

A Tale of Two Time Scales: Determining integrated volatility with noisy high-frequency data *

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Abstract

It is a common financial practice to estimate volatility from the sum of frequently-sampled squared returns. However market microstructure poses challenge to this estimation approach, as evidenced by recent empirical studies in finance. This work attempts to lay out theoretical grounds that reconcile continuous-time modeling and discrete-time samples. We propose an estimation approach that takes advantage of the rich sources in tick-by-tick data while preserving the continuous-time assumption on the underlying returns. Under our framework, it becomes clear why and where the “usual” volatility estimator fails when the returns are sampled at the highest frequency.

KEY WORDS: Measurement error; Subsampling; Market Microstructure; Martingale; Bias-correction; Realized volatility.

1 INTRODUCTION

In the analysis of high frequency financial data, a major problem concerns the nonparametric determination of the volatility of an asset return process. A common practice is to estimate volatility from the sum of the frequently-sampled squared returns. Though this approach is justified under the assumption of a continuous stochastic model, it meets the challenge from market microstructure in real applications. We argue that this customary way of estimating volatility is flawed in that it overlooks observation error. The usual mechanism for dealing with the problem is to throw away some data. We here propose a statistically sounder device. Our device is model-free, it takes advantage of the rich sources in tick-by-tick data, and to a great extent it corrects the effect of the

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microstructure on volatility estimation. In the course of constructing our volatility estimator, it becomes clear why and where the “usual” volatility estimator fails when the returns are sampled at the highest frequency.

Our interest is about using high frequency intra-day data to estimate the integrated volatility over some time periods. To fix the idea, let $\{S_t\}$ denote the price process of a security, suppose the return process $\{X_t\}$, where $X_t = \log S_t$, follows an Ito process

$$X_t = \mu_t dt + \sigma_t dB_t \quad (1)$$

where B_t is a standard Brownian motion. The standard model for the volatility σ_t^2 of a security price process is to take σ^2 to be the instantaneous variance (or squared diffusion coefficient) of the return process $\{X_t\}$. The parameter of our interest is the integrated (cumulative) volatility over one or successive time periods, $\int_0^{T_1} \sigma_t^2 dt$, $\int_{T_1}^{T_2} \sigma_t^2 dt \dots$. A natural way to estimate the cumulative volatility, say, a single time interval from 0 to T , is to use the the sum of squared incremental returns,

$$\sum_{t_i} (X_{t_{i+1}} - X_{t_i})^2 \approx \int_0^T \sigma_t^2 dt, \quad (2)$$

where the X_{t_i} 's are all the observations of the return process in the time interval from 0 to T . The estimator $\sum_{t_i} (X_{t_{i+1}} - X_{t_i})^2$ is so commonly used that it is broadly referred to as the “realized volatility”.

For a sample of the recent literature in integrated volatility, see Hull and White (1987), Jacod and Protter (1998), Gallant, Hsu and Tauchen (1999), Chernov and Ghysels (2000), Gloter (2000), Andersen, Bollerslev, Diebold, and Labys (2001), Barndorff-Nielsen and Shephard (2001), Mykland and Zhang (2002) and others.

Under model (1), the approximation in (2) is justified by the theoretical results in stochastic process which states

$$plim \sum_{t_i} (X_{t_{i+1}} - X_{t_i})^2 = \int_0^T \sigma_t^2 dt. \quad (3)$$

In other words, as the sampling frequency increases, the estimation error of the realized volatility diminishes. According to (3), realized volatility computed from the highest frequency data ought to provide the best possible estimate for $\int_0^T \sigma_t^2 dt$ the integrated volatility. However, this is not the general viewpoint from the finance literature.

It is generally held that the returns process X_t should not be sampled too often, regardless of the fact that the asset prices can often be observed with extremely high frequency, such as several times per second. It has been found empirically that the estimator is not robust when the sampling interval is quite small. Issues including bigger bias in the estimate and non-robustness to changes in sampling interval have been reported (Brown (1990), Campbell, Lo and MacKinlay (1997), Figlewski (1997), Bai, Russell and Tiao (2000), Andersen et al (2001)). The main explanation for this phenomenon is the so-called “market microstructure”, in particular bid-ask spread. When the

prices are sampled at finer intervals, the microstructure becomes more pronounced. It is suggested that the bias induced by the market microstructure make the most finely sampled data unusable for the calculation, and authors prefer to sample over longer time horizon to obtain reasonable estimates (Figlewski (1997), Andersen et al (2001)).

This approach to handling the data poses a conundrum from the statistical point of view. We argue that sampling over longer horizon merely reduces the impact of microstructure, rather than quantifying and correcting the effect from the microstructure for volatility estimation. And it goes against the grain to throw away data. On the other hand, market microstructure may pose so many problems that subsampling is the only way out.

In this paper we develop a method to estimate integrated volatility in a way which lessens this conflict. Our contention in the following is that the contamination due to market microstructure is, to first order, the same as what statisticians usually call “observation error”. We shall incorporate the observation error into the estimating procedure for integrated volatility. In other words, we shall suppose that the return process as observed at the sampling times is of the form $\log S_{t_i} = Y_{t_i}$ where

$$Y_{t_i} = X_{t_i} + \epsilon_{t_i}. \quad (4)$$

Here X_t is a latent “true return” process, and the ϵ_{t_i} s is independent noise around the true return. A similar structure was used in a parametric context by Gloter (2000) and Aït-Sahalia and Mykland (2003).

We show in Section 2.2 that, if the data have a structure of the form (4), this would have a devastating effect on the use of the “realized volatility”. Instead of (2), one gets

$$\sum_{t_i, t_{i+1} \in [0, T]} (Y_{t_{i+1}} - Y_{t_i})^2 = 2n \text{Var}(\epsilon) + O_p(n^{1/2}) \quad (5)$$

where the errors ϵ_{t_i} 's are iid with mean 0, and n is the number of sampling intervals over $[0, T]$.

The results from equation (5) suggest that the realized volatility no longer estimates the true integrated volatility, but rather the variance of the contamination noise. In fact, the true integrated volatility, which is $O_p(1)$, is even dwarfed by the magnitude of the asymptotically Gaussian $O_p(n^{1/2})$ term in (5). See Section 2 for details.

Of course, the model (4) may also not be correct. When made the basis of inference, it could still occur that one does not wish to sample as frequently as the data would permit. It may, however, make it possible to use substantially larger amounts of data than what would be possible under (2). Also, any subsampling based scheme can be made to incorporate all the data by the same construction as in Section 3 below.

In seeking to create an inference procedure under measurement error, we have sought to draw some lessons from the empirical practice that one should not use all the data, while at the same time not violating basic statistical principles. Our procedure estimates parameters at two different frequencies of sampling, and then by cancellation removes the effect of the ϵ s to the required order. We show in Section 4 that this leads to a variance-variance trade-off between the effect in (5) and an effect due to the sampling frequencies.

The theory, including also asymptotic distributions, is developed in Section 2-4 for the case of one time period $[0, T]$. The multi-period problem is treated in Section 5. Section 6 discusses how to estimate the asymptotic variance for equidistant sampling. A procedure for unequidistant case is being developed in a forthcoming paper, Zhang and Mykland (2003). And Section 7 discusses what to do if one really wants to use the customary realized volatility.

2 ANALYSIS OF THE “REALIZED VOLATILITY”

2.1 Set-up

To spell out the model above, we let Y be the logarithm of the price of an asset, observed at times $0 = t_0, t_1, \dots, t_n = T$. We assume that at these times, Y is related to a latent true price X (also in logarithm scale) through equation (4). The latent price X is then given in (1). The noise ϵ_{t_i} satisfies the following assumption,

$$\epsilon_{t_i} \text{ i.i.d. with } E\epsilon_{t_i} = 0, \text{ and } Var(\epsilon_{t_i}) = \nu. \text{ Also } \epsilon \perp\!\!\!\perp X \text{ process} \quad (6)$$

where $\perp\!\!\!\perp$ denotes independence between two random quantities. Our modeling as in (4) does not require that ϵ_t exists for every t , in other words, our interest in the noise is only at the observation times t_i 's.

For the moment, we focus on determining the integrated volatility of X for the entire time period $[0, T]$. This is also known as the continuous quadratic variation $\langle X, X \rangle$ of X . In other words,

$$\langle X, X \rangle_T = \int_0^T \sigma_t^2 dt \quad (7)$$

To describe succinctly the realized volatility, we use the notion of observed quadratic variation $[\cdot, \cdot]$. Given the grid $\mathcal{G} = \{t_0, \dots, t_n\}$, the observed quadratic variation for a generic process Z is

$$[Z, Z]_t = \sum_{t_{i+1} \leq t} (\Delta Z_{t_i})^2 \quad (8)$$

where $\Delta Z_{t_i} = Z_{t_{i+1}} - Z_{t_i}$. We shall later have occasion to vary the grid \mathcal{G} . Quadratic covariations are similarly defined. See Karatzas and Shreve (1991) for more details on quadratic variations.

Our interest is to assess how well the realized volatility, $[Y, Y]_T$, approximates the integrated volatility $\langle X, X \rangle_T$ of the true returns process. In our asymptotic considerations, we shall always assume that the number of observations in $[0, T]$ goes to infinity, and also that the maximum distance in time between two observations goes to zero:

$$\max_i \Delta t_i \rightarrow 0 \text{ as } n \rightarrow \infty \quad (9)$$

2.2 The Realized Volatility: An Estimator of the Spread of the Noise?

Under the additive model $Y_{t_i} = X_{t_i} + \epsilon_{t_i}$, the realized volatility based on the observed returns Y_{t_i} now has the form

$$[Y, Y]_T = [X, X]_T + 2[X, \epsilon]_T + [\epsilon, \epsilon]_T.$$

This gives the conditional mean and variance of $[Y, Y]_T$, given the process of latent true prices X . As derived in the Appendix,

$$E([Y, Y]_T \mid X \text{ process}) = [X, X]_T + 2n\nu, \quad (10)$$

under assumption (6) and the definition of $[\cdot, \cdot]$ in (8). Similarly,

$$\text{Var}([Y, Y]_T \mid X \text{ process}) = 4nE\epsilon^4 + O_p(1), \quad (11)$$

subject to condition (6) and $E\epsilon_{t_i}^4 < \infty$, for all i . The exact expression for the variance is given in the Appendix A.1.

Following the discussion in Appendix A.2, it is also the case that as $n \rightarrow \infty$, the distribution of $n^{-1/2}([Y, Y]_T - 2n\nu)$ becomes normal conditionally on the X process, with the mean 0 and variance $4E\epsilon^4$.

Equations (10) and (11) suggest that in a discrete-time world, realized volatility $[Y, Y]_T$ is not a reliable estimator for the true variation $[X, X]_T$ of the returns. *For large n , realized volatility could have little to do with the true returns.* As seen in (10), $[Y, Y]_T$ has a positive bias whose magnitude increases linearly with the sample size n . If one really wants to live with this bias – which we do not recommend – and use the customary realized volatility as a measure of variation, the above provides theoretical evidence for not sampling too often. See also Section 7.

Interestingly, apart from revealing the biased nature of $[Y, Y]_T$ in the high frequency setup, our analysis also delivers an estimator for the spread of the noise term. In other words, the realized volatility $[Y, Y]_T$ yields a consistent and asymptotically normal estimator of noise spread ν , namely $\hat{\nu} = \frac{1}{2n}[Y, Y]_T$. We have, for a fixed true return process X ,

$$n^{1/2}(\hat{\nu} - \nu) \rightarrow N(0, E\epsilon^4), \quad \text{as } n \rightarrow \infty, \quad (12)$$

cf. Theorem A.1 in the Appendix.

3 SAMPLING SPARSELY WHILE USING ALL THE DATA: ANALYSIS IN THE MULTIPLE GRID CASE

3.1 Multiple Grids and Sufficiency

We have argued in the previous section that one can indeed benefit from using infrequent data (see also Section 7). And yet, one of the most basic lessons of statistics is that one should not do

this. We here present two ways of tackling the problem. Both are based on selecting a number of subgrids of the original $\mathcal{G} = \{t_0, \dots, t_n\}$, and then averaging the estimators derived from the subgrids. The principle is that to the extent that there is a benefit to subsampling, this benefit can now be retained, while the variation of the estimator can be lessened by the averaging. The benefit of the averaging is clear from sufficiency considerations, and many statisticians would say that subsampling without subsequent averaging is inferentially incorrect.

In the following, we first introduce a set of notations, and then move to studying the realized volatility in the multi-grid context. In Section 4, we show how to explicitly estimate the model (4) by using a combination of the single grid \mathcal{G} and the multiple grids.

3.2 Notation for the Multiple Grids

We specifically suppose that the total grid \mathcal{G} , $\mathcal{G} = \{t_0, \dots, t_n\}$ as before, is partitioned into K non-overlapping subgrids $\mathcal{G}^{(k)}$, $k = 1, \dots, K$, in other words,

$$\mathcal{G} = \cup_{k=1}^K \mathcal{G}^{(k)} \text{ where } \mathcal{G}^{(k)} \cap \mathcal{G}^{(l)} = \emptyset \text{ when } k \neq l.$$

For most purposes, the natural way to select the k 'th subgrid $\mathcal{G}^{(k)}$ is to start with t_{k-1} and then pick every K th sample point after that, until T . That is to say that

$$\mathcal{G}^{(k)} = (t_{k-1}, t_{k-1+K}, t_{k-1+2K}, \dots, t_{k-1+n_k K}) \quad (13)$$

for $k = 1, \dots, K$, and n_k is the integer making $t_{k-1+n_k K}$ the last element in $\mathcal{G}^{(k)}$. We shall refer to this as *regular* allocation of sample points to subgrids.

Whether the allocation is regular or not, we let n_k be such that subgrid $\mathcal{G}^{(k)}$ has $n_k + 1$ elements. As before, the number of elements in the total grid \mathcal{G} is $n + 1$.

More general schemes for allocating sample points to grids can also be used, subject to the restrictions in Theorem A.1 in Appendix A.2.

The realized volatility based on all observation points \mathcal{G} , so far denoted $[Y, Y]_T$, will now for clarity be written as $[Y, Y]_T^{(all)}$. Meanwhile, if one uses only the subsampled observations Y_t , $t \in \mathcal{G}^{(k)}$, the realized volatility will be denoted as $[Y, Y]_T^{(k)}$, with the form of

$$[Y, Y]_T^{(k)} = \sum_{t_j, t_{j,+} \in \mathcal{G}^{(k)}} (Y_{t_{j,+}} - Y_{t_j})^2.$$

By definition, if $t_i \in \mathcal{G}^{(k)}$, then $t_{i,-}$ and $t_{i,+}$ are, respectively, the previous and next element in $\mathcal{G}^{(k)}$.

A natural competitor to $[Y, Y]_T^{(all)}$ is then given by

$$[Y, Y]_T^{(\cdot)} = \frac{1}{K} \sum_{k=1}^K [Y, Y]_T^{(k)}, \quad (14)$$

and this is the statistic we analyze in the following.

As before, we fix T and use only the observations within the time period $[0, T]$. Asymptotics will still be under (9) and under (15) below,

$$\text{as } n \rightarrow \infty, n/K \rightarrow \infty. \quad (15)$$

In general, the n_k need not be the same across k . We define

$$\bar{n} = \frac{1}{K} \sum_{k=1}^K n_k = \frac{n - K + 1}{K}. \quad (16)$$

3.3 Error Due to the Noise ϵ

Recall that we are interested in determining the integrated volatility $\langle X, X \rangle_T$, or quadratic variation, of the true returns. As an intermediate step, we study in this subsection how well the “pooled” realized volatility $[Y, Y]_T^{(\cdot)}$ approximates $[X, X]_T^{(\cdot)}$, where the latter is the “pooled” true integrated volatility when X is considered only at discrete time scale. $[X, X]$ has a form as defined in equation (8).

From (10) and (14),

$$E([Y, Y]_T^{(\cdot)} \mid X \text{ process}) = [X, X]_T^{(\cdot)} + 2\bar{n}\nu. \quad (17)$$

Also, since $\{\epsilon_t, t \in \mathcal{G}^{(k)}\}$ are independent for different k ,

$$\begin{aligned} \text{Var}([Y, Y]_T^{(\cdot)} \mid X \text{ process}) &= \frac{1}{K^2} \sum_{k=1}^K \text{Var}([Y, Y]_T^{(k)} \mid X \text{ process}) \\ &= \frac{1}{K^2} \sum_{k=1}^K n_k 4E\epsilon^4 + \text{Remainder}, \\ &= 4\frac{\bar{n}}{K} E\epsilon^4 + O_p\left(\frac{1}{K}\right) \end{aligned} \quad (18)$$

in the same way as in (11). The order of the remainder follows as in the single grid case, cf. (A.3) in the Appendix.

By Theorem A.1 in Appendix A.2, the conditional asymptotics of the estimator $[Y, Y]_T^{(\cdot)}$ is as follows

Theorem 1. *Suppose X is an Ito process of form (1). Suppose Y is related to X through model (4), and that (6) is satisfied with $E\epsilon^4 < \infty$. Also suppose that t_i and t_{i+1} is not in the same subgrid for any i . Under assumption (15), as $n \rightarrow \infty$*

$$\sqrt{\frac{K}{\bar{n}}}([Y, Y]_T^{(\cdot)} - [X, X]_T^{(\cdot)} - 2\nu\bar{n}) \xrightarrow{\mathcal{L}} 2\sqrt{E\epsilon^4}Z_\epsilon^{(\cdot)}, \quad (19)$$

conditional on the X process, where $Z_\epsilon^{(\cdot)}$ is standard normal.

This can be compared with the earlier result which is stated below in equation (49) in Section 7. Notice that $Z_\epsilon^{(\cdot)}$ in (19) is almost never the same as Z_ϵ in (49), in particular, $Cov(Z_\epsilon, Z_\epsilon^{(\cdot)}) = Var(\epsilon^2)/E\epsilon^4$, based on the proof in Theorem A.1 in the Appendix.

In comparison to the realized volatility using full grid \mathcal{G} , the aggregated estimator $[Y, Y]_T^{(\cdot)}$ provides an improvement in that both the asymptotic bias and variance are of smaller order of n . Cf. equations (10) and (11) in the preceding section. We shall use this in Section 4, and also in Section 7 below.

3.4 Error Due to the Discretization Effect: $[X, X]_T^{(\cdot)} - \langle X, X \rangle_T$

In this subsection, we study the impact of the time discretization. In other words, we investigate the deviation of $[X, X]_T^{(\cdot)}$ from the integrated volatility $\langle X, X \rangle_T$ of the true process. Denote the discretization effect as D_T , where

$$\begin{aligned} D_t &= [X, X]_t^{(\cdot)} - \langle X, X \rangle_t \\ &= \frac{1}{K} \sum_{k=1}^K ([X, X]_t^{(k)} - \langle X, X \rangle_t) \end{aligned} \quad (20)$$

with

$$[X, X]_t^{(k)} = \sum_{t_i \in \mathcal{G}^{(k)}: t_i, + \leq t} (X_{t_i, +} - X_{t_i})^2 \quad (21)$$

We consider in the following the asymptotics of D_T . The problem is similar to that of finding the limit of $[X, X]_T^{(all)} - \langle X, X \rangle_T$, cf. equation (50) below. This current case, however, is more complicated due to the multiple grids.

We suppose in the following that the sampling points are allocated to subgrids as described by equation (A.20) in Appendix A.3. In particular, this covers the regular allocation, as defined in Section 3.2. We also assume that

$$\max_i |\Delta t_i| = O\left(\frac{1}{n}\right). \quad (22)$$

Define the weight function

$$h_i = \frac{8n}{TK^3} \sum_{l=1}^K [t_i - t_i^{(l)} + \frac{1}{2} \Delta t_i [\#\{k : t_i^{(k)} > t_i^{(l)}\} + \frac{1}{2}]] \quad (23)$$

In the case where the t_i are equidistant, and under regular allocation of points to subgrids, $\Delta t_i = \Delta t$, and so all the h_i are equal, and

$$h_i = \frac{2}{\Delta t K^3} \Delta t (1 + 3^2 + \dots + (2K - 1)^2) = \frac{2K(2K - 1)(2K + 1)}{3K^3} = \frac{8}{3} + o(1). \quad (24)$$

More generally, assumptions (22) and (A.20) assure that

$$\sup_i h_i = O(1). \quad (25)$$

We take $\langle D, D \rangle_T$ to be the quadratic variation of D_t when viewed as a continuous time process (20). This gives the best approximation to the variance of D_T .

We show the following results in Appendix A.3.

Theorem 2. *Suppose X is an Ito process of the form (1), with drift coefficient μ_t and diffusion coefficient σ_t , both continuous almost surely. Assume (22) and (A.20). Then the quadratic variation of D_T is approximately*

$$\langle D, D \rangle_T = \frac{TK}{n} \eta_n^2 + o_p\left(\frac{K}{n}\right) \quad (26)$$

where

$$\eta_n^2 = \sum_i h_i (\langle X, X \rangle'_{t_i})^2 \Delta t_i. \quad (27)$$

In particular, $D_T = O_p((K/n)^{1/2})$. From this, we shall derive a variance-variance trade-off between the two effects that have been discussed – noise and discretization.

First, however, we discuss the asymptotic law of D_T . Stable convergence is discussed at the end of this section.

Theorem 3. *Assume the conditions of Theorem 2, and also that*

$$\eta_n^2 \xrightarrow{P} \eta^2 \quad (28)$$

Also assume Condition E in Appendix A.3. Then

$$D_T / (K/n)^{1/2} \xrightarrow{\mathcal{L}} \eta Z, \quad (29)$$

where Z is standard normal, and independent of the data. The convergence in law is stable.

In other words, $D_T / (K/n)^{1/2}$ can be taken to be asymptotically mixed normal “ $N(0, \eta^2)$.”

For most of our discussion, it is most convenient to suppose (28), and this is satisfied in many cases. For example, when the t_i are equidistant, and under regular allocation of points to subgrids,

$$\eta^2 = \frac{8}{3} \int_0^T \sigma_t^4 dt, \quad (30)$$

following (24). One does not need to rely on (28); we argue in Section A.3 that without this condition, one can take $D_T / (K/n)^{1/2}$ to be approximately $N(0, \eta_n^2)$.

For estimation of η^2 or η_n^2 , see Section 6.

Finally, *stable convergence* (Renyi (1963), Aldous and Eagleson (1978), Chapter 3 of Hall and Heyde (1980)) means for our purposes that the left hand side of (29) converges to the right hand side jointly with the X process, and that Z is independent of X . This is slightly weaker than convergence conditional on X , but serves the same function of permitting the incorporation of conditionality-type phenomena into arguments and conclusions, cf. the following sections.

3.5 Combining the Two Sources of Error

One can combine the error term from discretization and that from the observation noise. It follows from Theorems 1 and 3 that

$$[Y, Y]_T^{(\cdot)} - \langle X, X \rangle_T - 2\nu\bar{n} \stackrel{\mathcal{L}}{\approx} \xi Z_{\text{total}}, \quad (31)$$

where Z_{total} is a standard normal random variable independent of the X process, and

$$\xi^2 = 4\frac{\bar{n}}{K}E\epsilon^4 + \frac{T}{\bar{n}}\eta^2. \quad (32)$$

Here, the symbol “ $\stackrel{\mathcal{L}}{\approx}$ ” means that when multiplied by a suitable factor, the convergence is in law (and stable, by the preceding results). Cf. also the proof of Theorem 4 in the next section.

It is easily seen that if one takes $K = cn^{2/3}$, both components in ξ^2 will be present in the limit, otherwise one of them will dominate.

Based on (31), $[Y, Y]_T^{(\cdot)}$ is yet a biased estimator of the quadratic variation $\langle X, X \rangle_T$ of the true return process. In particular, the bias $2\nu\bar{n}$ still increases with the number of the sub-samples. One can recognize that, as far as the asymptotic bias is concerned, $[Y, Y]_T^{(\cdot)}$ is a better estimator than $[Y, Y]_T^{(all)}$, since $\bar{n} \leq n$, suggesting that the bias in the subsampled estimator $[Y, Y]_T^{(\cdot)}$ increases in a slower pace than the full-sampled estimator. One can also construct a bias-adjusted estimator from (31), and this further development would involve the higher order analysis between the bias and the subsampled estimator. We show the methodology of bias correction in Section 4.

4 ESTIMATION FOR THE MODEL WITH MEASUREMENT ERROR: COMBINING TWO SAMPLING FREQUENCIES

4.1 The Estimator: Main Result

In previous sections, we have seen that the multigrid estimator $[Y, Y]^{(\cdot)}$ is yet another biased estimator of the true integrated volatility $\langle X, X \rangle$. In this section we improve the multigrid estimator by adopting bias adjustment.

To access the bias, one utilizes the full grid. As mentioned from equation (12) in single-grid case (Section 2), ν can be consistently approximated by $\hat{\nu}$,

$$\hat{\nu} = \frac{1}{2n}[Y, Y]_T^{(all)}. \quad (33)$$

Hence the bias of $[Y, Y]^{(\cdot)}$ can be consistently estimated by $2\bar{n}\hat{\nu}$. A bias-adjusted estimator for $\langle X, X \rangle$ can thus be obtained by

$$\langle \widehat{X}, \widehat{X} \rangle_T = [Y, Y]_T^{(\cdot)} - 2\bar{n}\hat{\nu}. \quad (34)$$

To study the asymptotic behavior of $\langle \widehat{X}, \widehat{X} \rangle_T$, note first that under the conditions of Theorem A.1 in Appendix A.2

$$\begin{aligned} \left(\frac{K}{\bar{n}}\right)^{1/2} \left(\langle \widehat{X}, \widehat{X} \rangle_T - [X, X]_T^{(\cdot)}\right) &= \left(\frac{K}{\bar{n}}\right)^{1/2} \left([Y, Y]_T^{(\cdot)} - [X, X]_T^{(\cdot)} - 2\nu\bar{n}\right) \\ &\quad - 2(K\bar{n})^{1/2}(\hat{\nu} - \nu) \\ &\xrightarrow{\mathcal{L}} N(0, 8\nu^2), \end{aligned} \quad (35)$$

where the convergence in law is conditional on X .

We can now combine this with the results of Section 3.4 to determine the optimal choice of K as $n \rightarrow \infty$:

$$\begin{aligned} \langle \widehat{X}, \widehat{X} \rangle_T - \langle X, X \rangle_T &= (\langle \widehat{X}, \widehat{X} \rangle_T - [X, X]_T^{(\cdot)}) + ([X, X]_T^{(\cdot)} - \langle X, X \rangle_T) \\ &= O_p\left(\frac{\bar{n}^{1/2}}{K^{1/2}}\right) + O_p\left(\bar{n}^{-1/2}\right). \end{aligned} \quad (36)$$

The error is minimized by equating the two terms on the right hand side of (36), yielding that the optimal sampling step for $[Y, Y]_T^{(\cdot)}$ is $K = O(n^{2/3})$. The right hand side of (36) then has order $O_p(n^{-1/6})$.

In particular, if we take

$$K = cn^{2/3}, \quad (37)$$

we find the limit in (36), as follows.

Theorem 4. *Suppose X is an Ito process of form (1), and assume the conditions of Theorem 3 in Section 3.4. Suppose Y is related to X through model (4), and that (6) is satisfied with $E\epsilon^2 < \infty$. Also suppose that t_i and t_{i+1} is not in the same subgrid for any i . Under assumption (37),*

$$\begin{aligned} n^{1/6} \left(\langle \widehat{X}, \widehat{X} \rangle_T - \langle X, X \rangle_T\right) &\xrightarrow{\mathcal{L}} N(0, 8c^{-2}\nu^2) + \eta\sqrt{T}N(0, c) \\ &= (8c^{-2}\nu^2 + c\eta^2T)^{1/2} N(0, 1), \end{aligned} \quad (38)$$

where the convergence is stable in law (see Section 3.4).

Proof of Theorem 4. Note that the first normal distribution comes from equation (35) and the second from Theorem 3 in Section 3.4. The two normal distributions are independent since the convergence of the first term in (36) is conditional of the X process, which is why they can be amalgamated as stated. The requirement that $E\epsilon^4 < \infty$ (Theorem A.1 in the appendix) is not needed since only a law of large number is required for $M_T^{(1)}$ (see the proof of that theorem) when considering the difference in (35) above. This finishes the proof. \blacksquare

The estimation of the asymptotic spread $s^2 = 8c^{-2}\nu^2 + c\eta^2T$ of $\langle \widehat{X}, \widehat{X} \rangle_T$ is deferred to Section 6. Also, note that, by Theorem A.1 and the same methods as in Appendix A.2, a consistent estimator of the asymptotic variance of $\hat{\nu}$ is given by

$$\frac{1}{2} \frac{1}{n} \sum_i (\Delta Y_{t_i})^4 - 3\hat{\nu}^2. \quad (39)$$

4.2 Properties of $\langle \widehat{X}, \widehat{X} \rangle_T$: Optimal Sampling, and Bias Adjustment

To further pin down the optimal sampling frequency K one can minimize the expected asymptotic variance in (38) to obtain

$$c = \left(\frac{16\nu^2}{TE\eta^2} \right)^{1/3} \quad (40)$$

which can be consistently estimated from data in past time periods (before time $t_0 = 0$), using $\hat{\nu}$ and an estimator of η^2 , cf. Section 6. As mentioned in Section 3.4, η^2 can be taken to be independent of K so long as one allocates sampling points to grids regularly, as defined in Section 3.2. Hence one can choose c , and so also K , based on past data.

Example 1. If σ_t^2 is constant, and for equidistant sampling and regular allocation to grids, $\eta^2 = \frac{8}{3}\sigma^4 T$, and the asymptotic variance in equation (38) is

$$8c^{-2}\nu^2 + c\eta^2 T = 8c^{-2}\nu^2 + \frac{8}{3}c\sigma^4 T^2$$

and the optimal choice of c becomes

$$c_{opt} = \left(\frac{6\nu^2}{T^2\sigma^4} \right)^{1/3}. \quad (41)$$

In this case, the asymptotic variance is

$$4(6\nu^2)^{1/3}(\sigma^2 T)^{4/3}.$$

One can also, of course, estimate c to minimize the actual asymptotic variance in (38) from data in the current time period ($0 \leq t \leq T$). It is beyond the scope of this paper to consider whether such a device of selecting frequency has any impact on our asymptotic results.

In addition to large sample arguments, one can study $\langle \widehat{X}, \widehat{X} \rangle_T$ from a “smallish” sample point of view. We argue in the following that one can apply a bias type adjustment to get

$$\langle \widehat{X}, \widehat{X} \rangle_T^{(adj)} = \left(1 - \frac{\bar{n}}{n}\right)^{-1} \langle \widehat{X}, \widehat{X} \rangle_T. \quad (42)$$

The difference from the estimator in (34) is of order $O_p(\bar{n}/n) = O_p(K^{-1})$, and thus the two estimators behave the same to the asymptotic order that we consider. The estimator (42), however, has the appeal of being, in a certain way, “unbiased”, as follows. Consider all estimators of the form

$$\langle \widehat{X}, \widehat{X} \rangle_T^{(adj)} = a[Y, Y]_T^{(\cdot)} - 2b\hat{\nu}\bar{n},$$

then, from (10) and (17),

$$\begin{aligned} E(\langle \widehat{X}, \widehat{X} \rangle_T^{(adj)} | X \text{ process}) &= a([X, X]_T^{(\cdot)} + 2\bar{n}\nu) - b\frac{\bar{n}}{n}([X, X]_T^{(all)} + 2n\nu) \\ &= a[X, X]_T^{(\cdot)} - b\frac{\bar{n}}{n}[X, X]_T^{(all)} + 2(a - b)\bar{n}\nu. \end{aligned}$$

It is natural to choose $a = b$ to completely remove the effect of ν . Also, following Section 3.4, both $[X, X]_T^{(\cdot)}$ and $[X, X]_T^{(all)}$ are asymptotically unbiased estimators of $\langle X, X \rangle_T$. Hence one can argue that one should take $a(1 - \bar{n}/n) = 1$, yielding (42).

Similarly, an adjusted estimator of ν is given by

$$\hat{\nu}^{(adj)} = \frac{1}{2}(n - \bar{n})^{-1} \left([Y, Y]_T^{(all)} - [Y, Y]_T^{(\cdot)} \right), \quad (43)$$

which satisfies that $E(\hat{\nu}^{(adj)} | X \text{ process}) = \nu + \frac{1}{2}(n - \bar{n})^{-1} \left([X, X]_T^{(all)} - [X, X]_T^{(\cdot)} \right)$, and is therefore unbiased to high order. As for the asymptotic distribution, One can see from Theorem A.1 in the Appendix that

$$\begin{aligned} \hat{\nu}^{(adj)} - \nu &= (\hat{\nu} - \nu)(1 + O(K^{-1})) + O_p(Kn^{-3/2}) \\ &= \hat{\nu} - \nu + O_p(n^{-1/2}K^{-1}) + O_p(Kn^{-3/2}) \\ &= \hat{\nu} - \nu + O_p(n^{-5/6}) \end{aligned}$$

from (37). It follows that $n^{1/2}(\hat{\nu} - \nu)$ and $n^{1/2}(\hat{\nu}^{(adj)} - \nu)$ have the same asymptotic distribution.

5 MULTIPLE PERIOD INFERENCE

For a given family $\mathcal{A} = \{\mathcal{G}^{(k)}, k = 1, \dots, K\}$, we denote by

$$\langle \widehat{X}, \widehat{X} \rangle_t = [Y, Y]_t^{(\cdot)} - \frac{\bar{n}}{n}[Y, Y]_t \quad (44)$$

where, as usual, $[Y, Y]_t = \sum_{t_{i+1} \leq t} \Delta Y_{t_i}^2$ and $[Y, Y]_t^{(\cdot)} = \frac{1}{K} \sum_{k=1}^K [Y, Y]_t^{(k)}$, with

$$[Y, Y]_t^{(k)} = \sum_{t_i \in \mathcal{G}^{(k)}: t_i \leq t} (Y_{t_i, +} - Y_{t_i})^2.$$

In order to estimate $\langle X, X \rangle$ for several discrete time periods, say $[0, T_1], [T_1, T_2], \dots, [T_{M-1}, T_M]$, where M is fixed, this amounts to estimating $\langle X, X \rangle_{T_m} - \langle X, X \rangle_{T_{m-1}} = \int_{T_{m-1}}^{T_m} \sigma_u^2 du$, for $m = 1, \dots, M$, and the obvious estimator is $\langle \widehat{X}, \widehat{X} \rangle_{T_m} - \langle \widehat{X}, \widehat{X} \rangle_{T_{m-1}}$.

To carry out the asymptotics, let n_m be the number of points in the m^{th} time segment, and similarly let $K_m = c_m n_m^{2/3}$, where c_m is a constant. Then $\{n_m^{1/6}(\langle \widehat{X}, \widehat{X} \rangle_{T_m} - \langle \widehat{X}, \widehat{X} \rangle_{T_{m-1}} - \int_{T_{m-1}}^{T_m} \sigma_u^2 du), m = 1, \dots, M\}$ converge stably to $\{(8c_m^{-2}\nu^2 + c_m\eta_m^2(T_m - T_{m-1}))^{1/2}Z_m\}$, where the Z_m are iid standard normals, independent of the underlying process, and η_m^2 is the limit η^2 (Theorem 3) for time period m . In the case of equidistant t_i and regular allocation of sample points to grids, $\eta_m^2 = \frac{8}{3} \int_{T_{m-1}}^{T_m} \sigma_u^4 du$.

In other words, the one period asymptotics generalizes straightforwardly to the multiperiod case. This is because $\langle \widehat{X}, \widehat{X} \rangle_{T_m} - \langle \widehat{X}, \widehat{X} \rangle_{T_{m-1}} - \int_{T_{m-1}}^{T_m} \sigma_u^2 du$ has, to first order, a martingale structure. This can be seen from the Appendix.

An advantage of our proposed estimator is that if ϵ_{t_i} has different variance in different time segments, say $Var(\epsilon_{t_i}) = \nu_m$ for $t_i \in (T_{m-1}, T_m]$, then both consistency and asymptotic (mixed) normality continue to hold, provided that one replaces ν by ν_m . This adds a measure of robustness to the procedure. If one were convinced that ν is the same across time segments, an alternative estimator has the form

$$\langle \widehat{X}, \widehat{X} \rangle_t = [Y, Y]_t^{(\cdot)} - \left(\frac{1}{K} \#\{t_{i+1} \leq t\} - 1 \right) \frac{1}{n} [Y, Y]_t^{(all)}. \quad (45)$$

The errors $\langle \widehat{X}, \widehat{X} \rangle_{T_m} - \langle \widehat{X}, \widehat{X} \rangle_{T_{m-1}} - \int_{T_{m-1}}^{T_m} \sigma_u^2 du$, however, are in this case not asymptotically independent. Note that for $T = T_m$, both candidates (44) and (45) for $\langle \widehat{X}, \widehat{X} \rangle_t$ coincide with the quantity in (34).

6 DETERMINING THE ASYMPTOTIC VARIANCE

In the one period case, the main goal is to find the spread $s^2 = 8c^{-2}\nu^2 + c\eta^2T$, cf. (37)-(38). The multigrid case is a straightforward generalization, as indicated in Section 5.

Here, we shall only be concerned with the case where the points t_i are equally spaced ($\Delta t_i = \Delta t$), and are regularly allocated to the grids $\mathcal{A}_1 = \{\mathcal{G}^{(k)}, k = 1, \dots, K_1\}$. The more general case, and proof of the method, is treated in Zhang and Mykland (2003).

A richer set of ingredients are required to find the spread than just to estimate $\langle \widehat{X}, \widehat{X} \rangle_T$. To implement the estimator, create an additional family $\mathcal{A}_2 = \{\mathcal{G}^{(k,i)}, k = 1, \dots, K_1, i = 1, \dots, I\}$ of grids where $\mathcal{G}^{(k,i)}$ contains every i -th point of $\mathcal{G}^{(k)}$. We assume that $K_1 \sim c_1 n^{2/3}$. The new family then consists of $K_2 \sim c_2 n^{2/3}$ grids, where $c_2 = c_1 I$.

In addition, we need a division of the time line into segments $(T_{m-1}, T_m]$, where $T_m = \frac{m}{M}T$. For the purposes of this discussion, M is large but finite.

We now get an initial estimator of spread as

$$\hat{s}_0^2 = n^{1/3} \sum_{m=1}^M \left(\langle \widehat{X}, \widehat{X} \rangle_{T_m}^{K_1} - \langle \widehat{X}, \widehat{X} \rangle_{T_{m-1}}^{K_1} - (\langle \widehat{X}, \widehat{X} \rangle_{T_m}^{K_2} - \langle \widehat{X}, \widehat{X} \rangle_{T_{m-1}}^{K_2}) \right)^2$$

where $\langle \widehat{X}, \widehat{X} \rangle_t^{K_i}$ is the estimator (44) using grid family i , $i = 1, 2$.

Using the discussion in Section 5, one can see that

$$\hat{s}_0^2 \approx s_0^2, \quad (46)$$

where, for $c_1 \neq c_2$ ($I \neq 1$),

$$\begin{aligned} s_0^2 &= 8\nu^2(c_1^{-2} + c_2^{-2} - c_1^{-1}c_2^{-1}) + (c_1^{1/2} - c_2^{1/2})^2 T\eta^2 \\ &= 8\nu^2 c_1^{-2} (1 + I^{-2} - I^{-1}) + c_1 (I^{1/2} - 1)^2 T\eta^2. \end{aligned} \quad (47)$$

In (46), the symbol \approx means first convergence in law as $n \rightarrow \infty$, and then a limit in probability as $M \rightarrow \infty$. Since ν can be estimated by $\hat{\nu} = [Y, Y]^{(all)}/2n$, one can put hats on s_0^2 , ν^2 , and η^2 in (47) to obtain an estimator of η^2 . Similarly,

$$\begin{aligned} s^2 &= 8\nu^2 \left(c^{-2} - \frac{c(c_1^{-2} + c_2^{-2} - c_1^{-1}c_2^{-1})}{(c_1^{1/2} - c_2^{1/2})^2} \right) + \frac{c}{(c_1^{1/2} - c_2^{1/2})^2} s_0^2 \\ &= 8 \left(c^{-2} - cc_1^{-3} \frac{I^{-2} - I^{-1} + 1}{(I^{1/2} - 1)^2} \right) \nu^2 + \frac{c}{c_1 (I^{1/2} - 1)^2} s_0^2, \end{aligned} \quad (48)$$

where $c \sim Kn^{-2/3}$ where K is the number of grids used originally to estimate $\langle X, X \rangle_T$.

Normally, one would take $c_1 = c$. Hence an estimator \hat{s}^2 can be found from \hat{s}_0^2 and $\hat{\nu}$.

When $c_1 = c$, we argue that the optimal choice is $I = 3$ or 4 , as follows. The coefficients in (48) become

$$\begin{aligned} \text{coeff}(s_0^2) &= (I^{1/2} - 1)^{-2} \\ \text{coeff}(\nu^2) &= 8c^{-2}(I^{1/2} - 1)^{-2} f(I) \end{aligned}$$

where $f(I) = I - 2I^{1/2} - I^{-2} + I^{-1}$. For $I \geq 2$, $f(I)$ is increasing, and $f(I)$ crosses 0 for I between 3 and 4. These, therefore, are the two integer values of I which give the lowest ratio of $\text{coeff}(\nu^2)/\text{coeff}(s_0^2)$. Using $I = 3$ or 4 , therefore, would maximally insulate against $\hat{\nu}^2$ dominating over \hat{s}_0^2 . This is desirable as \hat{s}_0^2 is the estimator of carrying the information about η^2 . Numerical values for the coefficients are given in Table 1. If c is such that $\hat{\nu}^2$ still overwhelms \hat{s}_0^2 , then a choice of $c_1 \neq c$ should be considered.

Table 1. Coefficients of $\hat{\nu}^2$ and \hat{s}^2 when $c_1 = c$

I	$\text{coeff}(s_0^2)$	$\text{coeff}(\nu^2)$
3	1.866	$-3.611c^{-2}$
4	1.000	$1.5000c^{-2}$

7 THE BENEFITS OF SAMPLING SPARSELY

In the above we have constructed a method to directly estimate the integrated volatility of the process X , by combining different sampling frequencies. If one really insists, however, one can pretend that the noise term ϵ is so negligible that one can ignore it. In the following, we discuss whether this approach can possibly have at least some merit.

7.1 The Single Grid Case

In Section 2, we argued that the realized volatility estimates the wrong quantity. This problem only gets worse when observations are sampled more frequently. Its financial interpretation boils

down to market micro-structure, measured by ϵ in (4). As the data record is sampled finely, the change in true returns gets smaller while the microstructure noise, such as bid-ask spread and transaction cost, remains at the same magnitude. In other words, when the sampling frequency is extremely high, the observed fluctuation in the returns process is more heavily contaminated by microstructure noise and becomes less representative of the true variation $\langle X, X \rangle_T$ of the returns. Along this line of discussion, the broad opinion in financial application (see, for example, Figlewski (1997), Bai et al (2000), Andersen et al (2001)) is not to sample too often, at least when using realized volatility. We now discuss how this can be viewed in the context of the model (4).

Intuitively, suppose that ν is small. It could formally be taken to tend to zero as $n \rightarrow \infty$, along with $E\epsilon^4$. The asymptotic normality in Section 2.2 then takes the form

$$[Y, Y]_T \stackrel{\mathcal{L}}{\approx} [X, X]_T + 2\nu n + 2\sqrt{nE\epsilon^4}Z_\epsilon, \quad (49)$$

where the symbol “ $\stackrel{\mathcal{L}}{\approx}$ ” is used in a similar way to that of Section 3.5. Here Z_ϵ is standard normal, the subscript ϵ indicates that the randomness comes from the noise, or the deviation of the observables Y from the true process X . The convergence in law is conditional on the X process.

For small ν , one now has a chance at estimating $\langle X, X \rangle_T$. Following Rootzen (1980), Jacod and Protter (1998) and Mykland and Zhang (2002), and under the conditions stated in these papers, one can show that

$$\left(\frac{n}{T}\right)^{1/2} ([X, X]_T - \langle X, X \rangle_T) \xrightarrow{\mathcal{L}} \left(\int_0^T 2H'(t)\sigma_t^4 dt\right)^{1/2} \times Z_{\text{discr}}, \quad (50)$$

stably in law (see the end of Section 3.4). Z_{discr} is standard normal random variable, the subscript indicates that the randomness is due to the discretization effect in $[X, X]_T$ when evaluating $\langle X, X \rangle_T$. $H(t)$ is the asymptotic quadratic variation of time, as discussed in an earlier paper (Mykland and Zhang (2002)). In the case of equidistant observations $\Delta t_0 = \dots = \Delta t_{n-1} = \Delta t$, $H'(t) = 1$. For the irregularly spaced case, we refer to our earlier paper.

Again the convergence is in law, and it is stable, cf. the end of Section 3.4. Since the ϵ s are independent of the X process, Z_ϵ is independent of Z_{discr} .

It then follows from (49)-(50) that

$$[Y, Y]_T \stackrel{\mathcal{L}}{\approx} \langle X, X \rangle_T + 2\nu n + \Upsilon Z_{\text{total}}, \quad (51)$$

in the sense of stable convergence, where Z_{total} is standard normal, and where the variance has the form

$$\Upsilon^2 = 4nE\epsilon^4 + \frac{T}{n} \int_0^T 2H'(t)\sigma_t^4 dt \quad (52)$$

Seen from this angle, there is scope for using the realized volatility $[Y, Y]$ to estimate $\langle X, X \rangle$. It is with bias $2\nu n$, but the bias goes down if one uses fewer observations. This, then, is consistent with the practice in empirical finance.

As can be seen from (52), there is, however, a trade-off between sampling too often and too rarely. Consider the simple case where the observation times are equidistant, so that $H'(t) = 1$ independently of the sampling frequency. It is then natural to minimize the mean squared error

$$MSE = (2\nu n)^2 + \Upsilon^2, \quad (53)$$

which means that one should choose n to satisfy $\partial MSE/\partial n \approx 0$, in other words,

$$8\nu^2 n + 4E\epsilon^4 - \frac{T}{n^2} \int_0^T 2H'(t)\sigma_t^4 dt \approx 0. \quad (54)$$

To solve for n , we suppose as mentioned above that $\nu \rightarrow 0$ as $n \rightarrow \infty$, and we suppose that $E(\epsilon^4)/(E(\epsilon^2))^2$ is of order $O(1)$. Thus

$$n^3 + \frac{1}{2}n^2 \frac{E(\epsilon^4)}{(E(\epsilon^2))^2} - \nu^{-2} \frac{T}{8} \int_0^T 2H'(t)\sigma_t^4 dt \approx 0. \quad (55)$$

Hence, finally,

$$n = \nu^{-2/3} \left(\frac{T}{8} \int_0^T 2H'(t)\sigma_t^4 dt \right)^{1/3} + o(\nu^{-2/3}) \text{ as } \nu \rightarrow 0. \quad (56)$$

The equation (56) is the formal statement saying that one can sample more frequently when the error spread is small. Note that to first order, the final trade-off is between the bias $2\nu n$ and the variance due to discretization. The effect of the variance associated with Z_ϵ is of lower order when comparing n and ν .

It should be emphasized that (56) is a feasible way of choosing n . One can estimate ν using all the data following the procedure in Section 2.2. The integral $\int_0^T 2H'(t)\sigma_t^4 dt$ can be estimated by the methods discussed in Section 6 below. For a general procedure, see our forthcoming paper, Zhang and Mykland (2003).

We can do better, however, than using the ‘‘realized volatility’’, as we shall see in the following.

7.2 The Multiple Grid Case

Following the development in Section 3, one can go to the multi-grid case and search for an optimal frequency \bar{n} for subsampling to balance the coexistence of the bias and the variance in (31). To reduce the mean squared error of $[Y, Y]_T^{(\cdot)}$, we set $\partial MSE/\partial \bar{n} = 0$. From (32)-(31), $bias = 2\nu\bar{n}$ and $\xi^2 = 4\frac{\bar{n}}{K}E\epsilon^4 + \frac{T}{\bar{n}}\eta^2$, then

$$MSE = bias^2 + \xi^2 = 4\nu^2\bar{n}^2 + 4\frac{\bar{n}}{K}E\epsilon^4 + \frac{T}{\bar{n}}\eta^2 = 4\nu^2\bar{n}^2 + \frac{T}{\bar{n}}\eta^2 \text{ (to first order),}$$

thus the optimal \bar{n}^* satisfies that

$$\bar{n}^* = \left(\frac{T\eta^2}{8\nu^2} \right)^{1/3}.$$

Therefore, assuming the estimator $[Y, Y]_T^{(\cdot)}$ is adopted, one could benefit from a minimum MSE if one subsamples \bar{n}^* data in an equidistant fashion. In other words, all n observations can be used if one uses K^* , $K^* \approx n/\bar{n}^*$, subgrids. This is in contrast to the drawback of using all the data in the single grid case. The subsampling coupled with aggregation brings out the advantage of using the entire data. Of course, for the asymptotics to work, we need $\nu^2 \rightarrow 0$.

Our recommendation, however, is to use the methods in Sections 4 -6.

8 CONCLUSION

In this work, we have quantified and corrected the effect of noise on the nonparametric assessment of integrated volatility. In the setting of high frequency data, the usual financial practice is to use sparse sampling, in other words, throwing away most of the available data. We have argued that this is caused by not incorporating the noise in the model. While it is statistically unsound to throw away data, we have shown that it is possible to build on this practice to construct estimators that make statistical sense.

Specifically, we have found that the usual realized volatility mainly estimates the magnitude of the noise term rather than anything to do with volatility. An approach that is built on separating the observations into multiple “grids” lessens this problem. We found that the best results can be obtained by combining the usual (“single grid”) realized volatility with the multiple grid based device. This gives an estimator which is approximately unbiased, and we have also shown how to assess the (random) variance of this estimator. Most of the development is in the context of finding the integrated volatility over one time period; at the end, we extend this to multiple periods. Also, in the case where the noise can be taken to be almost negligible, we provide a way of optimizing the sampling frequency if one wishes to use the classical “realized volatility” or its multi-grid extension.

One important message of the paper: Any time one has an impulse to sample sparsely, one can always do better with a multi-grid method. No matter what the model is, no matter what quantity is being estimated.

APPENDIX: PROOFS OF RESULTS

When the total grid \mathcal{G} is considered, we use $\sum_{i=1}^{n-1}$, $\sum_{t_{i+1} \leq T}$ and $\sum_{t_i \in \mathcal{G}}$ interchangeably in the following proofs.

A.1 Variance of $[Y, Y]_T$ Given the X Process

We here calculate explicitly the variance in equation (11), from which the stated approximation follows. The explicit remainder term is also used for equation (18).

Let a partition of $[0, T]$ be $0 = t_0 \leq t_1 \leq \dots \leq t_n = T$. Under assumption (6),

$$\begin{aligned} & \text{Var}([Y, Y]_T | X \text{ process}) \\ &= \text{Var}\left[\sum_{t_{i+1} \leq T} (\Delta Y_{t_i})^2 | X \text{ process}\right] \\ &= \underbrace{\sum_{t_{i+1} \leq T} \text{Var}[(\Delta Y_{t_i})^2 | X \text{ process}]}_{I_T} + 2 \underbrace{\sum_{t_{i+1} \leq T} \text{Cov}[(\Delta Y_{t_{i-1}})^2, (\Delta Y_{t_i})^2 | X \text{ process}]}_{II_T} \end{aligned}$$

since $\Delta Y_{t_i} = \Delta X_{t_i} + \Delta \epsilon_{t_i}$ is 1-dependent given X process.

$$\begin{aligned} & \text{Var}[(\Delta Y_{t_i})^2 | X \text{ process}] \\ &= \kappa_4(\Delta Y_{t_i} | X \text{ process}) + 2[\text{Var}(\Delta Y_{t_i} | X \text{ process})]^2 + 4[E(\Delta Y_{t_i} | X \text{ process})]^2 \text{Var}(\Delta Y_{t_i} | X \text{ process}) \\ & \quad + 4E(\Delta Y_{t_i} | X \text{ process})\kappa_3(\Delta Y_{t_i} | X \text{ process}) \\ &= \kappa_4(\Delta \epsilon_{t_i}) + 2[\text{Var}(\Delta \epsilon_{t_i})]^2 + 4(\Delta X_{t_i})^2 \text{Var}(\Delta \epsilon_{t_i}) + 4(\Delta X_{t_i})\kappa_3(\Delta \epsilon_{t_i}), \quad \text{under assumption (6)} \\ &= 2\kappa_4(\epsilon) + 8\nu^2 + 8(\Delta X_{t_i})^2 \nu \end{aligned}$$

since $\kappa_3(\Delta \epsilon_{t_i}) = 0$. The κ s are the cumulants of the relevant order.

$$\text{So, } I_T = n(2\kappa_4(\epsilon) + 8\nu^2) + 8\nu[X, X]_T.$$

Similarly, for the covariance,

$$\begin{aligned} & \text{Cov}[(\Delta Y_{t_{i-1}})^2, (\Delta Y_{t_i})^2 | X \text{ process}] \\ &= \text{Cov}[(\Delta \epsilon_{t_{i-1}})^2, (\Delta \epsilon_{t_i})^2] + 4(\Delta X_{t_{i-1}})(\Delta X_{t_i})\text{Cov}(\Delta \epsilon_{t_{i-1}}, \Delta \epsilon_{t_i}) \\ & \quad + 2(\Delta X_{t_{i-1}})\text{Cov}[\Delta \epsilon_{t_{i-1}}, (\Delta \epsilon_{t_i})^2] + 2(\Delta X_{t_i})\text{Cov}[(\Delta \epsilon_{t_{i-1}})^2, \Delta \epsilon_{t_i}] \\ &= \kappa_4(\epsilon) + 2\nu^2 - 4(\Delta X_{t_{i-1}})(\Delta X_{t_i})\kappa_2(\epsilon) - 2(\Delta X_{t_i})\kappa_3(\epsilon) + 2(\Delta X_{t_{i-1}})\kappa_3(\epsilon) \end{aligned} \quad (\text{A.1})$$

because $\kappa_1(\epsilon) = 0$, $\kappa_2(\epsilon) = \text{Var}(\epsilon) = E(\epsilon^2)$, $\kappa_3(\epsilon) = E\epsilon^3$, and $\kappa_4(\epsilon) = E(\epsilon^4) - 3\nu^2$.

Thus, assuming the coefficients in (A.1)

$$\begin{aligned} II_T &= 2(n-1)(\kappa_4(\epsilon) + 2\nu^2) \\ & \quad - 8\nu \sum_{t_{i+1} \leq T} (\Delta X_{t_{i-1}})(\Delta X_{t_i}) - 4\kappa_3(\epsilon)(\Delta X_{t_{n-1}} - \Delta X_{t_0}) \end{aligned}$$

Amalgamating the two expressions one obtains

$$\begin{aligned} \text{Var}([Y, Y]_T | X \text{ process}) &= n(2\kappa_4(\epsilon) + 8\nu^2) + 8\nu[X, X]_T + 2(n-1)(\kappa_4(\epsilon) + 2\nu^2) \\ & \quad - 8\nu \sum (\Delta X_{t_{i-1}})(\Delta X_{t_i}) - 4\kappa_3(\epsilon)(\Delta X_{t_{n-1}} - \Delta X_{t_0}) \\ &= 4nE\epsilon^4 + R_n, \end{aligned} \quad (\text{A.2})$$

where the remainder term R_n satisfies

$$\begin{aligned} |R_n| &\leq 8\nu[X, X]_T + 2(\kappa_4(\epsilon) + 2\nu^2) \\ & \quad + 8\nu \left| \sum (\Delta X_{t_{i-1}})(\Delta X_{t_i}) \right| + 4|\kappa_3(\epsilon)|(|\Delta X_{t_{n-1}}| + |\Delta X_{t_0}|) \\ &\leq 16\nu[X, X]_T + 2(\kappa_4(\epsilon) + 2\nu^2) + 2|\kappa_3(\epsilon)|(2 + [X, X]_T) \end{aligned} \quad (\text{A.3})$$

by the Cauchy-Schwartz inequality and since $|x| \leq (1 + x^2)/2$.

Since $[X, X]_T = O_p(1)$, (11) follows. Similarly, (18) follows since $[X, X]_T^{(\cdot)} = O_p(1)$.

A.2 Relevant Central Limit Theorem

Lemma A.1. *Suppose X is an Ito process. Suppose Y is related to X through model (4). Then under assumption (6) and definitions (8) and (14),*

$$[Y, Y]_T^{(all)} = [\epsilon, \epsilon]_T^{(all)} + O_p(1), \text{ and } [Y, Y]_T^{(\cdot)} = [\epsilon, \epsilon]_T^{(\cdot)} + [X, X]_T^{(\cdot)} + O_p\left(\frac{1}{\sqrt{K}}\right) \quad \square$$

Proof of Lemma A.1

(a) one grid case:

$$[Y, Y]_T^{(all)} = [X, X]_T^{(all)} + [\epsilon, \epsilon]_T^{(all)} + 2[X, \epsilon]_T^{(all)} \quad (\text{A.4})$$

We show:

$$E\left(\left([X, \epsilon]_T^{(all)}\right)^2 \mid X \text{ process}\right) = O_p(1) \quad (\text{A.5})$$

and in particular

$$[X, \epsilon]_T^{(all)} = O_p(1) \quad (\text{A.6})$$

To see (A.5):

$$\begin{aligned} [X, \epsilon]_T^{(all)} &= \sum_{i=0}^{n-1} (\Delta X_{t_i})(\Delta \epsilon_{t_i}) \\ &= \sum_{i=0}^{n-1} (\Delta X_{t_i})\epsilon_{t_{i+1}} - \sum_{i=0}^{n-1} (\Delta X_{t_i})\epsilon_{t_i} \\ &= \sum_{i=1}^{n-1} (\Delta X_{t_{i-1}} - \Delta X_{t_i})\epsilon_{t_i} + \Delta X_{t_{n-1}}\epsilon_{t_n} - \Delta X_{t_0}\epsilon_{t_0} \end{aligned}$$

Since $E([X, \epsilon]_T^{(all)} \mid X \text{ process}) = 0$ and ϵ_{t_i} iid for different t_i , we get

$$\begin{aligned} E\left(\left([X, \epsilon]_T^{(all)}\right)^2 \mid X \text{ process}\right) &= \text{Var}([X, \epsilon]_T^{(all)} \mid X \text{ process}) \\ &= \nu \left[\sum_{i=1}^{n-1} (\Delta X_{t_{i-1}} - \Delta X_{t_i})^2 + \Delta X_{t_{n-1}}^2 + \Delta X_{t_0}^2 \right] \\ &= 2\nu[X, X]_T - 2\nu \sum_{i=1}^{n-1} (\Delta X_{t_{i-1}})(\Delta X_{t_i}) \\ &\leq 4\nu[X, X]_T \quad (\text{A.7}) \end{aligned}$$

by Cauchy-Schwartz Inequality, from which and from $[X, X]^{(all)}$ being of order $O_p(1)$, (A.5) follows. Hence (A.6) follows by Markov Inequality.

(b) multiple grid case:

notice that

$$[Y, Y]^{(\cdot)} = [X, X]_T^{(\cdot)} + [\epsilon, \epsilon]_T^{(\cdot)} + 2[X, \epsilon]_T^{(\cdot)} \quad (\text{A.8})$$

(A.8) strictly follows from model (4) and the definitions of grids and $[\cdot, \cdot]_t^{(\cdot)}$, see Section 3.2.

Need to show:

$$E([X, \epsilon]_T^{(\cdot)} | X \text{ process}) = O_p\left(\frac{1}{K}\right) \quad (\text{A.9})$$

in particular

$$[X, \epsilon]_T^{(\cdot)} = O_p\left(\frac{1}{K^{1/2}}\right). \quad (\text{A.10})$$

and $Var([X, \epsilon]_T^{(\cdot)} | X \text{ process}) = E([X, \epsilon]_T^{(\cdot)})^2 | X \text{ process}]$.

To show (A.9), note that $E([X, \epsilon]_T^{(\cdot)} | X \text{ process}) = 0$,

$$\begin{aligned} E([X, \epsilon]_T^{(\cdot)})^2 | X \text{ process}] &= Var([X, \epsilon]_T^{(\cdot)} | X \text{ process}) \\ &= \frac{1}{K^2} \sum_{k=1}^K Var([X, \epsilon]_T^{(k)} | X \text{ process}) \\ &\leq \frac{4\nu}{K} [X, X]_T^{(\cdot)} = O_p\left(\frac{1}{K}\right) \end{aligned}$$

where the second equality follows from the disjointness of different grids as well as $\epsilon \perp\!\!\!\perp X$. The inequality follows from the same argument as in (A.7). Then the order follows since $[X, X]_T^{(\cdot)} = O_p(1)$. Cf. the method in Mykland and Zhang (2002) if one wants a rigorous development for the order of $[X, X]_T^{(\cdot)}$. \blacksquare

Theorem A.1. *Suppose X is an Ito process of form (1). Suppose Y is related to X through model (4), and that (6) is satisfied with $E\epsilon^4 < \infty$. Also suppose that t_i and t_{i+1} is not in the same subgrid for any i . Under assumption (15), as $n \rightarrow \infty$, $(\sqrt{n}(\hat{\nu} - \nu), \sqrt{\frac{K}{n}}([Y, Y]_T^{(\cdot)} - [X, X]_T^{(\cdot)} - 2\nu\bar{n}))$ converges in law to a bivariate normal, with mean 0 and covariance matrix*

$$\begin{pmatrix} E\epsilon^4 & 2Var(\epsilon^2) \\ 2Var(\epsilon^2) & 4E\epsilon^4 \end{pmatrix} \quad (\text{A.11})$$

conditional on X process. where the limiting random variable is independent of the X process. \square

Proof of Theorem A.1:

By Lemma A.1, need the distribution of $[\epsilon, \epsilon]^{(\cdot)}$ and $[\epsilon, \epsilon]^{(all)}$.

First, we explore the convergence of

$$\frac{1}{\sqrt{n}} \left([\epsilon, \epsilon]_T^{(all)} - 2\nu n, [\epsilon, \epsilon]_T^{(\cdot)} K - 2\nu \bar{n} K \right) \quad (\text{A.12})$$

Recall that all the sampling points t_0, t_1, \dots, t_n are within $[0, T]$. We use \mathcal{G} to denote the time points in the full sampling, as in the single grid. $\mathcal{G}^{(k)}$ denotes the subsamplings from k-th grid.

As before, if $t_i \in \mathcal{G}^{(k)}$, then $t_{i,-}$ and $t_{i,+}$ are, respectively, the previous and next element in $\mathcal{G}^{(k)}$. $\epsilon_{t_i,-} = 0$ for $t_i = \min \mathcal{G}^{(k)}$ and $\epsilon_{t_i,+} = 0$ for $t_i = \max \mathcal{G}^{(k)}$.

Set

$$\begin{aligned} M_T^{(1)} &= \frac{1}{\sqrt{n}} \sum_{t_i \in \mathcal{G}} (\epsilon_{t_i}^2 - \nu) \\ M_T^{(2)} &= \frac{1}{\sqrt{n}} \sum_{t_i \in \mathcal{G}} \epsilon_{t_i} \epsilon_{t_{i-1}} \\ M_T^{(3)} &= \frac{1}{\sqrt{n}} \sum_{k=1}^K \sum_{t_i \in \mathcal{G}^{(k)}} \epsilon_{t_i} \epsilon_{t_{i,-}} \end{aligned} \quad (\text{A.13})$$

We first find the asymptotic distribution of $(M_T^{(1)}, M_T^{(2)}, M_T^{(3)})$ using the martingale central limit theorem, and then we use the result to find the limit of (A.12).

Note that $(M_T^{(1)}, M_T^{(2)}, M_T^{(3)})$ are the end points of martingales with respect to filtration $\mathcal{F}_i = \sigma(\epsilon_{t_j}, j \leq i, X_t, \text{ all } t)$. We now derive its (discrete-time) predictable quadratic variation $\langle M^{(l)}, M^{(k)} \rangle$, $l, k = 1, 2, 3$. (Discrete time predictable quadratic variations are only used in this proof, and are different from the continuous time quadratic variations in (7)).

$$\begin{aligned} \langle M^{(1)}, M^{(1)} \rangle_T &= \frac{1}{n} \sum_{t_i \in \mathcal{G}} \text{Var}(\epsilon_{t_i}^2 - \nu \mid \mathcal{F}_{t_{i-1}}) = \text{Var}(\epsilon^2) \\ \langle M^{(2)}, M^{(2)} \rangle_T &= \frac{1}{n} \sum_{t_i \in \mathcal{G}} \text{Var}(\epsilon_{t_i} \epsilon_{t_{i-1}} \mid \mathcal{F}_{t_{i-1}}) = \frac{\nu}{n} \sum_{t_i \in \mathcal{G}} \epsilon_{t_{i-1}}^2 = \nu^2 + o_p(1) \\ \langle M^{(3)}, M^{(3)} \rangle_T &= \frac{1}{n} \sum_{k=1}^K \sum_{t_i \in \mathcal{G}^{(k)}} \text{Var}(\epsilon_{t_i} \epsilon_{t_{i,-}} \mid \mathcal{F}_{t_{i-1}}) = \frac{\nu}{n} \sum_{k=1}^K \sum_{t_i \in \mathcal{G}^{(k)}} \epsilon_{t_{i,-}}^2 = \nu^2 + o_p(1) \end{aligned}$$

by the law of large numbers.

Similarly, for the predictable quadratic covariations,

$$\begin{aligned}
\langle M^{(1)}, M^{(2)} \rangle_T &= \frac{1}{n} \sum_{t_i \in \mathcal{G}} \text{Cov}(\epsilon_{t_i}^2 - \nu, \epsilon_{t_i} \epsilon_{t_{i-1}} \mid \mathcal{F}_{t_{i-1}}) = E \epsilon^3 \frac{1}{n} \sum_{t_i \in \mathcal{G}} \epsilon_{t_{i-1}} = o_p(1) \\
\langle M^{(1)}, M^{(3)} \rangle_T &= \frac{1}{n} \sum_{k=1}^K \sum_{t_i \in \mathcal{G}^{(k)}} \text{Cov}(\epsilon_{t_i}^2 - \nu, \epsilon_{t_i} \epsilon_{t_{i,-}} \mid \mathcal{F}_{t_{i-1}}) = E \epsilon^3 \frac{1}{n} \sum_{k=1}^K \sum_{t_i \in \mathcal{G}^{(k)}} \epsilon_{t_{i,-}} = o_p(1) \\
\langle M^{(2)}, M^{(3)} \rangle_T &= \frac{1}{n} \sum_{k=1}^K \sum_{t_i \in \mathcal{G}^{(k)}} \text{Cov}(\epsilon_{t_i} \epsilon_{t_{i-1}}, \epsilon_{t_i} \epsilon_{t_{i,-}} \mid \mathcal{F}_{t_{i-1}}) \\
&= \frac{\nu}{n} \sum_{k=1}^K \sum_{t_i \in \mathcal{G}^{(k)}} \epsilon_{t_{i-1}} \epsilon_{t_{i,-}} = o_p(1)
\end{aligned}$$

since t_{i+1} is not in the same grid as t_i .

Since the ϵ_{t_i} 's are iid and $E \epsilon_{t_i}^4 < \infty$, the conditional Lindeberg conditions are satisfied. Hence by the martingale CLT (see condition 3.1, p. 58 of Hall and Heyde (1980)), $(M^{(1)}, M^{(2)}, M^{(3)})$ are asymptotically normal, with covariance matrix as the asymptotic value of $\langle M^{(l)}, M^{(k)} \rangle$. In other words, asymptotically, $(M^{(1)}, M^{(2)}, M^{(3)})$ are independent normal with respective variances $\text{Var}(\epsilon)$, ν^2 , and ν^2 .

Returning to (A.12),

$$[\epsilon, \epsilon]^{(all)} - 2n\nu = 2 \sum_{i \neq 0, n} (\epsilon_{t_i}^2 - \nu) + (\epsilon_{t_0}^2 - \nu) + (\epsilon_{t_n}^2 - \nu) - 2 \sum_{t_i > 0} \epsilon_{t_i} \epsilon_{t_{i-1}} = 2\sqrt{n}(M^{(1)} - M^{(2)}) + O_p(1) \quad (\text{A.14})$$

Meanwhile:

$$\begin{aligned}
[\epsilon, \epsilon]^{(k)} - 2n_k\nu &= \sum_{\substack{t_i \in \mathcal{G}^{(k)} \\ t_i \neq \max \mathcal{G}^{(k)}}} (\epsilon_{t_{i,+}} - \epsilon_{t_i})^2 - 2n_k\nu \\
&= 2 \sum_{t_i \in \mathcal{G}^{(k)}} (\epsilon_{t_i}^2 - \nu) - (\epsilon_{\min \mathcal{G}^{(k)}}^2 - \nu) - (\epsilon_{\max \mathcal{G}^{(k)}}^2 - \nu) - 2 \sum_{t_i \in \mathcal{G}^{(k)}} \epsilon_{t_i} \epsilon_{t_{i,-}} \quad (\text{A.15})
\end{aligned}$$

where $n_k + 1$ is the total number of sampling points in $\mathcal{G}^{(k)}$.

Hence,

$$[\epsilon, \epsilon]_T^{(1)} K - 2\bar{n}\nu K = \sqrt{\bar{n}}(2M^{(1)} - 2M^{(3)}) - R = 2\sqrt{\bar{n}}(M^{(1)} - M^{(3)}) + O_p(K^{1/2}), \quad (\text{A.16})$$

since $R = \sum_{k=1}^K [(\epsilon_{\min \mathcal{G}^{(k)}}^2 - \nu) + (\epsilon_{\max \mathcal{G}^{(k)}}^2 - \nu)]$ satisfying $ER^2 = \text{Var}(R) \leq 4K\text{Var}(\epsilon^2)$.

Since $n^{-1}K \rightarrow 0$, and since the error terms in (A.14) and (A.15) are uniformly integrable, it follows that

$$(\text{A.12}) = 2(M^{(1)} - M^{(2)}, M^{(1)} - M^{(3)}) + o_p(1) \quad (\text{A.17})$$

Hence, (A.12) is also asymptotically normal with covariance matrix

$$\begin{pmatrix} 4E\epsilon^4 & 4Var(\epsilon^2) \\ 4Var(\epsilon^2) & 4E\epsilon^4 \end{pmatrix}.$$

By Lemma A.1, and as $n^{-1}K \rightarrow 0$,

$$\frac{1}{\sqrt{n}} \left([Y, Y]_T^{(all)} - 2\nu n, K([Y, Y]_T^{(\cdot)} - [X, X]_T^{(\cdot)} - 2\nu\bar{n}) \right) | X \text{ process}$$

is asymptotically normal,

$$\begin{aligned} \frac{1}{\sqrt{n}} \left(\begin{array}{c} [Y, Y]_T^{(all)} - 2\nu n \\ [Y, Y]_T^{(\cdot)} K - 2\nu\bar{n}K - [X, X]_T^{(\cdot)} K \end{array} \middle| X \text{ process} \right) &= 2 \begin{pmatrix} M^{(1)} - M^{(2)} \\ M^{(1)} - M^{(3)} \end{pmatrix} + o_p(1) \\ &\xrightarrow{\mathcal{L}} 2N \left(0, \begin{pmatrix} E\epsilon^4 & Var(\epsilon^2) \\ Var(\epsilon^2) & E\epsilon^4 \end{pmatrix} \right). \end{aligned} \quad (\text{A.18})$$

Since

$$\hat{\nu} = \frac{1}{2n} [Y, Y]_T^{(all)}, \text{ and } \frac{K}{\sqrt{n}} = \sqrt{\frac{K}{\bar{n}}} (1 + o(1)), \quad (\text{A.19})$$

Theorem A.1 follows.

A.3 Asymptotics of D_T

For transparency of notation, we take $\bar{\Delta}t = T/n$, in other words, the average of the Δt_i .

For given $s \in [0, T]$, let $s_-^{(k)}$ be the closest point on grid $\mathcal{G}^{(k)}$ smaller than s , *i.e.*, $s_-^{(k)} = \max\{u \leq s : u \in \mathcal{G}^{(k)}\}$. In particular, for grid points t_i , let $t_i^{(k)}$ be the closest point on grid $\mathcal{G}^{(k)}$ smaller than t_i , *i.e.*, $t_i^{(k)} = \max\{u \leq t_i : u \in \mathcal{G}^{(k)}\}$. Observe that $t_i^{(k)} = (t_i)_-^{(k)}$.

We here do not assume regular allocation of sample points to subgrids, but instead that

$$\max_i \sum_{l=1}^K \{\#k : t_i^{(k)} > t_i^{(l)}\}^2 = O(K^3). \quad (\text{A.20})$$

Note that $\{\#k : t_i^{(k)} > t_i^{(l)}\}$ is the number of points in the total grid \mathcal{G} between $t_i^{(l)}$ and t_i . The requirement (A.20) is satisfied under regular allocation of sample points to subgrids, as defined in Section 3.2, in other words, $\mathcal{G}^{(l)} = \{t_{l-1}, t_{K+l-1}, \dots\}$.

Proof of Theorem 2. Rewrite

$$\begin{aligned}
D_T &= [X, X]_T^{(\cdot)} - \langle X, X \rangle_T \\
&= \frac{1}{K} \sum_{k=1}^K ([X, X]_T^{(k)} - \langle X, X \rangle_T) \\
&= \frac{1}{K} \sum_{k=1}^K \sum_{\{t_i, t_{i,+} \in \mathcal{G}^{(k)}\}} 2 \int_{t_i}^{t_{i,+}} (X_s - X_{t_i}) dX_s \quad (\text{by Ito's formula}) \\
&= 2 \int_0^T \frac{1}{K} \sum_{k=1}^K (X_s - X_{s_-^{(k)}}) dX_s
\end{aligned}$$

Denote the integrand as Z_s . We can write $Z_s = X_s - \frac{1}{K} \sum_{k=1}^K X_{s_-^{(k)}}$

Following the arguments in Mykland and Zhang (2002), the quadratic variation of D_T is

$$\begin{aligned}
\langle D, D \rangle_T &= 4 \int_0^T Z_s^2 d\langle X, X \rangle_s \\
&= 4 \int_0^T \langle Z, Z \rangle_s d\langle X, X \rangle_s + o_p\left(\frac{K}{n}\right) \\
&= 4 \int_0^T \langle Z, Z \rangle_s \langle X, X \rangle'_s ds + o_p\left(\frac{K}{n}\right) \\
&= 4 \sum_i \int_{t_i}^{t_{i,+}} \langle Z, Z \rangle_s \langle X, X \rangle'_s ds + o_p\left(\frac{K}{n}\right),
\end{aligned}$$

where the sum is over all (except the last) observation points t_i . To calculate the integrand, note that for $t_i \leq s < t_{i,+}$,

$$\begin{aligned}
\langle Z, Z \rangle_s \langle X, X \rangle'_s &= \frac{1}{K^2} \sum_{k=1}^K \sum_{l=1}^K (\langle X, X \rangle_s - \langle X, X \rangle_{s_-^{(k)} \wedge s_-^{(l)}}) \langle X, X \rangle'_s \\
&= \frac{1}{K^2} \sum_{l=1}^K (\langle X, X \rangle_s - \langle X, X \rangle_{s_-^{(l)}}) (2\{\#k : s_-^{(k)} > s_-^{(l)}\} + 1) \langle X, X \rangle'_s \\
&= \frac{1}{K^2} \sum_{l=1}^K (s - t_i^{(l)}) (2\{\#k : t_i^{(k)} > t_i^{(l)}\} + 1) (\langle X, X \rangle'_{t_i})^2 + o_p\left(\frac{K}{n}\right),
\end{aligned}$$

since the $s_-^{(l)} = t_i^{(l)}$ as s varies over the relevant time interval $[t_i, t_{i+1})$. Hence

$$\begin{aligned}
 & \int_{t_i}^{t_{i+1}} \langle Z, Z \rangle_s \langle X, X \rangle'_s ds \\
 &= \frac{1}{K^2} \sum_{l=1}^K \left(\int_{t_i}^{t_{i+1}} (s - t_i^{(l)}) ds \right) (2\{\#k : t_i^{(k)} > t_i^{(l)}\} + 1) (\langle X, X \rangle'_{t_i})^2 + o_p\left(\frac{K}{n^2}\right) \\
 &= \frac{1}{K^2} \sum_{l=1}^K \left(\frac{1}{2} \Delta t_i^2 + (t_i - t_i^{(l)}) \Delta t_i \right) (2\{\#k : t_i^{(k)} > t_i^{(l)}\} + 1) (\langle X, X \rangle'_{t_i})^2 + o_p\left(\frac{K}{n^2}\right) \\
 &= \frac{1}{4} \bar{\Delta} t K h_i (\langle X, X \rangle'_{t_i})^2 \Delta t_i + o_p\left(\frac{K}{n^2}\right)
 \end{aligned}$$

where the h_i are defined by (23). Hence, since the error term above is uniform in i (in probability),

$$\langle D, D \rangle_T = \bar{\Delta} t K \sum_i h_i (\langle X, X \rangle'_{t_i})^2 \Delta t_i + o_p\left(\frac{K}{n}\right) \quad (\text{A.21})$$

thus showing Theorem 2.

We now proceed to the asymptotic distribution of D_T . We first state a technical condition on the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ to which X_t and μ_t (but not the ϵ 's) are assumed to be adapted.

Condition E (*Description of the filtration*): *There is a continuous multidimensional P -local martingale $\mathcal{X} = (\mathcal{X}^{(1)}, \dots, \mathcal{X}^{(p)})$, any p , so that \mathcal{F}_t is the smallest sigma-field containing $\sigma(\mathcal{X}_s, s \leq t)$ and \mathcal{N} , where \mathcal{N} contains all the null sets in $\sigma(\mathcal{X}_s, s \leq T)$.*

For example, \mathcal{X} can be a collection of Brownian motions.

Proof of Theorem 3. One shows by methods similar to those in the proof of Theorem 2 that if L is any martingale adapted to the filtration generated by \mathcal{X} , then

$$\sup_t \left| \frac{1}{\bar{\Delta} t K} \langle D, L \rangle_t \right| \xrightarrow{p} 0, \quad (\text{A.22})$$

The stable convergence with respect to the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ then follows in view of Rootzen (1980) or Jacod and Protter (1998). This ends the proof of Theorem 3.

Finally, in the case where η^2 does not converge, one can still use the mixed normal with variance η_n^2 . This is because every subsequence of η_n^2 has a further subsequence which does converge in probability to some η^2 in probability, and hence for which the assumption (28) in Theorem 3 would be satisfied.

The reason for this is that one can define the distribution function of a finite measure by

$$G_n(t) = \sum_{t_{i+1} \leq t} h_i \Delta t_i \quad (\text{A.23})$$

Since $G_n(t) \leq T \sup_i h_i$, it follows from (25) that the sequence G_n is weakly compact in the sense of weak convergence (see Helly's Theorem, e.g. Billingsley (1995) p. 336). For any convergent

subsequence $G_n \rightarrow G$, we then get that

$$\eta_n^2 = \int_0^T (\langle X, X \rangle'_t)^2 dG_n(t) \rightarrow \int_0^T (\langle X, X \rangle'_t)^2 dG(t). \quad (\text{A.24})$$

almost surely, since we have assumed $\langle X, X \rangle'_t$ to be a continuous function of t . One then defines η^2 to be the (subsequence dependent) right hand side of (A.24).

To further proceed with the asymptotics, continue the subsequence from above, and note that

$$\begin{aligned} \frac{1}{\Delta t K} \langle D, D \rangle_t &\approx \int_0^t (\langle X, X \rangle'_s)^2 dG_n(s) \\ &\rightarrow \int_0^t (\langle X, X \rangle'_s)^2 dG(s). \end{aligned}$$

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