Optimal Interest-Rate Smoothing *

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Abstract

This paper considers the desirability of the observed tendency of central banks to adjust interest rates only gradually in response to changes in economic conditions. It shows, in the context of a simple model of optimizing private-sector behavior, that assignment of an interest-rate smoothing objective to the central bank may be desirable, even when reduction of the magnitude of interest-rate changes is not a social objective in itself. This is because a response of policy to "irrelevant" lagged variables may be desirable owing to the way it steers private-sector expectations of future policy.

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1 Optimal Monetary Policy Inertia

Many students of central bank behavior have noted that the level of nominal interest rates in the recent past appears to be an important determinant of where the central bank will set its interest-rate instrument in the present. Changes in observed conditions, such as in the rate of inflation or in the level of economic activity, result in changes in the level of the central bank's operating target for the short-term interest rate that it controls, but these changes typically occur through a series of small adjustments in the same direction, drawn out over a period of months, rather than through an immediate once-and-for-all response to the new development. This type of behavior is especially noticeable in the case of the Federal Reserve in the U.S., but characterizes many other central banks to at least some extent as well.[1]

Such behavior may be rationalized on the ground that central banks seek to "smooth" interest rates, in the sense that they seek to minimize the variability of interest-rate changes, in addition to other objectives of policy such as inflation stabilization. Yet it remains unclear why it should be desirable for central banks to pursue such a goal. There are several plausible reasons why policymakers should prefer policies that do not require the *level* of short-term interest rates to be too variable. On the one hand, the zero nominal interest-rate floor (resulting from the availability of cash as a riskless, perfectly liquid zero-return asset) means that rates *cannot* be pushed below zero. This means that a policy consistent with a low average rate of inflation, which implies a low average level of nominal interest rates, cannot involve interest-rate reductions in response to deflationary shocks that are ever too large. And at the same time, high nominal interest rates always imply distortions, as resources are wasted on unnecessary efforts to economize on cash balances. Friedman (1969) stresses that this is a reason to prefer a regime with low average inflation, or even moderate deflation;

[1]See, *e.g.*, Cook and Hahn (1989), Rudebusch (1995), Goodhart (1996), and Sack (1998a, 1998b). The presence of lagged interest rates in estimated central-bank reaction functions (*e.g.*, Judd and Rudebusch, 1998; Sack, 1998b; or Clarida *et al.*, 1998, 2000) is often interpreted in terms of partial-adjustment dynamics for the gap between the actual level of the interest-rate instrument and a desired level that depends on variables such as current inflation and real activity.

but it is actually the level of nominal interest rates that directly determines the size of the distortion, and the argument applies as much to short-run variation in nominal interest rates as to their average level. Thus it is also desirable on this ground for policy not to raise nominal interest rates too much in response to inflationary shocks.[2] But while it makes a great deal of sense for a central bank to seek to achieve its other aims in a way consistent with as low as possible a variance of the *level* of short-run nominal rates, this in no way implies a direct concern with the variability of interest-rate *changes*.

Nonetheless, I shall argue that a concern with interest-rate smoothing on the part of a central bank can have desirable consequences. This is because such an objective can result in *history-dependent* central-bank behavior which, when anticipated by the private sector, can serve the bank's stabilization objectives through the effects upon current outcomes of anticipated future policy.

If the private sector is forward-looking, so that the effects of policy depend to an important extent on expectations regarding future policy, it is well known that discretionary minimization of a loss function representing true social objectives will generally lead to a (Markov) equilibrium which is suboptimal from the point of view of those same objectives. The reason is that a central bank that optimizes under discretion neglects at each point in time the effects that anticipations of its current actions have had upon equilibrium determination at earlier dates, as these past expectations can no longer be affected at the time that the bank decides how to act. Yet a different systematic pattern of conduct, justifying different expectations, might have achieve a better outcome in terms of the bank's own objectives.

As a consequence, a better outcome can often be obtained (in the Markov equilibrium associated with discretionary optimization) if the central bank is assigned an objective *different* from the true social objective; the problem of choosing an appropriate objective is sometimes called the problem of "optimal delegation". Famous examples include the pro-

[2] Both of these grounds for inclusion of a quadratic stabilization objective for a short-term nominal interest rate in the objective function that monetary policy should be designed to mininimize are analyzed in detail in Woodford (2002, chap. 6).

posal by Rogoff (1985) that a central banker should be chosen who is "conservative", in the sense of placing a greater weight on inflation stabilization than does the social loss function, or the proposal by King (1997) that the central bank should aim to stabilize the output gap around the level consistent with achieving its inflation target on average, even when a higher level of output relative to potential would be socially optimal. In both cases, modification of the central bank's objective can eliminate the bias toward higher-than-optimal average inflation resulting from discretionary policy when the central bank seeks to minimize the true social loss function.

In these examples, the central bank's assigned loss function is still a quadratic function of the same *target variables* as is the true social loss function; the assigned *target values* for these variables may be changed (as in the King proposal), or the relative weights on alternative stabilization objectives may be altered (as in the Rogoff proposal), but the variables that one wishes to stabilize are not changed. However, in general, there will also be advantages to introducing *new* target variables into the central bank's assigned loss function. This is the argument given here for assigning a central bank an interest-rate smoothing objective.

In particular, it will often be desirable to assign the central bank a loss function that involves lagged endogenous variables that are irrelevant to the computation of true social losses in a given period, as a way of causing policy to be *history-dependent*. In the case of any loss function that is a function of the same target variables as the true social loss function, and no others, policy must be *purely forward-looking* in a Markov equilibrium resulting from discretionary optimization by the bank. This means that at each point in time, policy (and the resulting values of the target variables) depend *only* on those aspects of the state of the world that define the set of feasible paths for the target variables from the present time onward. Yet in general, optimal policy is not purely forward-looking (Woodford, 2000). This is shown in the example considered in this paper through explicit computation of the optimal state-contingent evolution of the economy subject to the constraint that policy be purely forward-looking, and comparing this with the optimal state-contingent evolution when this constraint is relaxed.

It might seem that familiar "dynamic programming" arguments imply that optimal policy *should* be purely forward-looking. But such arguments apply to the optimal control of backward-looking systems of the kind considered in the engineering literature, and not to the control of a forward-system of the kind that a central bank is concerned with, as a result of private-sector optimization (under rational expectations). In a case of the latter sort, the evolution of the target variables depends not only the central bank's current actions, but also upon how the private sector *expects* monetary policy to be conducted in the future. It follows from this that a more desirable outcome may be achieved if it can be arranged for private sector expectations of future policy actions to adjust in an appropriate way in response to shocks. But if the private sector has rational expectations, it is not possible to arrange for expectations to respond to shocks in a desired way unless *subsequent policy* is affected by those past shocks in the way that one would like the private sector to anticipate. This will generally require that the central bank's behavior be history-dependent — that it *not* depend solely upon current conditions and the bank's current forecast of future conditions, but also upon past conditions, to which it was desirable for the private sector to be able to count upon the central bank's subsequent response.

The essential insight into why interest-rate smoothing by a central bank may be desirable is provided by a suggestion of Goodfriend (1991), also endorsed by Rudebusch (1995). Goodfriend argues that output and prices do not respond to daily fluctuations in the (overnight) federal funds rate, but only variations in *longer-term* interest rates. The Fed can thus achieve its stabilization goals only insofar as its actions affect these longer-term rates. But long rates should be determined by market expectations of future short rates. Hence an effective response by the Fed to inflationary pressures, say, requires that the private sector be able to believe that the entire *future path* of short rates has changed. A policy that maintains interest rates at a higher level for a period of time once they are raised — or even following initial small interest-rate changes by further changes in the same direction, in the absence of a change in conditions that makes this unnecessary — is one that, if understood by the private sector, will allow a moderate adjustment of current short rates to have a significant effect on long rates. Such a policy offers the prospect of significant effects of central bank policy upon aggregate demand, without requiring excessively volatile short-term interest rates.

This paper offers a formal analysis of the benefits of inertial behavior along essentially the lines sketched by Goodfriend, in the context of a simple, and now rather standard, forward-looking macro model, with clear foundations in optimizing private-sector behavior. Section 2 presents the model, poses the problem of optimal monetary policy, and derives the optimal state-contingent responses of endogenous variables, including nominal interest rates, to shocks under an optimal regime.[3] Section 3 highlights the need for policy to be history-dependent, by contrasting the fully optimal responses with the optimal responses subject to the constraint that policy be non-inertial. Section 4 then considers the optimal delegation problem, showing that it is desirable for the central bank's loss function to include an interest-rate smoothing objective, even though the true social loss function does not. Section 5 concludes.

2 Optimal Responses to Fluctuations in the Natural Rate of Interest

In order to illustrate more concretely the themes of the preceding discussion, it is useful to introduce a simple optimizing model of inflation and output determination under alternative monetary policies, where monetary policy is specified in terms of a feedback rule for a short-term nominal interest rate instrument. The model is similar, if not identical, to the small forward-looking models used in a number of recent analyses of monetary policy rules, including Kerr and King (1996), Bernanke and Woodford (1997), Rotemberg and Woodford (1997, 1999), McCallum and Nelson (1999a, 1999b), and Clarida *et al.* (1999). As is explained in Woodford (2002, chap. 4), the model's equations can be derived as log-linear

[3] The analysis of optimal state-contingent policy follows Woodford (1999a), which also contrasts optimal state-contingent policy under commitment with the Markov equilibrium associated with discretionary minimization of the true social loss function. The analysis of optimal forward-looking policy is new here, as is the treatment of the optimal delegation problem.

approximations to the equilibrium conditions of a simple intertemporal general equilibrium model with sticky prices. While the model is quite simple, it incorporates forward-looking private sector behavior in three respects, each of which is surely of considerable importance in reality, and would therefore also be present in some roughly similar form in any realistic model.

The model's two key equations are an *intertemporal IS equation* (or Euler equation for the intertemporal allocation of private expenditure) of the form

$$x_t = E_t x_{t+1} \sigma[i_t - E_t \pi_{t+1} - r_t^n], \tag{2.1}$$

and an *aggregate supply equation* of the form

$$\pi_t = \kappa x_t + \beta E_t \pi_{t+1}, \tag{2.2}$$

where x_t is the deviation of the log of real output from its natural rate, π_t is the rate of inflation (first difference of the log of the price level), and i_t is the deviation of the short-term nominal interest rate (the central bank's policy instrument) from its steady-state value in the case of zero inflation and steady output growth. These two equations, together with a rule for the central bank's interest-rate policy, determine the equilibrium evolution of the three endogenous variables π_t, x_t, and i_t.

The exogenous disturbance r_t^n corresponds to Wicksell's "natural rate of interest", the interest rate (determined by purely *real* factors) that would represent the equilibrium real rate of return under flexible prices, and that corresponds to the nominal interest rate consistent with an equilibrium with constant prices.[4] In our simple model, disturbances to the natural rate represent a useful summary statistic for *all* non-monetary disturbances that matter for the determination of inflation and the output gap, for no other disturbance term enters either equation (2.2) or (2.1), once they are written in terms of the output gap x_t as opposed to the level of output. Hence if, as we shall suppose, the goals of stabilization policy can be described in terms of the paths of the inflation rate, the output gap, and interest rates alone,

[4]See Woodford (2002, chap. 4) for further discussion of the importance of this concept for monetary policy.

then the problem of optimal monetary policy may be formulated as a problem of the optimal response to disturbances to the natural rate of interest.

We shall assume that the objective of monetary policy is to minimize the expected value of a loss criterion of the form

$$W = E_0 \left\{ \sum_{t=0}^{\infty} \beta^t L_t \right\}, \tag{2.3}$$

where $0 < \beta < 1$ is a discount factor, and the loss each period is given by

$$L_t = \pi_t^2 + \lambda_x x_t^2 + \lambda_i i_t^2, \tag{2.4}$$

for some weights $\lambda_x, \lambda_i > 0$. The assumed form of (2.4) is relatively conventional, except that an interest-rate stabilization objective is included, for either or both of the reasons discussed in the introduction.[5] Note that an interest-rate "smoothing" objective is *not* assumed. The "target values" of each of the target variables are assumed to be those associated with a steady state with zero inflation in the absence of real disturbances. Thus target values are not assumed that result in any bias in the *average* rate of inflation, or in the average values of other state variables, under discretionary policy; the reason for assigning the central bank a loss function other than the social loss function has solely to do with the sub-optimality of the dynamic responses to shocks under discretionary minimization of (2.4).[6]

I begin by characterizing the dynamic responses to shocks that would occur under an optimal commitment, and comparing these to the consequences of discretionary policy when the central bank seeks to minimize the true social loss function. Our problem is to choose stochastic processes π_t, x_t, and i_t — specifying each of these variables as a function of a random state I_t that includes not only the complete history of the exogenous disturbances $(r_t^n, r_{t-1}^n, \ldots, r_0^n)$, but also all public information at date t about the future evolution of the natural rate — in order to minimize the criterion defined by (2.3) and (2.4), subject to

[5] A welfare-theoretic justification for this objective function, in the context of the microfoundations of the structural model behind equations (2.1) – (2.2), is presented in Woodford (2002, chap. 6).

[6] Allowing for different target values would affect only the optimal long-run average values of the endogenous variables, and not the nature of optimal responses to shocks. Since our concern here is with stabilization issues, we abstract from any complications that may be involved in bringing about a desirable long-run average state.

the constraint that the processes satisfy equilibrium conditions (2.2) and (2.1) at all dates $t \geq 0$. We shall imagine in this calculation that a policymaker can choose the entire future (state-contingent) evolutions of these variables, once and for all, at date zero. Once this benchmark has been characterized, we can then consider the problem of implementation of such an optimal plan.

2.1 Characterization of the Optimal Plan

This sort of linear-quadratic optimization problem can be treated using methods that are by now familiar.[7] It is useful to write a Lagrangian of the form[8]

$$E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \{(1/2) L_t + \phi_{1t}[x_t - x_{t+1} + \sigma(i_t - \pi_{t+1} - r_t^n)] + \phi_{2t}[\pi_t - \kappa x_t - \beta \pi_{t+1}]\} \right\}. \quad (2.5)$$

An optimal plan then must satisfy the first-order conditions

$$\pi_t - \beta^{-1}\sigma\phi_{1t-1} + \phi_{2t} - \phi_{2t-1} = 0, \quad (2.6)$$

$$\lambda_x x_t + \phi_{1t} - \beta^{-1}\phi_{1t-1} - \kappa\phi_{2t} = 0, \quad (2.7)$$

$$\lambda_i i_t + \sigma\phi_{1t} = 0, \quad (2.8)$$

obtained by differentiating the Lagrangian with respect to π_t, x_t, and i_t respectively. Each of conditions (2.6) – (2.8) must hold at each date $t \geq 1$, and the same conditions also must hold at date $t = 0$, where however one adds the stipulation that

$$\phi_{1,-1} = \phi_{2,-1} = 0. \quad (2.9)$$

We may omit consideration of the transversality conditions, as we shall consider only bounded solutions to these equations, which necessarily satisfy the transversality conditions. A (bounded) optimal plan is then a set of bounded processes $\{\pi_t, x_t, r_t, \phi_{1t}, \phi_{2t}\}$ for dates $t \geq 0$, that satisfy (2.2), (2.1), and (2.6) – (2.8) at all of these dates, consistent with the initial conditions (2.9).

[7] See, e.g., Backus and Driffill (1986) for treatment of a general linear-quadratic problem. See Woodford (1999b) for further discussion of the optimal plan for this model.

[8] Note that conditional expectations are dropped from the way in which the constraints are written inside the square brackets, because the expectation E_0 at the front of the entire expression makes them redundant.

If the optimal plan is bounded (which is the only case in which our log-linear approximations to the model structural equations and our quadratic approximation to the social welfare function can be expected to accurately characterize it), one can show that this system of equations has a unique bounded solution of the form

$$z_t = G\phi_{t-1} - H\sum_{j=0}^{\infty} \tilde{A}^{-(j+1)} a E_t r^n_{t+j}, \tag{2.10}$$

where $z'_t \equiv [\pi_t \; x_t \; i_t]$ is the vector of endogenous variables and $\phi'_t \equiv [\phi_{1t} \; \phi_{2t}]$ is the vector of Lagrange multipliers, for certain matrices of coefficients that depend on the model parameters. The eigenvalues of the matrix $\tilde{A}$ lie outside the unit circle, so that the infinite sum converges in the case of any bounded process for the natural rate. The corresponding solution for the Lagrange multipliers is of the form

$$\phi_t = N\phi_{t-1} - C\sum_{j=0}^{\infty} \tilde{A}^{-(j+1)} a E_t r^n_{t+j}, \tag{2.11}$$

where the eigenvalues of the matrix N lie inside the unit circle. This property of the matrix N implies that (2.11) defines a bounded stochastic process for the multipliers ϕ_t, given any bounded process for the natural rate.

It is obvious that such an optimal plan will, in general, not be *time consistent*, in the sense discussed by Kydland and Prescott (1977). For a policymaker that solves a corresponding problem starting at some date $T > 0$ will choose processes for dates $t \geq T$ that satisfy equations (2.10) – (2.11), but starting from initial conditions

$$\phi_{1,T-1} = \phi_{2,T-1} = 0$$

corresponding to (2.9). Yet these last conditions will, in general, not be satisfied by the optimal plan chosen at date zero, according to solution (2.11) for the evolution of the Lagrange multipliers. This is why discretionary optimization leads to a different equilibrium outcome than the one characterized here.

The presence of the lagged Lagrange multipliers in (2.10) is also the reason why optimal policy cannot be implemented through any purely forward-looking procedure. These terms imply that the endogenous variables at date t – and in particular, the central bank's setting of the interest rate at that date – should not depend solely upon current and forecasted future values of the natural rate of interest. They should also depend upon the predetermined state variables ϕ_{t-1}, which represent an additional source of inertia in optimal monetary policy, independent of any inertia that may be present in the exogenous disturbance process r_t^n. The additional terms represent the way in which policy should deviate from what would be judged optimal simply taking into account the current outlook for the economy, in order to follow through upon *commitments* made at an earlier date. It is the desirability of the central bank's being able to credibly commit itself in this way that makes it desirable for monetary policy to be somewhat inertial.

The extent to which these equations imply inertial behavior of the nominal interest rate can be clarified by writing a law of motion for the interest rate that makes no reference to the Lagrange multipliers. Let us suppose that the relevant information at date t about the future evolution of the natural rate can be summarized by an exogenous state vector s_t, with law of motion

$$s_{t+1} = Ts_t + \epsilon_{t+1}, \tag{2.12}$$

where ϵ_{t+1} is a vector of exogenous disturbances unforecastable at t, and let the natural rate be given by some linear function of these states,

$$r_t^n = k's_t. \tag{2.13}$$

Equation (2.11) can then be written in the form

$$\phi_t = N\phi_{t-1} + ns_t, \tag{2.14}$$

for a certain matrix of coefficients n.

The endogenous variable ϕ_{2t} can then be eliminated from the system of equations (2.14), yielding an equation with instead two lags of ϕ_{1t}. Then using (2.8) to substitute out ϕ_{1t}, we obtain the law of motion

$$Q(L)i_t = R(L)s_t \tag{2.15}$$

for the nominal interest rate, where

$$Q(L) \equiv \det[I - NL], \qquad R(L) \equiv -(\lambda_r \sigma)^{-1}[n_1' + (N_{12}n_2' - N_{22}n_1')L]. \tag{2.16}$$

The degree of persistence in the *intrinsic* dynamics of the nominal interest rate under the optimal plan, unrelated to any persistence in the fluctuations in the exogenous states s_t is determined by the roots μ_i of the characteristic equation

$$Q(\mu) = 0,$$

which roots are just the eigenvalues of the matrix N. These roots are determined by factors independent of the dynamics of the exogenous disturbances. Thus it may be optimal for nominal interest rates to exhibit a great deal of persistence, regardless of the degree of persistence of the fluctuations in the natural rate.

2.2 A Simple Limiting Case

The extent to which the equations just derived imply behavior that might appear to involve interest-rate "smoothing" can be clarified by considering a limiting case, in which a closed-form solution is possible. This is the limiting case in which the value of the parameter κ (the slope of the "short-run Phillips curve") approaches zero. In this limit, variations in output relative to potential cause no change in the level of real marginal cost, and firms accordingly have no reason to change their prices at any time. Hence $\pi_t = 0$ at all times, regardless of monetary policy. We shall assume that the values of all other parameters are unchanged.

In this limiting case, the $\kappa\phi_{2t}$ term in (2.7) can be neglected, so that it becomes possible to solve for the variables x_t, i_t, and ϕ_{1t} using only equations (2.1), (2.7), and (2.8). We can furthermore use two of these equations to eliminate x_t and ϕ_{1t}, leaving the equation

$$E_t i_{t+1} - \left(1 + \beta^{-1} + \frac{\lambda_x}{\lambda_i}\sigma^2\right) i_t + \beta^{-1} i_{t-1} = -\frac{\lambda_x}{\lambda_i}\sigma^2 r_t^n \tag{2.17}$$

for the optimal interest-rate dynamics.

One observes that the characteristic equation associated with (2.17) necessarily has two real roots, satisfying

$$0 < \mu_1 < 1 < \beta^{-1} < \mu_2,$$

and that $\mu_2 = (\beta\mu_1)^{-1}$. Because exactly one root is inside the unit circle, (2.17) has a unique bounded solution, given by

$$i_t = \mu_1 i_{t-1} + \sigma^2(\lambda_x/\lambda_i)\sum_{j=0}^{\infty}\mu_2^{-(j+1)}E_t r_{t+j}^n. \tag{2.18}$$

This gives us a law of motion of the form (2.15), but in this limiting case, a representation is possible in which $Q(L)$ is only of first order, and $R(L)$ is a constant (there are no lags at all). In fact, one can easily show that (2.18) is a partial-adjustment equation of the form

$$i_t = \theta i_{t-1} + (1-\theta)\bar{\imath}_t, \tag{2.19}$$

where the inertia coefficient $\theta = \mu_1$, and the time-varying interest-rate "target" is given by[9]

$$\bar{\imath}_t = (1-\mu_2^{-1})\sum_{j=0}^{\infty}\mu_2^{-j}E_t r_{t+j}^n. \tag{2.20}$$

Thus the optimal interest-rate dynamics are described by partial adjustment toward a moving average of current and expected future natural rates of interest.

In the case that the natural rate is a simple first-order autoregressive process,

$$r_{t+1}^n = \rho r_t^n + \epsilon_{t+1} \tag{2.21}$$

for some $0 \le \rho < 1$, the target rate is just a function of the current natural rate of interest, although (because of expected mean-reversion of the natural rate in the future) it varies less than does the natural rate itself. Specifically, we have

$$\bar{\imath}_t = k r_t^n, \tag{2.22}$$

where $k \equiv (\mu_2 - 1)/(\mu_2 - \rho)$, so that $0 < k < 1$. If the fluctuations in the natural rate are largely transitory, the elasticity k may be quite small, though it is any event necessarily greater than $1-\beta$. If the fluctuations in the natural rate are nearly a random walk (ρ is near one), the elasticity k instead approaches one. In this case, interest rates eventually change by nearly as much as the (nearly permanent) change that has occurred in the natural rate;

[9]Here we use the fact that $\sigma^2\lambda_x/\lambda_i$ is equal to $(1-\mu_1)(\mu_2-1)$.

but even in this case, the change in the level of nominal interest rates is delayed. As a result, an innovation in the natural rate is followed by a series of interest rate changes in the same direction, as in the characterizations of actual central-bank behavior by Rudebusch (1995) and Goodhart (1996).

While this partial-adjustment representation of optimal interest-rate dynamics is only exactly correct in an unrealistic limiting case, it provides considerable insight into the optimal interest-rate responses in more realistic cases. This is shown through numerical analysis of a case with $\kappa > 0$ in the next section.

3 The Value of Interest-Rate Inertia

A central theme of this paper is the desirability of assigning to the central bank an objective which makes lagged nominal interest rates relevant to the bank's evaluation of possible current states. In order to show the need for an objective of that form, it is appropriate to consider the degree to which responses similar to those associated with the optimal plan can be achieved through choice of a suitable central-bank loss function that does *not* depend on any such lagged variables. To consider this question, we need not consider the Markov equilibria associated with alternative central-bank loss functions at all. Instead, we may simply consider what the best possible equilibrium would be like that can be achieved by *any* purely forward-looking decision procedure. To the extent that that pattern of responses to shocks — the optimal non-inertial plan — remains substantially inferior to the optimal plan in the absence of such a restriction, there are clear benefits to the introduction of history-dependence of the proper sort into the central bank's decision procedures.

3.1 The Optimal Non-Inertial Plan

By a purely forward-looking procedure we mean one that makes the bank's policy decision a function solely of the set of possible equilibrium paths for the economy from the present date onward. In a Markov equilibrium associated with any such procedure, the endogenous variables must be functions *only* of the state vector s_t. Hence we may proceed by optimizing over possible state-contingent evolutions of the economy that satisfy this restriction. We call the optimal pattern of responses to disturbances subject to this restriction the *optimal non-inertial plan.*

We shall simplify by here considering only the case in which the natural rate evolves according to (2.21). In this case, non-inertial plans are those in which each endogenous variable y_t is a time-invariant linear function[10] of the current natural rate of interest,

$$y_t = f_y r_t^n. \tag{3.1}$$

Substituting the representation (3.1) for each of the variables $y = \pi, x, i$ into (2.2) – (2.1), we find that feasible non-inertial plans correspond to coefficients f_y that satisfy

$$(1 - \beta\rho) f_\pi = \kappa f_x, \tag{3.2}$$

$$(1 - \rho) f_x = -\sigma(f_i - 1 - \rho f_\pi). \tag{3.3}$$

Among these plans, we seek the one that minimizes $E[W]$, the unconditional expectation of (2.3), taking the unconditional expectation over the stationary distribution of possible initial exogenous states r_0^n. We take this unconditional expectation so that our choice of the optimal plan does not depend upon the state that the economy happens to be in at the time that the commitment is made.[11]

Given our restriction to non-inertial plans, minimization of $E[W]$ is equivalent to minimization of $E[L]$, the unconditional expectation of the period loss (2.4). Thus we seek to

[10] We could also allow for a non-zero constant term in (3.1), but it is easily seen that in the present example the optimal long-run value of each of the variables is zero.

[11] If instead we were to minimize W, conditioning upon the state of the economy at the time of choice as in Levine (1991), the exact non-inertial plan that would be chosen would in general depend upon that state. This is because the choice of how the variables should depend upon r^n would be distorted by the desire to obtain an initial (unexpected) inflation, without creating expectations of a similar rate of inflation on average in the future; this could be done by exploiting the fact that r_0^n is known to have a value different from its expected value in the future (which is near zero eventually). By instead defining the optimal non-inertial policy as we do, we obtain a unique policy of this kind, and associated unique values for statistics such as the variability of inflation under this policy. Also, under our definition, unlike Levine's, the optimal "simple" plan is *certainty-equivalent,* just like the fully optimal plan and the time-consistent optimizing plan. That is, the optimal long-run average values of the variables are the same as for a certainty problem, while the optimal response coefficients f_y are independent of the variance of the disturbance process.

minimize

$$E[L] = [f_\pi^2 + \lambda_x f_x^2 + \lambda_i f_i^2]\mathrm{var}(r^n), \tag{3.4}$$

subject to the linear constraints (3.2) – (3.3).The first-order conditions for optimal choice of the f_y imply that

$$f_i = \frac{\kappa(1-\beta\rho)^{-1} f_\pi + \lambda_x f_x}{[(1-\rho)\sigma^{-1} - \rho\kappa(1-\beta\rho)^{-1}]\lambda_i}. \tag{3.5}$$

This condition along with (3.2) – (3.3) determines the optimal response coefficients.

3.2 A Numerical Example

To consider what degree of interest-rate inertia might be optimal in practice, it is useful to consider a numerical example, "calibrated" to match certain quantitative features of the Rotemberg and Woodford (1997, 1999) analysis of optimal monetary policy for the U.S. economy.[12] The numerical values that we shall use are given in Table 1. We assume an AR(1) process for the fluctuations in the natural rate of interest as in (2.21), so that we need only calibrate a single parameter ρ. The value of ρ chosen here implies a degree of concern for reduction of interest-rate variability similar to that obtained by Rotemberg and Woodford in their estimated model, though their estimate disturbance processes are more complex.

For the parameter values in Table 1, the matrix N is given by

$$N = \begin{bmatrix} .4611 & .0007 \\ -.7743 & .6538 \end{bmatrix},$$

and its eigenvalues are found to be approximately .65 and .46. Both of these are substantial positive quantities, suggesting that once interest rates are perturbed in response to some shock, it should take several quarters for them to be restored to nearly their normal level, even if the shock is completely transitory.

Figure 1 illustrates this by showing the optimal responses of inflation, the output gap, and the short-term nominal interest rate to a unit positive innovation ϵ_t in the natural-rate process. The natural rate of interest is made higher by $(0.35)^j$ percentage points in quarter

[12]Details of the justification for this calibration are set out in Woodford (1999b).

$t+j$ by this disturbance. The figure shows the dynamic responses of the endogenous variables in quarters $t+j$, for $j=0$ through 10, both under both the optimal plan and the optimal non-inertial plan.

Under the optimal non-inertial plan (dash-dot lines in the figure), the nominal interest rate is raised in response to the real disturbance, but only by about two-thirds the amount of the increase in the natural rate. As a result, monetary policy does not fully offset the inflationary pressure created by the disturbance, and both inflation and the output gap increase;[13] this is optimal within the class of non-inertial policies because it involves less interest-rate variability than would be required to completely stabilize inflation and the output gap (by perfectly tracking the variations in the natural rate). Because the policy is non-inertial, inflation, the output gap, and the nominal interest rate all decay back to their long-run average values at exactly the same rate as the real disturbance itself decays, *i.e.*, in proportion to $(0.35)^j$.

Under the fully optimal plan (solid lines in the figure), instead, the nominal interest rate is raised by less at the time of the shock. But the increase is more *persistent* than is the disturbance to the natural rate of interest, so that policy is expected to be tighter under this policy than under the optimal non-inertial plan from quarter $t+2$ onward. Thus interest rates are more inertial under the optimal plan, both in the sense that the central bank is slow to raise rates when the natural rate unexpectedly increases, and also in the sense that it is slow to bring them back down when the natural rate returns to its normal level.

The advantages of more inertial adjustment of the interest rate can be seen in the other two panels of the figure. Despite the gentler immediate interest-rate response, the initial increase in the output gap is no greater under this policy, because spending is restrained by the anticipation of tight policy farther into the future; and the output gap returns to its normal level much more rapidly under this policy, as interest rates are kept relatively high despite the decay of the natural rate back toward its normal level. Because the output

[13] See Woodford (2002, chap. 4) for discussion of the effects on inflation and output of fluctuations in the natural rate of interest in a model like this one.

stimulus is expected to be short-lived (the output gap is actually expected to undershoot its normal level by the quarter after the shock), the increase in inflation resulting from the shock is minimal under the optimal policy. Thus monetary policy is as successful at stabilizing inflation and the output gap under this policy as under the optimal non-inertial plan (actually, somewhat more successful overall), yet the desired result is achieved with much less variability of interest rates, owing to a commitment to adjust them in a smoother way.

Statistics regarding the variability of the various series under the two plans are reported in Table 2. Here independent drawings from the same distribution of shocks ϵ_t are assumed to occur each period, and infinite-horizon stochastic equilibria are computed under each policy. The measure of variability reported for each variable z_t is

$$V[z] \equiv E[E_0\{(1-\beta)\sum_{t=0}^{\infty}\beta^t \hat{z}_t^2\}], \tag{3.6}$$

where the outer (unconditional) expectation is over possible initial states of the economy r_0^n at the time that policy is chosen, computed using the stationary distribution associated with the exogenous process (2.21) for the natural rate. The unconditional expectation allows us a measure that is independent of the economy's initial state. Except for the discounting, $E[z]$ corresponds to the unconditional variance of z_t, and in the case of non-inertial plans, it is equal to the unconditional variance even though $\beta < 1$. In the case of the optimal plan, the discounted measure is of greater interest, because our loss measure $E[W]$ — the unconditional expectation of (2.3), integrating over the stationary distribution for the initial state r_0^n — is in that case just a weighted sum of the previous three columns. For purposes of comparison, the table also presents statistics for the Markov equilibrium resulting from discretionary minimization of the true social loss function.[14]

The table shows that for the calibrated parameter values, there is a substantial gain from commitment to an inertial policy, relative to the best possible non-inertial policy. This is primarily due to the lower volatility of nominal interest rates under the optimal plan ($V[i]$

[14]See Woodford (1999a) for further discussion of the differences between the optimal plan and the outcome of discretionary policy.

is reduced by more than 70 percent), although the central bank's other stabilization goals are better served as well ($V[\pi] + \lambda_x V[x]$ is also reduced).

Visual inspection of the optimal interest-rate dynamics in Figure 1 suggests that partial adjustment of the nominal adjustment toward a level determined by the current natural rate of interest, just as in the limiting case analyzed in section 2.2, gives a reasonable approximation to optimal interest-rate dynamics. This is because the element N_{12} of the matrix N is quite small. In the case that N_{12} were exactly equal to zero, $Q(L)$ and $R(L)$ would contain the common factor $(1 - N_{22}L)$. Removing this factor from both sides of (2.15), one would obtain interest-rate dynamics of the form (2.19), prescribing partial adjustment toward a time-varying "target" interest rate equal to

$$\bar{\imath}_t = -(\sigma/\lambda_i)(1 - N_{11})^{-1} n_1' s_t, \tag{3.7}$$

with an inertia coefficient of $\theta = N_{11}$. In our numerical example, the target rate (3.7) would be given by $\bar{\imath}_t = .52 r_t^n$, while the inertia coefficient is equal to $\theta = .46$, indicating that interest rates should be adjusted only about half of the way toward the current target level (implied by the natural rate) within the quarter.

4 Advantages of a Central Bank Smoothing Objective

We turn now to the question of the type of objective that should be assigned to the central bank in order to bring about equilibrium responses to shocks similar to those associated with the optimal plan.[15] We thus wish to address what is sometimes called the problem of *optimal delegation* of authority to conduct monetary policy. In such an analysis, one asks what objective the central bank should be charged with, understanding that the details of the pursuit of the goal on a day-to-day basis should then be left to the bank, and expecting that the bank will then act as a discretionary minimizer of its assigned loss function.[16] Our results in the previous section suggest that equilibrium responses to shocks can be improved if the

[15]This is not, of course, the only way that one might seek to bring about the desired type of equilibrium responses. See Woodford (1999b), Svensson and Woodford (1999) and Giannoni and Woodford (2002) for discussion of alternative approaches in the context of similar models.

central bank is assigned an interest-rate *smoothing* objective, leading to partial-adjustment dynamics for the bank's interest-rate instrument, even though the lagged nominal interest rate is irrelevant to both the true social objective function (2.4) and the structural equations of our model.

4.1 Markov Equilibrium with a Smoothing Objective

Let us consider the consequences of delegating the conduct of monetary policy to a central banker that is expected to seek to minimize the expected value of a criterion of the form (2.3), where however (2.4) is replaced by a function of the form

$$L_t^{cb} = \pi_t^2 + \hat{\lambda}_x x_t^2 + \hat{\lambda}_i i_t^2 + \lambda_\Delta (i_t - i_{t-1})^2. \tag{4.1}$$

Here we allow the weights $\hat{\lambda}_x, \hat{\lambda}_i$ to differ from the weights λ_x, λ_i associated with the true social loss function. We also allow for the existence of a term that penalizes interest-rate *changes,* not present in the true social loss function (2.4).

The time-consistent optimizing plan associated with such a loss function can be derived using familiar methods, expounded for example in Soderlind (1998). Note that the presence of a term involving the lagged interest rate in the period loss function (4.1) means that in a Markov equilibrium, outcomes will depend on the lagged interest rate. In such an equilibrium, the central bank's value function in period t is given by a time-invariant function $V(i_{t-1}; r_t^n)$.[17]

Standard dynamic programming reasoning implies that the value function must satisfy the Bellman equation

$$\begin{aligned} V(i_{t-1}; r_t^n) \quad = \quad & \min_{(i_t, \pi_t, x_t)} E_t \left\{ \frac{1}{2} [\pi_t^2 + \hat{\lambda}_x x_t^2 + \hat{\lambda}_i i_t^2 + \lambda_\Delta (i_t - i_{t-1})^2] \right. \\ & \left. + \beta V(i_t; r_{t+1}^n) \right\}, \end{aligned} \tag{4.2}$$

[16] Alternatively, the question is sometimes framed as asking what *type* of central banker (or monetary policy committee) should be appointed, taking it as given that the central banker will seek to maximize the good as he or she personally conceives it, again optimizing under discretion.

[17] Here we simplify by assuming that the natural rate of interest is itself Markovian, with law of motion (2.21), though we could easily generalize our results to allow for more complicated linear state-space models.

where the minimization is subject to the constraints

$$\begin{aligned}\pi_t &= \kappa x_t + \beta E_t[\pi(i_t; r_{t+1}^n)],\\ x_t &= E_t[x(i_t; r_{t+1}^n) - \sigma(i_t - r_t^n - \pi(i_t; r_{t+1}^n))].\end{aligned}$$

Here the functions $\pi(i_t; r_{t+1}^n)$, $x(i_t; r_{t+1}^n)$ describe the equilibrium that the central bank *expects* to occur in period $t+1$, conditional upon the exogenous state r_{t+1}^n. This represents the consequence of discretionary policy at that date and later, that the current central banker regards him or herself as unable to change.

We shall furthermore restrict attention to solutions of the Bellman equation in which the value function is a *quadratic* function of its arguments, and the solution functions for i, π, and x are each linear functions of their arguments. The solution functions can accordingly be written

$$i(i_{t-1}; r_t^n) = i_i i_{t-1} + i_n r_t^n, \tag{4.3}$$

$$\pi(i_{t-1}; r_t^n) = \pi_i i_{t-1} + \pi_n r_t^n, \tag{4.4}$$

$$x(i_{t-1}; r_t^n) = x_i i_{t-1} + x_n r_t^n, \tag{4.5}$$

where i_i, i_n, and so on are constant coefficients to be determined by solving a fixed-point problem. Note also that differentiation of (4.2) using the envelope theorem implies that the partial derivative of the value function with respect to its first argument must satisfy

$$V_1(i_{t-1}; r_t^n) = \lambda_\Delta[i_{t-1} - i(i_{t-1}; r_t^n)]. \tag{4.6}$$

Thus linearity of the solution function i guarantees the linearity of the function V_1 as well.

We turn now to the fixed-point problem for the constant coefficients in the solution functions. First of all, substitution of the assumed linear solution functions into the two constraints following (4.2), and using

$$E_t r_{t+1}^n = \rho r_t^n, \tag{4.7}$$

allows us to solve for x_t and π_t as linear functions of i_t and r_t^n. (Let the coefficients on i_t in the solutions for x_t and π_t be denoted X_i and Π_i respectively. These coefficients are themselves linear combinations of the coefficients x_i and π_i introduced in (4.4) – (4.5).) Requiring the solution functions defined in (4.3) – (4.5) to satisfy these linear restrictions yields a set of four nonlinear restrictions on the coefficients x_i, x_n and so on.

Substituting these solutions for x_t and π_t into the right-hand side of (4.2), the expression inside the minimization operator can be written as a function of i_t and r_t^n. This expression is quadratic in i_t, and so it achieves a minimum if and only if both first and second-order conditions are satisfied. Substituting (4.6) for the derivative of the value function, the first-order condition may be written as

$$\Pi_i \pi_t + \hat{\lambda}_x X_i x_t + \hat{\lambda}_i i_t + \lambda_\Delta (i_t - i_{t-1}) + \beta \lambda_\Delta (i_t - E_t i_{t+1}) = 0, \tag{4.8}$$

while the second-order condition is

$$\Omega \equiv \Pi_i^2 + \hat{\lambda}_x X_i^2 + \hat{\lambda}_i + \lambda_\Delta + \beta \lambda_\Delta (1 - i_i) \geq 0. \tag{4.9}$$

Requiring that the solutions defined in (4.3) – (4.5) always satisfy the linear equation (4.8) gives us another set of two nonlinear restrictions on the constant coefficients of the solution functions. We thus have a set of six nonlinear equations to solve for the six coefficients of equations (4.3) – (4.5). A set of coefficients satisfying these equations, and also satisfying the inequality (4.9), represent a linear Markov equilibrium for the central bank objective (4.1).

We shall as usual be interested solely in the case of a *stationary* equilibrium, so that fluctuations in i_t, π_t and x_t are bounded if the fluctuations in r_t^n are bounded.[18] It can be shown that this is true if and only if

$$|i_i| < 1. \tag{4.10}$$

Thus we are interested in solutions to the six nonlinear equations that satisfy both inequalities (4.9) and (4.10).

[18] Once again, this is the case in which our linear-quadratic approximations are justifiable in terms of a Taylor series approximation to the exact conditions associated with private-sector optimization, in the case of small enough exogenous disturbances.

In the case that $\hat{\lambda}_x, \hat{\lambda}_i, \lambda_\Delta \geq 0$, it will be observed that (4.10) implies condition (4.9), so that we need not concern ourselves with the second-order condition in that case. However, non-negativity of these weights in the central-bank objective is not *necessary* for convexity of the central bank's optimization problem, and it is of some interest to consider delegation to a central banker with a *negative* weight on some term. Such preferences need not result in a violation of convexity; the second-order condition will still be satisfied as long as the other four terms together outweigh the negative $\hat{\lambda}_i$ term.

We turn now to the question of what loss function the central bank should be assigned to minimize, if a Markov equilibrium of this kind is assumed to result from delegation of such an objective. We first note that setting $\lambda_\Delta > 0$ results in inertial interest-rate responses to fluctuations in the natural rate. For the first-order condition (4.8) implies that

$$\Omega i_i = \lambda_\Delta$$

in any solution; thus if $\lambda_\Delta > 0$, both Ω and i_i must be non-zero, and of the same sign. The second-order condition (4.9) then implies that in any equilibrium, both quantities must be positive. It follows that in any stationary equilibrium,

$$0 < i_i < 1, \tag{4.11}$$

so that the law of motion (4.3) for the nominal interest rate implies partial adjustment toward a time-varying target that is a linear function of the current natural rate of interest.

One may wonder whether it is possible to choose the weights in the central bank's loss function so as to completely eliminate the distortions associated with discretion, and achieve the same responses as under an optimal commitment. It should be immediately apparent that it is not in general possible to achieve this outcome exactly. For we have shown in section 2.1 that the optimal interest-rate dynamics have a representation of the form (2.15), where in general $Q(L)$ is of second order and $R(L)$ is of first order; thus they generally do not take a form as simple as (4.3). Nonetheless, we can show that exact implementation of the optimal plan is possible at least in a limiting case. And we can also show that it is possible to achieve a pattern of responses nearly as good as the optimal plan, in the calibrated numerical example of section 3.2. These points are taken up in succession in the next two subsections.

4.2 Optimal Delegation in a Limiting Case

Here we consider again the limiting case with $\kappa = 0$ taken up in section 2.2. We have shown there that in this special case, the optimal interest-rate and output dynamics do take the form given by (4.3) and (4.5) Hence we may ask whether it is possible to choose the weights in (4.1) so that the Markov equilibrium just characterized involves the optimal dynamics described in section 2.2. Note that only the ratios of the weights in the policy objective, $\hat{\lambda}_i/\hat{\lambda}_x$ and $\lambda_\Delta/\hat{\lambda}_x$, rather than the absolute size of the three weights, matter for this calculation. (Inflation variations are negligible under any policy regime, so the relative weight on inflation variability no longer matters.) Hence we may, without loss of generality, suppose that $\hat{\lambda}_x = \lambda_x$, the weight in the true social objective function.

Starting from the linear dynamics obtained in section 2.2, we need only check whether these can be consistent with (4.8) and (4.9) for some weights in the assigned loss function. Under the above choice of $\hat{\lambda}_x$, we find that there are unique values of $\hat{\lambda}_i$ and λ_Δ that render the optimal equilibrium responses consistent with (4.8). These are given by

$$\lambda_\Delta = \lambda_i \frac{\lambda_i(\beta^{-1} - \mu_1) + \lambda_x \sigma^2}{(1 - \beta\rho\mu_1)\beta\lambda_x\sigma^2} > 0, \tag{4.12}$$

$$\hat{\lambda}_i = -(1 - \beta\rho)(1 - \beta\mu_1)\lambda_\Delta < 0, \tag{4.13}$$

where μ_1 is again the smaller root of the characteristic equation associated with (2.17).

While one finds that the kind of partial-adjustment interest-rate dynamics associated with the optimal plan do require $\lambda_\Delta > 0$, as conjectured, one finds that they cannot be exactly matched through delegation to a central banker with discretion unless in addition $\hat{\lambda}_i < 0$. But as noted earlier, a negative value for $\hat{\lambda}_i$ does not necessarily imply violation of the convexity condition (4.9) needed for central-bank optimization. In fact, we have shown above that the second-order condition holds in the case of any solution to the first-order condition with $\lambda_\Delta > 0$ and $i_i > 0$. As we have shown in section 2.2 that $i_i > 0$, and (4.12) implies that $\lambda_\Delta > 0$, the above assumed central-bank objective does result in a convex optimization problem for the central bank. Thus the optimal pattern of responses to shocks can in this case be supported as an equilibrium outcome under discretion, as long as the central bank is charged with pursuit of an objective that involves interest-rate smoothing.

4.3 Optimal Delegation in a Numerical Example

When $\kappa > 0$, assignment of an objective from the simple class (4.1) does not suffice to implement the precise optimal plan characterized in section 2. Nonetheless, it is possible to achieve quite a good approximation to the optimal pattern of responses to shocks, in the case of plausible parameter values. We demonstrate this through numerical analysis, using once again the calibrated parameter values specified in Table 1. To begin, we shall assume that $\hat{\lambda}_x = \lambda_x = .048$ (the value in Table 1), and consider only the consequences of variation in $\hat{\lambda}_i$ and λ_Δ.

We first note that the nonlinear equations referred to above do not always have a unique solution for the coefficients i_i, i_n, and so on. It can be shown that given a value for i_i consistent with these equations, a unique solution can be obtained, generically, for the other coefficients. However, i_i solves a quintic equation, which equation may have as many as five real roots. For example, Figure 2 plots the solutions to this equation, as a function of $\hat{\lambda}_i$, in the case that $\lambda_\Delta = 0$. One observes that there is a unique real root, $i_i = 0$, in the case of any $\hat{\lambda}_i > 0$; but for $\hat{\lambda}_i < 0$, there are multiple solutions, and given the results of the previous sub-section, we are interested in considering loss functions of this kind.[19]

In the figure, solutions that also satisfy conditions (4.9) and (4.10), and so correspond to stationary equilibria, are indicated by solid lines, while additional branches of solutions that do not correspond to stationary equilibria are indicated by dashed lines.[20] We observe that

[19]When λ_Δ is exactly zero, only the root $i_i = 0$ is actually a Markov equilibrium, since in this special case the lagged nominal interest rate is an irrelevant state variable. However, for small non-zero values of λ_Δ, the graph of the solutions is similar, and all solutions count as Markov equilibria.

[20]Technically, when $\lambda_\Delta = 0$, the second-order condition is (weakly) satisfied even by solutions in which $i_i < 0$. But our real interest is in the set of solutions that exist for small positive values of λ_Δ. The solutions shown in Figure 2 with $i_i < 0$ also correspond to solutions with $i_i < 0$ in the case of small positive λ_Δ, and under that perturbation these solutions cease to satisfy the second-order condition. Hence we show these

while there exist multiple solutions to the nonlinear equations for all $\hat{\lambda}_i < 0$, there is still a unique stationary equilibrium involving optimization under discretion for all $\hat{\lambda}_i > -1$.[21] Only for even larger negative values do we actually have multiple stationary Markov equilibria. The same turns out to be true for $\lambda_\Delta > 0$ as well, at least in the case of the moderate values of λ_Δ that we shall consider here.[22]

We next consider how the properties of the stationary Markov equilibrium vary with the parameters $\hat{\lambda}_i$ and λ_Δ. In Figure 3, the white region indicates the set of loss function weights for which there is a unique stationary equilibrium of the linear form characterized above. In this region, the contour lines plot the value of $E[W]$ for this equilibrium. The grey region indicates weights for which there are multiple stationary equilibria. Here we plot the lowest possible value of $E[W]$. As it turns out, the best equilibrium that is attainable corresponds to weights in the white region, so that we do not have to face the question of whether one should choose weights that are consistent with *one* good equilibrium but also with other bad ones.

Four sets of policy weights marked on Figure 5 are of particular interest. The numerical values of the weights are given in Table 3, along with properties of the resulting Markov equilibrium. The X (corresponding to line 1 of Table 3) indicates the weights in the true social loss function; but charging a discretionary central bank to minimize this objective does not lead to the best equilibrium, under this same criterion. The large black dot (line 2 of Table 3) instead indicates the weights that lead to the best outcome, when one still restricts attention to central bank loss functions with no smoothing objective ($\lambda_\Delta = 0$). This corresponds to a weight $\hat{\lambda}_i$ that implements the optimal non-inertial plan, characterized in section 3.[23] It involves a value $\hat{\lambda}_i < \lambda_i$, so that interest rates respond more vigorously to

branches of solutions with dashed lines.

[21]To be more precise, for any small enough value $\lambda_\Delta > 0$, there exists a unique stationary equilibrium for all $\hat{\lambda}_i > -1$. This identifies the boundary of the white region in Figure 3 near the horizontal axis. It is interesting to note that for values of $\hat{\lambda}_i$ below a critical value, approximately -0.02, the unique stationary equilibrium no longer corresponds to the "minimum state variable solution", *i.e.,* the solution in which lagged interest rates are irrelevant.

[22]For very high values of $\lambda_\Delta > 0$, not shown in Figure 3 below, there exist multiple equilibria even for higher values of $\hat{\lambda}_i$.

[23]Compare the second line of Table 3 with the second line of Table 2.

variations in the natural rate of interest than occurs under discretion when the central bank seeks to minimize the true social loss function.

The circled star, or wheel (line 4 of Table 3), instead indicates the minimum achievable value of $E[W]$, among time-consistent equilibria of this kind. These weights therefore solve the optimal delegation problem, if we restrict ourselves to central-bank objectives of the form (4.1). As in the limiting case solved explicitly above, the optimal weights involve $\lambda_\Delta > 0, \hat{\lambda}_i < 0$.

We note that minimum value of $E[W]$ shown in Figure 5 is the same, to three significant digits, as that shown in Table 2 for the optimal plan under commitment. Thus more than 99.9 percent of the reduction in expected loss (relative to the outcome under discretionary minimization of the true social loss function) that is possible in principle, through an optimal commitment, can be achieved through an appropriate choice of objective for a discretionary central bank.[24] The exact optimal pattern of responses could presumably be supported as a time-consistent equilibrium if we were to consider more complex central bank loss functions; but our analysis here suffices to indicate the desirability of assigning the central bank an interest-rate smoothing objective.

It may not be thought possible, in practice, to assign the central bank a smoothing objective that involves a negative weight on one of the "stabilization" objectives. The star without a circle in Figure 3 (third line of Table 3) indicates the best Markov equilibrium that can be achieved subject to the constraint that $\hat{\lambda}_i \geq 0$. This point corresponds to a point of tangency between an isoquant of $E[W]$ and the vertical axis at $\hat{\lambda}_i = 0$. In this case, it is still desirable to direct the central bank to penalize large interest-rate changes, though the optimal λ_Δ is smaller than if it were possible to choose $\hat{\lambda}_i < 0$.

Thus far, we have assumed that the relative weight on output gap variability, $\hat{\lambda}_x$, equals the weight in the true social loss function, λ_x, given in Table 1. In fact, consideration of values $\hat{\lambda}_x \neq \lambda_x$ allows us to do no better, either in the case of loss functions with no smoothing

[24] The equilibrium achieved in this way is also very similar in other respects, such as the other statistics for the optimal plan reported in Table 2.

objective, or in the case of the fully unconstrained family. This is not because $\hat{\lambda}_x = \lambda_x$ is a uniquely optimal value in either case, but rather because we can find weights that support the optimal plan for an arbitrary value of $\hat{\lambda}_x$, so that the constraint that $\hat{\lambda}_x = \lambda_x$ has no cost. For example, it would also be possible to impose the constraint that $\hat{\lambda}_x = 0$, so that there is no output-gap term in the central bank loss function at all. The optimal weights in this case are given on the fifth line of Table 3. Note that again $\hat{\lambda}_i < 0$, $\lambda_\Delta > 0$.

This ceases to be true if we impose the constraint that all weights be non-negative. In this case, the constraint that $\hat{\lambda}_i \geq 0$ binds. But if we must set $\hat{\lambda}_i = 0$, the additional degree of freedom allowed by varying $\hat{\lambda}_x$ does allow some improvement of the time-consistent equilibrium, in general. In fact, for the numerical parameter values used above, the optimal $\hat{\lambda}_x$ is infinite; that is, the relative weight on the inflation term is best set to zero. To analyze this case, it is thus convenient to adopt an alternative normalization for the central bank loss function,

$$L_t^{cb} = x_t^2 + \tilde{\lambda}_\pi \pi_t^2 + \tilde{\lambda}_i i_t^2 + \tilde{\lambda}_\Delta (i_t - i_{t-1})^2. \quad (4.14)$$

In terms of this alternative normalization, the loss function described on the fourth line of Table 3 is instead described as on the sixth line of the table. The optimal objective in the family (4.14), when we impose the constraint that $\tilde{\lambda}_i \geq 0$, is instead given on the seventh line of the table. Once again we find that a positive weight on the smoothing objective is desirable, though the constrained-optimal central-bank objective puts no weight on either inflation stabilization or on reducing variation in the level of nominal interest rates.[25]

5 Conclusion

Even if there is no intrinsic benefit to minimizing the size of changes in the central bank's interest-rate instrument, it can be desirable for a central bank to seek to minimize a loss function that includes a smoothing objective. For pursuit of such an objective will lead a central bank that optimizes under discretion to adjust interest rates in a more inertial

[25] See Woodford (1999b) for further discussion of optimal delegation under this constraint.

fashion, and interest-rate dynamics of this kind are desirable for the sake of objectives that *are* important for monetary policy — namely, achieving a greater degree of stability of inflation and the output gap, without requiring so much variation in the level of interest rates.

Of course, the assignment to the central bank of an objective different from the true social loss function, in the expectation that it will pursue that objective with discretion, is not the only possible approach to the achievement of a desirable pattern of responses to disturbances. One defect of the "optimal delegation" approach considered here is that it presumes that the stationary Markov equilibrium associated with a particular distorted objective will be realized. Yet there may well be other possible rational expectations equilibria consistent with discretionary optimization by the central bank, "reputational" equilibria in which the bank may do a *better* job of minimizing the objective it has been assigned, but as a consequence bring about a pattern of responses that is *less* desirable from the point of view of the true social objective.

An alternative approach that would not raise these difficulties would be a commitment by the central bank to conduct policy according to an interest-rate feedback rule along the lines of the "Taylor rule". Interest-rate rules that would implement the optimal plan in the context of the model considered here are discussed in Woodford (1999b) and Giannoni and Woodford (2002). Under this approach as well, an optimal rule makes the current interest rate setting a function of the recent past level of interest rates. A purely contemporaneous rule — one that makes the current nominal interest rate a linear function of the current inflation rate and current output gap only, as proposed by Taylor (1993) — can at best implement only the optimal non-inertial plan. The more inertial interest rate dynamics shown in section 2 to characterize the optimal plan instead require a feedback rule that responds to lagged endogenous variables; in particular, the rule must specify that the level chosen for the current nominal interest rate will be higher the higher nominal interest rates already are. Thus this feature of estimated central-bank reaction functions can also be justified as a characteristic of optimal policy.

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Table 1: "Calibrated" parameter values.

Structural parameters	
β	0.99
σ^{-1}	.157
κ	.024
Shock process	
ρ	0.35
sd(r^n)	3.72
Loss function	
λ_x	.048
λ_i	.236

Table 2: Statistics for alternative policies.

Policy	V[π]	V[x]	V[i]	E[W]
Discretion	.487	22.95	4.023	2.547
Non-Inertial	.211	9.92	6.720	2.279
Optimal	.130	10.60	1.921	1.097

Table 3: Stationary Markov equilibria with alternative policy weights.

Policy Weights				Equilibrium Statistics			
$\hat{\lambda}_\pi$	$\hat{\lambda}_x$	$\hat{\lambda}_i$	λ_Δ	V[π]	V[x]	V[i]	E[W]
1	.048	.236	0	.487	22.95	4.023	2.547
1	.048	.120	0	.211	9.92	6.720	2.279
1	.048	0	.282	.082	11.73	2.907	1.337
1	.048	-.439	.807	.135	10.50	1.922	1.097
1	0	-.232	.204	.135	10.50	1.922	1.097
20.7	1	-9.09	16.7	.135	10.50	1.922	1.097
0	1	0	5.51	.077	11.95	2.649	1.281