\
 & r_{j-} \swarrow & \downarrow h_{j-} & & \downarrow h_{j+} & \searrow r_{j+} & \\
W_j^{res} & & W_{j-}^{tor} & \overset{\varphi_j^{tor}}{\dashrightarrow} & W_{j+}^{tor} & & W_{j+1}^{res} \\
\downarrow & f_{j-} \swarrow & & & & \searrow f_{j+} & \downarrow \\
W_j & & & \overset{\varphi_j}{\dashrightarrow} & & & W_{j+1}
\end{array}$$

where

(4.2.1) the canonical principalizations r_{j-} and r_{j+} are composites of blowings up with smooth centers,

(4.2.2) h_{j-} and h_{j+} are toroidal by Proposition 4.1.2, and

(4.2.3) φ_j^{tor} is toroidal by Proposition 3.2.1 and Lemma 2.6.2 applied in the toroidal case.

It follows that φ_j^{can} is toroidal. By the strong factorization theorem of toroidal birational maps φ_j^{can} is a composite of toroidal blowings up and blowings down with smooth centers. Thus we get a factorization

$$\phi : X_1 = W_1^{res} \dashrightarrow W_2^{res} \dashrightarrow \cdots W_j^{res} \dashrightarrow W_{j+1}^{res} \cdots W_{l-1}^{res} \dashrightarrow W_l^{res} = X_2,$$

where all W_i^{res} are nonsingular, and the birational maps are composed of a sequence of blowings up followed by a sequence of blowings down. We do not touch the open subset $U \subset X_1$ on which ϕ is an isomorphism. The projectivity over X_2 follows from Proposition 2.5.2 and the construction. This completes the proof of Theorem 1. $\square$

5. Problems

5.1. **Strong factorization.**

Problem 5.1.1. *Can one extend the torific ideal to all of B in such a way that the strong factorization conjecture follows?*

In the construction of the torific ideal in 3.6 and the analysis of its blowup in 3.2 and 4.1, the assumption of the cobordism B_{a_j} being quasi-elementary is essential. Actually we can extend the ideal to the one over the entire cobordism B by taking its Zariski closure, but its behavior at $B - B_{a_j}$ seems to be subtle.

5.2. **Toroidalization.**

Problem 5.2.1 (Toroidalization conjecture)**.** *Let $\phi : X \to Y$ be a surjective proper morphism between complete nonsingular varieties. Do there exist sequences of blowups with smooth centers $\nu_X : \tilde{X} \to X$ and $\nu_Y : \tilde{Y} \to Y$ so that the induced map $\tilde{\phi} : \tilde{X} \dashrightarrow \tilde{Y}$ is a toroidal morphism?*

We note that the toroidalization conjecture concerns not only birational morphisms ϕ but also generically finite morphisms or morphisms with $\dim X > \dim Y$. The solution to the above conjecture would immediately lead to that of the strong factorization conjecture.

At present the authors know a complete proof only in dimension 2, where the generically finite case already requires a simple but nontrivial analysis.

References

[1] D. Abramovich and A. J. de Jong *Smoothness, semistability, and toroidal geometry*, J. Alg. Geom. 6 1997, p. 789-801.

[2] D. Abramovich and K. Karu *Weak semistable reduction in characteristic 0*, preprint `alg-geom/9707012`.

[3] D. Abramovich, K. Matsuki and S. Rashid *A note on the factorization theorem of toric birational maps after Morelli and its toroidal extension*, preprint `math.AG/9803126`.

[4] D. Abramovich and J. Wang *Equivariant resolution of singularities in characteristic 0*, Math. Res. Letters (to appear) 1997.

[5] S. Abhyankar *On the valuations centered in a local domain,* Amer. J. Math. **78** (1956), 321–348.

[6] S. Akbulut and H. King *Topology of algebraic sets*, MSRI publications 25.

[7] E. Bierstone and D. Milman *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant*, Invent. math. 128, 1997, p. 207-302.

[8] M. Brion and C. Procesi, *Action d'un tore dans une variété projective,* in *Operator algebras, unitary representations, enveloping algebras, and invariant theory (Paris, 1989)*, 509–539, Progr. Math., 92, Birkhäuser, Boston, Boston, MA, 1990.

[9] C. Christensen *Strong domination/weak factorization of three dimensional regular local rings*, Journal of the Indian Math. Soc. 45, 1981, p. 21-47.

[10] A. Corti *Factorizing birational maps of 3-folds after Sarkisov*, J. Alg. Geom. 4, 1995, p. 23-254.

[11] B. Crauder *Birational morphisms of smooth algebraic threefolds collapsing three surfaces to a point*, Duke Math. J. 48, 1981, p. 589-632.

[12] S. D. Cutkosky *Local factorization of birational maps*, Advances in math. 132, 1997, p. 167-315.

[13] S. D. Cutkosky *Local factorization and monomialization of morphisms*, preprint 1997, p. 1-133.

[14] S. D. Cutkosky *Local factorization and monomialization of morphisms*, preprint `math.AG/9803078`.

[15] V. I. Danilov *The geometry of toric varieties*, Russian Math. Surveys1978, 33:2, p. 97-154.

[16] V. I. Danilov *Birational geometry of toric 3-folds* 1983, 21, p. 269-280.

[17] C. De Concini and C. Procesi *Complete Symmetric Varieties II 1985*, in Algebraic Groups and Related Topics (R. Hotta, ed.) Adv. Studies in Pure Math. 6, p. 481-513, Kinokuniya, Tokyo and North Holland, Amsterdam, New York, Oxford.

[18] G. Ewald *Blow-ups of smooth toric 3-varieties*, Abh. Math. Sem. Univ. Hamburg, 57, 1987, p. 193-201.

[19] W. Fulton *Introduction to Toric varieties*, Annals of Mathematics studies 131 Princeton University press 1993.

[20] R. Hartshorne *Algebraic Geometry*, Graduate Texts in Mathematics 52, Springer-Verlag, 1977.

[21] H. Hironaka *On the theory of birational blowing-up*, Thesis (Harvard) 1960.

[22] H. Hironaka *An example of a non-Kälerian complex-analytic deformation of Kählerian complex structure*, Annals of Math. (2), 75, 1962, p. 190-208.

[23] H. Hironaka *Resolution of singularities of an algebraic variety over a field of characteristic zero*, Annals of Math. vol 79, 1964, p. 109-326.

[24] Y. Hu *The geometry and topology of quotient varieties of torus actions,* Duke Math. J. **68** (1992), no. 1, 151–184; Erratum, Duke Math. J. **68** (1992), no. 3, 609.

[25] Y. Hu *Relative geometric invariant theory and universal moduli spaces,* Internat. J. Math. **7** (1996), no. 2, 151–181.

[26] S. Iitaka *Algebraic Geometry (An Introduction to Birational geometry of Algebraic Varieties)*, Graduate Texts in Math. 76, 1982.

[27] A. J. de Jong *Smoothness, semistability, and alterations*, Publ. Math. I.H.E.S. 83, 1996, p. 51-93.

[28] K. Kato *Toric singularities*, Amer. J. Math. 116, 1994, p. 1073-1099.

[29] Y. Kawamata *On the finiteness of generators of a pluricanonical ring for a* 3*-fold of general type,* Amer. J. Math. **106** (1984), no. 6, 1503-1512.

[30] Y. Kawamata *The cone of curves of algebraic varieties,* Ann. of Math. (2) **119** (1984), no. 3, 603–633.

[31] Y. Kawamata *Crepant blowing-ups of three dimensional canonical singularities and its application to degenerations of surfaces,* Ann. of Math. **127** (1988), 93-163.

[32] Y. Kawamata, K. Matsuda and K. Matsuki *Introduction to the minimal model problem*, Advanced Studies in Pure Mathematics 10, 1987, p. 283-360.

[33] G. Kempf, F. Knudsen, D. Mumford and B. Saint-Donat *Toroidal embeddings I*, 1973, 339, Lecture Notes in Mathematics, Springer.

[34] H. King *Resolving Singularities of Maps*, Real algebraic geometry and topology (East Lansing, MI 1993) 1995 Contemp. Math., Amer. Math. Soc.

[35] H. King *Electronic mail to Morelli* 1996.

[36] S. Kleiman *Toward a numerical theory of ampleness*, Annals of Math. 84, 1966, p. 293-344.

[37] J. Kollár *The cone theorem.* Note to a paper: "The cone of curves of algebraic varieties" ([30] by Y. Kawamata, Ann. of Math. (2) **120** (1984), no. 1, 1-5.

[38] V.S. Kulikov *Decomposition of a birational map of three-dimensional varieties outside codimension 2*, Math. USSR Izvestiya 21, 1983, p. 187-200.

[39] K. Matsuki *Introduction to the Mori program*, the manuscript of a textbook to appear from Springer-Verlag 1998.

[40] K. Matsuki and R. Wentworth *Mumford-Thaddeus principle on the moduli space of vector bundles on an algebraic surface*, Int. J. of Math. Volume 8, No. 1 (1997) 97-148.

[41] J. Milnor *Morse Theory*, Annals of Math. Stud. 51, Princeton Univ. Press 1963.

[42] B. Moishezon *On n-dimensional compact varieties with n algebraically independent meromorphic functions*, Amer. Math. Soc. Trans. 63, 1967, p. 51-177.

[43] R. Morelli *The birational geometry of toric varieties*, J. Alg. Geom. 5 1996, p. 751-782.

[44] R. Morelli *Correction to "The birational geometry of toric varieties"* , homepage at the Univ. of Utah 1997, p. 767-770.

[45] S. Mori *Threefolds whose canonical bundles are not numerically effective*, Annals of Math. 1982, 116, p. 133-176.

[46] S. Mori *Flip theorem and the existence of minimal models for 3-folds*, Journal of AMS 1, 1988, p. 117-253.

[47] D. Mumford *Abelian Varieties*, Oxford University Press 1970.

[48] D. Mumford, J. Fogarty and F. Kirwan *Geometric Invarinat Theory (Third Enlarged Edition)*, Ergebnisse der Mathematikund ihrer Grenzgebiete, 34, Springer-Verlag 1992.

[49] T. Oda, *Torus embeddings and applications*, Based on joint work with Katsuya Miyake. Tata Inst. Fund. Res., Bombay, 1978.

[50] T. Oda *Convex Bodies and Algebraic Geometry*, Springer-Verlag, 15, 1988.

[51] R. Pandharipande *A compactification over $\overline{M}_g$ of the universal moduli space of slope-semistable vector bundles*, J. Amer. Math. Soc. **9** (1996), no. 2, 425–471.

[52] H. Pinkham *Factorization of birational maps in dimension 3*, Proceedings of Symposia in Pure Math. 40, 1983.

[53] M. Reid *Canonical threeefolds*, Géométrie Algémétrie Algébrique Angers 1979, A. Beauville ed. Sijthoff and Nordhoff 1980, p. 273-310.

[54] M. Reid *Minimal models of canonical 3-folds*, Adv. Stud. in Pure Math. 1, 1983, p. 131-180.

[55] M. Reid *Decomposition of Toric Morphisms*, Arithmetic and Geometry, papers dedicated to I. R. Shafarevich on the occasion of his 60th birthday, vol. II, Progress in Math. (M. Artin and J. Tate eds.) 36, 1983, p. 395-418.

[56] M. Reid *Birational geometry of 3-folds according to Sarkisov*, preprint 1991.

[57] J. Sally *Regular overrings of regular local rings* , Trans. Amer. math. Soc. 171, 1972, p. 291-300.

[58] V. G. Sarkisov *Birational maps of standard $\mathbb{Q}$-Fano fiberings*, I. V. Kurchatov Institute Atomic Energy preprint, 1989.

[59] M. Schaps *Birational morphisms of smooth threefolds collapsing three surfaces to a curve*, Duke Math. J. 48, 1981, p. 401-420.

[60] I. R. Shafarevich *Basic Algebraic Geometry*, 1977 Springer-Verlag.

[61] D. L. Shannon *Monoidal transforms*, Amer. J. Math. 1973, 45, p. 284-320.

[62] V. V. Shokurov *A nonvanishing theorem,* Izv. Akad. Nauk SSSR Ser. Mat. **49** (1985), no. 3, 635–651.

[63] H. Sumihiro *Equivariant Completion I, II*, J. Math. Kyoto Univ. 14, 15, 1974, 1975, p. 1-28, 573-605.

[64] M. Thaddeus *Stable pairs, linear systems and the Verlinde formula*, Invent. Math. 117, 1994, p. 317-353.

[65] M. Thaddeus *Geometric invariant theory and flips*, J. Amer. Math. Soc. 9, 1996, p. 691-723.

[66] O. Villamayor *Constructiveness of Hironaka's resolution.* Ann. Sci. École Norm. Sup. (4) 22 (1989), no. 1, 1–32.

[67] J. Włodarczyk *Decomposition of Birational Toric Maps in Blow-Ups and Blow-Downs. A Proof of the Weak Oda Conjecture 1997* , Transactions of the AMS 349, pp. 373-411.

[68] J. Włodarczyk *Birational cobordism and factorization of birational maps*, preprint `math.AG/9904074`.

[69] J. Włodarczyk *Combinatorial structures on toroidal varieties and a proof of the weak Factorization Theorem*, preprint `math.AG/9904076`.

[70] O. Zariski *Algebraic Surfaces* Springer-Verlag 1934.

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