

The plank problem for symmetric bodies

by

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Abstract. Given a symmetric convex body C and n hyperplanes in an Euclidean space, there is a translate of a multiple of C , at least $\frac{1}{n+1}$ times as large, inside C , whose interior does not meet any of the hyperplanes. The result generalizes Bang's solution of the plank problem of Tarski and has applications to Diophantine approximation.

AMS 1980 Subject classification: 52A37, 46A22

⁽¹⁾Partially supported by NSF DMS-8807243

§1. Introduction and preliminary observations.

In the 1930's, Tarski posed what came to be known as the plank problem. A *plank* in \mathbb{R}^d is the region between two distinct parallel hyperplanes. Tarski conjectured that if a convex body of minimum width w is covered by a collection of planks in \mathbb{R}^d , then the sum of the widths of these planks is at least w . Tarski himself proved this for the disc in \mathbb{R}^2 . The problem was solved in general by Bang [B]. At the end of his paper, Bang asked whether his theorem could be strengthened by asking that the width of each plank should be measured relative to the width of the convex body being covered, in the direction of the normal to the plank. This affine invariant plank problem has a number of natural formulations: in particular, as the multi-dimensional “pigeon-hole principle” stated in the abstract. The history of the affine plank problem from Bang’s paper to the present, together with many interesting remarks can be found in the papers [Gr], [R] and especially [Ga].

In the case of symmetric bodies, the problem is perhaps most naturally stated in terms of normed spaces. Let X be a normed space. A plank in X , is a region of the form

$$\{x \in X: |\phi(x) - m| \leq w\}$$

where ϕ is a functional in X^* , m a real number and w a positive number. If ϕ is taken to be a functional of norm 1, w is said to be the *half-width* of the plank. The theorem proved here is the following.

Theorem 1. If the unit ball of a Banach space X is covered by a (countable) collection of planks in X , then the sum of the half-widths of these planks is at least 1.

The theorem is obviously best possible in the sense that for every unit vector $\phi \in X^*$, the ball of X can be covered by one (or more) planks, perpendicular to ϕ , whose half-widths add up to 1.

The infinite-dimensional case of Theorem 1 does not follow formally from the finite-dimensional: it will be discussed and proved in Section 3 of the paper. For finite-dimensional spaces, one can restate Theorem 1, with the aid of compactness, as follows.

Theorem 2. If $(\phi_i)_1^n$ is a sequence of unit functionals on a finite-dimensional normed space

$X, (m_i)_1^n$ is a sequence of reals and $(w_i)_1^n$ a sequence of positive numbers with $\sum w_i = 1$ then there is a point x in the unit ball of X for which

$$|\phi_i(x) - m_i| \geq w_i \quad \text{for every } i.$$

The question answered by Theorem 2 arises quite naturally in the theory of badly approximable numbers. In his paper [D], Davenport made use of the following observation. If C is a cube in \mathbb{R}^d and $(H_i)_1^n$ are n hyperplanes, then there is a cube C' at least 2^{-n} times as large as C , inside C , with faces parallel to those of C , whose interior is not met by any H_i . This pigeon-hole principle can be strengthened considerably if Theorem 2 is invoked. (This was already noticed by Alexander in [A].) The estimate below immediately transfers to give sharper estimates in Davenport's theorems.

Corollary. If C is a convex body, with a center of symmetry, in \mathbb{R}^d and $(H_i)_1^n$ are hyperplanes, then there is a set of the form $x + \frac{1}{n+1}C$ inside C , whose interior is not met by any H_i . The result is obviously sharp for every n and C .

Proof. Assume that C is centered at the origin and let X be the normed space represented on \mathbb{R}^d with unit ball C . For each i , choose a functional ϕ_i of norm 1 in X^* and a real number m_i so that

$$H_i = \{x \in \mathbb{R}^d: \phi_i(x) = m_i\}.$$

By Theorem 2 there is a point $x \in \frac{n}{n+1}C$ with

$$|\phi_i(x) - m_i| \geq \frac{1}{n+1} \quad \text{for each } i.$$

Then the set

$$x + \frac{1}{n+1}C \subset C$$

and for every y in $x + \frac{1}{n+1}C$, $\|y - x\| \leq \frac{1}{n+1}$ so that for each i , $|\phi_i(y) - \phi_i(x)| \leq \frac{1}{n+1}$: hence $\phi_i(y) - m_i$ has the same sign as $\phi_i(x) - m_i$. Thus, for each i , the whole of $x + \frac{1}{n+1}C$ lies on the same side of H_i as x does. \square

Theorem 2 is readily reduced to a combinatorial theorem concerning matrices. For a sequence $(\phi_i)_1^n$ of norm 1 functionals on X , construct a matrix $A = (a_{ij})$ given by

$$a_{ij} = \phi_i(x_j), \quad 1 \leq i, j \leq n$$

where for each j , x_j is a point in the unit ball of X at which ϕ_j attains its norm; i.e. $\phi_j(x_j) = \|x_j\| = 1$. If $(\lambda_j)_1^n$ is a sequence of reals with

$$\sum |\lambda_j| \leq 1$$

then the vector $x = \sum \lambda_j x_j$ has norm at most 1 and for each i ,

$$\phi_i(x) = \sum_j a_{ij} \lambda_j.$$

Thus, Theorem 2 follows from

Theorem 2'. Let $A = (a_{ij})$ be an $n \times n$ matrix whose diagonal entries equal 1, $(m_i)_1^n$ a sequence of reals and $(w_i)_1^n$ a sequence of non-negative numbers with $\sum_i^n w_i \leq 1$. Then there is a sequence $(\lambda_j)_1^n$ with

$$\sum_j |\lambda_j| \leq 1$$

and, for each i ,

$$\left| \sum_j a_{ij} \lambda_j - m_i \right| \geq w_i.$$

It is also easy to see that Theorem 2' follows immediately from Theorem 2 by regarding the rows of such a matrix as unit vectors in ℓ_∞^n . Bang effectively proved Theorem 2' for symmetric matrices: his elegant argument is reproduced here as a lemma, since the precise statement will be needed later.

Lemma 3 (Bang). Let $H = (h_{ij})$ be a real, symmetric $n \times n$ matrix with 1's on the diagonal, $(\mu_i)_1^n$ a sequence of reals and $(\theta_i)_1^n$ a sequence of non-negative numbers. Then there is a sequence of signs $(\varepsilon_j)_1^n$ ($\varepsilon_j = \pm 1$ for each j) so that for each i ,

$$\left| \sum_j h_{ij} \varepsilon_j \theta_j - \mu_i \right| \geq \theta_i.$$

Proof. Choose signs $(\varepsilon_j)_1^n$ so as to maximise

$$\sum_{i,j} h_{ij} \varepsilon_i \varepsilon_j \theta_i \theta_j - 2 \sum_i \varepsilon_i \theta_i \mu_i.$$

Fix k ($1 \leq k \leq n$) and define $(\delta_j)_1^n$ by

$$\delta_j = \begin{cases} \varepsilon_j & \text{if } j \neq k \\ -\varepsilon_j & \text{if } j = k. \end{cases}$$

Then

$$0 \leq \sum_{i,j} h_{ij} \varepsilon_i \varepsilon_j \theta_i \theta_j - 2 \sum_i \varepsilon_i \theta_i \mu_i - \left(\sum_{i,j} h_{ij} \delta_i \delta_j \theta_i \theta_j - 2 \sum_i \delta_i \theta_i \mu_i \right)$$

and since H is symmetric this expression is

$$4\varepsilon_k \theta_k \sum_{j \neq k} h_{kj} \varepsilon_j \theta_j - 4\varepsilon_k \theta_k \mu_k.$$

Since $h_{kk} = 1$, the latter is

$$-4\theta_k^2 + 4\varepsilon_k \theta_k \sum_j h_{kj} \varepsilon_j \theta_j - 4\varepsilon_k \theta_k \mu_k,$$

and so

$$\begin{aligned} 4\theta_k^2 &\leq 4\varepsilon_k \theta_k \left(\sum_j h_{kj} \varepsilon_j \theta_j - \mu_k \right) \\ &\leq 4\theta_k \left| \sum_j h_{kj} \varepsilon_j \theta_j - \mu_k \right|. \end{aligned}$$

Since this holds for each k , the result is proved. □

Note that the hypothesis of symmetry cannot be dropped from Lemma 3: consider, for example, the matrix $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ for $\theta_1 = \theta_2 = 1$ and $\mu_1 = \mu_2 = 0$.

In the proof of Theorem 2' it may be assumed that $w_i = \frac{1}{n}$, $1 \leq i \leq n$ since planks of varying widths can be almost covered by slightly overlapping "sheets", all of the same width. (This "change of density" argument is not really needed but simplifies the succeeding arguments.) It will be shown that in this situation, Theorem 2' can be strengthened: the sequence $(\lambda_j)_1^n$ to be chosen will actually satisfy

$$\sum_j \lambda_j^2 \leq \frac{1}{n}$$

(which implies that $\sum_j |\lambda_j| \leq 1$ by the Cauchy-Schwartz inequality). This stronger statement can be attacked by Hilbert space methods: if the statement is true for AU where U is an orthogonal matrix, then it is true for A . Unfortunately, it is not the case that for every matrix A with 1's on the diagonal, there is an orthogonal matrix U with AU both symmetric and having large diagonal. For example, if

$$A = \begin{pmatrix} 1 & 1 \\ \frac{1}{2} & 1 \end{pmatrix}$$

then the only symmetric matrices of the form AU are

$$\pm \begin{pmatrix} 1 & 1 \\ 1 & \frac{1}{2} \end{pmatrix} \quad \text{and} \quad \pm \begin{pmatrix} \frac{5}{\sqrt{17}} & \frac{3}{\sqrt{17}} \\ \frac{3}{\sqrt{17}} & \frac{7}{2\sqrt{17}} \end{pmatrix}.$$

Nevertheless, Theorem 2' is proved by using a modified matrix A .

§2. Symmetrisations of matrices and the proof of the main theorem.

The modification of a matrix, needed for the proof, is described by the following lemma. From now on, if H is a matrix, H will be said to be positive if it is symmetric and positive semi-definite.

Lemma 4. Let A be an $n \times n$ matrix of reals, each of whose rows is non-null. Then there is a sequence $(\theta_i)_1^n$ of positive numbers and an orthogonal matrix U so that the matrix $H = (h_{ij})$ given by

$$h_{ij} = \theta_i(AU)_{ij}$$

is positive and has 1's on the diagonal.

Lemma 4 can be proved using a fixed point theorem or other topological methods. However it has an elementary proof which provides an alternative description of the sequence $(\theta_i)_1^n$. Recall that for a matrix B , the trace-class, or nuclear, norm $\|B\|_{C_1}$ of B , is $\text{tr}(H)$, where H is the positive square root of BB^* . By the Cauchy-Schwartz inequality

$$\|B\|_{C_1} = \max\{\text{tr}(BU): U \text{ orthogonal}\}.$$

Also by the Cauchy-Schwartz inequality, if B and C are $n \times n$ matrices then

$$\|BC\|_{C_1} \leq (\text{tr}(BB^*))^{1/2}(\text{tr}(CC^*))^{1/2}.$$

Before the proof of Lemma 4 it will be convenient to prove the lemma that really forms the crux of the proof of Theorem 2. The estimate is somewhat unusual since it involves the sum of the squares of the diagonal entries of a matrix: nevertheless, it is a consequence of the matrix Cauchy-Schwartz inequality.

Lemma 5. If $H = (h_{ij})$ is a positive matrix with non-zero diagonal entries and U is orthogonal then

$$\sum_i \frac{(HU)_{ii}^2}{h_{ii}} \leq \sum_i h_{ii}.$$

Proof. For each i let $\gamma_i = \frac{(HU)_{ii}}{h_{ii}}$ and let D be the diagonal matrix, $\text{diag}(\gamma_i)_1^n$ and T , the positive square root of H . Then

$$\begin{aligned}
\sum_i \frac{(HU)_{ii}^2}{h_{ii}} &= \sum_i \gamma_i (HU)_{ii} \\
&= \text{tr } DHU \leq \|DH\|_{C_1} \\
&= \|(DT)T\|_{C_1} \\
&\leq [\text{tr } DT(DT)^*]^{\frac{1}{2}} [\text{tr } TT^*]^{\frac{1}{2}} \\
&= [\text{tr } DHD]^{\frac{1}{2}} [\text{tr } H]^{\frac{1}{2}} \\
&= \left(\sum_i \gamma_i^2 h_{ii} \right)^{\frac{1}{2}} \left(\sum_i h_{ii} \right)^{\frac{1}{2}} \\
&= \left(\sum_i \frac{(HU)_{ii}^2}{h_{ii}} \right)^{\frac{1}{2}} \left(\sum_i h_{ii} \right)^{\frac{1}{2}}.
\end{aligned}$$

Hence

$$\left(\sum_i \frac{(HU)_{ii}^2}{h_{ii}} \right)^{\frac{1}{2}} \leq \left(\sum_i h_{ii} \right)^{\frac{1}{2}}.$$

□

Lemma 5 immediately implies:

Lemma 6. If $H = (h_{ij})$ is a positive $n \times n$ matrix with non-zero diagonal entries, then

$$\left\| \left(\frac{1}{\sqrt{h_{ii}}} h_{ij} \right) \right\|_{C_1} \leq \sqrt{n} \|H\|_{C_1}^{1/2}.$$

Proof. There is some orthogonal matrix U for which

$$\left\| \left(\frac{1}{\sqrt{h_{ii}}} h_{ij} \right) \right\|_{C_1} = \sum_i \frac{1}{\sqrt{h_{ii}}} (HU)_{ii}.$$

By the Cauchy-Schwartz inequality this is at most

$$\sqrt{n} \left(\sum_i \frac{(HU)_{ii}^2}{h_{ii}} \right)^{\frac{1}{2}}$$

and this is at most $\sqrt{n} (\sum h_{ii})^{\frac{1}{2}} = \sqrt{n} \|H\|_{C_1}$ by Lemma 5. □

Proof of Lemma 4. Plainly it suffices to find $(\theta_i)_1^n$ positive and U orthogonal so that $(\theta_i (AU)_{ij})$ is positive and has constant, non-zero, diagonal.

Since the rows of A are non-null, there is a constant $c > 0$ so that if $(\theta_i)_1^n$ is a sequence of positive numbers

$$\|(\theta_i a_{ij})\|_{C_1} \geq c \max_i \theta_i.$$

Since $\|(\theta_i a_{ij})\|_{C_1}$ is continuous as a function of $(\theta_i)_1^n$, there is a sequence $(\theta_i)_1^n$ of positive numbers which minimises

$$\|(\theta_i a_{ij})\|_{C_1}$$

subject to the condition $\prod_i \theta_i = 1$. Let $H = (h_{ij})$ be the positive square root of $(\theta_i (AA^*)_{ij} \theta_j)$, for this particular sequence, and note that there is an orthogonal matrix U for which

$$h_{ij} = \theta_i (AU)_{ij}, \quad 1 \leq i, j \leq n.$$

Again, since A has non-null rows, the diagonal entries of H are non-zero. For each i , let

$$\gamma_i = \frac{1}{\sqrt{h_{ii}}} \left(\prod_{j=1}^n \sqrt{h_{jj}} \right)^{1/n}.$$

Since $\prod_i \gamma_i = 1$, the matrix $(\gamma_i \theta_i a_{ij})$ has nuclear norm at least that of $(\theta_i a_{ij})$, the latter being $\|H\|_{C_1}$. So

$$\begin{aligned} \|H\|_{C_1} &\leq \|(\gamma_i \theta_i a_{ij})\|_{C_1} \\ &= \|(\gamma_i h_{ij})\|_{C_1} \\ &= \left(\prod_{k=1}^n \sqrt{h_{kk}} \right)^{\frac{1}{n}} \left\| \left(\frac{1}{\sqrt{h_{ii}}} h_{ij} \right) \right\|_{C_1} \\ &\leq \sqrt{n} \|H\|_{C_1}^{\frac{1}{2}} \left(\prod_k \sqrt{h_{kk}} \right)^{\frac{1}{n}} \end{aligned}$$

by Lemma 6. So

$$\left(\frac{1}{n} \sum_i h_{ii}\right)^{\frac{1}{2}} \leq \left(\prod_k h_{kk}\right)^{\frac{1}{2n}}$$

implying that the h_{ii} 's are all the same. \square

Proof of Theorem 2'. The statement to be proved is that if $A = (a_{ij})$ is a real $n \times n$ matrix with 1's on the diagonal and $(m_i)_1^n$ is a sequence of reals, then there is a sequence $(\lambda_j)_1^n$ of reals with

$$\sum_j \lambda_j^2 \leq \frac{1}{n}$$

and, for every i

$$\left| \sum_j a_{ij} \lambda_j - m_i \right| \geq \frac{1}{n}.$$

Using Lemma 4, choose a sequence $(\theta_j)_1^n$ of positive numbers and an orthogonal matrix U , so that if

$$H = (\theta_i (AU)_{ij}), \tag{2}$$

then H is positive and has 1's on the diagonal.

By Lemma 3, there is a choice of signs $(\varepsilon_j)_1^n$ so that for each i ,

$$\left| \sum_j h_{ij} \varepsilon_j \theta_j - n \theta_i m_i \right| \geq \theta_i. \tag{3}$$

From (2) and (3), one has that for each i ,

$$\left| \theta_i \sum_j (AU)_{ij} \varepsilon_j \theta_j - n \theta_i m_i \right| \geq \theta_i,$$

and hence

$$\left| \sum_k a_{ik} \left(\frac{1}{n} \sum_j u_{kj} \varepsilon_j \theta_j \right) - m_i \right| \geq \frac{1}{n}.$$

For each k set

$$\lambda_k = \frac{1}{n} \sum_j u_{kj} \varepsilon_j \theta_j.$$

It remains to check that $\sum \lambda_k^2 \leq \frac{1}{n}$. But $\sum \lambda_k^2 = \frac{1}{n^2} \sum \theta_j^2$ since U is orthogonal and so what is needed is

$$\sum_j \theta_j^2 \leq n.$$

From (2),

$$\theta_i a_{ij} = (HU^*)_{ij} \quad \text{for all } i \text{ and } j$$

and so in particular, since $a_{ii} = 1$ for each i ,

$$\theta_i = (HU^*)_{ii}, \quad 1 \leq i \leq n.$$

Now since $h_{ii} = 1$ for each i , Lemma 5 shows that

$$\begin{aligned} \sum_i \theta_i^2 &= \sum_i (HU^*)_{ii}^2 = \sum_i \frac{(HU^*)_{ii}^2}{h_{ii}} \\ &\leq \sum_i h_{ii} = n. \end{aligned} \quad \square$$

§3. The infinite-dimensional case.

Theorem 2 and weak*-compactness immediately imply the following “multiple Hahn-Banach” theorem.

Theorem 7. Let $(x_i)_1^\infty$ be a sequence of unit vectors in a normed space X , $(m_i)_1^\infty$ a sequence of real numbers and $(w_i)_1^\infty$ a sequence of non-negative reals with $\sum_1^\infty w_i \leq 1$. Then there is a linear functional ϕ of norm at most 1 in X^* with

$$|\phi(x_i) - m_i| \geq w_i \quad \text{for every } i.$$

For reflexive spaces, Theorem 1 follows immediately from Theorem 2. For general spaces, Theorem 1 is a little more delicate. It can be regarded as a quantitative strengthening of the Banach-Steinhaus theorem. If Φ is an unbounded subset of the dual X^* of a Banach space X , then there are elements ϕ_1, ϕ_2, \dots of Φ with (say)

$$\sum_1^\infty n \|\phi_n\|^{-1} < 1.$$

By Theorem 1, there is a point $x \in X$ of norm at most 1 so that for each n ,

$$\left| \frac{\phi_n}{\|\phi_n\|}(x) \right| > n \|\phi_n\|^{-1},$$

i.e. $|\phi_n(x)| > n$.

To prove Theorem 1 it is necessary to examine the proof of Theorem 2 more closely. The change of density argument in Section 1 and the proof in Section 2 actually yield the following strong form of Theorem 2'. Theorem 1 will be deduced from this.

Theorem 8. Let (a_{ij}) be a real, $n \times n$ matrix with 1's on the diagonal, $(m_i)_1^n$ a sequence of real numbers and $(w_i)_1^n$, a sequence of positive numbers. Then there is a sequence $(\lambda_j)_1^n$ with

$$\sum_j w_j^{-1} \lambda_j^2 \leq \sum_j w_j$$

and, for every i ,

$$\left| \sum_j a_{ij} \lambda_j - m_i \right| \geq w_i. \quad \square$$

Proof of Theorem 1. Suppose $(\phi_i)_1^\infty$ are unit functionals in X^* , $(m_i)_1^\infty$ are real numbers and $(w_i)_1^\infty$ are non-negative numbers with $\sum_i w_i < 1$. The problem is to find a point x in the unit ball of X with

$$|\phi_i(x) - m_i| > w_i \quad \text{for each } i.$$

Choose a sequence $(v_i)_1^\infty$ with

$$v_i > w_i \geq 0 \quad \text{for each } i$$

but

$$\sum_i v_i = 1 - \varepsilon < 1.$$

For each i , choose a point x_i of norm at most 1 in X with $\phi_i(x_i) = 1 - \varepsilon$. Applying Theorem 8 to the first n functionals and vectors one obtains, for each n , a sequence $(\lambda_j^{(n)})_{j=1}^n$ satisfying

$$\sum_{j=i}^n v_i^{-1} (\lambda_j^{(n)})^2 \leq (1 - \varepsilon)^{-2} \sum_{j=i}^n v_j < (1 - \varepsilon)^{-1} \quad (5)$$

and for $1 \leq i \leq n$

$$\left| \phi_i \left(\sum_{j=i}^n \lambda_j^{(n)} x_j \right) - m_i \right| \geq v_i.$$

Regard $(\lambda_j^{(n)})$ as an infinite sequence by filling out with zeroes. From (5), for each n ,

$$\begin{aligned} \sum_{j=1}^{\infty} |\lambda_j^{(n)}| &\leq \left(\sum_{j=1}^{\infty} v_j \right)^{1/2} \left(\sum_{j=1}^{\infty} v_j^{-1} (\lambda_j^{(n)})^2 \right)^{1/2} \\ &< (1 - \varepsilon)^{1/2} (1 - \varepsilon)^{-1/2} = 1. \end{aligned} \quad (6)$$

Moreover, for each m and n ,

$$\sum_{j=m}^{\infty} |\lambda_j^{(n)}| \leq \left(\sum_{j=m}^{\infty} v_j \right)^{1/2} (1 - \varepsilon)^{-1/2}.$$

Since the right-hand side $\rightarrow 0$ as $m \rightarrow \infty$, the sequences $(\lambda_j^{(n)})_{j=1}^{\infty}$ are uniformly summable, so the collection has a (norm) limit point $(\lambda_j)_1^{\infty}$ (say) in ℓ_1 . From (6), the point $x = \sum_j \lambda_j x_j \in X$ has norm at most 1 and clearly

$$|\phi_i(x) - m_i| \geq v_i > w_i \quad \text{for every } i. \quad \square$$

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