A non-standard Finite Element Method based on boundary integral operators

Clemens Hofreither   Ulrich Langer   Clemens Pechstein

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Outline

Motivation

Let $\Omega \subset \mathbb{R}^d$ a domain. We want to solve the PDE

$$-\text{div}(a(x)\nabla u(x)) = f(x), \quad x \in \Omega$$

plus suitable boundary conditions on $\Gamma = \partial\Omega$.

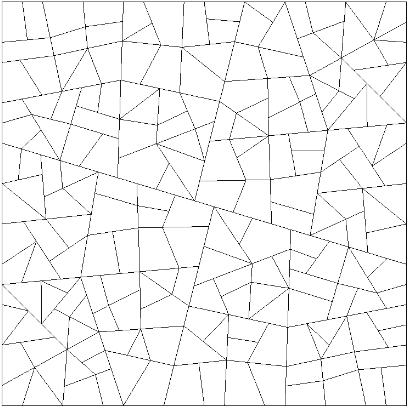
Discretize the domain by a mesh consisting of N polygonal/polyhedral elements T_i:

$$\overline{\Omega} = \bigcup_{i=1}^{N} \overline{T}_i$$

Meshes

We want to treat general polygonal/polyhedral meshes:

- arbitrary and mixed element shapes
- hanging nodes supported naturally



Motivation

These kinds of meshes arise, e.g., in

- reservoir simulation,

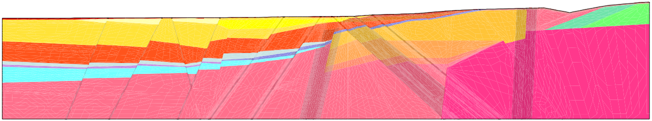
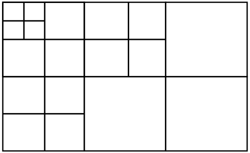


Image: rock permeability $\alpha(x)$ at Sellafield, UK. (c) NIREX UK Ltd.

- adaptive mesh refinement.



Another advantage: automatic mesh manipulation (join, split, manipulate elements)

Other possible approaches

Mimetic finite difference method

- Kuznetsov, Lipnikov, Shashkov:
  The mimetic finite difference method on polygonal meshes for diffusion-type problems, 2004
- Brezzi, Lipnikov, Shashkov:
  Convergence of Mimetic Finite Difference Method for Diffusion Problems on Polyhedral Meshes, 2004

Discontinuous Galerkin method

- Dolejsi, Feistauer, Sobotikova:
  Analysis of the discontinuous Galerkin method for nonlinear convection-diffusion problems, 2005

Our approach

Has its roots in symmetric BEM Domain Decomposition method.

1. Use piecewise PDE-harmonic ansatz functions in every element (local Trefftz method).
2. Reduce the variational formulation to the *skeleton* of the mesh.
3. Use boundary integral operators to solve local Dirichlet-to-Neumann problems.

Prior work

A selection of previous work related to this approach:

- Hsiao, Wendland:
  Domain Decomposition in Boundary Element Methods, 1991.
- Hsiao, Steinbach, Wendland:
  Domain decomposition methods via boundary integral equations, 2000.
- Copeland, Langer, Pusch:
  From the boundary element method to local Trefftz finite element methods on polyhedral meshes, 2009.

Model problem

Our method is designed to treat piecewise constant coefficients:

$$a(x) \equiv a_i \quad \text{for } x \in T_i$$

For simplicity, we assume $a(x) \equiv 1$, $f \equiv 0$, i.e. Laplace equation:

$$\begin{aligned} -\Delta u &= 0 \quad \text{in } \Omega, \\ u &= g \quad \text{on } \Gamma = \partial\Omega. \end{aligned}$$

Variational formulation

For all $v \in H^1_0(\Omega)$,

$$0 = \int_\Omega \nabla u \cdot \nabla v$$

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\begin{aligned}
0 &= \int_\Omega \nabla u \cdot \nabla v \\
&= \sum_i \int_{T_i} \nabla u \cdot \nabla v
\end{aligned}
$$

Variational formulation

For all $v \in H_0^1(\Omega)$,

$$0 = \int_\Omega \nabla u \cdot \nabla v = \sum_i \int_{T_i} \nabla u \cdot \nabla v = \sum_i \left[\int_{\Gamma_i} \frac{\partial u}{\partial n_i} v - \int_{T_i} \underbrace{\Delta u}_{=0} v \right]$$

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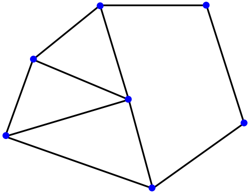
&= \sum_i \left[\int_{\Gamma_i} \frac{\partial u}{\partial n_i} v - \int_{T_i} \underbrace{\Delta u}_{=0} v \right] \\
&= \sum_i \int_{\Gamma_i} \frac{\partial u}{\partial n_i} v.
\end{aligned}
$$

Is defined purely on the element boundaries $(\Gamma_i)_{i=1,\ldots,N}$.

Function spaces on the skeleton

Let $\Gamma_S := \bigcup_i \Gamma_i$ denote the *skeleton* of the mesh.



Let $H^{1/2}(\Gamma_S)$ be the trace space of $H^1(\Omega)$ onto Γ_S.

Let

$$W = \{v \in H^{1/2}(\Gamma_S) : v|_\Gamma = 0\}$$

be the space of skeletal functions which are zero on the boundary of Ω.

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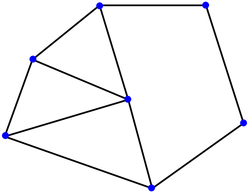

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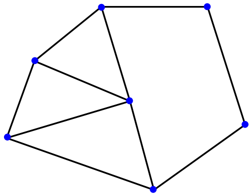

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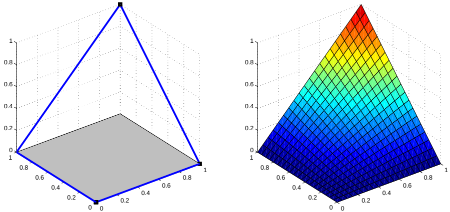
Harmonic extension

For any (sufficiently regular) function on the boundary of a domain, there is a unique way to extend it to a harmonic function ($\Delta u = 0$) in the domain:

$$\mathcal{H}_i : H^{1/2}(\Gamma_i) \to H^1(T_i)$$



We also define $\mathcal{H}_S : H^{1/2}(\Gamma_S) \to H^1(\Omega)$ by piecing together the local harmonic extensions.

Skeletal variational formulation

Recall:

$$\sum_i \int_{\Gamma_i} \frac{\partial u}{\partial n_i} v = 0 \quad \forall v \in H_0^1(\Omega). \tag{1}$$

Making use of the bijection

$$u_i = \gamma_{T_i} u \quad \longleftrightarrow \quad u = \mathcal{H}_i u_i \text{ on } T_i,$$

we can rewrite (1) as

$$\sum_i \int_{\Gamma_i} \frac{\partial}{\partial n_i} (\mathcal{H}_i u_i) v_i = 0 \quad \forall v \in W. \tag{2}$$

With the *Dirichlet-to-Neumann map* $S_i u_i := \frac{\partial}{\partial n_i}(\mathcal{H}_i u_i)$, we get

$$\sum_i \int_{\Gamma_i} S_i u_i \, v_i = 0 \quad \forall v \in W. \tag{3}$$

Relation to standard domain VF

Standard variational formulation

Find $u_\Omega \in H^1(\Omega)$ with $u_\Omega|_\Gamma = g$ such that for all $v_\Omega \in H^1_0(\Omega)$,

$$\int_\Omega \nabla u_\Omega \cdot \nabla v_\Omega \, dx = 0.$$

Skeletal variational formulation

Find $u \in H^{1/2}(\Gamma_S)$ with $u|_\Gamma = g$ such that for all $v \in W$,

$$\sum_{i=1}^{N} \langle S_i u_i, v_i \rangle_{\Gamma_i} = 0.$$

The formulations are equivalent: $u_\Omega = \mathcal{H}_i u_i$ on T_i.

Evaluating the Dirichlet-to-Neumann map

Evaluating $S_i u_i = \frac{\partial}{\partial n_i}(\mathcal{H}_i u_i)$:

1. solve a Laplace problem on T_i with Dirichlet data u_i,
2. obtain Neumann data of solution.

More directly:

$$S_i = D_i + (\frac{1}{2}I + K_i')V_i^{-1}(\frac{1}{2}I + K_i)$$

with the boundary integral operators

- V_i, single layer potential operator,
- K_i, double layer potential operator,
- D_i, hypersingular operator.

Defined in terms of the *fundamental solution* of the differential operator.

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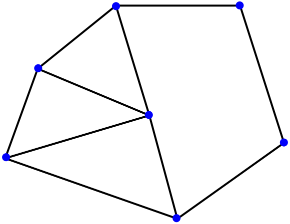
Discretization

A Galerkin method is obtained by choosing a finite-dimensional subspace $W_h \subset W$.

E.g.: piecewise linear, continuous functions (on the skeleton). Degrees of freedom are chosen in the vertices of the elements.



Discretizing the Dirichlet-to-Neumann map

Symmetric representation of Steklov-Poincaré operator:

$$S_i u_i = D_i u_i + (\frac{1}{2} I + K_i') w_i, \quad \text{where}$$
$$V_i w_i = (\frac{1}{2} I + K_i) u_i$$

Problem: We cannot compute $w_i \in H^{-1/2}(\Gamma_i)$ exactly. Instead, perform a local Galerkin projection $w_{h,i} \approx w_i$ to some finite-dimensional space and compute

$$\tilde{S}_i u_i = D_i u_i + (\frac{1}{2} I + K_i') w_{h,i}.$$

Natural choice: space of piecewise constant boundary functions, since w_i represents a normal derivative.

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Discretization of the variational problem

$$\sum_{i=1}^{N} \langle S_i u_i, v_i \rangle = 0 \quad \forall v \in W$$

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$$\downarrow \quad \text{Galerkin}$$

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$\downarrow$ D2N approximation

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This creates a s.p.d. and sparse stiffness matrix.
For purely tetrahedral mesh: identical to FEM.

Skeletal energy norm

We introduce the skeletal energy norm on $H^{1/2}(\Gamma_S)$,

$$|||v|||_S := \left(\sum_{i=1}^{N} \langle S_i v_i, v_i \rangle \right)^{1/2} = \left(\sum_{i=1}^{N} |\mathcal{H}_i v_i|^2_{H^1(T_i)} \right)^{1/2}.$$

Céa-type error estimate

Theorem

$$|u_\Omega - \mathcal{H}_S u_h|_{H^1(\Omega)} \leq C_S \left(\inf_{v_h \in W_h} \|u - v_h\|_S + \sqrt{\sum_{i=1}^{N} \inf_{z_{h,i} \in Z_{h,i}} \left\| \frac{\partial u_\Omega}{\partial n_i} - z_{h,i} \right\|_{V_i}^2} \right),$$

where C_S is a mesh-dependent constant.

Proof: Using the first lemma of Strang.

For a proper error estimate, we have to:

- bound the mesh constant C_S,
- estimate the Dirichlet approximation error,
- estimate the Neumann approximation error.

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Mesh constants

The constant C_S depends only on

$$c_{0,i} = \inf_{v \in H_*^{1/2}(\Gamma_i)} \frac{\langle D_i v, v \rangle}{\langle V_i^{-1} v, v \rangle} \in (0, \tfrac{1}{4}) \qquad \forall i \in \{1, \ldots, N\}.$$

Recent results by C. Pechstein:

$c_{0,i}$ may be bounded away from zero solely in terms of

- the Jones parameter $C_U(T_i)$,
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Estimating the approximation errors

Dirichlet approximation error:
Introduce a regular local triangulation of T_i and use standard FEM approximation results.

Neumann approximation error:

- recent results by C. Pechstein:
  bound $\|\cdot\|_{V_i}$ by $C(C_U(T_i), C_P(T_i)) \, \|\cdot\|_{H_\star^{-1/2}(\Gamma_i)}$
- again use a local triangulation of T_i composed of regular tetrahedra
- estimate error piecewise on each triangle on the surface of these tetrahedra

Main result

Theorem

If every polyhedral element T_i

- *has a regular tetrahedral triangulation,*
- *has a uniform upper bound on the number of its surface triangles,*
- *has a uniform upper bound on its Jones and Poincaré constants,*

and if $u_\Omega \in H^2(\Omega)$ is the exact solution, we have

$$|u_\Omega - \mathcal{H}_S u_h|_{H^1(\Omega)} \leq C \left(\sum_i h_i^2 |u_\Omega|^2_{H^2(T_i)} \right)^{\frac{1}{2}} \leq Ch|u_\Omega|_{H^2(\Omega)}.$$

A 3D numerical example

- implementation in C++ based on PARMAX framework by Pechstein & Copeland
- generation of BEM matrix entries: OSTBEM (Steinbach et al.)
- solution of linear system by non-preconditioned CG method

Problem specification

Solve the pure Dirichlet Laplace equation,

$$\begin{aligned} -\Delta u &= 0 && \text{in } \Omega, \\ u(x) &= g(x) && \forall x \in \partial\Omega, \end{aligned}$$

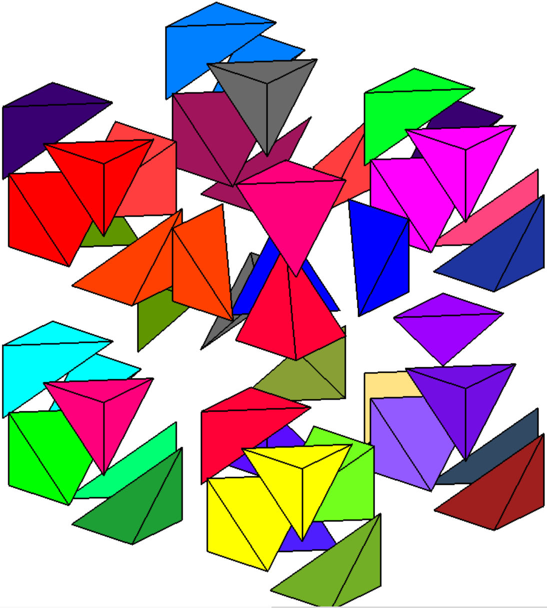
on the unit cube, $\Omega = (0,1)^3$.

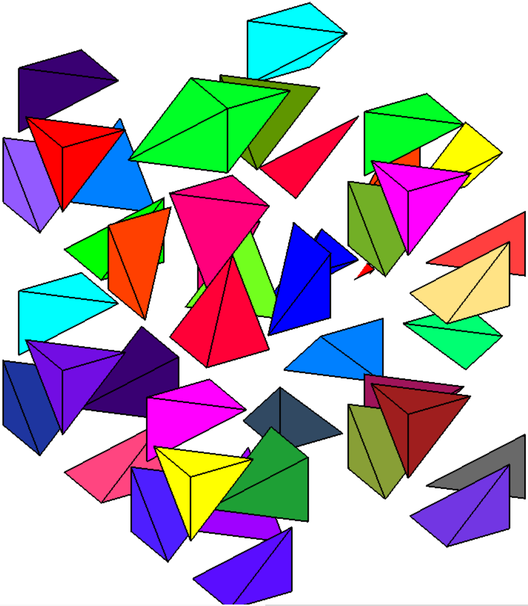
Exact solution: $u(x,y,z) = \exp(x)\cos(y)(1+z)$

Two sets of meshes:

- a purely tetrahedral mesh,
- a mixed mesh consisting of tetrahedra and polyhedra

Number of degrees of freedom is identical.



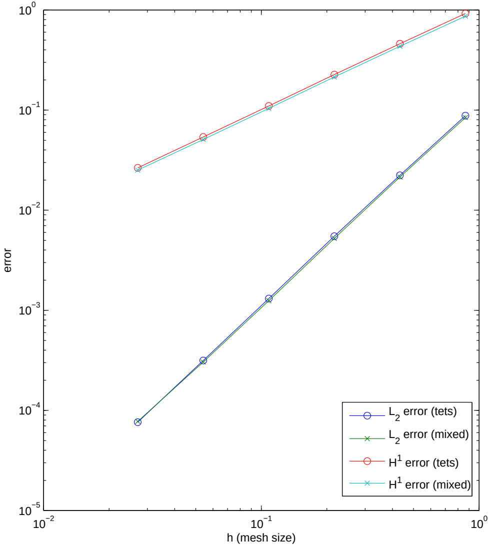


Tetrahedral mesh:

mesh size h	H^1 error	L^2 error	#tets
0.866025	0.923507	0.0879679	48
0.433013	0.459565	0.0223147	384
0.216506	0.226186	0.00549834	3.072
0.108253	0.109806	0.00131165	24.576
0.0541266	0.0537825	0.000315016	196.608
0.0270633	0.0264988	7.62441e-05	1.572.864

Mixed mesh:

mesh size h	H^1 error	L^2 error	#tets	#polys
0.866025	0.867685	0.0842554	40	4
0.433013	0.433557	0.0214242	258	63
0.216506	0.214188	0.00522372	2.044	514
0.108253	0.103955	0.00124863	15.822	4.377
0.0541266	0.0508436	0.000304395	125.350	35.629
0.0270633	0.0251327	7.76704e-05	996.390	288.237


error
h (mesh size)
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-2}
10^{-1}
10^0
L_2 error (tets)
L_2 error (mixed)
H^1 error (tets)
H^1 error (mixed)

Outlook

- treat convection-diffusion-reaction problems:

$$-\mathrm{div}(a(x)\nabla u(x)) + b(x)\cdot\nabla u(x) + c(x)u(x) = f(x), \quad x \in \Omega$$

- DG discretization
- develop optimal solvers
- higher order schemes: $\rightarrow$ Levy functions (Wendland)

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Boundary integral operators

Fundamental solution of 3D Laplace operator: $U^*(x,y) = \frac{1}{4\pi|x-y|}$.

$V_i : H^{-1/2}(\Gamma_i) \to H^{1/2}(\Gamma_i)$:

$$(V_i v)(y) = \int_{\Gamma_i} U^*(x,y) v(x)\, ds_x$$

$K_i : H^{1/2}(\Gamma_i) \to H^{1/2}(\Gamma_i)$:

$$(K_i u)(y) = \int_{\Gamma_i} \frac{\partial U^*}{\partial n_x}(x,y) u(x)\, ds_x$$

$K_i' : H^{-1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i)$:

$$(K_i' v)(y) = \int_{\Gamma_i} \frac{\partial U^*}{\partial n_y}(x,y) v(x)\, ds_x$$

$D_i : H^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i)$:

$$(D_i u)(y) = -\frac{\partial}{\partial n_y} \int_{\Gamma_i} \frac{\partial U^*}{\partial n_x}(x,y) \big(u(x) - u(y)\big)\, ds_x$$

Norms for error estimates

We use harmonic extensions $\mathcal{H}_i$ heavily.

Natural norms to work in:

$$|v_i|_{H^{1/2}(\Gamma_i)} := |\mathcal{H}_i v_i|_{H^1(T_i)} = \inf_{\substack{\phi \in H^1(T_i) \\ \phi|_{\Gamma_i} = v_i}} |\phi|_{H^1(T_i)},$$

$$\|v_i\|^2_{H^{1/2}(\Gamma_i)} := \frac{1}{\operatorname{diam}(T_i)^2} \|\mathcal{H}_i v_i\|^2_{L_2(T_i)} + |\mathcal{H}_i v_i|^2_{H^1(T_i)}.$$

Easily extended to the skeleton Γ_S:

$$|\!|\!| v |\!|\!|_S := \Big(\sum_{i=1}^{N} |v_i|^2_{H^{1/2}(\Gamma_i)} \Big)^{1/2} = |\mathcal{H}_S v|_{H^1(\Omega)}.$$

Jones parameter

Definition (Uniform domain)

A bounded and connected set $D \subset \mathbb{R}^d$ is called a *uniform domain* if there exists a constant $C_U(D)$ such that any two points $x_1, x_2 \in D$ can be joined by a rectifiable curve $\gamma(t) : [0, 1] \to D$ with $\gamma(0) = x_1$ and $\gamma(1) = x_2$ such that

$$\begin{aligned} \ell(\gamma) &\leq C_U(D)\,|x_1 - x_2|, \\ \min_{i=1,2} |x_i - \gamma(t)| &\leq C_U(D)\,\operatorname{dist}(\gamma(t), \partial D) \quad \forall t \in [0, 1]. \end{aligned}$$

We call the smallest such $C_U(D)$ the *Jones parameter* of D.

Any Lipschitz domain is a uniform domain.

Poincaré constant

Definition (Poincaré constant)

Let D be a uniform domain. The Poincaré constant $C_P(D)$ is the smallest constant such that

$$\inf_{c\in\mathbb{R}} \|u - c\|_{L_2(D)} \leq C_P(D)\,\mathrm{diam}(D)\,|u|_{H^1(D)} \quad \forall u \in H^1(D).$$

$C_P(D)$ can be bounded using the constant in an isoperimetric inequality.

For convex domains, $C_P(D) \leq \frac{1}{\pi}$ (Payne & Weinberger 1960; Bebendorf 2003).

Mesh constants

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$$c_{0,i} = \inf_{v \in H_*^{1/2}(\Gamma_i)} \frac{\langle D_i v, v \rangle}{\langle V_i^{-1} v, v \rangle} \in (0, \tfrac{1}{4}) \qquad \forall i \in \{1, \ldots, N\}.$$

Recent results by C. Pechstein:
For each element, fix a ball B_i enclosing it:

$$B_i \supset \overline{T_i}, \qquad \operatorname{dist}(\partial B_i, \partial T_i) \ \geq \ \tfrac{1}{2}\operatorname{diam}(T_i).$$

$c_{0,i}$ may be bounded away from zero solely in terms of

- the Jones parameters $C_U(T_i)$ and $C_U(B_i \setminus \overline{T_i})$,
- the Poincaré constants $C_P(T_i)$ and $C_P(B_i \setminus \overline{T_i})$.

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- the Jones parameters $C_U(T_i)$ and $C_U(B_i \setminus \overline{T_i})$,
- the Poincaré constants $C_P(T_i)$ and $C_P(B_i \setminus \overline{T_i})$.

Mesh regularity (3D)

Standard regularity assumptions on every tetrahedron $\tau \in \Xi_i$: with J_τ Jacobian of affine mapping from unit tetrahedron to τ,

$$\underline{c}_1 (\operatorname{diam} \tau)^3 \le |\det J_\tau| \le \overline{c}_1 (\operatorname{diam} \tau)^3,$$
$$\|J_\tau\|_{\ell_2} \le \overline{c}_2 \operatorname{diam} \tau,$$
$$\|J_\tau^{-1}\|_{\ell_2} \le (\underline{c}_2 \operatorname{diam} \tau)^{-1}.$$

This implies regularity of the boundary faces: for all $f \in \mathcal{F}_i$ and the tetrahedron τ on which f lies,

$$\underline{c}_2 \operatorname{diam} \tau \le \operatorname{diam} f \le \operatorname{diam} \tau,$$
$$\frac{\underline{c}_1}{2\overline{c}_2} (\operatorname{diam} \tau)^2 \le |f| \le \frac{1}{2} (\operatorname{diam} \tau)^2.$$

Furthermore, we assume that the number of boundary faces per element is uniformly small.

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Estimating the Dirichlet approximation error

- Conforming global triangulation $\Xi = \bigcup_i \Xi_i$.
- Introduce globally cont., pw. linear FE space V_h over Ξ.
- Choose suitable FE interpolant $I_{V_h} u_\Omega \in V_h$.
- Skeletal interpolant $I_{W_h} u_\Omega := \gamma_S(V_h u_\Omega) \in W_h$.
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\begin{aligned}
\inf_{v_h \in W_h} \|u - v_h\|_S^2 &\leq \|u - I_{W_h} u_\Omega\|_S^2 \\
&= \sum_{i=1}^N |\mathcal{H}_i(u - I_{W_h} u_\Omega)|_{H^1(T_i)}^2 \\
&\leq \sum_{i=1}^N |u_\Omega - I_{V_h} u_\Omega|_{H^1(T_i)}^2 = |u_\Omega - I_{V_h} u_\Omega|_{H^1(\Omega)}^2
\end{aligned}
$$

Estimating the Neumann approximation error

Steps in this estimation are quite standard, but:

- choice of norms is delicate,
- results need to be explicit in the regularity parameters. (C is a generic constant depending only on the reg. param.)

$$\inf_{z_{h,i} \in Z_{h,i}} \left\| \frac{\partial u_\Omega}{\partial n_i} - z_{h,i} \right\|_{V_i}$$

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&\overset{\text{Approx. thm.}}{\leq} C\, h_i \left| \frac{\partial u_\Omega}{\partial n_i} \right|_{H^{1/2}_{\sim \mathrm{pw}}(\Gamma_i)}
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&\overset{\text{N. trace thm.}}{\leq} C\, h_i \left| u_\Omega \right|_{H^2(T_i)}.
\end{aligned}$$

On the Neumann approximation theorem

Standard approximation on pw. constant boundary function space (e.g. Steinbach):

$$\|v - Q_{h,i} v\|_{L_2(\Gamma_i)} \leq C \, (\mathrm{diam}\, T_i)^{1/2} \, |v|_{H^{1/2}_{\sim\mathrm{pw}}(\Gamma_i)}.$$

We use the norm relations (for $v \in H^{1/2}(\Gamma_i)$)

$$|v|_{H^{1/2}_{\sim\mathrm{pw}}(\Gamma_i)} \;\leq\; C \, |v|_{H^{1/2}_{\mathrm{pw}}(\Gamma_i)} \;\leq\; C \, \sqrt{\#\mathcal{F}_i} |v|_{H^{1/2}(\Gamma_i)}$$

to obtain

$$\|v - Q_h v\|_{L_2(\Gamma_i)} \leq C \, \sqrt{\#\mathcal{F}_i} (\mathrm{diam}\, T_i)^{1/2} \, \|v\|_{H^{1/2}(\Gamma_i)}.$$

If $\#\mathcal{F}_i$ bounded, this gives us the desired approximation result for $\| \cdot \|_{H^{-1/2}(\Gamma_i)}$ using a duality argument.

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