Temporal Conceptual Modelling with *DL-Lite*

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1 Introduction

Conceptual modelling formalisms such as the Entity-Relationship model (ER) and Unified Modelling Language (UML) have become a *de facto* standard in database design by providing visual means to describe application domains in a declarative and reusable way. On the other hand, both ER and UML turned out to be closely connected with description logics that are underpinned by formal semantics and thus capable of providing services for effective reasoning over conceptual models; see, e.g., [11, 4].

Temporal conceptual data models (TCMs) [18, 25] have been introduced in the context of temporal databases [20, 15, 13]. In this case, apart from the classical constructs—such as inheritance between classes and relationships, cardinality constraints restricting participation in relationships, and disjointness and covering constraints—temporal constructs are used to capture the temporal behaviour of various components of conceptual schemas. Such constructs can be grouped into 3 categories. *Timestamping constraints* discriminate between those classes, relationships and attributes that change over time and those that are time-invariant [28, 18, 16, 6, 25]. *Evolution constraints* control how domain elements evolve over time by 'migrating' from one class to another [19, 23, 26, 25, 3]. We distinguish between *qualitative* evolution constraints describing generic temporal behaviour, and *quantitative* ones specifying the exact moment of migration. *Temporal cardinality constraints* restrict the number of times an instance of a class participates in a relationship. *Snapshot* cardinality constraints do it at each moment of time, while *lifespan* cardinality constraints impose restrictions over the entire existence of the instance as a member of the class [27, 22].

Temporal conceptual data models can be encoded in various temporal description logics (TDLs), which have been designed and investigated since the seminal paper [24] with the aim of understanding the computational price of introducing a temporal dimension in DLs; see [21] for a recent survey. A general conclusion one can draw from the obtained results is that—as far as there is nontrivial interaction between the temporal and DL components—TDLs based on full-fledged DLs like $\mathcal{ALC}$ turn out to be too complex for effective reasoning (see the end of the introduction for details).

The aim of this paper is to tailor 'minimal' TDLs that are capable of representing various aspects of TCMs and investigate their computational behaviour. First of all, as the DL component we choose the 'light-weight' *DL-Lite* logic

DL-Lite$^{\mathcal{N}}_{bool}$, which was shown to be adequate for capturing conceptual models without relationship inheritance[1] [4], and its fragment *DL-Lite*$^{\mathcal{N}}_{core}$ with most primitive concept inclusions, which are nevertheless enough to represent almost all types of constraints (apart from covering). To discuss our choice of the temporal constructs, consider a toy TCM describing a company.

For the timestamping constraint 'employee is a *snapshot class*' (by the standard TCM terminology, such a class never changes in time) one can use the axiom $\mathsf{Employee} \sqsubseteq \boxast\, \mathsf{Employee}$ with the temporal operator $\boxast$ 'always.' Likewise, the constraint 'manager is a *temporary class*' in the sense that each of its instances must leave the class, the axiom $\mathsf{Manager} \sqsubseteq \diamondplus \neg \mathsf{Manager}$ is required, where $\diamondplus$ means 'some time.' Both of these axioms are regarded as *global*, i.e., applicable to all time points. Note that to express $\diamondplus$ using more standard temporal constructs, we need both 'some time in the past' $\Diamond_P$ and 'some time in the future' $\Diamond_F$: e.g., $\diamondplus = \Diamond_P \Diamond_F$. To encode a snapshot n-ary relationship, one can reify it into a snapshot class with n auxiliary *rigid*—i.e., time-independent—roles; for a temporary relationship, the reifying class is temporary and the roles are *local* [9, 7]. The qualitative evolution constraints 'each manager was once an employee' and 'a manager will always remain a manager' can be expressed by the axioms $\mathsf{Manager} \sqsubseteq \Diamond_P \mathsf{Employee}$ and $\mathsf{Manager} \sqsubseteq \Box_F \mathsf{Manager}$, while 'an approved project keeps its status until a later date when it actually starts' can be expressed using the 'until' operator: $\mathsf{ApprovedProject} \sqsubseteq \mathsf{ApprovedProject}\, \mathcal{U}\, \mathsf{Project}$. The quantitative evolution constraint 'each project must be finished in 3 years' requires the next-time operator $\bigcirc_F$: $\mathsf{Project} \sqsubseteq \bigcirc_F \bigcirc_F \bigcirc_F \mathsf{FinishedProject}$. The snapshot cardinality constraint 'an employee can work on at most 2 projects at each moment of time' can be expressed as $\mathsf{Employee} \sqsubseteq\ \leq 2\, \mathsf{worksOn}$, while the lifespan constraint 'over the whole career, an employee can work on at most 5 projects' requires temporal operators on roles: $\mathsf{Employee} \sqsubseteq\ \leq 5\, \diamondplus \mathsf{worksOn}$. Note that 'temporalised' roles of the form $\diamondplus R$ and $\boxast R$ are always rigid. To represent a temporal database instance of a TCM, we use assertions like $\bigcirc_P \mathsf{Manager}(\mathsf{bob})$ for 'Bob was a manager last year' and $\bigcirc_F \mathsf{manages}(\mathsf{bob}, \mathsf{cronos})$ for 'Bob will manage project Cronos next year.' As usual, n-ary tables are represented via reification.

These considerations lead us to TDLs based on *DL-Lite*$^{\mathcal{N}}_{bool}$ and *DL-Lite*$^{\mathcal{N}}_{core}$ and interpreted over the flow of time $(\mathbb{Z}, <)$, in which (1) the future and past temporal operators can be applied to concepts; (2) roles can be declared local or rigid; (3) the 'undirected' temporal operators 'always' and 'some time' can be applied to roles; (4) the concept inclusions (TBox axioms) are global and the database (ABox) assertions are specified to hold at particular moments of time.

To our surprise, the most expressive TDL based on *DL-Lite*$^{\mathcal{N}}_{bool}$ and featuring all of (1)–(4) turns out to be undecidable. As follows from the proof of Theorem 5 below, it is a subtle interaction of functionality constraints on temporalised roles with the next-time operator and full Booleans on concepts that causes undecidability. This 'negative' result motivates consideration of various fragments of our full TDL by restricting not only the DL but also the temporal component. The table below illustrates the expressive power of the resulting fragments in the context of TCMs. We also note that both *DL-Lite*$^{\mathcal{N}}_{bool}$ and *DL-Lite*$^{\mathcal{N}}_{core}$ with global

[1] *DL-Lite*$^{\mathcal{N}}_{bool}$ with relationship inclusions regains the full expressive power of $\mathcal{ALC}$.

axioms can capture snapshot cardinality constraints, while lifespan cardinality constraints require temporalised roles of the form $\diamondast R$ and $\boxast R$.

concept temporal operators	timestamping	evolution	
		qualitative	quantitative
$\mathcal{U}/\mathcal{S}$	+	+	+
$\Box_{F/P}, \bigcirc_{F/P}$	+	+	+
$\Box_{F/P}$	+	+	−
$\boxast, \bigcirc_{F/P}$	+	−	+
$\boxast$	+	−	−

The next table summarises the complexity results obtained in this paper for satisfiability of temporal knowledge bases formulated in our TDLs.

concept temporal operators	local & rigid roles only		temporalised roles
	$DL\text{-}Lite^{\mathcal{N}}_{bool}$	$DL\text{-}Lite^{\mathcal{N}}_{core}$	$DL\text{-}Lite^{\mathcal{N}}_{bool}$
$\mathcal{U}/\mathcal{S}$	PSPACE Thm. 1	PSPACE [8]	undec. Thm. 5
$\Box_{F/P}, \bigcirc_{F/P}$	PSPACE Thm. 2 (*ii*)	NP Thm. 3	undec. Thm. 5
$\Box_{F/P}$	NP Thm. 2 (*i*)	NP [8]	?
$\boxast, \bigcirc_{F/P}$	PSPACE Thm. 2 (*ii*)	NP Thm. 3	undec. Thm. 5
$\boxast$	NP Thm. 2 (*i*)	NLOGSPACE Thm. 4	NP Thm. 6

Apart from the undecidability result of Theorem 5, quite surprising is NP-completeness of the temporal extension of $DL\text{-}Lite^{\mathcal{N}}_{core}$ with the operators $\Box_F$ and $\bigcirc_F$ (and their past counterparts) on concepts provided by Theorem 3. Indeed, if full Booleans are available, even the propositional temporal logic with these operators is PSPACE-complete. Moreover, if the 'until' operator $\mathcal{U}$ is available in the temporal component, disjunction is expressible even with $DL\text{-}Lite^{\mathcal{N}}_{core}$ as the underlying DL, and the logic becomes PSPACE-complete [8]. In all other cases, the complexity of TDL reasoning coincides with the maximal complexity of reasoning in the component logics (despite nontrivial interaction between them, as none of our TDLs is a fusion of its components). It is also of interest to observe the dramatic increase of complexity caused by the addition of $\bigcirc_F$ to the logic in the lower right corner of the table (from NP to undecidability).

To put this paper in the more general context of temporal description logics, we note first that our TDLs extend those in [8] with the past-time operators $\mathcal{S}$, $\Box_P$, $\Diamond_P$, $\bigcirc_P$ over $\mathbb{Z}$ (which are essential for capturing timestamping constraints), universal modalities $\boxast$ and $\diamondast$, and temporalised roles. Temporal operators on *DL-Lite* axioms and concepts in the presence of rigid roles were investigated in [7], where it was shown that the resulting temporalisations of $DL\text{-}Lite^{\mathcal{N}}_{bool}$ and $DL\text{-}Lite^{\mathcal{N}}_{horn}$ are EXPSPACE-complete. Temporal extensions of the standard DL $\mathcal{ALC}$ feature the following computational behaviour: $\mathcal{ALC}$ with temporal operators on axioms, rigid concepts and roles is 2EXPTIME-complete [10]. It is EXPSPACE-complete if temporal operators on concepts and axioms are allowed but no rigid or temporalised roles are available [17], and EXPTIME-complete if the language allows only temporalised concepts and global axioms [24, 2]. Finally, the 'undirected' temporal operators $\boxast$ and $\diamondast$ on concepts and roles together with global axioms result in a 2EXPTIME-complete extension of $\mathcal{ALC}$ [9].

2 Temporal DLs based on $\textit{DL-Lite}^{\mathcal{N}}_{bool}$

The TDL $T_{\mathcal{US}}\textit{DL-Lite}^{\mathcal{N}}_{bool}$ is based on $\textit{DL-Lite}^{\mathcal{N}}_{bool}$ [1, 5], which, in turn, extends $\textit{DL-Lite}_{\sqcap,\mathcal{F}}$ [12] with full Booleans over concepts and cardinality restrictions over roles. The language of $T_{\mathcal{US}}\textit{DL-Lite}^{\mathcal{N}}_{bool}$ contains *object names* $a_0, a_1, \dots$, *concept names* $A_0, A_1, \dots$, *local role names* $P_0, P_1, \dots$ and *rigid role names* $G_0, G_1, \dots$. *Roles* R, *basic concepts* B and *concepts* C are defined as follows:

$$
\begin{array}{lll}
S ::= P_i \mid G_i, & & R ::= S \mid S^-, \\
B ::= \bot \mid A_i \mid \geq q\, R, & & \\
C ::= B \mid \neg C \mid C_1 \sqcap C_2 \mid C_1 \,\mathcal{U}\, C_2 \mid C_1 \,\mathcal{S}\, C_2, & &
\end{array}
$$

where $q \geq 1$ is a natural number (the results obtained below do not depend on whether q is given in unary or binary). A $T_{\mathcal{US}}\textit{DL-Lite}^{\mathcal{N}}_{bool}$ *interpretation* is a function $\mathcal{I}$ on the integers $\mathbb{Z}$ (the intended flow of time):

$$\mathcal{I}(n) = \big(\Delta^{\mathcal{I}}, a_0^{\mathcal{I}}, \dots, A_0^{\mathcal{I}(n)}, \dots, P_0^{\mathcal{I}(n)}, \dots, G_0^{\mathcal{I}(n)}, \dots\big),$$

where $\Delta^{\mathcal{I}}$ is a nonempty set, the (constant) domain of $\mathcal{I}$, $a_i^{\mathcal{I}} \in \Delta^{\mathcal{I}}$, $A_i^{\mathcal{I}(n)} \subseteq \Delta^{\mathcal{I}}$ and $P_i^{\mathcal{I}(n)}, G_i^{\mathcal{I}(n)} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ with $G_i^{\mathcal{I}(n)} = G_i^{\mathcal{I}(m)}$, for $i \in \mathbb{N}$ and $n, m \in \mathbb{Z}$. We adopt the unique name assumption according to which $a_i^{\mathcal{I}} \neq a_j^{\mathcal{I}}$, for $i \neq j$, although our complexity results would not change if we dropped it, apart from the NLogSpace bound of Theorem 4, which would increase to NP [5]. The role and concept constructs are interpreted in $\mathcal{I}$ as follows:

$$
\begin{array}{l}
(S^-)^{\mathcal{I}(n)} = \{(y,x) \mid (x,y) \in S^{\mathcal{I}(n)}\}, \quad \bot^{\mathcal{I}(n)} = \emptyset, \quad (\neg C)^{\mathcal{I}(n)} = \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}(n)}, \\
(C_1 \sqcap C_2)^{\mathcal{I}(n)} = C_1^{\mathcal{I}(n)} \cap C_2^{\mathcal{I}(n)}, \quad (\geq q\, R)^{\mathcal{I}(n)} = \big\{x \mid \sharp\{y \mid (x,y) \in R^{\mathcal{I}(n)}\} \geq q\big\}, \\
(C_1 \,\mathcal{U}\, C_2)^{\mathcal{I}(n)} = \bigcup_{k>n}\big(C_2^{\mathcal{I}(k)} \cap \bigcap_{n<m<k} C_1^{\mathcal{I}(m)}\big), \\
(C_1 \,\mathcal{S}\, C_2)^{\mathcal{I}(n)} = \bigcup_{k<n}\big(C_2^{\mathcal{I}(k)} \cap \bigcap_{n>m>k} C_1^{\mathcal{I}(m)}\big).
\end{array}
$$

Note that our *until* and *since* operators are 'strict' (i.e., do not include the current moment). We also use the temporal operators $\Diamond_F$ ('some time in the future'), $\Diamond_P$ ('some time in the past'), $\diamondsuit\!\!\!\!\!+$ ('some time'), their duals $\Box_F$, $\Box_P$ and $\boxast$, $\bigcirc_F$ ('next time') and $\bigcirc_P$ ('previous time'), which are all expressible by means of $\mathcal{U}$ and $\mathcal{S}$, e.g., $\Diamond_F C = \neg\bot\mathcal{U}C$, $\Box_F C = \neg\Diamond_F\neg C$, $\bigcirc_F C = \bot\mathcal{U}C$, $\diamondsuit\!\!\!\!\!+\, C = \Diamond_F\Diamond_P C$ and $\boxast\, C = \Box_F\Box_P C$. (Other standard abbreviations we use include $C_1 \sqcup C_2$, $\exists R$ and $\top = \neg\bot$.) Apart from full $T_{\mathcal{US}}\textit{DL-Lite}^{\mathcal{N}}_{bool}$, we consider a few of its sublanguages allowing only some of the (definable) temporal operators mentioned above:

- $T_{FP}\textit{DL-Lite}^{\mathcal{N}}_{bool}$, which allows only $\Diamond_F C$, $\Diamond_P C$ and their duals (but no $\bigcirc_F C$ or $C_1 \,\mathcal{U}\, C_2$), and its extension $T_{FPX}\textit{DL-Lite}^{\mathcal{N}}_{bool}$ with $\bigcirc_F C$ and $\bigcirc_P C$;
- $T_U\textit{DL-Lite}^{\mathcal{N}}_{bool}$, allowing only $\diamondsuit\!\!\!\!\!+\, C$ and $\boxast\, C$, and its extension $T_{UX}\textit{DL-Lite}^{\mathcal{N}}_{bool}$ with $\bigcirc_F C$ and $\bigcirc_P C$.

A *TBox*, $\mathcal{T}$, in any of our languages $\mathcal{L}$ is a finite set of *concept inclusions* (CIs) of the form $C_1 \sqsubseteq C_2$, where the C_i are $\mathcal{L}$-concepts. An *ABox*, $\mathcal{A}$, consists of assertions of the form $\bigcirc^n B(a)$ and $\bigcirc^n S(a,b)$, where B is a basic concept, S a (local or rigid) role name, a,b object names and $\bigcirc^n$, for $n \in \mathbb{Z}$, is a sequence of n operators $\bigcirc_F$ if $n \geq 0$ and $|n|$ operators $\bigcirc_P$ if $n < 0$. Taken together, the TBox $\mathcal{T}$ and ABox $\mathcal{A}$ form the *knowledge base* (KB) $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ in $\mathcal{L}$.

The *truth-relation* is defined as usual: $\mathcal{I} \models C_1 \sqsubseteq C_2$ iff $C_1^{\mathcal{I}(n)} \subseteq C_2^{\mathcal{I}(n)}$, for all $n \in \mathbb{Z}$, that is, we interpret concept inclusions globally, $\mathcal{I} \models \bigcirc^n B(a)$ iff $a^{\mathcal{I}} \in B^{\mathcal{I}(n)}$, and $\mathcal{I} \models \bigcirc^n S(a,b)$ iff $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in S^{\mathcal{I}(n)}$. We call $\mathcal{I}$ a *model* of a KB $\mathcal{K}$ and write $\mathcal{I} \models \mathcal{K}$ if $\mathcal{I} \models \alpha$ for all α in $\mathcal{K}$. If $\mathcal{K}$ has a model then it is said to be *satisfiable*. A concept C (role R) is *satisfiable* w.r.t. $\mathcal{K}$ if there are a model $\mathcal{I}$ of $\mathcal{K}$ and $n \in \mathbb{Z}$ such that $C^{\mathcal{I}(n)} \neq \emptyset$ (respectively, $R^{\mathcal{I}(n)} \neq \emptyset$). Clearly, the concept and role satisfiability problems are equivalent to KB satisfiability.

Our first result states that the satisfiability problem for $T_{\mathcal{US}}$*DL-Lite*$_{bool}^{\mathcal{N}}$ KBs is as complex as satisfiability in propositional temporal logic *LTL*.

Theorem 1. *Satisfiability of* $T_{\mathcal{US}}$*DL-Lite*$_{bool}^{\mathcal{N}}$ *KBs is* PSPACE*-complete.*

The proof is by a two-step (non-deterministic polynomial) reduction to *LTL*. First, we reduce satisfiability of a $T_{\mathcal{US}}$*DL-Lite*$_{bool}^{\mathcal{N}}$ KB $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ to satisfiability in the one-variable first-order temporal logic in a way similar to [8]. For each basic concept B ($\neq \bot$), we take a fresh *unary* predicate $B^*(x)$ and encode $\mathcal{T}$ as

$$\mathcal{T}^\dagger \quad = \bigwedge_{C_1 \sqsubseteq C_2 \in \mathcal{T}} \boxast \forall x \left(C_1^*(x) \to C_2^*(x) \right),$$

where the C_i^* are the results of replacing each B with $B^*(x)$ ($\sqcap$ with $\wedge$, etc.). We assume that $\mathcal{T}$ contains CIs of the form $\geq q\, R \sqsubseteq\, \geq q'\, R$, for $\geq q\, R$, $\geq q'\, R$ in $\mathcal{T}$ such that $q > q'$ and there is no q'' with $q > q'' > q'$ and $\geq q''\, R$ in $\mathcal{T}$. We also assume that $\mathcal{T}$ contains $\geq q\, R \equiv \boxast \geq q\, R$ if $\geq q\, R$ occurs in $\mathcal{T}$, for a rigid role R (i.e., for G_i or G_i^-). To take account of the fact that roles are *binary* relations, we add to $\mathcal{T}^\dagger$ the following formula, for each role name S:

$$\varepsilon_S \quad = \quad \boxast \big(\exists x\, (\exists S)^*(x) \leftrightarrow \exists x\, (\exists S^-)^*(x) \big)$$

(which says that at each moment of time the domain of S is nonempty iff its range is nonempty). The ABox $\mathcal{A}$ is encoded by a conjunction $\mathcal{A}^\dagger$ of ground atoms of the form $\bigcirc^m B^*(a)$ and $\bigcirc^n (\geq q\, R)^*(a)$ in the same way as in [8]. Thus, $\mathcal{K}$ is satisfiable iff the formula

$$\mathcal{K}^\dagger \quad = \quad \mathcal{T}^\dagger \quad \wedge \quad \bigwedge_S \varepsilon_S \quad \wedge \quad \mathcal{A}^\dagger$$

is satisfiable. The second step of our reduction is based on the observation that if $\mathcal{K}^\dagger$ is satisfiable then it can be satisfied in a model such that

(R) if $(\exists S)^*(x)$ is true at some moment (on some domain element) then it is true at all moments of time (perhaps on different domain elements).

Indeed, if $\mathcal{K}^\dagger$ is satisfied in $\mathcal{I}$ then it is satisfied in the disjoint union $\mathcal{I}^*$ of all $\mathcal{I}^n$, $n \in \mathbb{Z}$, obtained from $\mathcal{I}$ by shifting its time line n moments forward. It follows from **(R)** that $\mathcal{K}^\dagger$ is satisfiable iff there is a set Σ of role names such that

$$\mathcal{K}^{\dagger\Sigma} \;=\; \mathcal{T}^\dagger \;\wedge\; \bigwedge_{S\in\Sigma}\big((\exists S)^*(d_S)\wedge(\exists S^-)^*(d_{S^-})\big) \;\wedge\; \bigwedge_{S\notin\Sigma} \boxast\forall x\,\neg\big((\exists S)^*(x)\vee(\exists S^-)^*(x)\big) \;\wedge\; \mathcal{A}^\dagger$$

is satisfiable, where the d_S are fresh constants (informally, the roles in Σ are nonempty at some moment, whereas all other roles are always empty). Finally, as $\mathcal{K}^{\dagger\Sigma}$ contains no existential quantifiers, it can be regarded as an *LTL*-formula because all the universal quantifiers can be instantiated by all the constants in the formula, which results only in a polynomial blow-up of $\mathcal{K}^{\dagger\Sigma}$.

This reduction can also be used to obtain complexity results for the fragments of $T_{\mathcal{US}}$*DL-Lite*$^{\mathcal{N}}_{bool}$ mentioned above. Using the well-known facts that satisfiability in the fragments of *LTL* with $\Diamond_F/\Diamond_P$ and with $\diamondast$ is NP-complete, and that the extension of any of these fragments with $\bigcirc_F/\bigcirc_P$ becomes PSPACE-complete again, we obtain:

Theorem 2. (i) *Satisfiability of* T_{FP}*DL-Lite*$^{\mathcal{N}}_{bool}$ *and* T_U*DL-Lite*$^{\mathcal{N}}_{bool}$ *KBs is* NP-*complete.* (ii) *For* T_{FPX}*DL-Lite*$^{\mathcal{N}}_{bool}$ *and* T_{UX}*DL-Lite*$^{\mathcal{N}}_{bool}$ *KBs, satisfiability is* PSPACE-*complete.*

3 Temporal DLs based on *DL-Lite*$^{\mathcal{N}}_{core}$

So far, to decrease complexity we have restricted the expressive power of the temporal component of $T_{\mathcal{US}}$*DL-Lite*$^{\mathcal{N}}_{bool}$. But the underlying DL *DL-Lite*$^{\mathcal{N}}_{bool}$ also has some natural fragments of lower complexity [5]. In this section, we consider the simplest of them known as *DL-Lite*$^{\mathcal{N}}_{core}$ and containing only CIs of the form $B_1 \sqsubseteq B_2$ and $B_1 \sqcap B_2 \sqsubseteq \bot$, where the B_i are basic concepts. Satisfiability of *DL-Lite*$^{\mathcal{N}}_{core}$ KBs is NLOGSPACE-complete.

Let $T_{\mathcal{US}}$*DL-Lite*$^{\mathcal{N}}_{core}$ be the fragment of $T_{\mathcal{US}}$*DL-Lite*$^{\mathcal{N}}_{bool}$ with CIs of the form $D_1 \sqsubseteq D_2$ and $D_1 \sqcap D_2 \sqsubseteq \bot$, where the D_i are defined by the rule:

$$D \;::=\; B \;\mid\; B_1\,\mathcal{U}\,B_2 \mid\; B_1\,\mathcal{S}\,B_2.$$

By restricting D_1 and D_2 to concepts of the form

$$D \;::=\; B \;\mid\; \Diamond_F B \;\mid\; \Diamond_P B \;\mid\; \Box_F B \;\mid\; \Box_P B$$

we obtain T_{FP}*DL-Lite*$^{\mathcal{N}}_{core}$. These restrictions do not improve the complexity of reasoning: satisfiability of $T_{\mathcal{US}}$*DL-Lite*$^{\mathcal{N}}_{core}$ KBs is PSPACE-complete, while for T_{FP}*DL-Lite*$^{\mathcal{N}}_{core}$ it is NP-complete [8].

What is really surprising and nontrivial is that extending T_{FP}*DL-Lite*$^{\mathcal{N}}_{core}$ with the next- and previous-time operators does not increase the complexity; cf. Theorem 2 (ii). More formally, define T_{FPX}*DL-Lite*$^{\mathcal{N}}_{core}$ by restricting D_1 and D_2 to concepts of the form:

$$D \;::=\; B \;\mid\; \Diamond_F B \;\mid\; \Diamond_P B \;\mid\; \Box_F B \;\mid\; \Box_P B \;\mid\; \bigcirc_F B \;\mid\; \bigcirc_P B,$$

and let T_{UX}*DL-Lite*$^{\mathcal{N}}_{core}$ be the logic with the D_i of the form:

$$D \;::=\; B \;\mid\; \diamondast B \;\mid\; \boxast B \mid\; \bigcirc_F B \;\mid\; \bigcirc_P B.$$

Theorem 3. *Satisfiability of T_{FPX}DL-Lite$^{\mathcal{N}}_{core}$ and T_{UX}DL-Lite$^{\mathcal{N}}_{core}$ KBs is* NP-*complete.*

We present only a sketch of the proof here; the full proof can be found in Section A of the Appendix.

In a way similar to the proof of Theorem 1, one can (non-deterministically and polynomially) reduce satisfiability of a T_{FPX}*DL-Lite*$^{\mathcal{N}}_{core}$ KB to satisfiability of an *LTL*-formula $\varphi = \bigwedge_i \boxast(E_i \vee E'_i) \wedge \psi$, where the E_i and E'_i are of the form p, $\Diamond_F p$, $\Diamond_P p$, $\Box_F p$, $\Box_P p$, $\bigcirc_F p$, $\bigcirc_P p$ or a negation thereof, and ψ is a conjunction of formulas of the form $\bigcirc^n p$, p a propositional variable. Let Γ be the set of all subformulas of φ of the form $\Diamond_F p$, $\Diamond_P p$, $\Box_F p$ or $\Box_P p$. It should be clear that if φ is satisfied in an interpretation then the flow of time can be partitioned into $|\Gamma| + 1$ intervals $I_0, \dots, I_{|\Gamma|}$ such that, for each $\gamma \in \Gamma$ and each I_i, γ is true at *some* point in I_i iff γ is true at *every* point in I_i. The existence of such intervals can be expressed by certain syntactic conditions on their 'states,' the most crucial of which is satisfiability of a formula of the form

$$\chi \quad = \quad \Psi \wedge \Box^{\leq m}\Phi \wedge \bigcirc^m(\Psi' \wedge \bigcirc\Psi''),$$

for $\Phi = \bigwedge_i (D_i \vee D'_i)$, with each of the D_i and D'_i being a literal L (a propositional variable or its negation) or $\bigcirc L$, conjunctions Ψ, Ψ' and Ψ'' of literals, and $m \geq 0$, where $\bigcirc^n\Psi$ is the result of attaching n operators $\bigcirc$ to each literal in Ψ and $\Box^{\leq m}\Phi = \bigwedge_{0 \leq i \leq m} \bigcirc^i \Phi$. Intuitively, m is the number of distinct states in an interval I_i, Ψ and Ψ' are the first and the last states in I_i, Ψ'' is the first state of the next interval I_{i+1}, and Φ a set of binary clauses that describe possible transitions between the states. Let $cons^m_\Phi(\Psi)$ be the set of all literals L that are true at the moment $m \geq 0$ in every model of $\Psi \wedge \Box^{\leq m}\Phi$. As the formula $\Psi \wedge \Box^{\leq m}\Phi$ is essentially a 2CNF, one can compute $cons^m_\Phi(\Psi)$ inductively as follows:

$$cons^0_\Phi(\Psi) = \{L \mid \Phi \cup \Psi \models L\},$$
$$cons^m_\Phi(\Psi) = \{L \mid \Phi \models L' \to \bigcirc L, L' \in cons^{m-1}_\Phi(\Psi)\} \cup \{L \mid \Phi \models L\}.$$

For each L, construct a non-deterministic finite automaton $\mathfrak{A}_L = (Q, Q_0, \sigma, F_L)$ over the alphabet $\{0\}$ that accepts 0^m iff $L \in cons^m_\Phi(\Psi)$. Define the states in Q to be all the literals from χ, the set of initial states $Q_0 = cons^0_\Phi(\Psi)$, the accepting states $F_L = \{L\}$, and the transition relation

$$\sigma = \{(L'', L') \mid \Phi \models L'' \to \bigcirc L'\} \cup \{(L', L') \mid \Phi \models L'\}.$$

Then a state L is reachable in m σ-steps from a state in Q_0 iff $L \in cons^m_\Phi(\Psi)$, and so $\mathfrak{A}_L$ is as required. Every such $\mathfrak{A}_L$ can be converted into an equivalent automaton in the Chrobak normal form [14] using Martinez's algorithm [29], which gives rise to M_L-many arithmetic progressions $a^L_1 + b^L_1\mathbb{N}, \dots, a^L_{M_L} + b^L_{M_L}\mathbb{N}$, where $a + b\mathbb{N} = \{a + bn \mid n \in \mathbb{N}\}$, such that

(A$_1$) $M_L, a^L_i, b^L_i \leq |\Phi \cup \Psi|^2$, for $1 \leq i \leq M_L$, and
(A$_2$) $L \in cons^m_\Phi(\Psi)$ iff $m \in \bigcup_{i=1}^{M_L}(a^L_i + b^L_i\mathbb{N})$.

Satisfiability of χ can now be established by a polynomial-time algorithm which checks whether the following three conditions hold:

1. $p, \neg p \in cons^n_\Phi(\Psi)$, for no variable p and no $0 \leq n \leq m+1$;
2. $\neg L \notin cons^m_\Phi(\Psi)$, for all literals $L \in \Psi'$;
3. $\neg L \notin cons^{m+1}_\Phi(\Psi)$, for all literals $L \in \Psi''$.

To verify 1, we check, for each variable p, whether the linear Diophantine equations $a^p_i + b^p_i x = a^{\neg p}_j + b^{\neg p}_j y$, for $1 \leq i \leq M_p$ and $1 \leq j \leq M_{\neg p}$, have a solution (x_0, y_0) such that $0 \leq a^p_i + b^p_i x_0 \leq m+1$. Set $a = b^p_i$, $b = -b^{\neg p}_j$ and $c = a^{\neg p}_j - a^p_i$, which gives us the equation $ax + by = c$. If $a \neq 0$ and $b \neq 0$ then, by Bézout's lemma, it has a solution iff c is a multiple of the greatest common divisor d of a and b, which can be checked in polynomial time using the Euclidean algorithm (provided that the numbers are encoded in unary, which can be assumed in view of $(\mathbf{A}_1)$). Moreover, if the equation has a solution, then the Euclidean algorithm also gives us a pair (u_0, v_0) such that $d = au_0 + bv_0$, in which case all the solutions of the above equation form the set $\big\{\big((cu_0 + bk)/d, (cv_0 - ak)/d\big) \mid k \in \mathbb{Z}\big\}$. Thus, it remains to check whether a number between 0 and $m+1$ is contained in $a^p_i + b^p_i(a^{\neg p}_j - a^p_i)u_0/d + b^p_i b^{\neg p}_j/d\mathbb{N}$. The case $a = 0$ or $b = 0$ is left to the reader. To verify condition 2, we check, for each $L \in \Psi'$, whether m belongs to one of $a^{\neg L}_i + b^{\neg L}_i\mathbb{N}$, for $1 \leq i \leq M_L$, which can be done in polynomial time. Condition 3 is analogous. This gives us the NP upper bound for the logics mentioned in Theorem 3. The lower bound can be proved by reduction of the 3-colourability problem to satisfiability of T_{UX}*DL-Lite*$^{\mathcal{N}}_{core}$ KBs.

Theorem 3 shows that T_{FPX}*DL-Lite*$^{\mathcal{N}}_{core}$ can be regarded as a good candidate for representing temporal conceptual data models. Although not able to express covering constraints, T_{FPX}*DL-Lite*$^{\mathcal{N}}_{core}$ still appears to be a reasonable compromise compared to the full PSPACE-complete logic T_{FPX}*DL-Lite*$^{\mathcal{N}}_{bool}$.

By restricting the temporal constructs to the undirected universal modalities $⊞$ and $◈$, we obtain an even simpler logic:

Theorem 4. *Satisfiability of* T_U*DL-Lite*$^{\mathcal{N}}_{core}$ *KBs is* NLOGSPACE-*complete.*

The proof of the upper bound is by embedding into the universal Krom fragment of first-order logic.

4 Temporal DLs with Temporalised Roles

As we have seen before, in order to express lifespan cardinalities, temporal operators on roles are required. Modalised roles are known to be 'dangerous' and very difficult to deal with when temporalising expressive DLs such as $\mathcal{ALC}$ [17, Section 14.2]. To our surprise, even in the case of *DL-Lite*, temporal operators on roles may cause undecidability (while rigid roles are 'mostly harmless'). Denote by $T^{\mathcal{R}}_X$*DL-Lite*$^{\mathcal{N}}_{bool}$ the fragment of $T_{\mathcal{US}}$*DL-Lite*$^{\mathcal{N}}_{bool}$ with $\bigcirc_F$ as the only temporal operator over concepts and with roles R of the form

$$R \quad ::= \quad S \quad | \quad S^- \quad | \quad ◈ R \quad | \quad ⊞ R.$$

The extensions of $◈ R$ and $⊞ R$ in an interpretation $\mathcal{I}$ are defined as follows:

$$(◈ R)^{\mathcal{I}(n)} = \bigcup_{k \in \mathbb{Z}} R^{\mathcal{I}(k)} \quad \text{and} \quad (⊞ R)^{\mathcal{I}(n)} = \bigcap_{k \in \mathbb{Z}} R^{\mathcal{I}(k)}.$$

Theorem 5. *Satisfiability of $T_X^{\mathcal{R}}$DL-Lite$_{bool}^{\mathcal{N}}$ KBs is undecidable.*

The proof is by reduction of the $\mathbb{N} \times \mathbb{N}$-tiling problem: given a finite set T of tile types $t = (up(t), down(t), left(t), right(t))$, decide whether T can tile the $\mathbb{N} \times \mathbb{N}$-grid. We assume that the tiles use k colours numbered from 1 to k.

We construct a $T_X^{\mathcal{R}}$DL-Lite$_{bool}^{\mathcal{N}}$ KB $\mathcal{K}_T$ such that $\mathcal{K}_T$ is satisfiable iff T tiles $\mathbb{N} \times \mathbb{N}$. The temporal dimension clearly provides us with one of the two axes of the grid. The other axis is constructed from the domain elements: let R be a role such that $\geq 2 \diamondast R \sqsubseteq \bot$ and $\geq 2 \diamondast R^- \sqsubseteq \bot$. In other words, if xRy at some moment of time then there is no $y' \neq y$ with xRy' at any moment of time (and the same for R^-). We can generate an infinite sequence of the domain elements by saying that $\exists R^- \sqcap \bigcirc_F \exists R^-$ is nonempty and $\exists R^- \sqcap \bigcirc_F \exists R^- \sqsubseteq \exists R \sqcap \bigcirc_F \exists R$. (The reason for generating the R-arrows at two consecutive moments of time will become apparent below.) It should be also noted that the produced sequence may in fact be either a finite loop or an infinite sequence of distinct elements.

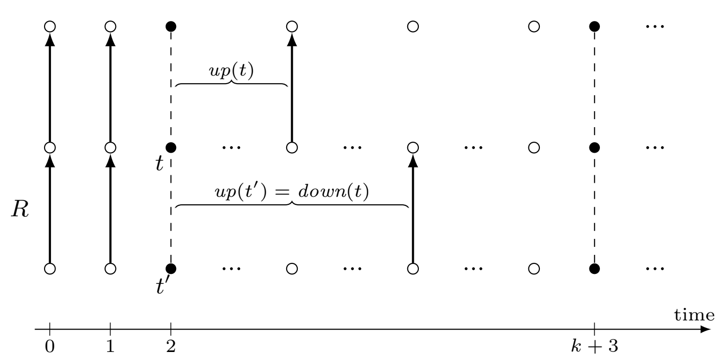
Now, let t be a fresh concept name, for each $t \in T$, and let tile types be disjoint, i.e., $t \sqcap t' \sqsubseteq \bot$ for $t \neq t'$. After the double R-arrows we place the first column of tiles, and every $k+1$ moments afterwards we place a column of tiles that matches the colours of the previous column:

$$\exists R^- \sqcap \bigcirc_F \exists R^- \sqsubseteq \bigsqcup_{t \in T} \bigcirc_F \bigcirc_F t, \qquad t \sqsubseteq \bigsqcup_{right(t)=left(t')} \bigcirc_F^{k+1} t', \quad \text{for each } t \in T.$$

It remains to ensure that the tiles are arranged in a proper grid and have matching top-bottom colours. It is for this purpose that we have (i) used the double R-arrows to generate the sequence of domain elements, and (ii) placed the columns of tiles every $k+1$ moments of time (not every moment). Consider the following CIs, for $t \in T$ and $1 \leq i \leq k$:

$$t \sqsubseteq \neg \exists R^-, \quad t \sqsubseteq \neg \bigcirc_F^i \exists R^- \ \ (\text{if } i \neq down(t)) \quad \text{and} \quad t \sqsubseteq \bigcirc_F^{up(t)} \exists R.$$

The first two CIs ensure that between any two tiles $k+1$ moments apart there may be only one incoming R-arrow. This, in particular, means that after the double R-arrows no other two consecutive R-arrows are possible, and thus the proper $\mathbb{N} \times \mathbb{N}$-grid is ensured. Moreover, the exact position of the incoming R-arrow is uniquely determined by the *down*-colour of the tile, which in conjunction with the last CI guarantees that this colour matches the tile below. The following picture illustrates the construction:



Note that the next-time operator $\bigcirc_F$ is heavily used in the encoding above. If we replace it with $\Diamond\!\!\!\!\diamond$ and $\boxast$ on concepts, then reasoning in the resulting logic $T_U^{\mathcal{R}}\mathit{DL\text{-}Lite}_{bool}^{\mathcal{N}}$ becomes much simpler:

Theorem 6. *Satisfiability of $T_U^{\mathcal{R}}\mathit{DL\text{-}Lite}_{bool}^{\mathcal{N}}$ KBs is* NP*-complete.*

This result is proved using a modification of the quasimodel construction from [7, 8]: we show that a KB is satisfiable iff there exists a *quasimodel* of polynomial size. In the *types* of our quasimodels, concepts $\geq q\, R$, $\geq q \diamondast R$ and $\geq q \boxast R$ reflect the number of R-successors of the element required, respectively, in the current moment of time, 'sometime' ($\diamondast R$-successors) and 'always' ($\boxast R$-successors). In order to deal with temporalised roles, we have to introduce the following conditions on quasimodels: (*i*) the numbers of $\diamondast R$-successors and $\boxast R$-successors in types do not change along a *run* (in other words, temporalised roles are rigid roles); (*ii*) the number of R-successors in every type is sandwiched between the number of $\boxast R$- and the number of $\diamondast R$-successors; (*iii*) if there is a run with more $\diamondast R$-successors than $\boxast R$-successors, then there is a run with more $\diamondast R^-$-successors than $\boxast R^-$-successors; (*iv*) in each run with more $\diamondast R$-successors than $\boxast R$-successors, not all R-successors are $\boxast R$-successors, and not all $\diamondast R$-successors are R-successors at all moments of time. Special conditions are also required for the runs on the objects in the ABox. Full details can be found in Section B in the Appendix.

5 Conclusion

From the complexity-theoretic point of view, the best candidates for reasoning about TCMs appear to be $T_{FPX}\mathit{DL\text{-}Lite}_{core}^{\mathcal{N}}$ and $T_{FPX}\mathit{DL\text{-}Lite}_{bool}^{\mathcal{N}}$: the former is NP-complete and the latter PSPACE-complete. Moreover, we believe that the reduction of $T_{FPX}\mathit{DL\text{-}Lite}_{core}^{\mathcal{N}}$ to *LTL* in the proof of Theorem 3 can be done deterministically, in which case one can use standard *LTL* provers for TCM reasoning. We also believe that $T_{FPX}\mathit{DL\text{-}Lite}_{core}^{\mathcal{N}}$ extended with temporalised roles can be decidable, which remains one of the most challenging open problems. But it seems to be next to impossible to reason in an effective way about all TCM constraints without any restrictions.

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Appendix

A Proof of Theorem 3

Theorem 3. *Satisfiability for* T_{FPX}*DL-Lite*$^{\mathcal{N}}_{core}$ *and* T_{UX}*DL-Lite*$^{\mathcal{N}}_{core}$ *KBs is* NP*-complete.*

Proof. Let φ be a propositional temporal formula of the form

$$\bigwedge_{1\leq i\leq n} \boxast(D_i \vee D_i') \quad \wedge \bigwedge_{1\leq j\leq k} \boxast(\overline{\Diamond L_j} \leftrightarrow \Diamond L_j) \quad \wedge \quad \psi,$$

where ψ is a propositional formula containing no temporal operators, the D_i and D_i' are temporal literals of the form

$$D \quad ::= \quad L \quad | \quad \bigcirc L,$$

L is a literal (i.e., a variable p or its negation $\neg p$) and $\overline{\Diamond L_j}$ is a variable, called the *surrogate* of $\Diamond L_j$, that can be used in the D_i and D_i'. It should be clear that if φ is satisfied in a model then the flow of time can be partitioned into $k+1$ intervals $I_0, \dots, I_k$ such that $\Diamond L_j$ is true at some point in I_i if and only if $\Diamond L_j$ is true at every point in I_i, for each interval I_i and each subformula $\Diamond L_j$ of φ. Therefore, φ is satisfiable if and only if there are $k+2$ natural numbers

$$m_0 < m_1 < \dots < m_{k+1},$$

and $k+3$ quadruples

$$(\Psi_0, \Xi_0, \Theta_0, \Psi_0'), \dots, (\Psi_{k+2}, \Xi_{k+2}, \Theta_{k+2}, \Psi_{k+2}')$$

of sets of literals such that

1. $m_0, \dots, m_{k+1}$ do not exceed $2^{|\varphi|}$;
2. for each $0 \leq i \leq k+2$ and
   - for each variable p in φ, either $p \in \Psi_i$ or $\neg p \in \Psi_i$ and either $p \in \Psi_i'$ or $\neg p \in \Psi_i'$;
   - for each $\overline{\Diamond L}$ in φ, either $\overline{\Diamond L} \in \Xi_i$ or $\neg\overline{\Diamond L} \in \Xi_i$;
3. for each $0 \leq i < k+1$ and each $\overline{\Diamond L}$ in φ,
   - if $\overline{\Diamond L} \in \Xi_{i+1}$ then $\overline{\Diamond L} \in \Xi_i$ and
   - if $\overline{\Diamond L} \in \Xi_i \setminus \Xi_{i+1}$ then $\bigcirc\neg L \in \Theta_{i+1}, \dots, \Theta_{k+1}$ and $L \in \Psi_{i+1}$;
4. for each $0 \leq i < k+2$,

$$\Psi_i \wedge \Box^{\leq m_i}(\Phi \wedge \Xi_i \wedge \Theta_i) \wedge \bigcirc^{m_i}\big(\Psi_i' \wedge \bigcirc\Psi_{i+1} \wedge \bigcirc\Xi_{i+1}\big),$$

   is satisfiable, where Φ is the set of $D_i \vee D_i'$, $1 \leq i \leq n$, $\bigcirc^\ell\Psi$ is the result of attaching ℓ operators $\bigcirc$ to each literal in Ψ and $\Box^{\leq m}\Psi = \bigwedge_{0\leq i\leq m} \bigcirc^i\Psi$;
5. $\psi \wedge \Psi_{i_0}$ is consistent, for some $0 \leq i_0 \leq k+2$;
6. $\Psi_{k+1}' = \Psi_{k+1}$ and the last two quadruples coincide.

The intuition behind the quadruples is as follows: Ψ_i and Ψ_i' are the first and the last state on the ith interval, respectively, Ξ_i are the surrogate literals ($\overline{\Diamond L}$ or $\neg\overline{\Diamond L}$) that are true throughout the ith interval and Θ_i are the literals that are true in the model starting from the second state of the ith interval (these sets are related to false subformulas $\Diamond L$). So, Condition 2 ensures that the sets of literals are saturated. Condition 3 guarantees that the surrogates $\overline{\Diamond L}$ behave exactly as the respective $\Diamond L$ subformulas. Condition 4 means that, for each i, there exists a snippet of the model of length $m_i + 1$ such that Ψ_i, Ψ_i' and Ψ_{i+1} hold at 0, m_i and $m_i + 1$, respectively, the snippet satisfies Φ throughout its length and has the required surrogates, Ξ_i and Ξ_{i+1}, and literals Θ_i true. It then follows that the last state of the ith snippet coincides with the first state of the $(i + 1)$th snippet and thus, they can be glued together in a sequence. Condition 5 guarantees that ψ is true somewhere in the resulting sequence and Condition 6 allows us to repeat the last snippet infinitely many times and thus to construct a model satisfying φ. It should be clear that Conditions 2, 3, 5 and 6 can be checked in non-deterministic polynomial time. So, it remains to show that Condition 4 can also be checked in polynomial time.

Thus, the problem is as follows: given a set Φ of formulas of the form $D_i \vee D_i'$, sets of literals Ψ, Ψ' and Ψ'' and $m \geq 0$, decide whether there is an interpretation satisfying

$$\Psi \wedge \Box^{\leq m}\Phi \wedge \bigcirc^m(\Psi' \wedge \bigcirc\Psi''). \tag{1}$$

First, we find a compact representation of the set $cons^m_\Phi(\Psi)$ of all literal consequences of Ψ with respect to $\Box^{\leq m}\Phi$ at moment $m \geq 0$, i.e., $L \in cons^m_\Phi(\Psi)$ if and only if L is true at m in any interpretation satisfying $\Psi \wedge \Box^{\leq m}\Phi$.

Lemma 1. *$cons^0_\Phi(\Psi) = \{L \mid \Phi \cup \Psi \models L\}$ and $cons^m_\Phi(\Psi) = \{L \mid \Phi \models L' \to \bigcirc L, \quad L' \in cons^{m-1}_\Phi(\Psi)\} \cup \{L \mid \Phi \models L\}$, for $m \geq 1$.*

Proof. The formula $\Psi \wedge \Box^{\leq m}\Phi$ may be regarded as a 2CNF (by treating each $\bigcirc^n L$ as a fresh propositional variable). Clearly, for each $L \in cons^m_\Phi(\Psi)$, $\bigcirc^m L$ is a logical consequence of this 2CNF. Conversely, let $\bigcirc^m L$ be a logical consequence of the 2CNF. We prove by induction on m that $L \in cons^m_\Phi(\Psi)$. The base of induction ($m = 0$) is clear. Suppose $m > 1$. Take a minimal set of binary clauses that implies $\bigcirc^m L$. If it involves only clauses from $\bigcirc^m\Phi$ then we are done. Otherwise, it must involve a clause of the form $L' \to \bigcirc L$ such that $\Phi \models L' \to \bigcirc L$ and $\bigcirc^{m-1}L'$ is a logical consequence of $\Psi \wedge \Box^{\leq m-1}\Phi$, which by IH, means $L' \in cons^{m-1}_\Phi(\Psi)$.

Now, for each literal L, we construct a non-deterministic finite automaton $\mathfrak{A}_L = (Q, Q_0, \sigma, F_L)$ over the alphabet $\{0\}$ that accepts 0^m if and only if $L \in cons^m_\Phi(\Psi)$. Define the states in Q to be all the literals, the set of initial states $Q_0 = cons^0_\Phi(\Psi)$, the accepting states $F_L = \{L\}$ and

$$\sigma = \{(L'', L') \mid \Phi \models L'' \to \bigcirc L'\} \cup \{(L', L') \mid \Phi \models L'\}.$$

By Lemma 1, a state L is reachable in m σ-steps from Q_0 if and only if $L \in cons^m_\Phi(\Psi)$, and so the automaton $\mathfrak{A}_L$ is as requited. Every such $\mathfrak{A}_L$ can be converted into an equivalent automaton in the Chrobak normal form [14] using Martinez's algorithm [29]. The automaton in the Chrobak normal form gives rise to M_L arithmetic progressions

$$a_1^L + b_1^L\mathbb{N}, \quad \dots, \quad a_{M_L}^L + b_{M_L}^L\mathbb{N},$$

where $a + b\mathbb{N} = \{a + bn \mid n \in \mathbb{N}\}$, such that

(A$_1$) $M_L \leq |\Phi \cup \Psi|^2$ and $a_i^L, b_i^L \leq |\Phi \cup \Psi|^2$, for $1 \leq i \leq M_L$, and
(A$_2$) $L \in \mathit{cons}_\Phi^m(\Psi)$ if and only if $m \in \bigcup_{i=1}^{M_L}(a_i^L + b_i^L\mathbb{N})$.

We are now in a position to present a non-deterministic polynomial-time algorithm that checks whether (1) is consistent. Clearly, (1) is satisfiable if and only if there is

- no variable p and no number n with $0 \leq n \leq m+1$ and $p, \neg p \in \mathit{cons}_\Phi^n(\Psi)$,
- no literal $L \in \Psi'$ with $\neg L \in \mathit{cons}_\Phi^m(\Psi)$ and
- no literal $L \in \Psi''$ with $\neg L \in \mathit{cons}_\Phi^{m+1}(\Psi)$.

To verify the first condition, we check, for each variable p, whether the linear Diophantine equations

$$a_i^p + b_i^p x = a_j^{\neg p} + b_j^{\neg p} y,$$

for $1 \leq i \leq M_p$ and $1 \leq j \leq M_{\neg p}$, have a solution (x, y) such that $a_i^p + b_i^p x \leq m+1$. Let $a = b_i^p$, $b = -b_j^{\neg p}$ and $c = a_j^{\neg p} - a_i^p$. We then have the following equation

$$ax + by = c.$$

If $a \neq 0$ and $b \neq 0$ then, by Bézout's lemma, the equation has a solution if and only if c is a multiple of the greatest common divisor d of a and b. The latter condition can be checked in polynomial time using the Euclidean algorithm (provided that the numbers are encoded in unary, which we can assume in view of **(A$_1$)**). Moreover, if the equation has a solution, then the Euclidean algorithm also gives us a pair (u_0, v_0) such that $d = au_0 + bv_0$, in which case all the solutions of the above equation form the set

$$\big\{\big((cu_0 + bk)/d, (cv_0 - ak)/d\big) \mid k \in \mathbb{Z}\big\}.$$

Thus, it remains to check whether a number between 0 and $m+1$ is contained in $a_i^p + b_i^p(a_j^{\neg p} - a_i^p)u_0/d + b_i^p b_j^{\neg p}/d\mathbb{N}$. Clearly, the above operations can be done in polynomial time.

To verify the second condition, we check whether, for each $L \in \Psi'$, m belongs to one of the sets $a_i^L + b_i^L\mathbb{N}$, for $1 \leq i \leq M_L$, which can clearly be done in polynomial time. The last condition is checked analogously.

B Proof of Theorem 6

Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be a $T_U^{\mathcal{R}}\mathit{DL}\text{-}\mathit{Lite}_{bool}^{\mathcal{N}}$ KB. In what follows, we write $(\geq q\, R)^{\mathcal{I}}$ instead of $(\geq q\, R)^{\mathcal{I}(x)}$, $x \in \mathbb{Z}$, for a *temporalised* role R because such a role is always rigid.

Denote by $ob(\mathcal{A})$ the set of all object names occurring in $\mathcal{A}$ and by $role^{\pm}(\mathcal{K})$ the set of role names in $\mathcal{K}$ and their inverses. Let $Q_{\mathcal{K}} \subseteq \mathbb{N}$ be the set consisting of

- all q such that at least one of $\geq q \boxast R$, $\geq q\, R$ or $\geq q \diamondast R$ occurs in $\mathcal{K}$, for $R \in role^{\pm}(\mathcal{K})$, and
- all the integers from 1 to $|\mathcal{A}|$.

Let $q_{\mathcal{K}} = \max Q_{\mathcal{K}}$.

Lemma 2. *If a $T_U^{\mathcal{R}}$DL-Lite$_{bool}^{\mathcal{N}}$ KB $\mathcal{K}$ is satisfiable then it can be satisfied in a model $\mathcal{I}$ such that $(\geq q_{\mathcal{K}} + 1 \boxast R)^{\mathcal{I}} = \emptyset$, for each $R \in role^{\pm}(\mathcal{K})$.*

Proof. Let $\mathcal{I} \models \mathcal{K}$. Without loss of generality we will assume that, for each $x \in \mathbb{Z}$, the set $\{y \in \mathbb{Z} \mid \mathcal{I}(x) = \mathcal{I}(y)\}$ is infinite and that the domain $\Delta^{\mathcal{I}}$ is at most countable. Suppose that $w \in \Delta^{\mathcal{I}}$ has $(\geq q_{\mathcal{K}} + 1)$ $\boxast R$-successors in $\mathcal{I}$, e.g., assume that the pairs

$$(w, w_1), \dots, (w, w_{q_{\mathcal{K}}}), (w, w_{q_{\mathcal{K}}+1}), \dots$$

are all in $(\boxast R)^{\mathcal{I}}$. We can also assume that if $w_n = a^{\mathcal{I}}$, for some object name a in $\mathcal{A}$, then $n \leq q_{\mathcal{K}}$.

Construct a new model $\mathcal{I}^*$ as follows. The domain of $\mathcal{I}^*$ is $\Delta^{\mathcal{I}} \times \mathbb{N}$ and $a^{\mathcal{I}^*} = (a^{\mathcal{I}}, 0)$ for all object names a. For a concept name A, role name P and $x \in \mathbb{Z}$, we set

$$A^{\mathcal{I}^*(x)} = \{(u, i) \mid u \in A^{\mathcal{I}(x)},\ i \in \mathbb{N}\},$$
$$P^{\mathcal{I}^*(x)} = \{((u, i), (v, i)) \mid (u, v) \in P^{\mathcal{I}(x)},\ i \in \mathbb{N}\}.$$

It should be clear that $\mathcal{I}^* \models \mathcal{K}$. We now remove and redirect some of the R-arrows of the form $((w, i), (w_n, i))$, for *all moments of time*, in the following manner. First, we remove $((w, i), (w_n, i))$ from $(\boxast R)^{\mathcal{I}^*}$, for all n and i such that $n > q_{\mathcal{K}}$ or $i > 0$. Note that this removal does not involve arrows to the interpretations of the ABox elements. To compensate for the loss, we then add new $\boxast R$-arrows of the form $((w, i), (w_n, j))$ to $(\boxast R)^{\mathcal{I}^*}$ in such a way that the following conditions are satisfied:

- for every (w_n, j), there is precisely one $\boxast R$-arrow of the form $((w, i), (w_n, j))$ and, moreover, $i = j$ iff $i = j = 0$ and $n \leq q_{\mathcal{K}}$;
- for every (w, i), there are precisely $q_{\mathcal{K}}$ $\boxast R$-arrows of the form $((w, i), (w_n, j))$.

Such a rearrangement is possible because $\mathcal{I}^*$ contains countably infinitely many copies of the original model $\mathcal{I}$. We leave it to the reader to check that the resulting interpretation is still a model of $\mathcal{K}$. The process is repeated for each point in $\mathcal{I}$.

We now define a notion of quasimodel for the KB $\mathcal{K}$. Let $Q \supseteq Q_{\mathcal{K}}$ be a finite set of natural numbers such that $\max Q = q_{\mathcal{K}}$,

$$Q^* = \{q + n > 0 \mid q \in Q \text{ and } -(|\mathcal{A}| + 1) \leq n \leq |\mathcal{A}| + 1\},$$

and let Σ be the set of concepts consisting of

– all subconcepts of concepts occurring in $\mathcal{K}$,
– for $R \in \mathit{role}^{\pm}(\mathcal{K})$, all concepts of the form

$$\geq q \boxast R \text{ and } \geq q \diamondast R, \text{ for } q \in Q, \qquad \text{and} \qquad \geq q\, R, \text{ for } q \in Q^*,$$

and all concepts of the form $\geq \omega\, R$ and $\geq \omega \diamondast R$,[2]
– the negations of the concepts in the two previous items (we assume that $\neg\neg C = C$).

By a ΣQ-*type* we mean a subset $\boldsymbol{t} \subseteq \Sigma$ such that

– $C \in \boldsymbol{t}$ iff $\neg C \notin \boldsymbol{t}$, for each $C \in \Sigma$,
– $C_1 \sqcap C_2 \in \boldsymbol{t}$ iff $C_1, C_2 \in \boldsymbol{t}$, for each $C_1 \sqcap C_2 \in \Sigma$,
– if $\geq q'\, R \in \boldsymbol{t}$ and $q' > q$ then $\geq q\, R \in \boldsymbol{t}$, for each $\geq q\, R \in \Sigma$,
– for all $R \in \mathit{role}^{\pm}(\mathcal{K})$ and $\geq q\, R \in \Sigma$,
  - if $\geq q \boxast R \in \boldsymbol{t}$ then $\geq q\, R \in \boldsymbol{t}$,
  - if $\geq q\, R \in \boldsymbol{t}$ and $\geq q \diamondast R \in \Sigma$ then $\geq q \diamondast R \in \boldsymbol{t}$.

Let $Z_{\mathcal{A}}$ be the set of all integers m such that $\bigcirc^m B(a)$ or $\bigcirc^m S(a,b)$ occurs in $\mathcal{A}$ and let $Z \supseteq Z_{\mathcal{A}}$ be a finite set of integers. By a $(Z, \Sigma Q)$-*run* r (or just a run if Z and ΣQ are clear from the context) we mean a function from Z into the set of ΣQ-types such that, for all $z \in Z$,

($\mathbf{r}_{\Box}$) $\boxast C \in r(z)$ iff $C \in r(z')$, for all $z' \in Z$,
($\mathbf{r}_{=}$) $\geq q\, R \in r(z)$ iff $\geq q\, R \in r(z')$, for all $z' \in Z$ and all rigid (in particular temporalised) roles R.

Let r be a run and $R \in \mathit{role}^{\pm}(\mathcal{K})$. The *required* R-*ranks* of r at $z \in Z$ for $\boxast R$, R and $\diamondast R$ are, respectively:

$$\begin{aligned} \varrho_r^{\Box R} &= \max\nolimits_0 \big\{ q \in Q \mid\, \geq q \boxast R \in r(z), z \in Z \big\}, \\ \varrho_r^{R}(z) &= \max\nolimits_0 \big\{ q \in Q^* \cup \{\omega\} \mid\, \geq q\, R \in r(z) \big\}, \\ \varrho_r^{\Diamond R} &= \max\nolimits_0 \big\{ q \in Q \cup \{\omega\} \mid\, \geq q \diamondast R \in r(z), z \in Z \big\}, \end{aligned}$$

where $\max_0 X = \max(\{0\} \cup X)$. It follows from the definition of Σ that $\varrho_r^{\Box R} \in Q \cup \{0\}$, $\varrho_r^{R}(z) \in Q^* \cup \{0, \omega\}$ and $\varrho_r^{\Diamond R} \in Q \cup \{0, \omega\}$. As both roles $\boxast R$ and $\diamondast R$ are rigid and in view of **($\mathbf{r}_{=}$)**, their ranks do not depend on z, and so we omitted the argument z.

A run r is said to be R-*saturated* if $\varrho_r^{\Diamond R} > \varrho^{\Box R}$ implies that

($\mathbf{r}_{+}$) there is $z' \in Z$ with $\varrho_r^{R}(z') > \varrho_r^{\Box R}$, and
($\mathbf{r}_{-}$) there is $z'' \in Z$ with $\varrho_r^{R}(z'') < \varrho_r^{\Diamond R}$ whenever $\varrho_r^{\Diamond R} < \omega$

(here we use the fact that $q+1, q-1 \in Q^*$, for $q \in Q$).

Let $\mathfrak{A}$ be an ABox extending $\mathcal{A}$ with assertions of the form $\bigcirc^z R(a,b)$, where $z \in Z$, $R \in \mathit{role}^{\pm}(\mathcal{K})$ and $a, b \in \mathit{ob}(\mathcal{A})$. We will require the following numbers:

$$\begin{aligned} N_{\mathfrak{A}}^{\Box R}(a) &= \sharp\{ b \mid \bigcirc^z R(a,b) \in \mathfrak{A}, \text{ for all } z \in Z \}, \\ N_{\mathfrak{A}}^{R}(a,z) &= \sharp\{ b \mid \bigcirc^z R(a,b) \in \mathfrak{A} \}, \\ N_{\mathfrak{A}}^{\Diamond R}(a) &= \sharp\{ b \mid \bigcirc^z R(a,b) \in \mathfrak{A}, \text{ for some } z \in Z \}. \end{aligned}$$

The extension of the ABox is required for the following two examples:

[2] We use the usual order: $1 < \cdots < q_{\mathcal{K}} < q_{\mathcal{K}} + 1 < \cdots < \omega$.

– Consider $\mathcal{K} = (\mathcal{T}, \mathcal{A})$, where

$$\begin{aligned}\mathcal{T} =& \{\geq 9 \diamondsuit R \sqsubseteq \bot\},\\ \mathcal{A} =& \{\bigcirc(\geq 5\, R)(a), R(a, b_1), R(a, b_2), R(a, b_3), \bigcirc R(a, b_1)\}.\end{aligned}$$

It follows that, in every model $\mathcal{I}$ of $\mathcal{K}$, we have either $\mathcal{I} \models \bigcirc R(a, b_2)$ or $\mathcal{I} \models \bigcirc R(a, b_3)$.

– Consider $\mathcal{K} = (\mathcal{T}, \mathcal{A})$, where

$$\begin{aligned}\mathcal{T} =& \{\geq 6 \diamondsuit R \sqsubseteq \bot,\ \top \sqsubseteq \geq 4 \boxast R\},\\ \mathcal{A} =& \{R(a, b_1), R(a, b_2)\}.\end{aligned}$$

Then in every model $\mathcal{I}$ of $\mathcal{K}$, either $\mathcal{I} \models \boxast R(a, b_1)$ or $\mathcal{I} \models \boxast R(a, b_2)$.

A $(Z, \Sigma Q)$-*quasimodel* $\mathfrak{Q}$ *for* $\mathcal{K}$ is a pair $(\mathfrak{R}, \mathfrak{A})$, where $\mathfrak{R}$ is a set of $(Z, \Sigma Q)$-runs that are R-saturated for each $R \in role^{\pm}(\mathcal{K})$ and $\mathfrak{A}$ a set of ABox assertions such that

(Q$_1$) $\mathfrak{A}$ is an extension of $\mathcal{A}$ as described above,

(Q$_2$) for each $a \in ob(\mathcal{A})$, there is $r_a \in \mathfrak{R}$ such that:

  (Q$_{2.1}$) $B \in r_a(z)$, for all $\bigcirc^z B(a) \in \mathfrak{A}$,

  (Q$_{2.2}$) for every $z \in Z$,
  - $\varrho_{r_a}^{\Box R} \geq N_{\mathfrak{A}}^{\Box R}(a)$,
  - $\varrho_{r_a}^{R}(z) \geq \varrho_{r_a}^{\Box R} + (N_{\mathfrak{A}}^{R}(a, z) - N_{\mathfrak{A}}^{\Box R}(a))$,
  - $\varrho_{r_a}^{\Diamond R} \geq \varrho_{r_a}^{R}(z) + (N_{\mathfrak{A}}^{\Diamond R}(a) - N_{\mathfrak{A}}^{R}(a, z))$ whenever $\varrho_{r_a}^{R}(z) < \omega$,

  (Q$_{2.3}$) if $\varrho_{r_a}^{\Box R} = \varrho_{r_a}^{\Diamond R}$ then $N_{\mathfrak{A}}^{\Box R}(a) = N_{\mathfrak{A}}^{\Diamond R}(a)$,

  (Q$_{2.4}$) if $\varrho_{r_a}^{\Box R} + (N_{\mathfrak{A}}^{\Diamond R}(a) - N_{\mathfrak{A}}^{\Box R}(a)) < \varrho_{r_a}^{\Diamond R}$ then
  - there is $z' \in Z$ such that $\varrho_{r_a}^{R}(z') > \varrho_{r_a}^{\Box R} + (N_{\mathfrak{A}}^{R}(a, z') - N_{\mathfrak{A}}^{\Box R}(a))$, and
  - there is $z'' \in Z$ such that $\varrho_{r_a}^{R}(z'') < \varrho_{r_a}^{\Diamond R} - (N_{\mathfrak{A}}^{\Diamond R}(a) - N_{\mathfrak{A}}^{R}(a, z''))$ whenever $\varrho_{r_a}^{\Diamond R} < \omega$;

(Q$_3$) for each $R \in role^{\pm}(\mathcal{K})$,
  - if there is $r \in \mathfrak{R}$ with $\varrho_{r}^{\Box R} \geq 1$ then there is $r' \in \mathfrak{R}$ with $\varrho_{r'}^{\Box R^-} \geq 1$;
  - if there is $r \in \mathfrak{R}$ with $\varrho_{r}^{\Box R} < \varrho_{r}^{\Diamond R}$ then there is $r' \in \mathfrak{R}$ with $\varrho_{r'}^{\Box R^-} < \varrho_{r'}^{\Diamond R^-}$.

Lemma 3. *Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be a $T_U^{\mathcal{R}}$DL-Lite$_{bool}^{\mathcal{N}}$ KB.*
(i) If $\mathcal{K}$ is satisfiable then there is a $(Z, \Sigma Q)$-quasimodel $\mathfrak{Q} = (\mathfrak{R}, \mathfrak{A})$ for $\mathcal{K}$ such that

- $|\mathfrak{R}| \leq |ob(\mathcal{A})| + 2|role^{\pm}(\mathcal{K})|$,
- $|Q| \leq |Q_{\mathcal{K}}| + |role^{\pm}(\mathcal{K})|$,
- $|\Sigma| \leq 2(|\mathcal{K}| + 4|role^{\pm}(\mathcal{K})| \cdot |Q| \cdot (|\mathcal{A}| + 1))$,
- $|Z| \leq |Z_{\mathcal{A}}| + |\mathfrak{R}| \cdot (|\mathcal{K}| + 2|role^{\pm}(\mathcal{K})|) + 2|ob(\mathcal{A})| \cdot |role^{\pm}(\mathcal{K})| \cdot (|ob(\mathcal{A})| + 1)$.

(ii) If there is a quasimodel for $\mathcal{K}$ then $\mathcal{K}$ is satisfiable.

Proof. (i) Let $\mathcal{I}$ be a model of $\mathcal{K}$. By Lemma 2, we may assume that every point in it has fewer than $(q_{\mathcal{K}}+1)$ $\boxast R$-successors, for each $R \in \mathit{role}^{\pm}(\mathcal{K})$. First we select a set D of elements $w \in \Delta^{\mathcal{I}}$ required to define runs and a set Q of numerical parameters. Let $D_0 = \mathit{ob}(\mathcal{K})$. For $w \in \Delta^{\mathcal{I}}$, set

$$\rho^{\Box R}(w) = \max\bigl\{0 \le q \le q_{\mathcal{K}} \mid w \in (\geq q \boxast R)^{\mathcal{I}}\bigr\},$$
$$\rho^{\Diamond R}(w) = \begin{cases} \omega, & w \in (\geq q_{\mathcal{K}}+1 \diamondast R)^{\mathcal{I}},\\ \max\bigl\{0 \le q \le q_{\mathcal{K}} \mid w \in (\geq q \diamondast R)^{\mathcal{I}}\bigr\}, & \text{otherwise.}\end{cases}$$

We proceed by extending D_0. Suppose that we have already defined D_m.

– If there is $w \in D_m$ with $\rho^{\Box R}(w) \geq 1$ and there is no $w' \in D_m$ with $\rho^{\Box R^-}(w') \geq 1$, we can find $w' \in \Delta^{\mathcal{I}}$ such that $w' \in (\exists \boxast R^-)^{\mathcal{I}}$.
– If there is $w \in D_m$ with $\rho^{\Box R}(w) < \rho^{\Diamond R}(w)$ but there is no $w' \in D_m$ with $\rho^{\Box R^-}(w') < \rho^{\Diamond R^-}(w')$, we can find $w' \in \Delta^{\mathcal{I}}$ such that $w' \in (\geq q \diamondast R^-)^{\mathcal{I}} \setminus (\geq q \boxast R^-)^{\mathcal{I}}$, for some $q \geq 0$. By Lemma 2, $q \leq q_{\mathcal{K}}+1$.

In either case we set $D_{m+1} = D_m \cup \{w'\}$. It should be clear that the above procedure cannot continue for longer than $|\mathit{role}^{\pm}(\mathcal{K})|$ steps. Denote the resulting set by D and let Q be the union of $Q_{\mathcal{K}}$ with all

$$\{\rho^{\Box R}(w) \mid w \in D\} \cup \{\rho^{\Diamond R}(w) \leq q_{\mathcal{K}} \mid w \in D\},$$

for $R \in \mathit{role}^{\pm}(\mathcal{K})$. Clearly, $|D| \leq |D_0| + 2|\mathit{role}^{\pm}(\mathcal{K})|$ and Q is as required.

We are now in a position to define the set Z of 'time slices,' which will be defined as the following union:

$$Z = Z_{\mathcal{A}} \cup Z_0 \cup Z_1 \cup Z_2 \cup Z_3.$$

Then the ABox $\mathfrak{A}$ consists of all the relations between elements of $\mathit{ob}(\mathcal{A})$ at moments in Z:

$$\mathfrak{A} = \{\bigcirc^z R(a,b) \mid (a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}(z)} \text{ for } z \in Z\},$$

We need Z_0 to satisfy $(\mathbf{r}_\Box)$, Z_1 to satisfy $(\mathbf{r}_-)$ and $(\mathbf{r}_+)$, Z_2 to guarantee that the numbers $N^{\Box R}_{\mathfrak{A}}(a)$ and $N^{\Diamond R}_{\mathfrak{A}}(a)$ are the true numbers of $\boxast R$- and $\diamondast R$-successors of a in the ABox $\mathfrak{A}$ defined on Z (so the notation $N^{\Box R}_{\mathfrak{A}}(a)$ and $N^{\Diamond R}_{\mathfrak{A}}(a)$ is well-defined), and Z_3 to satisfy $(\mathbf{Q}_{2.4})$:

– Let Z_0 be a minimal set containing, for every $w \in D$ and every $\boxast C \in \Sigma$ with $w \notin (\boxast \neg C)^{\mathcal{I}(x')}$, for some (all) $x' \in \mathbb{Z}$, a time point $x \in \mathbb{Z}$ such that $w \in (\neg C)^{\mathcal{I}(x)}$ (such an x exists as $\mathcal{I} \models \mathcal{K}$).
– Let Z_1 be a minimal set containing, for every $w \in D$ and every $R \in \mathit{role}^{\pm}(\mathcal{K})$ with $\rho^{\Box R}(w) < \rho^{\Diamond R}(w)$, time points $x', x'' \in \mathbb{Z}$ such that
  – $w \in (\geq q'+1\, R)^{\mathcal{I}(x')}$ with $q' = \rho^{\Box R}(w)$;
  – $w \notin (\geq q''\, R)^{\mathcal{I}(x'')}$ with $q'' = \rho^{\Diamond R}(w)$ if $\rho^{\Diamond R}(w) < \omega$

  (such x' and x'' exist because $\mathcal{I} \models \mathcal{K}$).

- Let Z_2 be a minimal set containing, for all $a, b \in ob(\mathcal{A})$ and $R \in role^{\pm}(\mathcal{K})$ with $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in (\Diamonddot R)^{\mathcal{I}}$, time points $x', x'' \in \mathbb{Z}$ such that
  - $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}(x')}$;
  - $(a^{\mathcal{I}}, b^{\mathcal{I}}) \notin R^{\mathcal{I}(x'')}$ whenever $(a^{\mathcal{I}}, b^{\mathcal{I}}) \notin (\boxdot R)^{\mathcal{I}}$

  (such x' and x'' exist by definition).
- Let Z_3 be a minimal set containing, for all $a \in ob(\mathcal{A})$ and $R \in role^{\pm}(\mathcal{K})$ with $\rho^{\Box R}(a) + (N_{\mathfrak{A}}^{\Diamond R}(a) - N_{\mathfrak{A}}^{\Box R}(a)) < \rho^{\Diamond R}(a)$, time points $x', x'' \in \mathbb{Z}$ such that
  - $a^{\mathcal{I}} \in (\geq q' + 1\, R)^{\mathcal{I}(x')}$, for $q' = \rho^{\Box R}(a) + (N(a, x') - N_{\mathfrak{A}}^{\Box R}(a))$;
  - $a^{\mathcal{I}} \notin (\geq q''\, R)^{\mathcal{I}(x'')}$, for $q'' = \rho^{\Diamond R}(a) - (N_{\mathfrak{A}}^{\Diamond R}(a) - N(a, x''))$, whenever $\rho^{\Diamond R}(a) < \omega$,

  where $N(a, x) = \{b \in ob(\mathcal{A}) \mid (a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}(x)}\}$. We show that such an x' exists. Suppose on the contrary that, for all $x' \in \mathbb{Z}$, we have $a^{\mathcal{I}} \notin (\geq q' + 1\, R)^{\mathcal{I}(x')}$, for $q' = \rho^{\Box R}(a) + (N(x') - N_{\mathfrak{A}}^{\Box R}(a))$. It follows that at all $x' \in \mathbb{Z}$, $a^{\mathcal{I}}$ has at most $(\rho^{\Box R}(a) + (N(x') - N_{\mathfrak{A}}^{\Box R}(a)))$ R-successors, which means that the number of R-successors of $a^{\mathcal{I}}$ that are not in the set $A = \{b^{\mathcal{I}} \mid b \in ob(\mathcal{A})\}$ cannot exceed $\rho^{\Box R}(a) - N_{\mathfrak{A}}^{\Box R}(a)$. So, at all $x' \in \mathbb{Z}$, every R-successor of a is either in A or is in fact a $\boxdot R$-successor, contrary to $\rho^{\Box R}(a) + (N_{\mathfrak{A}}^{\Diamond R}(a) - N_{\mathfrak{A}}^{\Box R}(a)) < \rho^{\Diamond R}(a)$. The existence of x'' is proved similarly.

Now the set of runs $\mathfrak{R}$ is $\{r_w \mid w \in D\}$, where each run r_w is defined by taking, for all $z \in Z$,

- $\geq \omega\, R \in r_w(z)$ iff $w \in (\geq (q_{\mathcal{K}} + |\mathcal{A}| + 1)\, R)^{\mathcal{I}(z)}$, for $R \in role^{\pm}(\mathcal{K})$,
- $\geq \omega \Diamonddot R \in r_w(z)$ iff $w \in (\geq q_{\mathcal{K}} + 1 \Diamonddot R)^{\mathcal{I}(z)}$, for $R \in role^{\pm}(\mathcal{K})$,
- $C \in r_w(z)$ iff $w \in C^{\mathcal{I}(z)}$, for all other concepts $C \in \Sigma$.

It can be easily seen that each $r_w \in \mathfrak{R}$ is a ΣQ-run (in particular, $(\mathbf{r}_{\Box})$ holds because $Z_0 \subseteq Z$). As $Z_1 \subseteq Z$, each $r_w \in \mathfrak{R}$ is saturated. Then $(\mathbf{Q}_1)$ and $(\mathbf{Q}_{2.1})$ hold because $\mathcal{I} \models \mathcal{A}$ and $Z_{\mathcal{A}} \subseteq Z$. We have $(\mathbf{Q}_{2.2})$ and $(\mathbf{Q}_{2.3})$ because $\mathcal{I} \models \mathfrak{A}$. Since $Z_3 \subseteq Z$, we have $(\mathbf{Q}_{2.4})$. Finally, $(\mathbf{Q}_3)$ holds by construction of D and the definition of Q.

(*ii*) Let $\mathfrak{Q} = (\mathfrak{R}, \mathfrak{A})$ be a quasimodel for $\mathcal{K}$. We define a model $\mathcal{I}$ of $\mathcal{K}$ over a domain $\Delta^{\mathcal{I}}$, which will be constructed as a union $\bigcup_{m=0}^{\infty} \Delta_m$. We set $a^{\mathcal{I}} = a$, for all $a \in ob(\mathcal{A})$, and $\Delta_0 = ob(\mathcal{A})$. Our construction extends each Δ_m to Δ_{m+1} by adding new points created from runs in $\mathfrak{Q}$. To keep track of the process of making copies of runs, we simultaneously construct a function $\kappa \colon \Delta^{\mathcal{I}} \to \mathfrak{R}$, which is also a union $\bigcup_{m=0}^{\infty} \kappa_m$ of functions $\kappa_m \colon \Delta_m \to \mathfrak{R}$. For the initial step, take $\kappa_0(a) = r_a$ provided by $(\mathbf{Q}_2)$, for each $a \in \Delta_0$.

We will also need to multiply and rearrange the time instances when creating a point from a run in $\mathfrak{R}$. To keep track of that, we introduce functions $\nu_w \colon \mathbb{Z} \to Z$, for $w \in \Delta^{\mathcal{I}}$, where $\nu_w(x) \in Z$ is the time instance of $\kappa(w)$ in $\mathfrak{Q}$ that serves as the original for the time instance $x \in \mathbb{Z}$ of w in $\mathcal{I}$. For the initial step, take $\nu_w(x) = \nu_0(x)$, where ν_0 is a function such that (*i*) $\nu_0^{-1}(z)$ is infinite for all $z \in Z_{\mathcal{K}}$ and (*ii*) $\nu_0(x) = x$, for $x \in Z_{\mathcal{A}}$. When the construction is complete, we will have

$$A^{\mathcal{I}(x)} = \{w \in \Delta^{\mathcal{I}} \mid A \in r(\nu_w(x)), r = \kappa_0(w)\},$$

for each concept name A. The interpretation of roles names $P^{\mathcal{I}(x)}$ will be constructed as a union $\bigcup_{m=0}^{\infty} P_m^x$, where $P_m^x \subseteq \Delta_m \times \Delta_m$. For $m = 0$, we set

$$P_0^x = \big\{(a, b) \in \Delta_0 \times \Delta_0 \mid \bigcirc^{\nu_0(x)} P(a, b) \in \mathfrak{A}\big\}$$

for all role names P and all $x \in \mathbb{Z}$. Here, to simplify notation, we do not distinguish between $P^-(b, a) \in \mathfrak{A}$ and $P(a, b) \in \mathfrak{A}$.

For $R \in \mathit{role}^{\pm}(\mathcal{K})$, the *actual m-step R-rank* of $w \in \Delta^{\mathcal{I}}$ at $x \in \mathbb{Z}$, for $R = \boxast P$, $R = P$ and $R = \diamondast P$, is defined as follows:

$$\begin{aligned}
\tau_w^{\Box P}(m) &= \sharp\{w' \in \Delta_m \mid (w, w') \in \textstyle\bigcap_{x\in\mathbb{Z}} P_m^x\},\\
\tau_w^{P}(x, m) &= \sharp\{w' \in \Delta_m \mid (w, w') \in P_m^x\},\\
\tau_w^{\Diamond P}(m) &= \sharp\{w' \in \Delta_m \mid (w, w') \in \textstyle\bigcup_{x\in\mathbb{Z}} P_m^x\};
\end{aligned}$$

for $R = \boxast P^-$, $R = P^-$ and $R = \diamondast P^-$, (w, w') should be replaced with (w', w).

We will use the following properties of the partial model we construct: for each $R \in \mathit{role}^{\pm}(\mathcal{K})$ and $w \in \Delta_m$,

(rn) for all $x \in \mathbb{Z}$,

$$0 \;\leq\; \varrho_{\kappa(w)}^{\Box R} - \tau_w^{\Box R}(m) \;\leq\; \varrho_{\kappa(w)}^{R}(\nu_w(x)) - \tau_w^{R}(x, m) \;\leq\; \varrho_{\kappa(w)}^{\Diamond R} - \tau_w^{\Diamond R}(m),$$

(fn) $\tau_{\kappa(w)}^{\Diamond R}(m) < \omega$,

(df) if $\varrho_{\kappa(w)}^{\Diamond R} - \tau_w^{\Diamond R}(m) > \varrho_{\kappa(w)}^{\Box R} - \tau_w^{\Box R}(m)$ then

- **(df$_0$)** $\varrho_{\kappa(w)}^{\Diamond R} > \varrho_{\kappa(w)}^{\Box R}$,
- **(df$_+$)** $\varrho_{\kappa(w)}^{R}(\nu_w(x)) - \tau_w^{R}(x, m) > \varrho_{\kappa(w)}^{\Box R} - \tau_w^{\Box R}(m)$ for infinitely many $x \in \mathbb{Z}$,
- **(df$_-$)** $\varrho_{\kappa(w)}^{R}(\nu_w(x)) - \tau_w^{R}(x, m) < \varrho_{\kappa(w)}^{\Diamond R} - \tau_w^{\Diamond R}(m)$ for infinitely many $x \in \mathbb{Z}$, whenever $\varrho_{\kappa(w)}^{R} < \omega$.

Here $\omega - n = \omega$, for all $n < \omega$.

First, we show that these properties hold for $a \in \Delta_0$. Observe that $\tau_a^{\Box R}(0) = N_{\mathfrak{A}}^{\Box R}(a)$, $\tau_a^{R}(x, 0) = N_{\mathfrak{A}}^{R}(\nu_a(x))$, for all $x \in \mathbb{Z}$, and $\tau_a^{\Diamond R}(0) = N_{\mathfrak{A}}^{\Diamond R}(a)$. Then **(rn)** follows from **(Q$_{2.2}$)**, and **(fn)** is trivial. To show **(df$_0$)**, suppose $\varrho_{\kappa(a)}^{\Diamond R} \leq \varrho_{\kappa(a)}^{\Box R}$. Then, by **(Q$_{2.3}$)**, $\tau_a^{\Diamond R}(0) = \tau_a^{\Box R}(0)$ contrary to $\varrho_{\kappa(a)}^{\Diamond R} - \tau_a^{\Diamond R}(0) > \varrho_{\kappa(a)}^{\Box R} - \tau_a^{\Box R}(0)$. Then **(df$_+$)** and **(df$_-$)** follow directly from **(Q$_{2.4}$)** and the fact that the preimage $\nu_a^{-1}(z)$ is infinite, for each $z \in Z$.

As follows from **(rn)**, for all $w \in \Delta_m$, we have

$$\tau_w^{\Box R}(m) \leq \varrho_{\kappa(w)}^{\Box R}, \qquad \tau_w^{R}(x, m) \leq \varrho_{\kappa(w)}^{R}(\nu_w(x)), \text{ for all } x \in \mathbb{Z}, \qquad \text{and } \tau_w^{\Diamond R}(m) \leq \varrho_{\kappa(w)}^{\Diamond R}.$$

If these inequalities are in fact equalities, then we are done. If this is not the case, let $w \in \Delta_m$, for some $m \geq 0$.

(1) If $\tau_w^{\Box P}(m) < \varrho_{\kappa(w)}^{\Box P}$, we have $\varrho_{\kappa(w)}^{\Box P} \geq 1$. By $(\mathbf{Q}_3)$, there is $r' \in \mathfrak{R}$ such that $\varrho_{r'}^{\Box P^-}$. We add $n = \varrho_{\kappa(w)}^{\Box P} - \tau_w^{\Box P}(m)$ copies $w_1, \dots, w_n$ of the run r' to Δ_{m+1}, set $\kappa_{m+1}(w_i) = r'$, add the pairs (w, w_i) to P_{m+1}^x, for all $x \in \mathbb{Z}$, and let $\nu_{w_i} \colon \mathbb{Z} \to Z$ be such that the pre-image $\nu_{w_i}^{-1}(z)$ is infinite, for each $z \in Z$. Observe that, for all w_i and $R \in \mathit{role}^{\pm}(\mathcal{K})$,

$$\tau_{w_i}^{\Box R}(m+1) = \tau_{w_i}^{R}(z, m+1) = \tau_{w_i}^{\Diamond R}(m+1)$$

with $\tau_{w_i}^{\Box R}(m+1) = 1$ if $R = P^-$, and $\tau_{w_i}^{\Box R}(m+1) = 0$ otherwise. So, **(rn)** follows, by the last item in the definition of ΣQ-type, from $\varrho_{\kappa(w_i)}^{\Box P^-} \geq 1$ and $\varrho_{\kappa(w_i)}^{\Box R} \geq 0$, for all other roles R. By definition, **(fn)**; $(\mathbf{df}_0)$ is immediate from the above equalities, whereas $(\mathbf{df}_+)$ and $(\mathbf{df}_-)$ follow from $(\mathbf{r}_+)$ and $(\mathbf{r}_-)$, respectively, and the fact that the pre-image $\nu_{w_i}^{-1}(z)$ is infinite, for each $z \in Z$.

(2) If $\varrho_{\kappa(w)}^{\Diamond P} - \tau_w^{\Diamond P}(m) = \varrho_{\kappa(w)}^{\Box P} - \tau_w^{\Box P}(m)$ we do not need to do anything for this w; otherwise let

$$K = \bigl\{ i \in \mathbb{N} \mid i < (\varrho_{\kappa(w)}^{\Diamond P} - \tau_w^{\Diamond P}(m)) - (\varrho_{\kappa(w)}^{\Box P} - \tau_w^{\Box P}(m)) \bigr\}.$$

Note that, by **(fn)**, $\tau_w^{\Diamond P}(m) < \omega$, and thus the expression is well-defined (it contains at most one ω). Note also that K is infinite in case $\varrho_{\kappa(w)}^{\Diamond P} = \omega$. We have to 'connect' $|K|$ new $\circledast P$-successors to w in such a way that the required ranks $\varrho_{\kappa(w)}^{\Box P}$, $\varrho_{\kappa(w)}^{P}(x)$ and $\varrho_{\kappa(w)}^{\Diamond P}$ are respected. By **(rn)** and **(df)**, there is a function $\tau \colon \mathbb{Z} \to 2^K$ such that

- for all $x \in \mathbb{Z}$, $|\tau(x)| = (\varrho_{\kappa(w)}^{P}(\nu_w(x)) - \tau_w^{P}(x, m)) - (\varrho_{\kappa(w)}^{\Box P} - \tau_w^{\Box P}(m))$ (this expression is also well-defined as it contains at most one ω);
- for every $i \in K$, there are infinitely many $x \in \mathbb{Z}$ with $i \in \tau(x)$, and infinitely many $x \in \mathbb{Z}$ with $i \notin \tau(x)$.

By $(\mathbf{df}_0)$, we have $\varrho_{\kappa(w)}^{\Diamond P} > \varrho_{\kappa(w)}^{\Box P}$, and so, by $(\mathbf{Q}_3)$, there exists $r' \in \mathfrak{R}$ with $\varrho_{r'}^{\Diamond P^-} > \varrho_{r'}^{\Box P^-}$. We add $|K|$ *fresh copies* $w_1, \dots w_{|K|}$ of r' to Δ_{m+1}, set $\kappa_{m+1}(w_i) = r'$ and add (w, w_i) to P_{m+1}^x iff $i \in \tau(x)$. It remains to define the ν_{w_i}. By $(\mathbf{r}_+)$,

$$Z_{>\Box} = \bigl\{ z \in Z \mid \geq (\varrho_{r'}^{\Box P^-} + 1)\, P^- \in r'(z) \bigr\} \neq \emptyset.$$

If $\varrho_{r'}^{\Diamond P^-} = \omega$ then we take $Z_{<\Diamond} = Z$, in which case, clearly, $Z_{>\Box} \cup Z_{<\Diamond} = Z$. Otherwise, as $\varrho_{r'}^{\Diamond P^-} + 1 \in Q^*$, by $(\mathbf{r}_-)$,

$$Z_{<\Diamond} = \bigl\{ z \in Z \mid \neg(\geq \varrho_{r'}^{\Diamond P^-}\, P^-) \in r'(z) \bigr\} \neq \emptyset.$$

It follows that $\varrho_{r'}^{\Diamond P^-} \geq \varrho_{r'}^{\Box P^-} + 1$, and so $Z_{>\Box} \cup Z_{<\Diamond} = Z_{\mathcal{K}}$.

For each $i \in K$, we take a function ν_{w_i} such that the pre-image $\nu_{w_i}^{-1}(z)$ is infinite, for all $z \in Z$, and

- if $z \in Z_{<\Diamond} \setminus Z_{>\Box}$ then $(w, w_i) \notin P_{m+1}^x$, for $x \in \nu_{w_i}^{-1}(z)$,
- if $z \in Z_{>\Box} \setminus Z_{<\Diamond}$ then $(w, w_i) \in P_{m+1}^x$, for $x \in \nu_{w_i}^{-1}(z)$,

– if $z \in Z_{>\Box} \cap Z_{<\Diamond}$ then $(w, w_i) \notin P^x_{m+1}$, for infinitely many $x \in \nu^{-1}_{w_i}(z)$, and $(w, w_i) \in P^x_{m+1}$, for infinitely many $x \in \nu^{-1}_{w_i}(z)$.

Observe that, for each w_i, we have:

– $\tau^{\Box R}_{w_i}(m+1) = \tau^R_{w_i}(z, m+1) = \tau^{\Diamond R}_{w_i}(m+1) = 0$, for all $R \neq P^-$;
– $\tau^{\Box P^-}_{w_i}(m+1) = 0$ and $\tau^{\Diamond P^-}_{w_i}(m+1) = 1$;

It remains to show that conditions **(rn)**, **(fn)** and **(df)** hold. Indeed, each $x \in \mathbb{Z}$ falls into one of the three groups:
(a) if $\nu_{w_i}(x) \in Z_{<\Diamond} \setminus Z_{>\Box}$ then

$$\varrho^{\Box P^-}_{\kappa(w_i)} = \varrho^{P^-}_{\kappa(w_i)}(\nu_{w_i}(x)) < \varrho^{\Diamond P^-}_{\kappa(w_i)} \text{ and } \tau^{P^-}_{w_i}(x, m+1) = 0;$$

(b) if $\nu_{w_i}(x) \in Z_{>\Box} \setminus X_{<\Diamond}$ then

$$\varrho^{\Box P^-}_{\kappa(w_i)} < \varrho^{P^-}_{\kappa(w_i)}(\nu_{w_i}(x)) = \varrho^{\Diamond P^-}_{\kappa(w_i)} \text{ and } \tau^{P^-}_{w_i}(x, m+1) = 1;$$

(c) otherwise, $\nu_{w_i}(x) \in Z_{>\Box} \cap Z_{<\Diamond}$, from which $\varrho^{\Diamond P^-}_{\kappa(w_i)} > \varrho^{\Box P^-}_{\kappa(w_i)}$ and there are infinitely many $x' \in \nu^{-1}_{w_i}(\nu_{w_i}(x))$ with

$$\varrho^{\Box P^-}_{\kappa(w_i)} < \varrho^{P^-}_{\kappa(w_i)}(\nu_{w_i}(x')) = \varrho^{\Diamond P^-}_{\kappa(w_i)} \text{ and } \tau^{P^-}_{w_i}(x', m+1) = 1,$$

as well as infinitely many $x'' \in \nu^{-1}_{w_i}(\nu_{w_i}(x))$ with

$$\varrho^{\Box P^-}_{\kappa(w_i)} = \varrho^{P^-}_{\kappa(w_i)}(\nu_{w_i}(x'')) < \varrho^{\Diamond P^-}_{\kappa(w_i)} \text{ and } \tau^{P^-}_{w_i}(x'', m+1) = 0.$$

This completes the inductive step in the construction of the model.