The Intrinsic Normal Cone

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Abstract

We suggest a construction of virtual fundamental classes of certain types of moduli spaces.

Contents

0 Introduction

Moduli spaces in algebraic geometry often have an expected dimension at each point, which is a lower bound for the dimension at that point. For instance, the moduli space of smooth, complex projective n-dimensional varieties with ample canonical class has expected dimension $h^1(V,T_V) - h^2(V,T_V)$ at a point $[V]$. In general, the expected dimension will vary with the point; however, in some significant cases it will stay constant on connected components. In the previous example, this is the case if $n \leq 2$, for then the expected dimension is $-\chi(V,T_V)$. In some cases the dimension coincides with the expected dimension, in others it does so under some genericity assumptions. However, it can happen that there is no way to get a space of the expected dimension; it is also possible that special cases with bigger dimension are easier to understand and to deal with than the generic case.

When we have a moduli space X which has a well-defined expected dimension, it can be useful to be able to construct in its Chow ring a class of the expected dimension. The main examples we have in mind are Donaldson theory (with X the moduli space of torsion-free, semi-stable sheaves on a surface) and the Gromov-Witten invariants (with X the moduli space of stable maps from curves of genus g to a fixed projective variety). In this paper we deal with the problem of defining such a class in a very general set-up; the construction is divided into two steps.

First, given any Deligne-Mumford stack X, we associate to it an algebraic stack $\mathfrak{C}_X$ over X of pure dimension zero, its *intrinsic normal cone*.

This has nothing to do with X being a moduli space; it is just an intrinsic invariant, whose structure is related to the singularities of X (see for instance Proposition 3.12).

Then, we define the concept of an obstruction theory and of a perfect obstruction theory for X. To say that X has an obstruction theory means, very roughly speaking, that we are given locally on X an (equivalence class of) morphisms of vector bundles such that at each point the kernel of the induced linear map of vector spaces is the tangent space to X, and the cokernel is a space of obstructions. Usually, if X is a moduli space then it has an obstruction theory, and if this is perfect then the expected dimension is constant on X. Once we are given an obstruction theory, with the additional (technical) assumption that it admits a global resolution, we can define a virtual fundamental class of the expected dimension.

An application of the results of this work is contained in a paper [3] by the first author. There Gromov-Witten invariants are constructed for any genus, any target variety and the axioms listed in [4] are verified.

We now give a more detailed outline of the contents of the paper. In the first section we recall what we need about cones and we introduce the notion of cone stacks over a Deligne-Mumford stack X. These are Artin stacks which are locally the quotient of a cone by a vector bundle acting on it. We call a cone *abelian* if it is defined as $\operatorname{Spec}\operatorname{Sym}\mathcal{F}$, where $\mathcal{F}$ is a coherent sheaf on X. Every cone is contained as a closed subcone in a minimal abelian one, which we call its *abelian hull.* The notions of being abelian and of abelian hull generalize immediately to cone stacks.

In the second section we construct, for a complex $E^\bullet$ in the derived category $D(\mathcal{O}_X)$ which satisfies some suitable assumptions (which we call Condition $(\star)$, see Definition 2.3), an associated abelian cone stack $h^1/h^0((E^\bullet)^\vee)$. In particular the cotangent complex $L_X^\bullet$ of X satisfies Condition $(\star)$, so we can define the abelian cone stack $\mathfrak{N}_X := h^1/h^0((L_X^\bullet)^\vee)$, the *intrinsic normal sheaf.*

The name is motivated in the third section, where $\mathfrak{N}_X$ is constructed more directly as follows: étale locally on X, embed an open set U of X in a smooth scheme W, and take the stack quotient of the normal sheaf (viewed as abelian cone) $N_{U/W}$ by the natural action of $T_W|_U$. One can glue these abelian cone stacks together to get $\mathfrak{N}_X$. The intrinsic normal cone $\mathfrak{C}_X$ is the closed subcone stack of $\mathfrak{N}_X$ defined by replacing $N_{U/W}$ by the normal cone $C_{U/W}$ in the previous construction.

In the fourth section we describe the relationship between the intrinsic normal sheaf of a Deligne-Mumford stack X and the deformations of affine

X-schemes, showing in particular that $\mathfrak{N}_X$ carries obstructions for such deformations. With this motivation, we introduce the notion of obstruction theory for X. This is an object $E^\bullet$ in the derived category together with a morphism $E^\bullet \to L_X^\bullet$, satisfying Condition ($\star$) and such that the induced map $\mathfrak{N}_X \to h^1/h^0((E^\bullet)^\vee)$ is a closed immersion.

An obstruction theory $E^\bullet$ is called perfect if $\mathfrak{E} = h^1/h^0((E^\bullet)^\vee)$ is smooth over X. So we have a vector bundle stack $\mathfrak{E}$ with a closed subcone stack $\mathfrak{C}_X$, and to define the virtual fundamental class of X with respect to $E^\bullet$ we simply intersect $\mathfrak{C}_X$ with the zero section of $\mathfrak{E}$. This construction requires Chow groups for Artin stacks, which we do not have at our disposal. There are several ways around this problem. We choose to assume that $E^\bullet$ is globally given by a homomorphism of vector bundles $F^{-1} \to F^0$. Then $\mathfrak{C}_X$ gives rise to a cone C in $F_1 = {F^{-1}}^\vee$ and we intersect C with the zero section of F_1.

Another approach, suggested by Kontsevich [11], is via virtual structure sheaves (see Remark 5.4). The drawback of that approach is that it requires a Riemann-Roch theorem for Deligne-Mumford stacks, for which we do not know a reference.

In the sixth section we give some examples of how this construction can be applied in some standard moduli problems. We consider the following cases: a fiber of a morphism between smooth algebraic stacks, the scheme of morphisms between two given projective schemes, a moduli space for Gorenstein projective varieties.

In the seventh section we give a relative version of the intrinsic normal cone and sheaf $\mathfrak{C}_{X/Y}$ and $\mathfrak{N}_{X/Y}$ for a morphism $X \to Y$ with unramified diagonal of algebraic stacks; we are mostly interested in the case where Y is smooth and pure-dimensional, which preserves many good properties of the absolute case (e.g., $\mathfrak{C}_{X/Y}$ is pure-dimensional). This is not needed in this paper, but will be applied by the first author to give an algebraic definition of Gromov-Witten classes for smooth projective varieties.

The starting point for this work was a talk by Jun Li at the AMS Summer Institute on Algebraic Geometry, Santa Cruz 1995, where he reported on joint work in progress with G. Tian. Their construction, in the complex analytic context, is based on the existence of the Kuranishi map; by using it they define, under suitable assumptions, a pure-dimensional cone in some bundle and get classes of the expected dimension by intersecting with the zero section.

Our construction owes its existence to theirs; we started by trying to understand and reformulate their results in an algebraic way, and found stacks to be a convenient, intrinsic language. In our opinion the introduction of stacks is very natural, and it seems almost surprising that the intrinsic normal cone was not defined before. We find it important to separate the construction of the cone, which can be carried out for any Deligne-Mumford stack, from its embedding in a vector bundle stack. We work completely in an algebraic context; of course the whole paper could be rewritten without changes over the category of analytic spaces.

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Notations and Conventions

Unless otherwise mentioned, we work over a fixed ground field k.

An *algebraic stack* is an algebraic stack over k in the sense of [1] or [12]. Unless mentioned otherwise, we assume all algebraic stacks (in particular all algebraic spaces and all schemes) to be quasi-separated and locally of finite type over k.

A *Deligne-Mumford stack* is an algebraic stack in the sense of [5], in other words an algebraic stack with unramified diagonal. For a Deligne-Mumford stack X we denote by X_{fl} the big fppf-site and by $X_{\text{ét}}$ the small étale site of X. The associated topoi of sheaves are denoted by the same symbols.

Recall that a complex of sheaves of modules is *of perfect amplitude contained in* $[a, b]$, where $a, b \in \mathbb{Z}$, if, locally, it is isomorphic (in the derived category) to a complex $E^a \to \ldots \to E^b$ of locally free sheaves of finite rank.

1 Cones and Cone Stacks

Cones

To fix notation we recall some basic facts about cones.

Let X be a Deligne-Mumford stack. Let

$$S = \bigoplus_{i \geq 0} S^i$$

be a graded quasi-coherent sheaf of $\mathcal{O}_X$-algebras such that $S^0 = \mathcal{O}_X$, S^1 is coherent and S is generated locally by S^1. Then the affine X-scheme

$C = \operatorname{Spec} S$ is called a *cone* over X. A *morphism* of cones over X is an X-morphism induced by a graded morphism of graded sheaves of $\mathcal{O}_X$-algebras. A *closed subcone* is the image of a closed immersion of cones. If

$$\begin{array}{ccc} & & C_2 \\ & & \downarrow \\ C_1 & \longrightarrow & C_3 \end{array}$$

is a diagram of cones over X, the fibered product $C_1 \times_{C_3} C_2$ is a cone over X.

Every cone $C \to X$ has a section $0 : X \to C$, called the *vertex* of C, and an $\mathbb{A}^1$-action (or a multiplicative contraction onto the vertex), that is a morphism

$$\gamma : \mathbb{A}^1 \times C \longrightarrow C$$

such that

1.
$$\begin{array}{ccc} C & \xrightarrow{(1,\mathrm{id})} & \mathbb{A}^1 \times C \\ & \mathrm{id} \searrow & \downarrow \gamma \\ & & C \end{array}$$

commutes,

2.
$$\begin{array}{ccc} C & \xrightarrow{(0,\mathrm{id})} & \mathbb{A}^1 \times C \\ & 0 \searrow & \downarrow \gamma \\ & & C \end{array}$$

commutes,

3.
$$\begin{array}{ccc} \mathbb{A}^1 \times \mathbb{A}^1 \times C & \xrightarrow{\mathrm{id} \times \gamma} & \mathbb{A}^1 \times C \\ {\scriptstyle m \times \mathrm{id}} \downarrow & & \downarrow \gamma \\ \mathbb{A}^1 \times C & \xrightarrow{\gamma} & C \end{array}$$

commutes, where $m : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ is multiplication, $m(x, y) = xy$.

The vertex of C is induced by the augmentation $S \to S^0$, the $\mathbb{A}^1$-action is given by the grading of S. In fact, the morphism $S \to S[x]$ giving rise to γ maps $s \in S^i$ to sx^i.

Note that a morphism of cones is just a morphism respecting 0 and γ.

Abelian Cones

If $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module we get an associated cone

$$C(\mathcal{F}) = \operatorname{Spec}\operatorname{Sym}(\mathcal{F}).$$

For any X-scheme T we have

$$C(\mathcal{F})(T) = \operatorname{Hom}(\mathcal{F}_T, \mathcal{O}_T),$$

so $C(\mathcal{F})$ is a group scheme over X. We call a cone of this form an *abelian cone*. A fibered product of abelian cones is an abelian cone. If E is a vector bundle over X, then $E = C(\mathcal{E}^\vee)$, where $\mathcal{E}$ is the coherent O_X-module of sections of E and $\mathcal{E}^\vee$ its dual.

Any cone $C = \operatorname{Spec}\bigoplus_{i\geq 0} S^i$ is canonically a closed subcone of an abelian cone $A(C) = \operatorname{Spec}\operatorname{Sym} S^1$, called the *associated abelian cone* or the *abelian hull* of C. The abelian hull is a vector bundle if and only if S^1 is locally free. Any morphism of cones $\phi : C \to D$ induces a morphism $A(\phi) : A(C) \to A(D)$, extending ϕ. Thus A defines a functor from cones to abelian cones called *abelianization*. Note that ϕ is a closed immersion if and only if $A(\phi)$ is.

Lemma 1.1 *A cone C over X is a vector bundle if and only if it is smooth over X.*

Proof. Let $C = \operatorname{Spec}\bigoplus_{i\geq 0} S^i$, and assume that $C \to X$ has constant relative dimension r. Then $S^1 = 0^*\Omega_{C/X}$ is a rank r vector bundle. C is a closed subcone of $A(C) = (S^1)^\vee$, hence by dimension reasons $C = A(C)$. □

If E and F are abelian cones over X, then any morphism of cones $\phi : E \to F$ is a morphism of X-group schemes. If E and F are vector bundles, then ϕ is a morphism of vector bundles.

Example If $X \to Y$ is a closed immersion with ideal sheaf $\mathcal{I}$, then

$$\bigoplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$$

is a sheaf of $\mathcal{O}_X$-algebras and

$$C_{X/Y} = \operatorname{Spec}\bigoplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$$

is a cone over X, called the *normal cone* of X in Y. The associated abelian cone $N_{X/Y} = \operatorname{Spec}\operatorname{Sym}\mathcal{I}/\mathcal{I}^2$ is also called the *normal sheaf* of X in Y.

More generally, any local immersion of Deligne-Mumford stacks has a normal cone whose abelian hull is its normal sheaf (see [14], definition 1.20).

Exact Sequences of Cones

Definition 1.2 A sequence of cone morphisms

$$0 \longrightarrow E \xrightarrow{i} C \longrightarrow D \longrightarrow 0$$

is *exact* if E is a vector bundle and locally over X there is a morphism of cones $C \to E$ splitting i and inducing an isomorphism $C \to E \times D$.

Remark Given a short exact sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow 0$$

of coherent sheaves on X, with $\mathcal{E}$ locally free, then

$$0 \longrightarrow C(\mathcal{E}) \longrightarrow C(\mathcal{F}') \longrightarrow C(\mathcal{F}) \longrightarrow 0$$

is exact, and conversely (see [6], Example 4.1.7).

Lemma 1.3 *Let $C \to D$ be a smooth, surjective morphism of cones, and let $E = C \times_{D,0} X$; then the sequence*

$$0 \longrightarrow E \longrightarrow C \longrightarrow D \longrightarrow 0$$

is exact.

Proof. Write $C = \operatorname{Spec} \bigoplus S^i$, $D = \operatorname{Spec} \bigoplus S'^i$. We start by proving that

$$0 \longrightarrow E \longrightarrow A(C) \longrightarrow A(D) \longrightarrow 0$$

is exact.

By base change we may assume $S'^i = 0$ for $i \geq 2$. The cone $E = \operatorname{Spec} \operatorname{Sym} \mathcal{E}$ is a vector bundle because it is smooth. On the other hand, $E = \operatorname{Spec} \bigoplus (S^i / S'^1 S^{i-1})$. As $C \to D$ is smooth and surjective, $S^1 \to S'^1$ is injective. So we get an exact sequence

$$0 \longrightarrow S^1 \longrightarrow S'^1 \longrightarrow \mathcal{E} \longrightarrow 0.$$

To complete the proof, remark that $C \to A(C) \times_{A(D)} D$ is a closed immersion, and both these schemes are smooth of the same relative dimension over C. □

E-Cones

If E is a vector bundle and $d : E \to C$ a morphism of cones, we say that C *is an E-cone*, if C is invariant under the action of E on $A(C)$. We denote the induced action of E on C by

$$\begin{array}{ccc} E \times C & \longrightarrow & C \\ (\nu, \gamma) & \longmapsto & d\nu + \gamma \end{array} .$$

A *morphism* ϕ from an E-cone C to an F-cone D (or a *morphism of vector bundle cones*) is a commutative diagram of cones

$$\begin{array}{ccc} E & \xrightarrow{d} & C \\ \phi \downarrow & & \downarrow \phi \\ F & \xrightarrow{d} & D. \end{array}$$

If $\phi : (E, d, C) \to (F, d, D)$ and $\psi : (E, d, C) \to (F, d, D)$ are morphisms, we call them *homotopic*, if there exists a morphism of cones $k : C \to F$, such that

1. $kd = \psi - \phi$,
2. $dk = \psi - \phi$.

Here the second condition is to be interpreted as saying that $\phi + dk = \psi$. (More precisely, we say that k is a *homotopy* from ϕ to ψ.)

Remark A sequence of cone morphisms with E a vector bundle

$$0 \longrightarrow E \xrightarrow{i} C \longrightarrow D \longrightarrow 0$$

is exact if and only if C is an E-cone, $C \to D$ is surjective, and the diagram

$$\begin{array}{ccc} E \times C & \xrightarrow{\sigma} & C \\ p \downarrow & & \downarrow \phi \\ C & \xrightarrow{\phi} & D \end{array}$$

is cartesian, where p is the projection and σ the action.

Proposition 1.4 *Let $(C, 0, \gamma)$ and $(D, 0, \gamma)$ be algebraic X-spaces with sections and $\mathbb{A}^1$-actions and let $\phi : C \to D$ be an $\mathbb{A}^1$-equivariant X-morphism, which is smooth and surjective. Let $E = C \times_{D,0} X$. Then C is an E-cone over X if and only if D is a cone over X. Moreover, C is abelian (a vector bundle) if and only if D is.*

PROOF. Let us first assume that C is an abelian cone, $C = \operatorname{Spec}\operatorname{Sym}\mathcal{F}$. The morphism $E \to C$ gives rise to $\mathcal{F} \to \mathcal{E}^\vee$, where $\mathcal{E}$ is the coherent $\mathcal{O}_X$-modules of sections of E. Note that $\mathcal{F} \to \mathcal{E}^\vee$ is an epimorphism, since $E \to C$ is injective. Let $\mathcal{G}$ be the kernel, so that

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{F} \longrightarrow \mathcal{E}^\vee \longrightarrow 0$$

is a short exact sequence. Then

$$0 \longrightarrow E \longrightarrow C \longrightarrow C(\mathcal{G}) \longrightarrow 0$$

is a short exact sequence of abelian cones over X, so $D \cong C(\mathcal{G})$ and so D is an abelian cone.

In general, $C \subset A(C)$ is defined by a homogeneous sheaf of ideals $\mathcal{I} \subset \operatorname{Sym}\mathcal{S}^1$, where $\mathcal{S} = \bigoplus \mathcal{S}^i$ and $C = \operatorname{Spec}\mathcal{S}$. Let $\mathcal{F} = \mathcal{S}^1$ and let $\mathcal{G}$ as above be the kernel of $\mathcal{F} \to \mathcal{E}^\vee$. Let $\mathcal{J} = \mathcal{I} \cap \operatorname{Sym}\mathcal{G}$, which is a homogeneous sheaf of ideals in $\operatorname{Sym}\mathcal{G}$, so $C' = \operatorname{Spec}\operatorname{Sym}\mathcal{G}/\mathcal{J}$ is a cone over X. By construction, C' is the scheme theoretic image of C in $C(\mathcal{G})$. Hence C' is the quotient of C by E and so $C' \cong D$ and D is a cone.

Now for the converse. The claim is local in X. So since D is affine over X we may assume that $C = D \times E$ as X-schemes with $\mathbb{A}^1$-action. Then we are done. □

Cone Stacks

Let X be, as above, a Deligne-Mumford stack over k. We need to define the 2-category of algebraic stacks with $\mathbb{A}^1$-action over X.

Definition 1.5 Let $\mathfrak{C}$ be an algebraic stack over X, together with a section $0 : X \to \mathfrak{C}$. An $\mathbb{A}^1$-*action* on $(\mathfrak{C}, 0)$ is given by a morphism of X-stacks

$$\gamma : \mathbb{A}^1 \times \mathfrak{C} \longrightarrow \mathfrak{C}$$

and three 2-isomorphisms θ_1, θ_0 and θ_γ between the 1-morphisms in the following diagrams.

1. $$\begin{array}{ccc} \mathfrak{C} & \stackrel{(1,\mathrm{id})}{\longrightarrow} & \mathbb{A}^1 \times \mathfrak{C} \\ & \mathrm{id} \searrow & \downarrow \gamma \\ & & \mathfrak{C} \end{array}$$

   and $\theta_1 : \mathrm{id} \to \gamma \circ (1, \mathrm{id})$.

2.

$$\begin{array}{ccc} \mathfrak{C} & \stackrel{(0,\mathrm{id})}{\longrightarrow} & \mathbb{A}^1 \times \mathfrak{C} \\ & {\scriptstyle 0} \searrow & \downarrow {\scriptstyle \gamma} \\ & & \mathfrak{C} \end{array}$$

and $\theta_0 : 0 \to \gamma \circ (0, \mathrm{id})$.

3.

$$\begin{array}{ccc} \mathbb{A}^1 \times \mathbb{A}^1 \times \mathfrak{C} & \stackrel{\mathrm{id} \times \gamma}{\longrightarrow} & \mathbb{A}^1 \times \mathfrak{C} \\ {\scriptstyle m \times \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \gamma} \\ \mathbb{A}^1 \times \mathfrak{C} & \stackrel{\gamma}{\longrightarrow} & \mathfrak{C} \end{array}$$

and $\theta_\gamma : \gamma \circ (m \times \mathrm{id}) \to \gamma \circ (\mathrm{id} \times \gamma)$.

The 2-isomorphisms θ_1, θ_0 and θ_γ are required to satisfy certain compatibilities which we leave to the reader to make explicit (see also Section 1.4 in Exposé XVIII of [2], where a similar problem, the definition of Picard stacks, is dealt with).

Let $(\mathfrak{C}, 0, \gamma)$ and $(\mathfrak{D}, 0, \gamma)$ be X-stacks with sections and $\mathbb{A}^1$-actions. Then an $\mathbb{A}^1$*-equivariant morphism* $\phi : \mathfrak{C} \to \mathfrak{D}$ is a triple $(\phi, \eta_0, \eta_\gamma)$, where $\phi : \mathfrak{C} \to \mathfrak{D}$ is a morphism of algebraic X-stacks and η_0 and η_γ are 2-isomorphisms between the morphisms in the following diagrams.

1.

$$\begin{array}{ccc} X & \stackrel{0}{\longrightarrow} & \mathfrak{C} \\ & {\scriptstyle 0} \searrow & \downarrow {\scriptstyle \phi} \\ & & \mathfrak{D} \end{array} \qquad (1)$$

and $\eta_0 : 0 \to \phi \circ 0$.

2.

$$\begin{array}{ccc} \mathbb{A}^1 \times \mathfrak{C} & \stackrel{\mathrm{id} \times \phi}{\longrightarrow} & \mathbb{A}^1 \times \mathfrak{D} \\ {\scriptstyle \gamma} \downarrow & & \downarrow {\scriptstyle \gamma} \\ \mathfrak{C} & \stackrel{\phi}{\longrightarrow} & \mathfrak{D} \end{array} \qquad (2)$$

and $\eta_\gamma : \phi \circ \gamma \to \gamma \circ (\mathrm{id} \times \phi)$.

Again, the 2-isomorphisms have to satisfy certain compatibilities we leave to the reader to spell out.

Finally, let $(\phi, \eta_0, \eta_\gamma) : \mathfrak{C} \to \mathfrak{D}$ and $(\psi, \eta_0', \eta_\gamma') : \mathfrak{C} \to \mathfrak{D}$ be two $\mathbb{A}^1$-equivariant morphisms. An $\mathbb{A}^1$*-equivariant isomorphism* $\zeta : \phi \to \psi$ is a 2-isomorphism $\zeta : \phi \to \psi$ such that the diagrams (notation compatible with (1) and (2))

1.

$$\begin{array}{ccc} 0 & \xrightarrow{\eta_0} & \phi \circ 0 \\ & \eta_0' \searrow & \downarrow \zeta \circ 0 \\ & & \psi \circ 0 \end{array}$$

2.

$$\begin{array}{ccc} \phi \circ \gamma & \longrightarrow & \gamma \circ (\mathrm{id} \times \phi) \\ \zeta \circ \gamma \downarrow & & \downarrow \gamma \circ (\mathrm{id} \times \zeta) \\ \psi \circ \gamma & \longrightarrow & \gamma \circ (\mathrm{id} \times \psi) \end{array}$$

commute.

If C is an E-cone, then since E acts on C, we may form the stack quotient of C by E over X, denoted $[C/E]$. For an X-scheme T, the groupoid of sections of $[C/E]$ over T is the category of pairs (P,f), where P is an E-torsor (a principal homogeneous E-bundle) over T and $f : P \to C$ is an E-equivariant morphism.

The X-stack $[C/E]$ comes with a section $0 : X \to [C,E]$ and an $\mathbb{A}^1$-action $\gamma : \mathbb{A}^1 \times [C/E] \to [C/E]$. The section 0 is given by the pair $(E_T, 0)$ over every X-scheme T; here E_T is the trivial E-bundle on T and $0 : E_T \to C$ is the vertex morphism. The $\mathbb{A}^1$-action of $\alpha \in \mathbb{A}^1(T) = \mathcal{O}_T(T)$ on the category $[C/E](T)$ is given by $\alpha \cdot (P,f) = (\alpha P, \alpha f)$, where $\alpha P = P \times_{E,\alpha} E$ and $\alpha f : P \times_{E,\alpha} E \to C$ is given by $[p,\nu] \mapsto \alpha f(p) + d(\nu)$.

If $\phi : (E,C) \to (F,D)$ is a morphism of vector bundle cones we get an induced $\mathbb{A}^1$-equivariant morphism $\widetilde{\phi} : [C/E] \to [D/F]$. A homotopy $k : \phi \to \psi$ gives rise to an $\mathbb{A}^1$-equivariant 2-isomorphism $\widetilde{k} : \widetilde{\phi} \to \widetilde{\psi}$ of $\mathbb{A}^1$-equivariant morphism of stacks with $\mathbb{A}^1$-action. (See Section 2 where these constructions are made explicit in a similar case.)

Lemma 1.6 *Let $\phi, \psi : (E,C) \to (F,D)$ be morphisms and $\zeta : \widetilde{\phi} \to \widetilde{\psi}$ an $\mathbb{A}^1$-equivariant 2-isomorphism between the associated $\mathbb{A}^1$-equivariant morphisms $[C/E] \to [D/F]$. Then $\zeta = \widetilde{k}$, for a unique homotopy $k : \phi \to \psi$.*

PROOF. We indicate how to construct $k : C \to F$. Given a section $c \in C(T)$ of C over the X-scheme T, we consider the induced object (E_T, c) of $[C/E](T)$. The associated F_T-torsors $E_T \times_{E_T,\phi^0} F_T$ and $E_T \times_{E_T,\psi^0} F_T$ are trivial, so that $\phi(T)(E_T,c)$ is a section of F over T. This section we define to be $k(c)$. □

Proposition 1.7 *Let C be an E-cone and D an F-cone. Let $\phi : (E, C) \to (F, D)$ be a morphism. If the diagram*

$$\begin{array}{ccc} E & \longrightarrow & C \\ \downarrow & & \downarrow \\ F & \longrightarrow & D \end{array}$$

is cartesian and $F \times C \to D; (\mu, \gamma) \mapsto d\mu + \phi(\gamma)$ is surjective, then $[C/E] \to [D/F]$ is an isomorphism of algebraic X-stacks with $\mathbb{A}^1$-action.

Proof. Similar to the proof of Proposition 2.1 below. $\square$

Definition 1.8 We call an algebraic stack $(\mathfrak{C}, 0, \gamma)$ over X with section and $\mathbb{A}^1$-action a *cone stack*, if, locally with respect to the étale topology on X, there exists a cone C over X and an $\mathbb{A}^1$-equivariant morphism $C \to \mathfrak{C}$ that is smooth and surjective.

The morphism $C \to \mathfrak{C}$, or by abuse of language C, is called a *local presentation* of $\mathfrak{C}$. The section $0 : X \to \mathfrak{C}$ is called the *vertex* of $\mathfrak{C}$.

Let $\mathfrak{C}$ and $\mathfrak{D}$ be cone stacks over X. A *morphism of cone stacks* $\phi : \mathfrak{C} \to \mathfrak{D}$ is an $\mathbb{A}^1$-equivariant morphism of algebraic X-stacks.

A *2-isomorphism of cone stacks* is just an $\mathbb{A}^1$-equivariant 2-isomorphism.

If $C \to \mathfrak{C}$ is a presentation of $\mathfrak{C}$, and $E = C \times_{\mathfrak{C},0} X$, then C is an E-cone and $\mathfrak{C} \cong [C/E]$ as stacks with $\mathbb{A}^1$-action (use Lemma 1.3 and Proposition 1.4).

If $\phi : \mathfrak{C} \to \mathfrak{D}$ is a morphism of cone stacks, then, locally with respect to the étale topology on X, ϕ is $\mathbb{A}^1$-equivariantly isomorphic to $[C/E] \to [D/F]$, where $E \to F$ is a morphism of vector bundles over X and $C \to D$ is a morphism from the E-cone C to the F-cone D.

A 2-isomorphism of cone stacks $\zeta : \phi \to \psi$, where $\phi, \psi : \mathfrak{C} \to \mathfrak{D}$, is locally over X given by a homotopy of morphisms of vector bundle cones. More precisely, one can find local presentations $\mathfrak{C} \cong [C/E]$ and $\mathfrak{D} \cong [D/F]$ such that both ϕ and ψ are induced by morphisms of vector bundle cones $\overline{\phi}, \overline{\psi} : (E, C) \to (F, D)$ and under these identifications ζ comes from a homotopy from $\overline{\phi}$ to $\overline{\psi}$. This follows from Lemma 1.6.

Remark Let $\mathfrak{C}$ be a cone stack over X. By Proposition 1.4 the fibered product over $\mathfrak{C}$ of any two local presentations is again a local presentation. Moreover, if $\mathfrak{C}$ is a representable cone stack over X, then $\mathfrak{C}$ is a cone. Every fibered product of cone stacks is a cone stack.

Examples All cones are cone stacks and all morphisms of cones are morphisms of cone stacks. For a vector bundle E on X, the classifying stack BE is a cone stack. Every homomorphism of vector bundles $\phi : E \to F$ gives rise to a morphism of cone stacks.

Definition 1.9 A cone stack $\mathfrak{C}$ over X is called *abelian*, if, locally in X, one can find presentations $C \to \mathfrak{C}$, where C is an abelian cone. A cone stack is a *vector bundle stack*, if one can find such local presentations such that C is a vector bundle. If $\mathfrak{C}$ is abelian (a vector bundle stack), then for every local presentation $C \to \mathfrak{C}$ the cone C will be abelian (a vector bundle).

Proposition 1.10 *Every cone stack is a closed subcone stack of an abelian cone stack. There exists a universal such abelian cone stack. It is called the* associated abelian cone stack *or the* abelian hull.

Proof. Just glue the stacks obtained from the abelian hulls of local presentations. $\square$

Definition 1.11 Let $\mathfrak{E}$ be a vector bundle stack and $\mathfrak{E} \to \mathfrak{C}$ a morphism of cone stacks. We say that $\mathfrak{C}$ is an $\mathfrak{E}$-*cone stack*, if $\mathfrak{E} \to \mathfrak{C}$ is locally isomorphic (as a morphism of cone stacks, i.e. $\mathbb{A}^1$-equivariantly) to the morphism $[E_1/E_0] \to [C/F]$ coming from a commutative diagram

$$\begin{array}{ccc} E_0 & \longrightarrow & F \\ \downarrow & & \downarrow \\ E_1 & \longrightarrow & C, \end{array}$$

where C is both an E_1- and an F-cone.

If $\mathfrak{C}$ is an $\mathfrak{E}$-cone stack, then there exists a natural morphism $\mathfrak{E} \times \mathfrak{C} \to \mathfrak{C}$ coming from the action $E_1 \times C \to C$ in a local presentation of $\mathfrak{E} \to \mathfrak{C}$ as above. We call $\mathfrak{E} \times \mathfrak{C} \to \mathfrak{C}$ the *action* of $\mathfrak{E}$ on $\mathfrak{C}$.

Definition 1.12 Let $\mathfrak{E} \to \mathfrak{C} \to \mathfrak{D}$ be a sequence of morphisms of cone stacks, where $\mathfrak{C}$ is an $\mathfrak{E}$-cone stack. If

1. $\mathfrak{C} \to \mathfrak{D}$ is a smooth epimorphism,
2. the diagram

$$\begin{array}{ccc} \mathfrak{E} \times \mathfrak{C} & \xrightarrow{\sigma} & \mathfrak{C} \\ {\scriptstyle p}\downarrow & & \downarrow \\ \mathfrak{C} & \longrightarrow & \mathfrak{D} \end{array}$$

(where p is the projection and σ the action) is cartesian, we call $0 \to \mathfrak{E} \to \mathfrak{C} \to \mathfrak{D} \to 0$ a *short exact sequence* of cone stacks. Note that this is equivalent to $\mathfrak{C}$ being locally isomorphic to $\mathfrak{E} \times \mathfrak{D}$.

Proposition 1.13 *The sequence $\mathfrak{E} \to \mathfrak{C} \to \mathfrak{D}$ of morphisms of cone stacks is exact if and only if locally in X there exist commutative diagrams*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_0 & \longrightarrow & F & \longrightarrow & G & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E_1 & \longrightarrow & C & \longrightarrow & D & \longrightarrow & 0, \end{array}$$

where the top row is a short exact sequence of vector bundles and the bottom row is a short exact sequence of cones, such that $\mathfrak{E} \to \mathfrak{C} \to \mathfrak{D}$ is isomorphic to $[E_1/E_0] \to [C/F] \to [D/G]$.

Proof. The statement is local on X. To prove the only if part we can assume $\mathfrak{C} = \mathfrak{E} \times \mathfrak{D}$, and then it's trivial. To prove the if part, note that both short exact sequences are locally split. □

2 Stacks of the Form h^1/h^0

The General Theory

We shall review here some aspects of the theory of Picard stacks developed by Deligne in Section 1.4 of Exposé XVIII in [2]. For the precise definition of Picard stack see [ibid.]. Roughly speaking, a Picard stack is a stack together with an 'addition' operation, that is both associative and commutative. An example would be the stack of torsors under a commutative group sheaf.

Let X be a topos and $d : E^0 \to E^1$ a homomorphism of abelian sheaves on X, which we shall consider as a complex of abelian sheaves on X. Via d, the abelian sheaf E^0 acts on E^1 and we may consider the stack-theoretic quotient of this action, denoted

$$h^1/h^0(E^\bullet) = [E^1/E^0],$$

which is a Picard stack on X. (See also [ibid.] 1.4.11, where $h^1/h^0(E^\bullet)$ is denoted $\mathrm{ch}(E^\bullet)$.) For an object $U \in \mathrm{ob}\, X$ the groupoid $h^1/h^0(E^\bullet)(U)$ of sections of $h^1/h^0(E^\bullet)$ over U is the category of pairs (P, f), where P is an E^0-torsor (principal homogeneous E^0-bundle) over U and $f : P \to E^1|U$ is an E^0-equivariant morphism of sheaves on U.

Now if $d : F^0 \to F^1$ is another homomorphism of abelian sheaves on X and $\phi : E^\bullet \to F^\bullet$ is a homomorphism of homomorphisms (or in other words a homomorphism of complexes), then we get an induced morphism of Picard stacks (an additive morphism in the terminology of [ibid.])

$$h^1/h^0(\phi) : h^1/h^0(E^\bullet) \longrightarrow h^1/h^0(F^\bullet).$$

For an object $U \in \operatorname{ob} X$ the functor $h^1/h^0(\phi)(U)$ maps the pair (P, f) to the pair $(P \times_{E^0,\phi^0} F^0, \phi^1(f))$, where $\phi^1(f)$ denotes the map

$$\begin{array}{rcl} \phi^1(f) : P \times_{E^0} F^0 & \longrightarrow & F^1 \\ {[p, \nu]} & \longmapsto & \phi^1(f(p)) + d(\nu). \end{array}$$

Now, if $\psi : E^\bullet \to F^\bullet$ is another homomorphism of complexes and $k : \phi \to \psi$ is a homotopy, i.e. a homomorphism of abelian sheaves $k : E^1 \to F^0$, such that

1. $kd = \psi^0 - \phi^0$,
2. $dk = \psi^1 - \phi^1$,

then we get an induced isomorphism $\theta : h^1/h^0(\phi) \to h^1/h^0(\psi)$ of morphisms of Picard stacks from $h^1/h^0(E^\bullet)$ to $h^1/h^0(F^\bullet)$. If $U \in \operatorname{ob} X$ is an object, then $\theta(U)$ is a natural transformation of functors from $h^1/h^0(\phi)(U)$ to $h^1/h^0(\psi)(U)$. For an object (P, f) of $h^1/h^0(E^\bullet)(U)$ the morphism $\theta(U)(P, f)$ is a morphism from $h^1/h^0(\phi)(U)(P, f)$ to $h^1/h^0(\psi)(U)(P, f)$ in the category $h^1/h^0(F^\bullet)(U)$. In fact, $\theta(U)(P, f)$ is the isomorphism of $F^0|U$-torsors

$$\begin{array}{rcl} \theta(U)(P, f) : P \times_{E^0,\phi^0} F^0 & \longrightarrow & P \times_{E^0,\psi^0} F^0 \\ {[p, \nu]} & \longmapsto & [p, kf(p) + \nu], \end{array} \tag{3}$$

such that the diagram of $F^0|U$-sheaves

$$\begin{array}{ccc} P \times_{E^0,\phi^0} F^0 & & \\ {\scriptstyle \theta(U)(P,f)} \downarrow & \searrow {\scriptstyle \phi^1(f)} & \\ P \times_{E^0,\psi^0} F^0 & \xrightarrow{\psi^1(f)} & F^1 \end{array}$$

commutes.

Proposition 2.1 *Let $\phi : E^\bullet \to F^\bullet$ be a homomorphism of homomorphisms of abelian sheaves on X, as above. If ϕ induces isomorphisms on kernels and cokernels (i.e. if ϕ is a quasi-isomorphism), then $h^1/h^0(\phi) : h^1/h^0(E^\bullet) \to h^1/h^0(F^\bullet)$ is an isomorphism of Picard stacks over X.*

PROOF. First let us treat the case that ϕ is a homotopy equivalence. Then, in fact, any homotopy inverse of ϕ will provide an inverse to $h^1/h^0(\phi)$, by the above remarks.

As a second case, let us assume that $\phi^\bullet : E^\bullet \to F^\bullet$ is an epimorphism (i.e. ϕ^0 and ϕ^1 are epimorphisms). In this case $E^1 \to [F^1/F^0]$ is an epimorphism, so for $[E^1/E^0]$ to be isomorphic to $[F^1/F^0]$, it is necessary and sufficient that

$$\begin{array}{ccc} E^0 \times E^1 & \stackrel{d+\mathrm{id}}{\longrightarrow} & E^1 \\ {\scriptstyle \mathrm{pr}} \downarrow & & \downarrow \\ E^1 & \longrightarrow & [F^1/F^0] \end{array}$$

be cartesian. This quickly reduces to proving that

$$\begin{array}{ccc} E^1 \times E^0 & \longrightarrow & E^1 \\ \downarrow & & \downarrow \\ E^1 \times F^0 & \longrightarrow & F^1 \end{array}$$

is cartesian, which, in turn, is equivalent to

$$\begin{array}{ccc} E^0 & \longrightarrow & E^1 \\ \downarrow & & \downarrow \\ F^0 & \longrightarrow & F^1 \end{array}$$

being cartesian, which is a consequence of the assumptions.

Finally, let us note that a general ϕ factors as a homotopy equivalence followed by an epimorphism. To see this consider $E^\bullet \oplus F^0$, which is homotopy equivalent to $E^\bullet$. Define a homomorphism $\psi : E^\bullet \oplus F^0 \to F^\bullet$ by $\psi^0(\nu, \mu) = \phi^0(\nu) + \mu$ and $\psi^1(\chi, \mu) = \phi^1(\chi) + d(\mu)$. Then ψ is surjective and $\phi = \psi \circ i$, where $i : E^\bullet \to E^\bullet \oplus F^0$ is given by $i = \mathrm{id} \oplus 0$. $\square$

If $E^\bullet$ is a complex of arbitrary length of abelian sheaves on X, let

$$\begin{aligned} Z^i(E^\bullet) &= \ker(E^i \to E^{i+1}) \\ C^i(E^\bullet) &= \operatorname{cok}(E^{i-1} \to E^i). \end{aligned}$$

The complex $E^\bullet$ induces a homomorphism

$$\tau_{[0,1]} E^\bullet = [C^0(E^\bullet) \to Z^1(E^\bullet)]$$

and we let $h^1/h^0(E^\bullet) = h^1/h^0(\tau_{[0,1]} E^\bullet)$.

Now let $\mathcal{O}_X$ be a sheaf of rings on X and $C(\mathcal{O}_X)$, $K(\mathcal{O}_X)$ and $D(\mathcal{O}_X)$ the category of complexes of $\mathcal{O}_X$-modules, the category of complexes of $\mathcal{O}_X$-modules up to homotopy and the derived category of the category $\mathrm{Mod}(\mathcal{O}_X)$ of $\mathcal{O}_X$-modules, respectively. Let $\phi : E^\bullet \to F^\bullet$ be a morphism in $D(\mathcal{O}_X)$. Let

$$\begin{array}{ccc} H^\bullet & \xrightarrow{\psi} & F^\bullet \\ {\scriptstyle\alpha}\downarrow & & \\ E^\bullet & & \end{array}$$

be a diagram in $C(\mathcal{O}_X)$ giving rise to ϕ, where α is a quasi-isomorphism. We get an induced diagram of Picard stacks

$$\begin{array}{ccc} h^1/h^0(H^\bullet) & \xrightarrow{h^1/h^0(\psi)} & h^1/h^0(F^\bullet) \\ {\scriptstyle h^1/h^0(\alpha)}\downarrow & & \\ h^1/h^0(E^\bullet), & & \end{array}$$

where $h^1/h^0(\alpha)$ is an isomorphism by Proposition 2.1. Choosing an inverse of $h^1/h^0(\alpha)$ induces a morphism

$$h^1/h^0(E^\bullet) \longrightarrow h^1/h^0(F^\bullet).$$

One checks that different choices of $(\alpha, H^\bullet, \psi)$ and $h^1/h^0(\alpha)^{-1}$ give rise to isomorphic morphisms $h^1/h^0(E^\bullet) \to h^1/h^0(F^\bullet)$. This proves in particular that if $E^\bullet$ and $F^\bullet$ are isomorphic in $D(\mathcal{O}_X)$, then the Picard X-stacks $h^1/h^0(E^\bullet)$ and $h^1/h^0(F^\bullet)$ are isomorphic.

Example If $d : E^0 \to E^1$ is a monomorphism then $h^1/h^0(E^\bullet) = \mathrm{cok}(d)$ is a sheaf over X.

If $d : E^0 \to E^1$ is an epimorphism then $h^1/h^0(E^\bullet) = B\ker(d)$ is a gerbe over X.

Lemma 2.2 *1. Let $\phi, \psi : E^\bullet \to F^\bullet$ be two morphisms in $D(\mathcal{O}_X)$. Then, if for some choice of $h^1/h^0(\phi)$ and $h^1/h^0(\psi)$ we have $h^1/h^0(\phi) \cong h^1/h^0(\psi)$ as morphisms of Picard stacks, then $\phi = \psi$.*

2. Let $0(E, F)$ be the zero morphism $0(E, F) : h^1/h^0(E^\bullet) \to h^1/h^0(F^\bullet)$. Then $\mathrm{Aut}(0(E, F)) = \mathrm{Hom}^{-1}_{D(\mathcal{O}_X)}(E^\bullet, F^\bullet)$.

PROOF. These are similar to Lemma 1.6. See also [ibid.]. □

Application to Schemes

Let X be a Deligne-Mumford stack. Consider the morphism of topoi

$$v : X_{\text{fl}} \longrightarrow X_{\text{ét}}.$$

The functor v_* restricts a sheaf on the big fppf-site to the small étale site and its left adjoint v^{-1} extends the embedding of the étale site into the flat site.

Let $\mathcal{O}_{X_{\text{fl}}}$ and $\mathcal{O}_{X_{\text{ét}}}$ denote the sheaves of rings induced by $\mathcal{O}_X$ on X_{fl} and $X_{\text{ét}}$, respectively. There is a canonical morphism of sheaves of rings $v^{-1}\mathcal{O}_{X_{\text{ét}}} \to \mathcal{O}_{X_{\text{fl}}}$, so that we have a morphism of ringed topoi

$$v : (X_{\text{fl}}, \mathcal{O}_{X_{\text{fl}}}) \to (X_{\text{ét}}, \mathcal{O}_{X_{\text{ét}}}).$$

The induced functor from $\text{Mod}(\mathcal{O}_{X_{\text{ét}}})$ to $\text{Mod}(\mathcal{O}_{X_{\text{fl}}})$ will be denoted by v^*:

$$v^*(M) = v^{-1}M \otimes_{v^{-1}\mathcal{O}_{X_{\text{ét}}}} \mathcal{O}_{X_{\text{fl}}}.$$

Since $\text{Mod}(\mathcal{O}_{X_{\text{ét}}})$ has enough flat modules we may derive the right exact functor v^* to get the functor $Lv^* : D^-(\mathcal{O}_{X_{\text{ét}}}) \to D^-(\mathcal{O}_{X_{\text{fl}}})$. To abbreviate notation, we write $M^\bullet_{\text{fl}} = Lv^*M^\bullet$ for $M^\bullet \in \text{ob}\, D^-(\mathcal{O}_{X_{\text{ét}}})$.

We shall also need to consider the functor

$$R\mathcal{H}om(\,\cdot\,, \mathcal{O}_{X_{\text{fl}}}) : D^-(\mathcal{O}_{X_{\text{fl}}}) \longrightarrow D^+(\mathcal{O}_{X_{\text{fl}}}).$$

It is defined using an injective resolution $\mathcal{O}_{X_{\text{fl}}} \xrightarrow{\sim} \mathcal{I}^\bullet$ of $\mathcal{O}_{X_{\text{fl}}}$, i.e.

$$R\mathcal{H}om(M^\bullet, \mathcal{O}_{X_{\text{fl}}}) = \text{tot}\, \mathcal{H}om(M^\bullet, \mathcal{I}^\bullet),$$

but if $M^\bullet$ happens to have a projective resolution $\mathcal{P}^\bullet \xrightarrow{\sim} M^\bullet$, then we have

$$R\mathcal{H}om(M^\bullet, \mathcal{O}_{X_{\text{fl}}}) \cong \mathcal{H}om(\mathcal{P}^\bullet, \mathcal{O}_{X\text{fl}}).$$

We shall abbreviate notation by writing

$$M^{\bullet\vee} = R\mathcal{H}om(M^\bullet, \mathcal{O}_{X_{\text{fl}}}).$$

We will be interested in the stack $h^1/h^0((M^\bullet_{\text{fl}})^\vee)$ associated to an object $M^\bullet \in \text{ob}\, D^-(\mathcal{O}_{X_{\text{ét}}})$. Note that for such $M^\bullet \in \text{ob}\, D^-(\mathcal{O}_{X_{\text{ét}}})$ we have

$$h^1/h^0((M^\bullet_{\text{fl}})^\vee) \cong h^1/h^0((\tau_{\geq -1}M^\bullet_{\text{fl}})^\vee).$$

Definition 2.3 We say that an object $L^\bullet$ of $D(\mathcal{O}_{X_{\text{ét}}})$ *satisfies Condition* $(\star)$ if

1. $h^i(L^\bullet) = 0$ for all $i > 0$,
2. $h^i(L^\bullet)$ is coherent, for $i = 0, -1$.

Proposition 2.4 *Let* $L^\bullet \in \operatorname{ob} D(\mathcal{O}_{X_{\text{ét}}})$ *satisfy Condition* $(\star)$. *Then the* X*-stack* $h^1/h^0((L^\bullet_{\text{fl}})^\vee)$ *is an algebraic* X*-stack, in fact an abelian cone stack over* X. *Moreover, if* $L^\bullet$ *is of perfect amplitude contained in* $[-1, 0]$*, then* $h^1/h^0((L^\bullet_{\text{fl}})^\vee)$ *is a vector bundle stack.*

Proof. The claim is local in X (with respect to the étale topology), so we may assume that $L^\bullet$ has a free resolution, or that $L^\bullet$ itself consists of free $\mathcal{O}_X$-modules. We may also assume that $L^i = 0$, for all $i > 0$ and that L^0 and L^{-1} have finite rank. Then $L^\bullet_{\text{fl}}$ is given by $L^\bullet$ itself, since a free sheaf is flat, and $(L^\bullet_{\text{fl}})^\vee$ is given by $L^{\vee\bullet}$, taking duals component-wise, since a free module is projective. Thus

$$h^1/h^0((L^\bullet_{\text{fl}})^\vee) = [Z^1(L^{\vee\bullet})/L^{\vee 0}],$$

which is the cone stack given by the homomorphism of abelian cones $L^{\vee 0} \to Z^1(L^{\vee\bullet}) = C(C^{-1}(L^\bullet))$.

If $L^\bullet$ is of perfect amplitude contained in $[-1, 0]$, then we may assume that in addition to the above assumptions $L^i = 0$, for all $i \leq -2$. Then $Z^1(L^{\vee\bullet}) = L^{\vee 1}$ is a vector bundle. □

So if $\phi : E^\bullet \to L^\bullet$ is a homomorphism in $D(\mathcal{O}_{X_{\text{ét}}})$, where $E^\bullet$ and $L^\bullet$ satisfy $(\star)$, then we get an induced morphism of algebraic stacks

$$\phi^\vee : h^1/h^0((L^\bullet_{\text{fl}})^\vee) \longrightarrow h^1/h^0((E^\bullet_{\text{fl}})^\vee).$$

Proposition 2.5 *The morphism* $\phi^\vee$ *is a morphism of abelian cone stacks. Moreover,* $h^0(\phi)$ *is surjective, if and only if* $\phi^\vee$ *is representable.*

Proof. The fact that $\phi^\vee$ is a morphism of abelian cone stacks is immediate from the definition. The second question is local in X, so we may assume that $E^\bullet$ and $L^\bullet$ are complexes of free $\mathcal{O}_X$-modules and that $E^i = L^i = 0$, for $i > 0$, and that L^0, L^{-1}, E^0 and E^{-1} are of finite rank. Consider the commutative diagram

$$\begin{array}{ccc} C^{-1}(E^\bullet) & \longrightarrow & E^0 \\ \downarrow & & \downarrow \\ C^{-1}(L^\bullet) & \longrightarrow & L^0 \end{array}$$

of coherent sheaves on X. Let F be the fibered product

$$\begin{array}{ccc} F & \longrightarrow & E^0 \\ \downarrow & & \downarrow \\ C^{-1}(L^\bullet) & \longrightarrow & L^0. \end{array}$$

The fact that $h^0(\phi)$ is surjective, is equivalent to saying that the sequence

$$0 \longrightarrow F \longrightarrow E^0 \oplus C^{-1}(L^\bullet) \longrightarrow L^0 \longrightarrow 0$$

is exact. Since L^0 is free, we get an induced exact sequence of cones

$$0 \longrightarrow L^{\vee 0} \longrightarrow E^{\vee 0} \oplus Z^1(L^{\vee\bullet}) \longrightarrow C(F) \longrightarrow 0.$$

Hence by Proposition 1.7 we have

$$[Z^1(L^{\vee\bullet})/L^{\vee 0}] \cong [C(F)/E^{\vee 0}].$$

In particular the diagram

$$\begin{array}{ccc} C(F) & \longrightarrow & Z^1(E^{\vee\bullet}) \\ \downarrow & & \downarrow \\ h^1/h^0((L^\bullet_{\text{fl}})^\vee) & \longrightarrow & h^1/h^0((E^\bullet_{\text{fl}})^\vee) \end{array}$$

is cartesian, hence $\phi^\vee$ is representable.

For the converse, note that $\phi^\vee$ representable implies that $L^{\vee 0} \to E^{\vee 0} \times Z^1(L^{\vee\bullet})$ is a closed immersion, which implies that $E^0 \oplus C^{-1}(L^\bullet) \to L^0$ is an epimorphism. □

Proposition 2.6 *The morphism $\phi^\vee$ is a closed immersion if and only if $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is surjective. Moreover, $\phi^\vee$ is an isomorphism if and only if $h^0(\phi)$ and $h^{-1}(\phi)$ are.*

Proof. Following the previous argument, $\phi^\vee$ is a closed immersion if and only if $C(F) \to Z^1(E^{\vee\bullet})$ is. This is equivalent to $C^{-1}(E^\bullet) \to F$ being surjective. A simple diagram chase shows that this is equivalent to $h^0(\phi)$ being an isomorphism and $h^{-1}(\phi)$ being surjective. The 'moreover' follows similarly. □

Proposition 2.7 *Let*

$$E^\bullet \longrightarrow F^\bullet \longrightarrow G^\bullet \longrightarrow E^\bullet[1]$$

be a distinguished triangle in $D(\mathcal{O}_{X_{\text{ét}}})$, where $E^\bullet$ and $F^\bullet$ satisfy ($\star$) and $G^\bullet$ is of perfect amplitude contained in $[-1,0]$. Then the induced sequence

$$h^1/h^0(G^\vee) \longrightarrow h^1/h^0(F^\vee) \longrightarrow h^1/h^0(E^\vee)$$

is a short exact sequence of abelian cone stacks over X.

PROOF. The question is local, so assume that E^i and F^i are 0 for $i>0$ and vector bundles for $i=0,-1$, and that $G^i = F^i \oplus E^{i+1}$. We have to prove that

$$0 \longrightarrow [Z^1(G^\vee)/G^{\vee 0}] \longrightarrow [Z^1(F^\vee)/F^{\vee 0}] \longrightarrow [Z^1(E^\vee)/E^{\vee 0}] \longrightarrow 0$$

is a short exact sequence of cone stacks. By Proposition 1.13, it is enough to prove that the exact sequence of sheaves

$$0 \longrightarrow C^{-1}(E^\bullet) \longrightarrow C^{-1}(F^\bullet) \oplus E^0 \longrightarrow C^{-1}(G^\bullet) \longrightarrow 0$$

is exact. This is then a straightforward verification. □

3 The Intrinsic Normal Cone

Normal Cones

Normal cones have the following functorial property. Consider a commutative diagram of (arbitrary) algebraic k-stacks

$$\begin{array}{ccc} X' & \xrightarrow{j} & Y' \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ X & \xrightarrow{i} & Y, \end{array} \qquad (4)$$

where i and j are local immersions. Then there is a natural morphism of cones over X'

$$\alpha : C_{X'/Y'} \longrightarrow u^* C_{X/Y}.$$

If (4) is cartesian, then α is a closed immersion. If, moreover, v is flat, then α is an isomorphism.

Proposition 3.1 *Consider a commutative diagram of Deligne-Mumford stacks*

$$\begin{array}{ccc} X & \xrightarrow{i'} & Y' \\ & {\scriptstyle i}\searrow & \downarrow{\scriptstyle f} \\ & & Y, \end{array}$$

where i and i' are local immersions and f is smooth. Then the sequence of morphisms of cones over X

$$i'^*T_{Y'/Y} \xrightarrow{\beta} C_{X/Y'} \xrightarrow{\alpha} C_{X/Y}, \tag{5}$$

where the maps α and β are the natural ones, is exact.

Proof. The question is local, so we can assume that X, Y and Y' are schemes and that i' and i are immersions. This is then Example 4.2.6 in [6]. □

Lemma 3.2 *Let*

$$U \xrightarrow{f} M$$

be a local immersion of affine k-schemes of finite type, where M is smooth over k. Then the normal cone $C_{U/M} \hookrightarrow N_{U/M}$ is invariant under the action of f^*T_M on $N_{U/M}$. In other words, $C_{U/M}$ is an f^*T_M-cone.

Proof. Let $p_i : M \times M \to M$, $i = 1, 2$, be the two projections. Each one gives rise to a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\Delta f} & M \times M \\ & f \searrow & \downarrow p_i \\ & & M, \end{array}$$

and hence to an exact sequence

$$0 \longrightarrow f^*T_M \xrightarrow{j_i} N_{U/M \times M} \xrightarrow{p_{i*}} N_{U/M} \longrightarrow 0$$

of abelian cones on U.

The diagonal gives rise to the commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{f} & M \\ & \Delta f \searrow & \downarrow \Delta \\ & & M \times M \end{array}$$

and hence to a homomorphism

$$N_{U/M} \xrightarrow{s} N_{U/M \times M}$$

of abelian cones on U.

Now s is a section of both p_{1*} and p_{2*}. Using (j_1, p_{1*}) we make the identification

$$N_{U/M\times M} = f^*T_M \times N_{U/M}. \tag{6}$$

Then p_{2*} is identified with the action of f^*T_M on $N_{U/M}$. Since the same functorialities of normal sheaves used so far are enjoyed by normal cones, we get that under the identification (6) the subcone $C_{U/M\times M} \subset N_{U/M\times M}$ corresponds to $f^*T_M \times C_{U/M}$ and the action $p_{2*} : f^*T_M \times N_{U/M} \to N_{U/M}$ restricts to $p_{2*} : f^*T_M \times C_{U/M} \to C_{U/M}$. □

The following is not used until Section 5.

Consider the diagram (4), assume it is cartesian and assume that v is a regular local immersion. Assume also that Y is smooth of constant dimension. Let $C = C_{X/Y}$ and $N = N_{Y'/Y}$. Then we get an induced cartesian diagram

$$\begin{array}{ccccc} N\times_Y C & \longrightarrow & u^*C & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ j^*N & \longrightarrow & X' & \stackrel{u}{\longrightarrow} & X \\ \downarrow & & {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle i} \\ N & \stackrel{\rho}{\longrightarrow} & Y' & \stackrel{v}{\longrightarrow} & Y. \end{array} \tag{7}$$

If Y is a scheme, Vistoli constructed in [14] a canonical rational equivalence $\beta(Y',X) \in W_*(N\times_Y C)$ such that

$$\partial\beta(Y',X) = [C_{u^*C/C}] - [\rho^*C_{X'/Y'}].$$

Note Let $0 : u^*C \to N\times_Y C$ be the zero section. Then

$$0^![C_{u^*C/C}] = v^![C] \in A_*(u^*C),$$

by the definition of $v^!$. On the other hand,

$$0^![\rho^*C_{X'/Y'}] = 0^!\rho^![C_{X'/Y'}] = [C_{X'/Y'}] \in A_*(u^*C).$$

So the existence of Vistoli's rational equivalence implies that

$$v^![C] = [C_{X'/Y'}].$$

Proposition 3.3 *Vistoli's rational equivalence commutes with any smooth base change $\phi : Y_1 \to Y$. More precisely, if we denote by a subscript $(\,\cdot\,)_1$ the base change via ϕ of any object in (7), then*

$$\phi^*\beta(Y',X) = \beta(Y_1',X_1) \in W_*(N_1\times_{Y_1} C_1).$$

PROOF. If ϕ is étale, this is Lemma 4.6(ii) in [14]. Vistoli's proof is based on the fact that the following commute with étale base change: blowing up a scheme along a closed subscheme; normalization; order of a Cartier divisor along an irreducible Weil divisor on a reduced, equidimensional scheme. But all these operations do in fact commute with smooth base change. $\square$

A first consequence of this proposition is that we may drop the assumption that Y be a scheme. We get $\beta(Y', X) \in W_*(N \times_Y C)$ for any situation (7). The consequence $v^![C] = [C_{X'/Y'}]$ holds if Y (and hence all other stacks in (7)) is of Deligne-Mumford type.

Now let us assume that $i : X \to Y$ factors as

$$\begin{array}{ccc} X & \xrightarrow{\widetilde{\imath}} & \widetilde{Y} \\ & {\scriptstyle i}\searrow & \downarrow{\scriptstyle \pi} \\ & & Y, \end{array}$$

where $\widetilde{\imath}$ is another local immersion and π is of relative Deligne-Mumford type (i.e. has unramified diagonal) and is smooth of constant fiber dimension. Then we construct the cartesian diagram

$$\begin{array}{ccc} \widetilde{Y}' & \xrightarrow{\widetilde{v}} & \widetilde{Y} \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ Y' & \xrightarrow{v} & Y \end{array}$$

and over

$$\begin{array}{ccc} X' & \xrightarrow{u} & X \\ {\scriptstyle \widetilde{\jmath}}\downarrow & & \downarrow{\scriptstyle \widetilde{\imath}} \\ \widetilde{Y}' & \xrightarrow{\widetilde{u}} & \widetilde{Y} \end{array}$$

we construct the analogue of (7):

$$\begin{array}{ccccc} N \times_Y \widetilde{C} & \longrightarrow & u^*\widetilde{C} & \longrightarrow & \widetilde{C} \\ \downarrow & & \downarrow & & \downarrow \\ j^*N & \longrightarrow & X' & \xrightarrow{u} & X \\ \downarrow & & {\scriptstyle \widetilde{\jmath}}\downarrow & & \downarrow{\scriptstyle \widetilde{\imath}} \\ \pi^*N & \xrightarrow{\widetilde{\rho}} & \widetilde{Y}' & \xrightarrow{\widetilde{v}} & \widetilde{Y}, \end{array} \tag{8}$$

i.e. $\widetilde{C} = C_{X/\widetilde{Y}}$. Diagrams (7) and (8) may be fused into one large diagram

$$\begin{array}{ccccc} N \times_Y \widetilde{C} & \longrightarrow & u^*\widetilde{C} & \longrightarrow & \widetilde{C} \\ \downarrow & & \downarrow & & \downarrow \alpha \\ N \times_Y C & \longrightarrow & u^*C & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ j^*N & \longrightarrow & X' & \stackrel{u}{\longrightarrow} & X \\ \downarrow & & \widetilde{\jmath}\downarrow & & \downarrow \widetilde{\imath} \\ \pi^*N & \stackrel{\widetilde{\rho}}{\longrightarrow} & \widetilde{Y}' & \stackrel{\widetilde{v}}{\longrightarrow} & \widetilde{Y} \\ \downarrow & & \downarrow & & \downarrow \pi \\ N & \stackrel{\rho}{\longrightarrow} & Y' & \stackrel{v}{\longrightarrow} & Y. \end{array} \qquad (9)$$

By Proposition 3.1 the morphism $\widetilde{C} \to C$ is a $T_{\widetilde{Y}/Y} \times_{\widetilde{Y}} C$-bundle.

Proposition 3.4 *We have $\alpha^*(\beta(Y', X)) = \beta(\widetilde{Y}', X)$ in $W_*(N \times_Y \widetilde{C})$.*

Proof. By the compatibilities of β proved in [14] we reduce to the case that $\widetilde{Y} = \mathbb{A}^n_Y$, $\pi : \mathbb{A}^n_Y \to Y$ is a relative affine n-space and $\widetilde{\imath} : Y \to \mathbb{A}^n_Y$ is the zero section. Then one checks that Vistoli's construction commutes with π. □

Proposition 3.5 *In the situation of Diagram (7) assume that Y is of Deligne-Mumford type. Vistoli's rational equivalence $\beta(Y', X) \in W_*(N \times_Y C)$ is invariant under the natural action of $j^*N \times_Y T_Y$ on $N \times_Y C$.*

Proof. The vector bundle i^*T_Y acts on the X-cone C by Lemma 3.2. Pulling back from X to j^*N gives the natural action of $j^*N \times_Y T_Y$ on $N \times_Y C$. Using the construction of the proof of Lemma 3.2 the claim follows from Proposition 3.4 applied to $\widetilde{Y} = Y \times Y$ and $\widetilde{\imath} = \Delta \circ i : X \to Y \times Y$. □

The Intrinsic Normal Cone

Let X be a Deligne-Mumford stack, locally of finite type over k. Let $L^\bullet_X$ be the cotangent complex of X relative to k. Then $L^\bullet_X \in \operatorname{ob} D(\mathcal{O}_{X_{\text{ét}}})$ and $L^\bullet_X$ satisfies $(\star)$.

Definition 3.6 We denote the algebraic stack $h^1/h^0(((L^\bullet_X)_{\text{fl}})^\vee)$ by $\mathfrak{N}_X$ and call it the *intrinsic normal sheaf* of X.

We shall now construct the intrinsic normal cone as a closed subcone stack of $\mathfrak{N}_X$.

Definition 3.7 A *local embedding* of X is a diagram

$$\begin{array}{ccc} U & \xrightarrow{f} & M \\ {\scriptstyle i}\downarrow & & \\ X & & \end{array},$$

where

1. U is an affine k-scheme of finite type,
2. $i : U \to X$ is an étale morphism,
3. M is a smooth affine k-scheme of finite type,
4. $f : U \to M$ is a local immersion.

By abuse of language we call the pair (U, M) a local embedding of X.

A morphism of local embeddings $\phi : (U', M') \to (U, M)$ is a pair of morphisms $\phi_U : U' \to U$ and $\phi_M : M' \to M$ such that

1. ϕ_U is an étale X-morphism,
2. ϕ_M is a smooth morphism such that

$$\begin{array}{ccc} U' & \xrightarrow{f'} & M' \\ {\scriptstyle \phi_U}\downarrow & & \downarrow{\scriptstyle \phi_M} \\ U & \xrightarrow{f} & M \end{array}$$

   commutes.

If (U', M') and (U, M) are local embeddings of X, then $(U' \times_X U, M' \times M)$ is naturally a local embedding of X which we call the *product* of (U', M') and (U, M), even though it may not be the direct product of (U', M') and (U, M) in the category of local embeddings of X.

Let

$$\begin{array}{ccc} U & \xrightarrow{f} & M \\ {\scriptstyle i}\downarrow & & \\ X & & \end{array}$$

be a local embedding of X. Let I/I^2 be the conormal sheaf of U in M. There is a natural homomorphism of coherent $\mathcal{O}_U$-modules $I/I^2 \to f^*\Omega_M$. Moreover, there exists a natural homomorphism

$$\phi : L^\bullet_X|U \longrightarrow [I/I^2 \to f^*\Omega_M]$$

in $D(\mathcal{O}_{U_{\text{ét}}})$, where we think of $[I/I^2 \to f^*\Omega_M]$ as a complex concentrated in degrees -1 and 0. Moreover, ϕ induces an isomorphism on h^{-1} and h^0 (see [9], Chapitre III, Corollaire 3.1.3). Hence by Proposition 2.6 we get an induced isomorphism of cone stacks

$$\phi^\vee : [N_{U/M}/f^*T_M] \longrightarrow i^*\mathfrak{N}_X,$$

where T_M is the tangent bundle of M and $N_{U/M}$ is the normal sheaf of the local embedding f. In other words, $N_{U/M}$ is a local presentation of the abelian cone stack $\mathfrak{N}_X$.

If $\chi : (U', M') \to (U, M)$ is a morphism of local embeddings we get an induced commutative diagram

$$\begin{array}{ccc} I/I^2|U' & \longrightarrow & f^*\Omega_M|U' \\ \downarrow & & \downarrow \\ I'/I'^2 & \longrightarrow & f'^*\Omega_{M'} \end{array} ,$$

in other words a homomorphism

$$\widetilde{\chi} : [I/I^2 \to f^*\Omega_M]|U' \longrightarrow [I'/I'^2 \to f'^*\Omega_{M'}] \quad .$$

We have $\widetilde{\chi} \circ \phi|U' = \phi'$ in $D(\mathcal{O}_{U'_{\text{ét}}})$, because of the naturality of ϕ. Thus the induced morphism

$$\widetilde{\chi}^\vee : [N_{U'/M'}/f'^*T_{M'}] \longrightarrow [N_{U/M}/f^*T_M]|U'$$

is compatible with the isomorphisms to $\mathfrak{N}_X$. Note that, in particular, $\widetilde{\chi}^\vee$ is an isomorphism of cone stacks over U'.

Recall Lemma 3.2. Let $\chi : (U', M') \to (U, M)$ be a morphism of local embeddings. Then we get an induced morphism from the $f'^*T_{M'}$-cone $C_{U'/M'}$ to the $f^*T_M|U'$-cone $C_{U/M}|U'$. Note that the kernel of $f'^*T_{M'} \to f^*T_M|U'$ is $f'^*T_{M'/M}$.

Lemma 3.8 *The pair $(C_{U/M} \hookrightarrow N_{U/M})|U'$ is the quotient of $(C_{U'/M'} \hookrightarrow N_{U'/M'})$ by the action of $f'^*T_{M'/M}$.*

PROOF. This follows immediately from Proposition 3.1. $\square$

Corollary 3.9 *The isomorphism*

$$\widetilde{\chi}^\vee : [N_{U'/M'}/f'^*T_{M'}] \longrightarrow [N_{U/M}/f^*T_M]|U'$$

identifies the closed subcone stack $[C_{U'/M'}/f'^*T_{M'}]$ *with the closed subcone stack* $[C_{U/M}/f^*T_M]|U'$.

By this corollary, there exists a unique closed subcone stack $\mathfrak{C}_X \hookrightarrow \mathfrak{N}_X$, such that for every local embedding (U, M) of X we have $\mathfrak{C}_X|U = [C_{U/M}/f^*T_M]$, or in other words that

$$\begin{array}{ccc} C_{U/M} & \longrightarrow & N_{U/M} \\ \downarrow & & \downarrow \\ \mathfrak{C}_X & \longrightarrow & \mathfrak{N}_X \end{array}$$

is cartesian.

Definition 3.10 The cone stack $\mathfrak{C}_X$ is called the *intrinsic normal cone* of X.

Theorem 3.11 *The intrinsic normal cone* $\mathfrak{C}_X$ *is of pure dimension zero. Its abelian hull is* $\mathfrak{N}_X$.

PROOF. The second claim follows because the normal sheaf is the abelian hull of the normal cone, for any local embedding. To prove the claim about the dimension of $\mathfrak{C}_X$, consider a local embedding (U, M) of X, giving rise to the local presentation $C_{U/M}$ of $\mathfrak{C}_X$. Assume that M is of pure dimension. We then have a cartesian and cocartesian diagram of U-stacks

$$\begin{array}{ccc} f^*T_M \times C_{U/M} & \longrightarrow & C_{U/M} \\ \downarrow & & \downarrow \\ C_{U/M} & \longrightarrow & [C_{U/M}/f^*T_M]. \end{array}$$

Thus $C_{U/M} \to [C_{U/M}/f^*T_M]$ is a smooth epimorphism of relative dimension $\dim M$. So since $C_{U/M}$ is of pure dimension $\dim M$ (see [6], B.6.6) the stack $[C_{U/M}/f^*T_M]$ has pure dimension $\dim M - \dim M = 0$. $\square$

Remark One may construct $\mathfrak{N}_X$ by simply gluing the various stacks $[N_{U/M}/f^*T_M]$, coming from the local embeddings of X. So one doesn't need the construction preceding Proposition 2.4 to define the intrinsic normal sheaf and the intrinsic normal cone. But for objects $E^\bullet$ of $D^-(\mathcal{O}_{X_{\text{ét}}})$ satisfying ($\star$) other than $L^\bullet_X$, we could not prove that such gluing works a priori. The problem is, that in general one does not have such a nice distinguished class of local resolutions of $E^\bullet$ (like the one coming from local embeddings for $L^\bullet_X$). In general, local (free) resolutions of $E^\bullet$ are only compatible up to homotopy.

Basic Properties

Proposition 3.12 (Local Complete Intersections) *The following are equivalent.*

1. *X is a local complete intersection,*
2. *$\mathfrak{C}_X$ is a vector bundle stack,*
3. *$\mathfrak{C}_X = \mathfrak{N}_X$.*

If, for example, X is smooth, we have $\mathfrak{C}_X = \mathfrak{N}_X = BT_X$.

Proof. (1)$\Longrightarrow$(3). If X is a local complete intersection, then local embeddings of X are regular immersions, but for regular immersions normal cone and normal sheaf coincide.

(3)$\Longrightarrow$(2). If for a local embedding normal cone and normal sheaf coincide, then it is a regular immersion. Thus X is a local complete intersection so that $\mathfrak{N}_X$ is a vector bundle stack.

(2)$\Longrightarrow$(1). If $\mathfrak{C}_X$ is a vector bundle stack it is equal to its abelian hull. Hence $\mathfrak{C}_X = \mathfrak{N}_X$ and X is a local complete intersection. □

Proposition 3.13 (Products) *Let X and Y be Deligne-Mumford stacks of finite type over k. Then*

$$\mathfrak{N}_{X\times Y} = \mathfrak{N}_X \times \mathfrak{N}_Y$$

and

$$\mathfrak{C}_{X\times Y} = \mathfrak{C}_X \times \mathfrak{C}_Y.$$

Proof. If $X \subset V$ and $Y \subset W$ are affine schemes, it is easy to check that there is a natural isomorphism $C_{X/V} \times C_{Y/W} \to C_{X\times Y/V\times W}$, compatible with étale base change; the same is true if we replace the normal cone by the normal sheaf.

If C is an E-cone and D is an F-cone, then $C \times D$ is an $E \times F$-cone and there is a canonical isomorphism of cone stacks $[C/E] \times [D/F] \to [C \times D/E \times F]$.

Putting together this remarks and verifying that the canonical isomorphisms glue completes the proof. □

Proposition 3.14 (Pullback) *Let $f : X \to Y$ be a local complete intersection morphism. Then we have a natural short exact sequence of cone stacks*

$$\mathfrak{N}_{X/Y} \longrightarrow \mathfrak{C}_X \longrightarrow f^*\mathfrak{C}_Y$$

over X, where $\mathfrak{N}_{X/Y} = h^1/h^0(T^{\bullet}_{X/Y})$.

Proof. We have a distinguished triangle in $D(\mathcal{O}_{X_{\text{ét}}})$

$$f^*L_Y \longrightarrow L_X \longrightarrow L_{X/Y} \longrightarrow f^*L_Y[1],$$

and $L_{X/Y}$ is of perfect amplitude contained in $[-1, 0]$. So by Proposition 2.7 we have a short exact sequence of abelian cone stacks

$$\mathfrak{N}_{X/Y} \longrightarrow \mathfrak{N}_X \longrightarrow f^*\mathfrak{N}_Y$$

on X. So the claim is local in X and we may assume that we have a diagram

$$\begin{array}{ccccc} X & \xrightarrow{i} & M'' & \longrightarrow & M' \\ & \searrow & \downarrow & & \downarrow \\ & & Y & \longrightarrow & M, \end{array}$$

where the square is cartesian, the vertical maps are smooth, the horizontal maps are local immersions, i is regular and M is smooth. Then we have a morphism of short exact sequences of cones on X:

$$\begin{array}{ccccc} i^*T_{M''/Y} & \longrightarrow & T_{M'}|X & \longrightarrow & T_M|X \\ \downarrow & & \downarrow & & \downarrow \\ N_{X/M''} & \longrightarrow & C_{X/M'} & \longrightarrow & C_{Y/M}|X. \end{array}$$

This is a local presentation for the short exact sequence

$$\mathfrak{N}_{X/Y} \longrightarrow \mathfrak{C}_X \longrightarrow f^*\mathfrak{C}_Y$$

of cone stacks. □

4 Obstruction Theory

The Intrinsic Normal Sheaf as Obstruction

A closed immersion $T \to \overline{T}$ of schemes is called a *square-zero extension* with ideal sheaf J if J is the ideal sheaf of T in $\overline{T}$ and $J^2 = 0$.

Let X be a Deligne-Mumford stack, $\mathfrak{N}_X$ its intrinsic normal sheaf. Let $T \to \overline{T}$ be a square zero extension with ideal sheaf J and $g : T \to X$ a morphism. By the functorialities of the cotangent complex we have a canonical homomorphism

$$g^* L_X^\bullet \longrightarrow L_T^\bullet \longrightarrow L_{T/\overline{T}}^\bullet \tag{10}$$

in $D(\mathcal{O}_{T_{\text{ét}}})$. Since $\tau_{\geq -1} L_{T/\overline{T}}^\bullet = J[1]$, this homomorphism may be considered as an element $\omega(g)$ of $\operatorname{Ext}^1(g^* L_X^\bullet, J)$. Recall the following basic facts of deformation theory. An extension $\overline{g} : \overline{T} \to X$ of g exists if and only if $\omega(g) = 0$ and if $\omega(g) = 0$ the extensions form a torsor under $\operatorname{Ext}^0(g^* L_X^\bullet, J) = \operatorname{Hom}(g^* \Omega_X, J)$.

These facts can be interpreted in terms of the intrinsic normal sheaf $\mathfrak{N}_X$ of X. To do this, note that (10) gives rise to a morphism

$$h^1/h^0(L_{T/\overline{T}}^\bullet) \longrightarrow h^1/h^0(g^* L_X^\bullet)$$

of cone stacks over T. Since $h^1/h^0(L_{T/\overline{T}}^\bullet) = C(J)$ and $h^1/h^0(g^* L_X^\bullet) = g^* \mathfrak{N}_X$ we have constructed a morphism $ob(g) : C(J) \to g^* \mathfrak{N}_X$. We also consider the morphism $0(g) : C(J) \to g^* \mathfrak{N}_X$ given as the composition of $C(J) \to X$ with the vertex of $g^* \mathfrak{N}_X$. By $\underline{\operatorname{Hom}}(ob(g), 0(g))$ we shall denote the sheaf of 2-isomorphisms of cone stacks from $ob(g)$ to $0(g)$, restricted to $T_{\text{ét}}$.

Given a square zero extension $T \to \overline{T}$ and a morphism $g : T \to X$, we denote the set of extensions $\overline{g} : \overline{T} \to X$ of g by $\operatorname{Ext}(g, \overline{T})$. These extensions in fact form a sheaf on $T_{\text{ét}}$ which we shall denote $\underline{\operatorname{Ext}}(g, \overline{T})$.

Proposition 4.1 *There is a canonical isomorphism*

$$\underline{\operatorname{Ext}}(g, \overline{T}) \xrightarrow{\sim} \underline{\operatorname{Hom}}_{\mathcal{O}_T}(ob(g), 0(g))$$

of sheaves on $T_{\text{ét}}$. In particular, extensions of g to $\overline{T}$ exist, if and only if $ob(g)$ is $\mathbb{A}^1$-equivariantly isomorphic to $0(g)$.

PROOF. Locally, we may embed X into a smooth scheme M and call the embedding $i : X \to M$, the conormal sheaf I/I^2. Then there always exist local extensions $h : \overline{T} \to M$ of $i \circ g : T \to M$.

$$\begin{array}{ccc} T & \longrightarrow & \overline{T} \\ g \downarrow & & \downarrow h \\ X & \stackrel{i}{\longrightarrow} & M \end{array}$$

Any such h gives rise to a homomorphism $h^\sharp : g^* I/I^2 \to J$, and hence to a realization of $ob(g)$ as the morphism of cone stacks induced by the homomorphism of complexes

$$h^\sharp : g^*[I/I^2 \to i^*\Omega_M] \longrightarrow [J \to 0].$$

Note that if $\widetilde{h}$ is another such extension, the difference between h and $\widetilde{h}$ induces a homomorphism $g^* i^* \Omega_M \to J$, which is in fact a homotopy from $h^\sharp$ to $\widetilde{h}^\sharp$.

Now let $\overline{g} : \overline{T} \to X$ be an extension of g. Then $(i \circ \overline{g})^\sharp = 0$, so that we get a homotopy from any local $h^\sharp$ as above to 0, or in other words a local $\mathbb{A}^1$-equivariant isomorphism from $ob(g)$ to $0(g)$. Since these local isomorphisms glue, we get the required map

$$\underline{\mathrm{Ext}}(g, \overline{T}) \longrightarrow \underline{\mathrm{Hom}}(ob(g), 0(g)).$$

To construct the inverse, let $\theta : ob(g) \to 0(g)$ be a 2-isomorphism of cone stacks. Note that θ defines for every local h as above an extension of $h^\sharp$ to $\overline{h}^\sharp : i^*\Omega_M \to J$ (use Lemma 1.6). Changing h by $\overline{h}^\sharp$ defines $h' : \overline{T} \to M$ such that $(h')^\sharp = 0$. Thus h' factors through X, and in fact these locally defined h' glue to give the required extension $\overline{g} : \overline{T} \to X$. □

Proposition 4.2 *There is a canonical isomorphism*

$$\underline{\mathrm{Aut}}(0(g)) \xrightarrow{\sim} \mathcal{H}om(g^*\Omega_X, J)$$

of sheaves on $T_{\text{ét}}$.

PROOF. Again, Lemma 1.6 shows that the automorphisms of $0(g)$ are (locally) the homomorphisms from $g^* i^* \Omega_M$ to J vanishing on $g^* I/I^2$. The exact sequence

$$I/I^2 \longrightarrow i^*\Omega_M \longrightarrow \Omega_X \longrightarrow 0$$

finishes the proof. See also Lemma 2.2. □

Corollary 4.3 *The sheaf* $\underline{\mathrm{Hom}}(\mathrm{ob}(g), 0(g))$ *is a formal* $\mathcal{H}om(g^*\Omega_X, J)$*-torsor. So if* $ob(g) \cong 0(g)$*, the set* $\mathrm{Hom}(ob(g), 0(g))$ *is a torsor under the group* $\mathrm{Hom}(g^*\Omega_X, J)$*.*

Note Combining this with Proposition 4.1 gives that $\mathrm{Ext}(g, \overline{T})$ is a $\mathrm{Hom}(g^*\Omega, J)$-torsor if the obstruction vanishes, reproving this fact from deformation theory alluded to above.

Obstruction Theories

Definition 4.4 Let $E^\bullet \in \mathrm{ob}\, D(\mathcal{O}_{X_{\text{ét}}})$ satisfy $(\star)$ (see Definition 2.3). Then a homomorphism $\phi : E^\bullet \to L^\bullet_X$ in $D(\mathcal{O}_{X_{\text{ét}}})$ is called an *obstruction theory* for X, if $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is surjective. By abuse of language we also say that $E^\bullet$ is an obstruction theory for X.

Note By Proposition 2.6 the homomorphism $\phi : E^\bullet \to L^\bullet_X$ is an obstruction theory if and only if

$$\phi^\vee : \mathfrak{N}_X \longrightarrow \mathfrak{E}$$

is a closed immersion, where $\mathfrak{E} = h^1/h^0((E^\bullet_{\text{fl}})^\vee)$. So if $E^\bullet$ is an obstruction theory and $\mathfrak{C}_X \subset \mathfrak{N}_X$ is the intrinsic normal cone of X, then $\phi^\vee(\mathfrak{C}_X)$ is a closed subcone stack of $\mathfrak{E}$ of pure dimension zero. We sometimes call $\phi^\vee(\mathfrak{C}_X)$ the *obstruction cone* of the obstruction theory $\phi : E^\bullet \to L^\bullet_X$.

Let $E^\bullet \in \mathrm{ob}\, E(\mathcal{O}_{X_{\text{ét}}})$ satisfy $(\star)$ and let $\phi : E^\bullet \to L^\bullet_X$ be a homomorphism. Let $\mathfrak{E} = h^1/h^0((E^\bullet_{\text{fl}})^\vee)$ and $\phi^\vee : \mathfrak{N}_X \to \mathfrak{E}$ the induced morphism of cone stacks. If $T \to \overline{T}$ is a square zero extension of k-schemes with ideal sheaf J and $g : T \to X$ is a morphism, then we denote by $\phi^*\omega(g)$ the image of the obstruction $\omega(g) \in \mathrm{Ext}^1(g^*L^\bullet_X, J)$ in $\mathrm{Ext}^1(g^*E^\bullet, J)$ and by $\phi^\vee(ob(g))$ the composition

$$C(J) \xrightarrow{ob(g)} g^*\mathfrak{N}_X \xrightarrow{g^*\phi^\vee} g^*\mathfrak{E}$$

of morphisms of cone stacks over T.

Theorem 4.5 *The following are equivalent.*

1. $\phi : E^\bullet \to L^\bullet_X$ *is an obstruction theory.*

2. $\phi^\vee : \mathfrak{N}_X \to \mathfrak{E}$ *is a closed immersion of cone stacks over* X*.*

3. *For any* $(T,\overline{T},g)$ *as above, the obstruction* $\phi^*(\omega(g)) \in \mathrm{Ext}^1(g^*E^\bullet,J)$ *vanishes if and only if an extension* $\overline{g}$ *of* g *to* $\overline{T}$ *exists; and if* $\phi^*(\omega(g))=0$*, then the extensions form a torsor under* $\mathrm{Ext}^0(g^*E^\bullet,J) = \mathrm{Hom}(g^*h^0(E^\bullet),J)$.

4. *For any* $(T,\overline{T},g)$ *as above, the sheaf of extensions* $\underline{\mathrm{Ext}}(g,\overline{T})$ *is isomorphic to the sheaf* $\underline{\mathrm{Hom}}(\phi^\vee(ob(g)),0)$ *of* $\mathbb{A}^1$*-equivariant isomorphism from* $\phi^\vee(ob(g)) : C(J) \to g^*\mathfrak{E}$ *to the vertex* $0 : C(J) \to g^*\mathfrak{E}$.

Proof. The equivalence of (1) and (2) has already been noted. In view of Proposition 4.1 it is clear that (2) implies (4). The implication (4)⇒(3) follows from Lemma 2.2. So let us prove that (3) implies (1).

To prove that $h^0(\phi)$ is an isomorphism we can assume that $X = \mathrm{Spec}\, R$ is an affine scheme (as the statement is local); let A be any R-algebra, M any A-module. Let $T = \mathrm{Spec}\, A$, $\overline{T} = \mathrm{Spec}(A \oplus M)$, where the ring structure is given by $(a,m)(a',m') = (aa', am' + a'm)$. Let $g : T \to X$ be the morphism induced by the R-algebra structure of A. Then g extends to $\overline{T}$, so there is a bijection $\mathrm{Hom}(h^0(L_X^\bullet) \otimes A, M) \to \mathrm{Hom}(h^0(E^\bullet)\otimes A, M)$. This implies easily that $h^0(\phi)$ is an isomorphism.

The fact that $h^{-1}(\phi)$ is surjective is local in the étale topology (and only depends on $\tau_{\geq -1}E^\bullet$). Assume therefore that X is an affine scheme, $i : X \to W$ a closed embedding in a smooth affine scheme W, and let I be the ideal of X in W. We can assume that $E^0 = f^*\Omega_W$ (see the proof of 2.5), that E^{-1} is a coherent sheaf, and that $E^i = 0$ for $i \neq 0,-1$.

We have to prove that $E^{-1} \to I/I^2$ is surjective; let M be its image. Let $T = X$, $\widetilde{M} \subset I$ the inverse image of M, and $\overline{T} \subset W$ the subscheme defined by $\widetilde{M}$; let $g : T \to X$ be the identity. We can extend g to the inclusion $\widetilde{g} : \overline{T} \to W$. Let $\pi : I/I^2 \to I/\widetilde{M}$ be the natural projection. By assumption π factors via E^0 if and only if g extends to a map $\overline{T} \to X$, if and only if $\pi \circ \phi^{-1} : E^{-1} \to I/\widetilde{M}$ factors via E^0. As $\pi\circ\phi^{-1}$ is the zero map, it certainly factors. Therefore π also factors. Consider now the commutative diagram with exact rows

$$\begin{array}{ccccccc} E^{-1} & \longrightarrow & E^0 & \longrightarrow & h^0(E^\bullet) & \longrightarrow & 0 \\ \phi\downarrow & & \| & & \| & & \\ I/I^2 & \longrightarrow & E^0 & \longrightarrow & h^0(E^\bullet) & \longrightarrow & 0. \end{array}$$

By an easy diagram chasing argument, the fact that π factors via E^0 together with $\pi \circ \phi^{-1} = 0$ implies $\pi = 0$, hence $\phi^{-1} : E^{-1} \to I/I^2$ is surjective. □

Obstructions for Small Extensions

Let Art be the category of local Artinian k-algebras with residue field k. A *small extension* will be a surjective morphism $A' \to A$ in Art with kernel J isomorphic to k. A *semi-small* extension is one with kernel isomorphic to a k-vector space as an A'-algebra.

Let $F : Art \to Sets$ be a pro-representable covariant functor (in the sense of [13]). An *obstruction space* for F is a set k-vector space T^2 and, for any semi-small extension $A' \to A$ with kernel J, an exact sequence

$$F(A') \longrightarrow F(A) \xrightarrow{ob} T^2 \otimes J.$$

This means that, for all $\xi \in F(A)$, ξ is in the image of $F(A')$ if and only if $ob(\xi) = 0$. It is also required that ob is functorial in the obvious sense (see [10]). We say that $v \in T^2$ *obstructs a small extension* $A' \to A$ if $ob(\xi) = v \otimes w$ for some $\xi \in F(A)$ and some nonzero $w \in J$.

Let X be a Deligne-Mumford stack, $p \in X$ a fixed point with residue field k. Let $h_p : Art \to Sets$ be the covariant functor associating to an object A of Art the set of morphisms $\operatorname{Spec} A \to X$ sending the closed point to p. The functor h_p is pro-representable, and it is unchanged if we replace X by any étale open neighborhood of p.

Let $N_p = p^*\mathfrak{N}_X$, and let $\overline{N}_p$ be the coarse moduli space of N_p. Note that $\overline{N}_p = T^1_{X,p}/T^0_{X,p}$, so that $\overline{N}_p$ is in fact a k-vector space. Here $T^i_{X,p} = h^i(p^*T^\bullet_X) = h^i(p^*L^\bullet_X)^\vee$ are the 'higher tangent spaces' of X at p. Let $\overline{C}_p \subset \overline{N}_p$ be the subcone coarsely representing $p^*\mathfrak{C}_X$. Proposition 4.1 implies that $\overline{N}_p$ is an obstruction space for h_X. The following is probably known but we include a proof for lack of a suitable reference; it is a version of Theorem 4.5 for semi-small extensions.

Lemma 4.6 *The space $\overline{N}_p$ is a universal obstruction space for h_p; that is, for any other obstruction space T^2, there is a unique injection $N_p \to T^2$ compatible with the obstruction maps.*

Proof. Let (U, W) be a local embedding for X near p. Assume that $W = \operatorname{Spec} P$, $U = \operatorname{Spec} R = \operatorname{Spec} P/I$; let $\mathfrak{m}$ be the maximal ideal of p in P, and assume that $I \subset \mathfrak{m}^2$. In this case $\overline{N}_p = (I/\mathfrak{m}I)^\vee$.

If n is sufficiently large, the natural map $I/\mathfrak{m}I \to (I+\mathfrak{m}^n)/(\mathfrak{m}I+\mathfrak{m}^n)$ is an isomorphism; choose such an n. Let $A'_n \to A_n$ be the extension $P/(\mathfrak{m}I + \mathfrak{m}^n) \to P/(I+\mathfrak{m}^n)$, and let $\xi_n \in h_p(A_n)$ be the natural quotient map. Then if T^2 is any obstruction space, the obstruction to ξ_n gives a linear map

$I/\mathfrak{m}I \to T^2$ which must be injective. It is easy to check by functoriality that taking a different n does not change the map. But given any semi-small extension $A' \to A$, there is always an extension of the type $A'_n \to A_n$ mapping to it, so one can apply functoriality again. $\square$

Proposition 4.7 *Every $v \in \overline{N}_p$ obstructs some small extension; it obstructs some small curvilinear extension if and only if $v \in \overline{C}_p$.*

PROOF. Let $v \in \overline{N}_p$, and view it as a linear map $I \to k$ having $\mathfrak{m}I$ in the kernel; we prove first that v is an obstruction for some small extension. Let $L = \ker v$, and choose n sufficiently large, so that $L + \mathfrak{m}^n \neq I + \mathfrak{m}^n$. Let $A = P/I + \mathfrak{m}^n$, and $A = P/L + \mathfrak{m}^n$; choose $\xi : R \to A$ to be the natural surjection. Let $J = \ker(A' \to A)$; J is naturally isomorphic to I/L. Then $\mathrm{ob}_\xi : I/\mathfrak{m}I \to J$ is the obvious map, and the image of the dual map in $\overline{N}_p$ is the vector space generated by v.

Choose a set of generators $f_1, \ldots, f_r$ of I inducing a basis for $I/\mathfrak{m}I$. This defines a map $f : W \to \mathbb{A}^r$ such that U is the fiber over the origin. Then $\overline{C}_p$ is the normal cone to the image of W in $\mathbb{A}^r$. The proof then follows the argument of Proposition 20.2 in [8]. $\square$

5 Obstruction Theories and Fundamental Classes

Virtual Fundamental Classes

As usual, let X be a Deligne-Mumford stack over k.

Definition 5.1 We call an obstruction theory $E^\bullet \to L_X^\bullet$ *perfect*, if $E^\bullet$ is of perfect amplitude contained in $[-1, 0]$.

Now assume that X is separated (or, more generally, satisfies the condition of Vistoli in [14]). We shall denote by $A_k(X)$ the rational Chow group of cycles of dimension k on X modulo rational equivalence tensored with $\mathbb{Q}$ (see [ibid]). We shall also use the corresponding bivariant groups $A^k(X \to Y)$, for morphisms $X \to Y$ of separated Deligne-Mumford stacks.

Let $E^\bullet$ be a perfect obstruction theory for X, and let $\mathfrak{C}_X \hookrightarrow h^1/h^0(E^\vee)$ be the intrinsic normal cone. We call $\mathrm{rk}\, E^\bullet$ the *virtual dimension* of X with respect to the obstruction theory $E^\bullet$. Recall that $\mathrm{rk}\, E^\bullet = \dim E^0 - \dim E^{-1}$, if locally $E^\bullet$ is written as a complex of vector bundles $[E^{-1} \to E^0]$. This is a well-defined locally constant function on X. We shall assume that the virtual dimension of X with respect to $E^\bullet$ is constant, equal to n.

To construct the *virtual fundamental class* $[X, E^\bullet] \in A_n(X)$ of X with respect to the obstruction theory $E^\bullet$, we would like to simply intersect the intrinsic normal cone $\mathfrak{C}_X$ with the vertex (zero section) of $h^1/h^0(E^\vee)$. Since $h^1/h^0(E^\vee)$ is smooth of relative dimension $-n$ over X, the codimension of X in $h^1/h^0(E^\vee)$ is $-n$, so that the dimension of the intersection of $\mathfrak{C}_X$ with X is $0 - (-n) = n$. Unfortunately, this construction would require Chow groups for Artin stacks, which we do not have at our disposal. This is why we shall make the assumption that $E^\bullet$ has global resolutions.

Definition 5.2 Let $F^\bullet = [F^{-1} \to F^0]$ be a homomorphism of vector bundles on X considered as a complex of $\mathcal{O}_X$-modules concentrated in degrees -1 and 0. An isomorphism $F^\bullet \to E^\bullet$ in $D(\mathcal{O}_{X_{\text{ét}}})$ is called a *global resolution* of $E^\bullet$.

Let $F^\bullet$ be a global resolution of $E^\bullet$. Then

$$h^1/h^0(E^\vee) = [F^{-1\vee}/F^{0\vee}],$$

so that $F_1 = F^{-1\vee}$ is a (global) presentation of $h^1/h^0(E^\vee)$. Let $C(F^\bullet)$ be the fibered product

$$\begin{array}{ccc} C(F^\bullet) & \longrightarrow & F_1 \\ \downarrow & & \downarrow \\ \mathfrak{C}_X & \longrightarrow & h^1/h^0(E^\vee). \end{array}$$

Then $C(F^\bullet)$ is a closed subcone of the vector bundle F_1. We define the *virtual fundamental class* $[X, E^\bullet]$ to be the intersection of $C(F^\bullet)$ with the zero section of F_1. Note that $C(F^\bullet) \to \mathfrak{C}_X$ is smooth of relative dimension $\operatorname{rk} F_0$ (where $F_0 = F^{0\vee}$), so that $C(F^\bullet)$ has pure dimension $\operatorname{rk} F_0$ and $[X, E^\bullet]$ then has degree

$$\operatorname{rk} F_0 - \operatorname{rk} F_1 = \operatorname{rk} E^\bullet = n.$$

Proposition 5.3 *The virtual fundamental class $[X, E^\bullet]$ is independent of the global resolution $F^\bullet$ used to construct it.*

Proof. Let $H^\bullet$ be another global resolution of $E^\bullet$. Without loss of generality assume that $H^\bullet \to E^\bullet$ and $F^\bullet \to E^\bullet$ are given by morphisms of complexes. Then we get an induced homomorphism $H^0 \oplus F^0 \to E^0$. So by constructing the cartesian diagram

$$\begin{array}{ccc} K^{-1} & \longrightarrow & H^0 \oplus F^0 \\ \downarrow & & \downarrow \\ E^{-1} & \longrightarrow & E^0, \end{array}$$

and letting $K^0 = H^0 \oplus F^0$, we get a global resolution $K^\bullet$ of $E^\bullet$ such that both $H^\bullet$ and $F^\bullet$ map to $K^\bullet$ by a strict monomorphism. So it suffices to compare $F^\bullet$ with $K^\bullet$. Dually, we have an epimorphism $K_1 \to F_1$. Consider the diagram

$$\begin{array}{ccccc} X & \xrightarrow{0} & C(H^\bullet) & \longrightarrow & C(F^\bullet) \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{0} & K_1 & \xrightarrow{\alpha} & F_1, \end{array}$$

in which both squares are cartesian. Note that α is smooth. The virtual fundamental class using $F^\bullet$ is equal to

$$(\alpha \circ 0)^![C(F^\bullet)] = 0^!\alpha^![C(F^\bullet)] = 0^![C(H^\bullet)],$$

which is the virtual fundamental class using $H^\bullet$. □

Example If X is a complete intersection, then $L_X^\bullet$ is of perfect amplitude contained in $[-1, 0]$, so that $L_X^\bullet$ itself is a perfect obstruction theory. Any embedding of X into a smooth Deligne-Mumford stack gives rise to a global resolution of $L_X^\bullet$.The virtual fundamental class $[X, L_X^\bullet]$ thus obtained is equal to $[X]$, the 'usual' fundamental class.

Remark 5.4 [Virtual Structure Sheaves] Let X be a Deligne-Mumford stack and let $\mathfrak{C} \hookrightarrow \mathfrak{E}$ be a closed subcone stack of a vector bundle stack. Then we define a graded commutative sheaf of coherent $\mathcal{O}_X$-algebras $\mathcal{O}_{(\mathfrak{C},\mathfrak{E})}$ as follows.

If $\mathfrak{E} \cong [E_1/E_0]$, then $\mathfrak{C}$ induces a cone C in E_1 and we set

$$\mathcal{O}^i_{(\mathfrak{C},\mathfrak{E})} = \mathcal{T}or_i^{\mathcal{O}_{E_1}}(\mathcal{O}_C, \mathcal{O}_X),$$

where we think of $\mathcal{O}_X$ as an $\mathcal{O}_{E_1}$-algebra via the zero section of E_1. Standard arguments show that

$$\mathcal{O}_{(\mathfrak{C},\mathfrak{E})} = \bigoplus_i \mathcal{O}^i_{(\mathfrak{C},\mathfrak{E})}$$

is independent of the choice of presentation $\mathfrak{E} \cong [E_1/E_0]$. Hence the locally defined sheaves glue, giving rise to a globally defined sheaf.

If $\mathfrak{C} = \mathfrak{C}_X$, $E^\bullet$ is a perfect obstruction theory of X and $\mathfrak{E} = h^1/h^0(E^{\bullet\vee})$, we call $\mathcal{O}_{(\mathfrak{C},\mathfrak{E})}$ the *virtual structure sheaf* of X with respect to the obstruction theory $E^\bullet$, denoted $\mathcal{O}_{(X,E^\bullet)}$. This seems to be the virtual structure sheaf proposed by Kontsevich in [11].

If one has on X a homological Chern character $\tau : K_0(X) \to A_*(X)$ one can define the virtual fundamental class of X with respect to $E^\bullet$ by

$$[X, E^\bullet] = \operatorname{td}(E^\bullet) \cap \tau(\mathcal{O}_{(X,E^\bullet)}).$$

This agrees with the above definition using global resolutions if they exist. In the absence of a general Riemann Roch theorem, we rather assume the existence of global resolutions.

Basic Properties

Proposition 5.5 (No obstructions) *If $E^\bullet$ is perfect, $h^0(E^\bullet)$ is locally free and $h^1(E^\bullet) = 0$, then X is smooth, the virtual dimension of X with respect to $E^\bullet$ is $\dim X$ and the virtual fundamental class $[X, E^\bullet]$ is just $[X]$, the usual fundamental class.* $\square$

Proposition 5.6 (Locally free obstructions) *Let X be smooth and $E^\bullet$ a perfect obstruction theory for X. If $h^0(E^\bullet)$ is locally free (or equivalently $h^1(E^{\bullet\vee})$ is locally free) then the virtual fundamental class is*

$$[X, E^\bullet] = c_r(h^1(E^{\bullet\vee})) \cdot [X],$$

where $r = \operatorname{rk} h^1(E^{\bullet\vee})$.

Proof. To see this, note that if $F^\bullet$ is a global resolution of $E^\bullet$, then $C(F^\bullet) = \operatorname{im}(F_0 \to F_1)$. $\square$

Proposition 5.7 (Products) *Let $E \to L_X$ be a perfect obstruction theory for X and $F \to L_Y$ a perfect obstruction theory for Y. Then $L_{X \times Y} = L_X \boxplus L_Y$. The induced homomorphism $E \boxplus F \to L_X \boxplus L_Y$ is a perfect obstruction theory for $X \times Y$. If E and F have global resolutions, then so does $E \boxplus F$ and we have*

$$[X \times Y, E \boxplus F] = [X, E] \times [Y, F]$$

in $A_{\operatorname{rk} E + \operatorname{rk} F}(X \times Y)$.

Proof. The statement about cotangent complexes is [9], Chapitre II, Corollaire 3.11. To prove the rest, use Proposition 3.13. $\square$

Consider a cartesian diagram of Deligne-Mumford stacks

$$\begin{array}{ccc} X' & \xrightarrow{u} & X \\ {\scriptstyle g} \downarrow & & \downarrow {\scriptstyle f} \\ Y' & \xrightarrow{v} & Y, \end{array} \tag{11}$$

where v is a local complete intersection morphism. Let $E \to L_X$ and $F \to L_{X'}$ be perfect obstruction theories for X and X', respectively.

Definition 5.8 A *compatibility datum* (relative to v) for E and F is a triple (ϕ, ψ, χ) of morphisms in $D(\mathcal{O}_{X'})$ giving rise to a morphism of distinguished triangles

$$\begin{array}{ccccccc} u^*E & \xrightarrow{\phi} & F & \xrightarrow{\psi} & g^*L_{Y'/Y} & \xrightarrow{\chi} & u^*E[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ u^*L_X & \longrightarrow & L_{X'} & \longrightarrow & L_{X'/X} & \longrightarrow & u^*L_X[1]. \end{array}$$

Given a compatibility datum, we call E and F *compatible* (over v).

Assume that E and F are endowed with such a compatibility datum. Then we get (Proposition 2.7) a short exact sequence of vector bundle stacks

$$g^*h^1/h^0(T^{\bullet}_{Y'/Y}) \longrightarrow h^1/h^0(F^{\vee}) \longrightarrow u^*h^1/h^0(E^{\vee})$$

which we shall abbreviate by

$$g^*\mathfrak{N}_{Y'/Y} \longrightarrow \mathfrak{F} \xrightarrow{\phi} u^*\mathfrak{E}.$$

If v is a regular local immersion, then $\mathfrak{N}_{Y'/Y} = N_{Y'/Y}$ is the normal bundle of Y' in Y. Its pullback to X' we shall denote by N.

Lemma 5.9 *If Y and Y' are smooth and v a regular local immersion, then there is a (canonical) rational equivalence $\beta(Y', X) \in W_*(N \times \mathfrak{F})$ such that*

$$\partial\beta(Y', X) = [\phi^*C_{u^*\mathfrak{c}_X/\mathfrak{c}_X}] - [N \times \mathfrak{C}_{X'}].$$

Proof. Let $X \to M$ be a local embedding, where M is smooth. We get an induced cartesian diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' \times M & \longrightarrow & Y \times M, \end{array}$$

which we enlarge to

$$\begin{array}{ccccc}
N \times_X C & \longrightarrow & u^*C & \longrightarrow & C \\
\downarrow & & \downarrow & & \downarrow \\
N & \longrightarrow & X' & \stackrel{u}{\longrightarrow} & X \\
\downarrow & & {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle i} \\
N_{Y'/Y} \times M & \stackrel{\rho}{\longrightarrow} & Y' \times M & \stackrel{v}{\longrightarrow} & Y \times M,
\end{array}$$

where C is the normal cone of X in $Y \times M$. As in Section 3 we have a canonical rational equivalence $\beta(Y' \times M, X) \in W_*(N \times_X C)$ such that

$$\partial\beta(Y' \times M, X) = [C_{u^*C/C}] - [N \times C_{X'/Y' \times M}].$$

By Proposition 3.5 $\beta(Y' \times M, X)$ is invariant under the action of $N \times u^*i^*T_{Y \times M}$ on $N \times_X C$. So it descends to $N \times_X \mathfrak{C}_X$. In particular, $\beta(Y' \times M, X)$ is invariant under the subsheaf $N \times j^*T_{Y' \times M}$ and thus descends to $N \times [u^*C/j^*T_{Y' \times M}]$. Note that $[u^*C/j^*T_{Y' \times M}] = \mathfrak{F} \times_{\mathfrak{E}} \mathfrak{C}_X$, which is a closed subcone stack of $\mathfrak{F}$. So pushing forward via this closed immersion, we get a rational equivalence on $N \times \mathfrak{F}$ which we denote by $\beta(Y', X)$. We have

$$\partial\beta(Y', X) = [\phi^* C_{u^*\mathfrak{C}_X/\mathfrak{C}_X}] - [N \times \mathfrak{C}_{X'}]$$

as required. Now use Proposition 3.4 to show that $\beta(Y', X)$ does not depend on the choice of the local embedding $X \to M$. So even if no global embedding exists, the locally defined rational equivalences glue, proving the lemma. □

Proposition 5.10 (Functoriality) *Let E and F be compatible perfect obstruction theories, as above. If E and F have global resolutions then*

$$v^![X, E] = [X', F]$$

holds in the following cases.

1. *v is smooth,*
2. *Y' and Y are smooth.*

Proof. First note that one may choose global resolutions $[E_0 \to E_1]$ of $E^\vee$ and $[F_0 \to F_1]$ of $F^\vee$ together with a pair of epimorphisms $\phi_0 : F_0 \to u^*E_0$ and $\phi_1 : F_1 \to u^*E_1$ such that

$$\begin{array}{ccc}
F_0 & \stackrel{\phi_0}{\longrightarrow} & u^*E_0 \\
\downarrow & & \downarrow \\
F_1 & \stackrel{\phi_1}{\longrightarrow} & u^*E_1
\end{array}$$

commutes. Letting G_i be the kernel of ϕ_i we get a short exact sequence of homomorphisms of vector bundles

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G_0 & \longrightarrow & F_0 & \longrightarrow & u^*E_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & G_1 & \longrightarrow & F_1 & \longrightarrow & u^*E_1 & \longrightarrow & 0. \end{array}$$

The induced short exact sequence

$$[G_1/G_0] \longrightarrow [F_1/F_0] \longrightarrow [u^*E_1/u^*E_0]$$

of vector bundle stacks is isomorphic to $g^*\mathfrak{N}_{Y'/Y} \to \mathfrak{F} \to \mathfrak{E}$. We let $C_1 = \mathfrak{C}_X \times_{\mathfrak{E}} E_1$ and $D_1 = \mathfrak{C}_{X'} \times_{\mathfrak{F}} F_1$. Then $[X, E] = 0^!_{E_1}[C_1]$ and $[X', F] = 0^!_{F_1}[D_1]$, where 0_{E_1} and 0_{F_1} are the zero sections of E_1 and F_1, respectively.

If v is smooth, then by Proposition 3.14 the diagram

$$\begin{array}{ccc} \mathfrak{C}_{X'} & \longrightarrow & u^*\mathfrak{C}_X \\ \downarrow & & \downarrow \\ \mathfrak{F} & \longrightarrow & u^*\mathfrak{E} \end{array}$$

is cartesian, which implies that

$$\begin{array}{ccc} D_1 & \longrightarrow & u^*C_1 \\ \downarrow & & \downarrow \\ F_1 & \longrightarrow & u^*E_1 \end{array}$$

is cartesian. Hence $0^!_{u^*E_1}[u^*C_1] = 0^!_{F_1}[D_1]$ and we have

$$\begin{aligned} v^![X, E] &= v^!0^!_{E_1}[C_1] \\ &= 0^!_{u^*E_1}[u^*C_1] \\ &= 0^!_{F_1}[D_1] \\ &= [X', F]. \end{aligned}$$

If Y' and Y are smooth, let us first treat the case that v is a regular local immersion. Then we may choose F_1 as the fibered product

$$\begin{array}{ccc} F_1 & \longrightarrow & E_1 \\ \downarrow & & \downarrow \\ \mathfrak{F} & \stackrel{\phi}{\longrightarrow} & \mathfrak{E}. \end{array}$$

Lifting the rational equivalence $\beta(Y', X)$ of Lemma 5.9 to $N \times F_1$ we get that

$$[N \times D_1] = \phi^*[C_{u^*C_1/C_1}]$$

in $A_*(N \times F_1)$. Then we have

$$\begin{aligned}
[X', F_1] &= 0^!_{F_1}[D_1] \\
&= 0^!_{N\times F_1}[N \times D_1] \\
&= 0^!_{N\times F_1}\phi^*[C_{u^*C_1/C_1}] \\
&= 0^!_{N\times u^*E_1}[C_{u^*C_1/C_1}] \\
&= 0^!_{*E_1}v^![C_1] \\
&= v^!0^!_{E_1}[C_1] \\
&= v^![X, E].
\end{aligned}$$

In the general case factor v as

$$Y' \xrightarrow{\Gamma_v} Y' \times Y \xrightarrow{p} Y.$$

Then Diagram 11 factors as

$$\begin{array}{ccccc}
X' & \longrightarrow & Y' \times X & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow \\
Y' & \xrightarrow{\Gamma_v} & Y' \times Y & \xrightarrow{p} & Y.
\end{array}$$

Since Y' is smooth it has a canonical obstruction theory, namely $\Omega_{Y'}$. As obstruction theory on $Y' \times X$ take $\Omega_{Y'} \boxplus E$. Then $\Omega_{Y'} \boxplus E$ is compatible with E over p and F is compatible with $\Omega_{Y'} \boxplus E$ over Γ_v. So combining Cases (1) and (2) yields the result. □

6 Examples

The Basic Example

Assume that

$$\begin{array}{ccc}
X & \xrightarrow{j} & V \\
g\downarrow & & \downarrow f \\
Y & \xrightarrow{i} & W
\end{array}$$

is a cartesian diagram of schemes, that V and W are smooth and that i is a regular embedding. Let $E^\bullet$ be the complex $[g^*N_{Y/W}{}^\vee \to j^*\Omega_V]$ (in degrees

-1 and 0), where the map is given by pulling back to X and composing $N_{Y/W}{}^{\vee} \to i^*\Omega_W$ with $f^*\Omega_W \to \Omega_V$. The complex $E^\bullet$ has a natural morphism to $L_X^\bullet$, induced by $g^*L_Y^\bullet \to L_X^\bullet$ and $j^*L_V^\bullet \to L_X^\bullet$ (note that $E^\bullet$ is the cokernel of $g^*i^*L_W^\bullet \to j^*L_V^\bullet \oplus g^*L_Y^\bullet$, where the first component is the negative of the canonical map).

This makes $E^\bullet$ into a perfect obstruction theory for X; the virtual fundamental class $[X, E^\bullet]$ is just $i^![V]$ as defined in [6], p. 98. The construction also works in case X, Y, V and W are assumed to be just Deligne-Mumford stacks.

Fibers of a Morphism between Smooth Stacks

Let $f : V \to W$ be a morphism of algebraic stacks. We shall assume that V and W are smooth over k and that f has unramified diagonal, so that V is a relative Deligne-Mumford stack over W. Let $w : \operatorname{Spec} k \to W$ be a k-valued point of W and let X be the fiber of f over w. In this situation X has an obstruction theory as follows.

Choose a smooth morphism $\widetilde{W} \to W$, with $\widetilde{W}$ a scheme, and a lifting $\widetilde{w} : \operatorname{Spec} k \to \widetilde{W}$ of w (assume k algebraically closed). Let $\widetilde{V}$ be the fiber product $V \times_W \widetilde{W}$; by the assumptions $\widetilde{V}$ is a smooth Deligne-Mumford stack. Then X is isomorphic to the fiber over $\widetilde{w}$ of $\widetilde{V} \to \widetilde{W}$, hence it has an obstruction theory as above.

To check that the obstruction theory so defined does not depend on the choices made, it is enough to compare two different ones induced by a smooth morphism of schemes $\widetilde{W}' \to \widetilde{W}$; this is then a straightforward verification. Similarly, one generalizes to the case of arbitrary ground field k.

See Example 7.6 for an alternative construction.

Moduli Stacks of Projective Varieties

Let M and X be Deligne-Mumford stacks. Let $p : M \to X$ be a flat, relatively Gorenstein projective morphism: by this we mean that it has constant relative dimension and that the relative dualizing complex $\omega^\bullet_{M/X}$ is a line bundle ω.

If $G^\bullet \in D^+(\mathcal{O}_X)$, we have $p^!G^\bullet = p^*G^\bullet \otimes \omega$. So for any complex $F^\bullet \in D^-(\mathcal{O}_M)$ we have natural isomorphisms

$$\operatorname{Ext}^k_{\mathcal{O}_M}(F^\bullet, p^*G^\bullet) \to \operatorname{Ext}^k_{\mathcal{O}_M}(F^\bullet \otimes \omega, p^!G^\bullet) \to \operatorname{Ext}^k_{\mathcal{O}_X}(Rp_*(F^\bullet \otimes \omega), G^\bullet).$$

In particular, the Kodaira-Spencer map $L_{M/X} \to p^*L_X[1]$ induces a map $E^\bullet \to L_X^\bullet$ (well-defined up to homotopy). Define the complex $E^\bullet$ on X to be $Rp_*(L_{M/X}^\bullet \otimes \omega)[-1]$.

Proposition 6.1 *Let $p : M \to X$ be a flat, projective, relatively Gorenstein morphism of Deligne-Mumford stacks, and assume that the family M is universal at every point of X (e.g., X is an open set in a fine moduli space and M is the universal family). Then $E^\bullet \to L_X^\bullet$ is an obstruction theory for X.*

Proof. Let T be a scheme, $f : T \to X$ a morphism, and consider the cartesian diagram

$$\begin{array}{ccc} N & \xrightarrow{g} & M \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ T & \xrightarrow{f} & X. \end{array}$$

If $T \to \bar{T}$ is a square zero extension with ideal sheaf $\mathcal{J}$, the obstruction to extending N to a flat family over $\bar{T}$ lies in $\mathrm{Ext}^2(L_{N/T}^\bullet, q^*\mathcal{J})$, and the extensions, if they exist, are a torsor under $\mathrm{Ext}^1(L_{N/T}^\bullet, q^*\mathcal{J})$. Now $L_{N/T}^\bullet = g^*L_{M/X}^\bullet$ because p is flat, hence

$$\mathrm{Ext}^k_{\mathcal{O}_N}(L_{N/T}^\bullet, q^*\mathcal{J}) = \mathrm{Ext}^k_{\mathcal{O}_M}(L_{M/X}^\bullet, Rg_*q^*\mathcal{J}) = \mathrm{Ext}^k_{\mathcal{O}_M}(L_{M/X}^\bullet, p^*Rf_*\mathcal{J}).$$

By the previous argument,

$$\mathrm{Ext}^k_{\mathcal{O}_M}(L_{M/X}^\bullet, p^*Rf_*\mathcal{J}) = \mathrm{Ext}^{k-1}_{\mathcal{O}_X}(E^\bullet, Rf_*\mathcal{J}) = \mathrm{Ext}^{k-1}_{\mathcal{O}_T}(f^*E^\bullet, \mathcal{J}).$$

Assume now that X is an open subset of a fine moduli space, that is the family M is universal at every point. This implies that the fibers of p have finite and reduced automorphism group, hence $E^\bullet$ satisfies ($\star$).

The map $E^\bullet \to L_X^\bullet$ induces morphisms

$$\phi_k : \mathrm{Ext}^k_{\mathcal{O}_N}(L_{N/T}^\bullet, q^*\mathcal{J}) = \mathrm{Ext}^{k-1}_{\mathcal{O}_T}(f^*E^\bullet, \mathcal{J}) \to \mathrm{Ext}^{k-1}_{\mathcal{O}_T}(f^*L_X^\bullet, \mathcal{J})$$

and the fact that X is a moduli space implies that ϕ_1 is an isomorphism and ϕ_2 is injective. By Theorem 4.5, this implies that $E^\bullet$ is an obstruction theory for X. □

Remark If p is smooth of relative dimension ≤ 2, then $E^\bullet$ is a perfect obstruction theory.

Spaces of Morphisms

Let C and V be projective k-schemes. Let $X = \mathrm{Mor}(C, V)$ be the k-scheme of morphisms from C to V (see [7]). Let $f : C \times X \to V$ be the universal morphism and $\pi : C \times X \to X$ the projection. By the functorial properties of the cotangent complex we get a homomorphism

$$f^* L_V^\bullet \longrightarrow L_{C\times X}^\bullet \longrightarrow L_{C\times X/C}^\bullet$$

and a homomorphism

$$\pi^* L_X^\bullet \longrightarrow L_{C\times X/C}^\bullet.$$

The latter is an isomorphism so that we get an induced homomorphism

$$e : f^* L_V^\bullet \longrightarrow \pi^* L_X^\bullet.$$

Assume that C has a dualizing complex ω_C. Then we get a homomorphism

$$e \otimes \omega_C : f^* L_V^\bullet \overset{L}{\otimes} \omega_C \longrightarrow \pi^* L_X^\bullet \overset{L}{\otimes} \omega_C = \pi^! L_X^\bullet$$

and by adjunction a homomorphism

$$\pi_*(e \otimes \omega_C) : R\pi_*(f^* L_V^\bullet \overset{L}{\otimes} \omega_C) \longrightarrow L_X^\bullet.$$

By duality we have

$$R\pi_*(f^* L_V^\bullet \overset{L}{\otimes} \omega_C) = (R\pi_*(f^* T_V^\bullet))^\vee.$$

Let us denote the resulting homomorphism by

$$\pi_*(e^\vee)^\vee : (R\pi_*(f^* T_V^\bullet))^\vee \longrightarrow L_X^\bullet.$$

Proposition 6.2 *Assume that C is Gorenstein. Then the homomorphism $\phi := \pi_*(e^\vee)^\vee$ is an obstruction theory for X. If C is a curve and V is smooth then this obstruction theory is perfect.*

Proof. Let T be an affine scheme, $g : T \to X$ a morphism, $\mathcal{J}$ a coherent sheaf on T; let $p : C \times T \to T$ be the projection, $h : C \times T \to V$ the morphism induced by g.

By an argument analogous to that in the previous example, we get

$$\mathrm{Ext}^k_{\mathcal{O}_{C\times T}}(h^* L_V^\bullet, p^* \mathcal{J}) = \mathrm{Ext}^k_{\mathcal{O}_C}(g^* E^\bullet, \mathcal{J}).$$

Apply now Theorem 4.5, more precisely the equivalence between (1) and (3). Choose any square zero extension $\bar{T}$ of T with ideal sheaf $\mathcal{J}$. Then g extends to $\bar{g} : \bar{T} \to X$ if and only if h extends to $\bar{h} : C \times \bar{T} \to V$, if and only if $\phi^* \omega(g)$ is zero in $\mathrm{Ext}^1_{\mathcal{O}_{C\times T}}(h^* L_V^\bullet, p^* \mathcal{J})$. The extensions, if they exist, form a torsor under $\mathrm{Hom}_{\mathcal{O}_{C\times T}}(h^* L_V^\bullet, p^* \mathcal{J})$. □

7 The Relative Case

Bivariant Theory for Artin Stacks

For what follows, we need a little bivariant intersection theory for algebraic stacks that are not necessarily of Deligne-Mumford type.

For simplicity, let us assume that $f : X \to Y$ is a morphism of algebraic k-stacks which is representable. This assumption implies that whenever

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X & \stackrel{f}{\longrightarrow} & Y \end{array}$$

is a cartesian diagram and Y' is a Deligne-Mumford stack satisfying the condition needed to define its Chow group (see [14]), then X' is of the same type. The following remarks can be generalized to any morphism f satisfying this property, e.g. any f which has finite unramified diagonal.

For such an $f : X \to Y$ we define bivariant groups $A^*(X \to Y)$ by using the same definition as Definition 5.1 in [14]. Then just as in [ibid.] one proves that the elements of $A^*(X \to Y)$ act on Chow groups of Deligne-Mumford stacks.

The same definition as [ibid.] Definition 3.10 applies in case $f : X \to Y$ is a regular local immersion, and defines a canonical element $[f] \in A^*(X \to Y)$ whose action on cycle classes is denoted by $f^!$. This is justified, since Theorems 3.11, 3.12, and 3.13 from [ibid.] hold with the same proofs in this more general context. In fact, $[f]$ even commutes with the Gysin morphism for any other local regular immersion of algebraic stacks.

Similarly, if $f : X \to Y$ is flat, flat pullback of cycles defines a canonical orientation $[f] \in A^*(X \to Y)$.

The Relative Intrinsic Normal Cone

We shall now replace the base $\operatorname{Spec} k$ by an arbitrary smooth (or more generally pure dimensional, but always of constant dimension) algebraic k-stack Y (not necessarily of Deligne-Mumford type). We shall consider algebraic stacks X over Y which are of relative Deligne-Mumford type over Y, i.e. such that the diagonal $X \to X \times_Y X$ is unramified. This assures that $h^i(L^\bullet_{X/Y}) = 0$, for all $i > 0$ (i.e. $h^1(L^\bullet_{X/Y}) = 0$), so that $L_{X/Y}$ satisfies Condition $(\star)$.

The *relative intrinsic normal sheaf* $\mathfrak{N}_{X/Y}$ is defined as

$$\mathfrak{N}_{X/Y} = h^1/h^0(T^\bullet_{X/Y}).$$

Using local embeddings of X into schemes smooth over Y, we construct as in the absolute case a subcone stack $\mathfrak{C}_{X/Y} \subset \mathfrak{N}_{X/Y}$ called the *relative intrinsic normal cone* of X over Y. If $n = \dim Y$, then $\mathfrak{C}_{X/Y}$ is of pure dimension n.

The definition of a *relative obstruction theory* is the same as Definition 4.4, with $L^\bullet_X$ replaced by $L^\bullet_{X/Y}$. As in the absolute case the relative intrinsic normal cone embeds as a closed subcone stack of a vector bundle stack

$$\mathfrak{C}_{X/Y} \subset h^1/h^0(E^\vee),$$

if E is a perfect relative obstruction theory. (Note that 'perfect' means 'absolutely perfect'.)

So let E be a perfect obstruction theory for X over Y admitting global resolutions. If X is a separated Deligne-Mumford stack then we get a virtual fundamental class $[X, E^\bullet] \in A_{n+\mathrm{rk}\, E}(X)$ by 'intersecting $\mathfrak{C}_X$ with the vertex of $h^1/h^0(E^\vee)$' as in the discussion preceding Proposition 5.3.

Consider the following diagram, where Y and Y' are smooth of constant dimension, v has finite unramified diagonal and X and X' are separated Deligne-Mumford stacks.

$$\begin{array}{ccc} X' & \stackrel{u}{\longrightarrow} & X \\ \downarrow & & \downarrow \\ Y' & \stackrel{v}{\longrightarrow} & Y \end{array} \qquad (12)$$

Proposition 7.1 *There is a natural morphism*

$$\alpha : \mathfrak{C}_{X'/Y'} \longrightarrow \mathfrak{C}_{X/Y} \times_Y Y'.$$

If (12) is cartesian, then α is a closed immersion. If, moreover, v is flat, then α is an isomorphism.

PROOF. Both statements follow immediately from the corresponding properties of normal cones for schemes. □

Proposition 7.2 (Pullback) *Let $E \to L_{X/Y}$ be a perfect obstruction theory for X over Y. If (12) is cartesian then u^*E is a perfect obstruction theory for X' over Y'. If E has global resolutions so does u^*E and for the induced virtual fundamental classes we have*

$$v^![X, E] = [X', u^*E],$$

at least in the following cases.

1. *v is flat,*

2. *v is a regular local immersion.*

Proof. Let $E^{-1} \to E^0$ be a global resolution of $E^\bullet$ and C the cone induced by $\mathfrak{C}_{X/Y}$ in E_1. Let $u^* E_i = E_i'$, and D the cone induced by $\mathfrak{C}_{X'/Y'}$ in E_1'.

If v is flat we have $\mathfrak{C}_{X'/Y'} = v^* \mathfrak{C}_{X/Y}$ and hence $D = v^* C$ by Proposition 7.1 and the statement follows ¿from the fact that $v^!$ is a bivariant class; in this case that $v^!$ commutes with $0^!_{E_1}$, where $0 : X \to E_1$ is the zero section.

If v is a regular local immersion, let $N = N_{Y'/Y}$ and use Vistoli's rational equivalence

$$\beta(Y', X) \in W_*(N \times_Y C)$$

(see Proposition 3.3) to prove that $v^![C] = [D]$. Then proceed as before. □

The following are relative versions of the basic properties of virtual fundamental classes from Section 5.

Proposition 7.3 (Locally free obstructions) *Let $E^\bullet$ be a perfect relative obstruction theory for X over Y such that $h^0(E^\bullet)$ is locally free. Assume that $E^\bullet$ has global resolutions and X is a separated Deligne-Mumford stack, so that the virtual fundamental class $[X, E^\bullet]$ exists.*

1. *If $h^{-1}(E^\bullet) = 0$, then X is smooth over Y and $[X, E^\bullet] = [X]$.*

2. *If X is smooth over Y, then $h^1(E^\vee)$ is locally free and $[X, E^\bullet] = c_r(h^1(E^\vee)) \cdot [X]$, where $r = \operatorname{rk} h^1(E^\vee)$.*

Proof. The proofs are the same as in the absolute case (Propositions 5.5and 5.6). □

Proposition 7.4 (Products) *Let E be a perfect relative obstruction theory for X over Y and F a perfect relative obstruction theory for X' over Y'. Then $E \boxplus F$ is a perfect relative obstruction theory for $X \times X'$ over $Y \times Y'$. If E and F have global resolutions and X and X' are separated Deligne-Mumford stacks, then $E \boxplus F$ has global resolutions and $X \times X'$ is a separated Deligne-Mumford stack and we have*

$$[X \times X', E \boxplus F] = [X, E] \times [X', F]$$

in $A_{\dim Y + \dim Y' + \operatorname{rk} E + \operatorname{rk} F}(X \times X')$.

Let E be a perfect relative obstruction theory for X over Y and F a perfect relative obstruction theory for X' over Y. Let $v : Z' \to Z$ be a local complete intersection morphism of Y-stacks that have finite unramified diagonal over Y. Let there be given a cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{u} & X \\ g \downarrow & & \downarrow f \\ Z' & \xrightarrow{v} & Z \end{array}$$

of Y-stacks. Then E and F are *compatible over* v if there exists a homomorphism of distinguished triangles

$$\begin{array}{ccccccc} u^*E & \longrightarrow & F & \longrightarrow & g^*L_{Z'/Z} & \longrightarrow & u^*E[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ u^*L_{X/Y} & \longrightarrow & L_{X'/Y} & \longrightarrow & L_{X'/X} & \longrightarrow & u^*L_{X/Y}[1]. \end{array}$$

in $D(\mathcal{O}_{X'})$.

Proposition 7.5 (Functoriality) *If E and F are compatible over v, then*

$$v^![X, E] = [X', F],$$

at least if v is smooth or Z' and Z are smooth over Y.

Proof. The proof is the same as that of Proposition 5.10. □

Example 7.6 Consider a cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{j} & V \\ g \downarrow & & \downarrow h \\ Y & \xrightarrow{i} & W \end{array}$$

of algebraic stacks, where i and j are local immersions and h has unramified diagonal. We have a canonical homomorphism

$$\phi : j^*L_{V/W} \longrightarrow L_{X/Y},$$

which makes $j^*L_{V/W}$ a relative obstruction theory for X over Y. To see this, it suffices to prove that $h^{-1}(F^\bullet) = h^0(F^\bullet) = 0$, where $F^\bullet$ is the cone of ϕ. But $F^\bullet$ is isomorphic to the cone of the homomorphism

$$g^*L_{Y/W} \longrightarrow L_{X/V},$$

so this is indeed true.

Now if V and W are smooth, then $h^i(L_{V/W}) = 0$ for all $i \neq -1, 0$ and $j^* L_{V/W}$ is a perfect obstruction theory. In particular, we get a virtual fundamental class

$$[X, j^* L_{V/W}] \in A_{\dim Y + \dim V - \dim W}(X),$$

if Y is pure dimensional, $j^* L_{V/W}$ has global resolutions and X is a separated Deligne-Mumford stack.

If, in addition, i is a regular local immersion with normal bundle $N_{Y/W}$, the normal cone $C_{X/V}$ of X in V is a closed subcone of $g^* N_{Y/W}$ and intersecting it with the zero section 0 of $g^* N_{Y/X}$ gives a class

$$0^![C_{X/V}] \in A_{\dim Y + \dim V - \dim W}(X).$$

The proof that

$$0^![C_{X/V}] = [X, j^* L_{V/W}]$$

is similar to the proof of Proposition 7.2.

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