Signal Processing 81 (2001) 1991–1995

SIGNAL PROCESSING

www.elsevier.com/locate/sigpro

The approximation of a morphological opening and closing in the presence of noise

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Received 4 May 1999; received in revised form 11 March 2001

Abstract

Morphological openings and closings form the cornerstones of a rich set of nonlinear operators in signal processing. In this paper, we show how a new morphological operator can be constructed that serves as an approximation to openings and closings for images corrupted by low levels of noise. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Openings; Closings; Mathematical morphology; Alternating rank filters

1. Introduction

Mathematical morphology plays an important role in nonlinear signal and image processing; it has its roots in probing an image (signal) by translations of a small set (see [7,8,3] for an overview). From an operational point of view, a main characteristic is the use of maximum and minimum operators instead of linear ones.

Among all morphological operators, the opening and closing operator are fundamental. They can be used for image description as well as image manipulation (with noise filtering as a special case). Given their sensibility to noise, in recent literature, several proposals were published to improve their robustness. The majority aim to improve their performance as a filter. One way is to make use of an explicit description of the statistical nature of the noise and to look for optimality with respect to a certain error measure (cf. [6,2]). A second approach is to embed morphological operators in a parameterized family of operators where the extremes correspond to erosion and dilation. The simplest example is [4] where one substitutes the *max* operator by the L_p norm or a weighted variant. Yuille et al. [10] elaborate how to embed morphological operators in a statistical physics formulation and relate it with Bayesian reconstruction. Based on this result, Regazzoni et al. [5] introduce statistical erosion and dilation.

In this paper, we follow a third path using a point of view based on set theory, much closer to the basic principles of morphology and conserving the typical two-step approach of mathematical morphology. We will introduce a new morphological operator that extends in a certain sense *Soft-Morphology* [1], combining statistical erosions and dilations. Remark that opposite to the previous approaches, we will consider openings and closings as tools for image description (e.g., the construction of a Pattern Spectrum) and not for filtering.

In what follows, X will denote an image; in order to simplify the notation, depending on the context,

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PII: S0165-1684(01)00060-3

X will represent the set of black pixels (as in the binary case) as well as the gray level function where X_x denotes the gray level value at position x. The cardinality of a set S is denoted by $|S|$. We will call S a symmetric set if $-S=S$. The translation of a binary image X by s, $\{x+s, x\in X\}$ is denoted by $X_{(s)}$.

In Section 2, we consider the case of binary images. In Section 3, we extend the results to gray scale images and discuss several experiments.

2. Approximations of openings and closings for binary images

The starting point is the classical definition of an opening by a symmetric set S (called *structural element*): a pixel x belongs to the opening of a binary image X by the structural element S if x belongs to at least *one* translation of S *totally* included in X.

A natural generalization is to weaken the total inclusion requirement to several partial ones, i.e., we require that there are at least r_2 translations of S covering x and that they have at least r_1 pixels in common with the image X.

For this purpose, we use the operator $\rho()$:

$$\rho_{r,S}(X)=\{x: |S_{(x)}\cap X|\geqslant r\}. \tag{1}$$

This allows us to formalize the above in the following way.

Definition 1. For a given symmetrical structural element S, we define the morphological operator $R_S^{r_1,r_2}()$ as:

$$R_S^{r_1,r_2}(X)=\rho_{r_2,S}(\rho_{r_1,S}(X)). \tag{2}$$

The restriction to a symmetrical structural element is only for reasons of simplification and can be omitted (as in classical morphology). We observe that $R_S^{|S|,1}(X)$ corresponds to a classical opening $X\circ S:=\delta_S(\varepsilon_S(X))$, where $\delta_S()$ and $\varepsilon_S()$ denote a morphological dilation resp. erosion. On the other hand, a closing, $X\bullet S:=\varepsilon_S(\delta_S(X))$, coincides with $R_S^{1,|S|}$. Among the main properties, we mention [9]:

Proposition 2. (1) If $X\subseteq Y$: $R_S^{r_1,r_2}(X)\subseteq R_S^{r_1,r_2}(Y)$.
(2) If $S\subseteq T$: $R_S^{r_1,r_2}(X)\subseteq R_T^{r_1,r_2}(X)$.
(3) If $r_1\leqslant s_1$, $r_2\leqslant s_2$: $R_S^{s_1,s_2}(X)\subseteq R_S^{r_1,r_2}(X)$. *Consequently* $\varepsilon_S(\varepsilon_S(X))\subseteq R_S^{s_1,s_2}(X)\subseteq\delta_S(\delta_S(X))$.
(4) $R_S^{s_1,s_2}(X)=R_S^{|S|+1-s_1,|S|+1-s_2}(X^c)$.
(5) $R_S^{r_1,r_2}(X)=\bigcap_j\bigcup_i\delta_{A_j}(\varepsilon_{B_i}(X))$, *with* $\{A_j\}$ *resp.* $\{B_i\}$ *all subsets of* S *with exactly* $|S|-r_2+1$ *resp.* r_1 *elements.*

As an illustration, Fig. 1(d) presents the results of the application of $R_S^{22,7}(X)$ with S a disk of radius 3, to the image of Fig. 1(a) corrupted with 5% pepper and salt noise (Fig. 1(c)). This is to be compared with the opening of the original image (Fig. 1(b)). In Fig. 1(e) the opening of the noisy image with S is shown.

3. Approximations of openings and closings for gray scale images

One way to extend the previous results for binary images to gray scale images is by making use of *slices* (thresholds):

Definition 3. We define for a gray scale image X and $u\in\mathbb{R}$:

$$T_u(X)=\{x: X_x\geqslant u\}.$$

Using the above, one generalization of (2) is:

Definition 4. If X is a gray scale image and S a subset of pixels, we define the morphological operator $R_S^{r_1,r_2}(X)$ as the image Z:

$$Z_x=\sup\{u: x\in\rho_{r_2,S}(\rho_{r_1,S}(T_u(X)))\}. \tag{3}$$

The above definition allows a nice geometric interpretation. The value of the robust operator at x corresponds to the value u of the highest slice $T_u(X)$, such that there are in that slice at least r_2 translations of S, containing x, and each one has at least r_1 pixels in common with $T_u(X)$. The next proposition offers another interpretation, relating the above operator with the well studied *rank filter* [9].

Proposition 5. *For a given structural element* S,

$$R_S^{r_1,r_2}(X)=Rank_S^{r_2}(Rank_S^{r_1}(X)),$$

where $Rank_S^r(X)$ *denotes the* r*—rank filter with window* S.

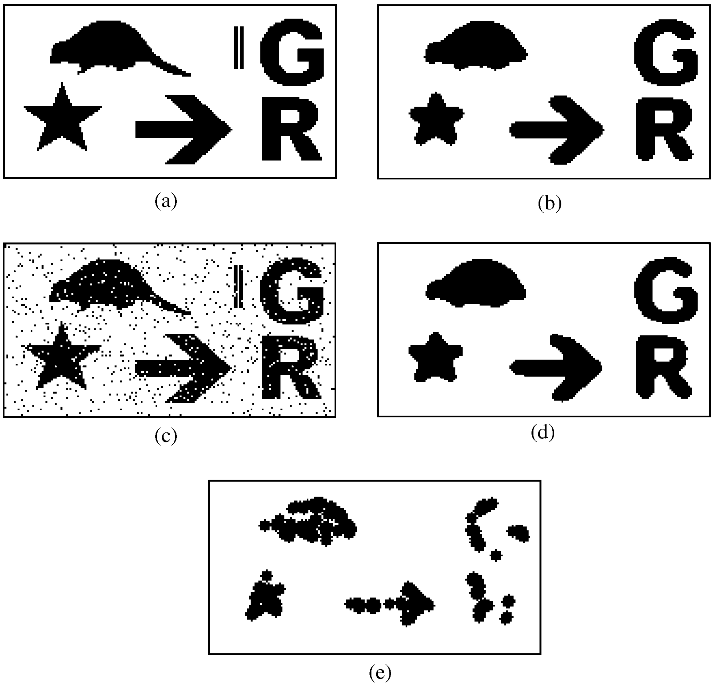

Fig. 1. (a) original image X; (b) $X \circ S$; (c) $N(X) := X$ corrupted with 5% pepper and salt noise; (d) $R_S^{25,5}(N(X))$; (e) $N(X) \circ S$.

Finally, we mention that convergence of repeated applications of $R_S^{r_1,r_2}(X)$ to a fixed image can be proved [9].

In Fig. 2 the results are shown of the application of the morphological operator $R_S^{22,3}()$ to an image corrupted with 1% bit noise (i.e., the value of each bit in each byte that decodes the gray level of a pixel is changed with probability 0.01) and S is a 5×5 square.

For the image X of Fig. 2(a), in Fig. 3 the average over 10 noisy versions of X, N(X), of the normalized L_1 difference mean absolute error (MAE) is shown between $X \circ S$ and $R_S^{|S|-r,r}(N(X))$ for different values of r and different levels of bit noise. Here S is a circle with radius 2. The lines $r1$, $r2$, $r3$, $r4$ correspond respectively to $R_S^{13-1,1}()$, $R_S^{13-2,2}()$, $R_S^{13-3,3}()$, $R_S^{13-4,4}()$.

To get a reference frame, we also implemented experiments where we first passed the image through a noise filter and afterwards we applied a standard opening. Although strictly speaking this procedure and many of its variants do not result in a morphological operator as is the case for $R_S^{|S|-r,r}()$, we plotted in the same Fig. 3

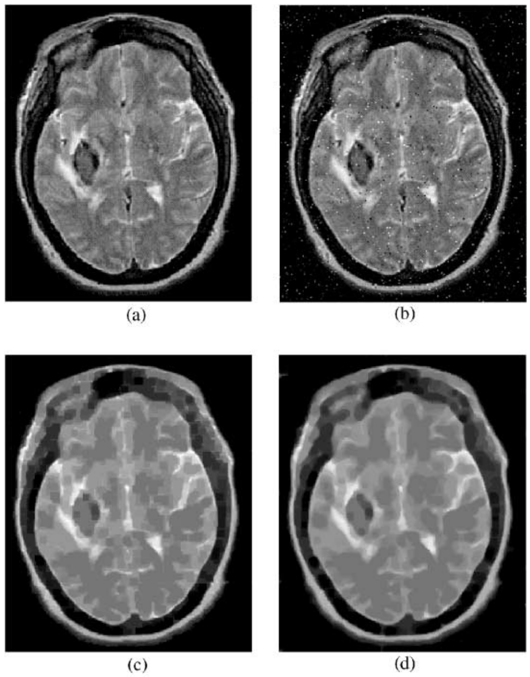

Fig. 2. (a) original image X; (b) $N(X)$:X corrupted by 1% bit noise; (c) $X \circ S$ with S a 5×5 square; (d) $R_S^{22,3}(N(X))$.

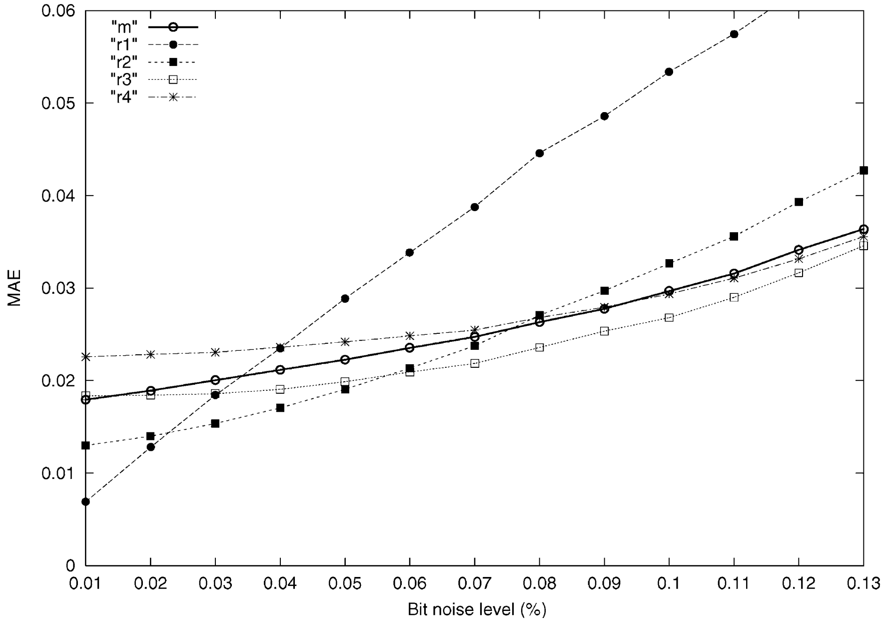

Fig. 3 . The average over 10 noisy versions of $X, N(X)$, of the Mean Absolute Error (MAE) between $X \circ S$ and $R_S^{|S|-r,r}(N(X))$ for different values of r and different levels of bit noise. The thick line m marks the MAE between $\text{MedianFilter}_W(N(X)) \circ S$ and $X \circ S$. W and S are circles of radius 1 resp. 2 (see text).

(thick line m) the MAE between $X \circ S$ and $\text{MedianFilter}_W(N(X)) \circ S$ where the window W is a circle of radius 1. As one observes, the lower the noise level, the more convenient it is to use $R_S^{|S|-r,r}()$ to approximate $X \circ S$. We also did experiments with other window sizes but the above choice outperformed all the others.

We repeated the above experiments for images corrupted with pepper and salt noise and different sizes of S. It turned out that the same conclusions hold.

Because of the symmetry in the definition of $R_S^{r_1,r_2}(X)$ and because of the nature of the noise, the above conclusions are also valid for the approximation of closings.

4. Conclusions

In this paper, we have presented a theoretical framework to construct a morphological operator to approximate an opening or closing in the presence of noise, following a point of view based on set theory.

Experiments confirm the quality of the introduced operators, at least for low noise levels. The statistical estimation of the optimal parameter seems to us an attractive point of departure for future research.

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