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U-Shaped Development in Math: 7-Year-Olds Outperform 9-Year-Olds on Equivalence Problems

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What is the nature of the association between age (7–11 years) and performance on mathematical equivalence problems (e.g., 7 + 4 + 5 = 7 + __)? Many prevailing theories suggest that there should be a positive association. However, change-resistance accounts (e.g., N. M. McNeil & M. W. Alibali, 2005b) predict a U-shaped association. The purpose of the present research was to test these differing predictions. Results from two studies supported a change-resistance account. In the first study ($N = 87$), performance on equivalence problems declined between the ages of 7 and 9 and improved between the ages of 9 and 11. The decrements in performance between the ages of 7 and 9 were then replicated in a second study ($N = 35$). Results suggest that the association between age and performance on equivalence problems is U-shaped.

Keywords: cognitive development, U-shaped development, mathematical problem solving

The focus of this article is on the development of children's performance on mathematical equivalence problems (e.g., 3 + 4 + 5 = 3 + __). Mathematical equivalence is a fundamental concept in algebra (Carpenter, Franke, & Levi, 2003; Knuth, Stephens, McNeil, & Alibali, 2006), and success in algebra is crucial to future educational and employment opportunities (Ladson-Billings, 1998; Moses & Cobb, 2001; National Research Council, 1998). Equally important, researchers have relied on detailed analyses of performance in well-defined areas of mathematics for decades as a means for evaluating general claims about developmental and individual differences in learning, memory, and problem solving (e.g., Anderson, Reder, & Lebiere, 1996; Dixon & Moore, 1996; Geary, Bow Thomas, Liu, & Siegler, 1996; Gelman, Meck, & Merkin, 1986; Goldin-Meadow, Alibali, & Church, 1993; Mix, Levine, & Huttenlocher, 1999; Qin et al., 2004; Schwartz & Black, 1996; Siegler & Stern, 1998; Swanson & Beebe-Frankenberger, 2004).

Many studies have shown that children have substantial difficulties with mathematical equivalence problems (e.g., Alibali, 1999; McNeil & Alibali, 2004; Perry, Church, & Goldin-Meadow, 1988; Rittle-Johnson & Alibali, 1999). However, little is known about how performance develops with age. To date, no theory has specifically addressed how performance on mathematical equivalence problems develops over the course of the elementary school years. However, several theories have focused on the development of quantitative skills more generally. Many of these theories suggest that performance on equivalence problems should improve across the elementary school years. For example, the Piagetian account suggests that performance should improve with age as children construct progressively more sophisticated logical structures for coordinating relationships of equivalence (Inhelder & Piaget, 1958). Similarly, theories that focus on the role of working memory capacity in math problem solving predict that performance should improve with age as executive processing capacity and phonological short-term memory mature (Adams & Hitch, 1997; Ashcraft, 1992; Barrouillet & Lepine, 2005; Bull & Johnston, 1997; Gathercole & Pickering, 2000; Hecht, Torgesen, Wagner, & Rashotte, 2001; Hitch, 1978; Hitch, Towse, & Hutton, 2001). Likewise, theories that stress the importance of basic number knowledge in higher level mathematics predict that performance should improve with age as proficiency with basic arithmetic facts increases (Haverty, 1999; Haverty, Koedinger, Klahr, & Alibali, 2000; see also Kotovsky, Hayes, & Simon, 1985). The prevalence of this "positive association" stance is not really surprising because "'performance improves with age' is as close to a law as any generalization that has emerged from the study of cognitive development" (Siegler, 2004, p. 2). Besides, it makes sense intuitively—children should get better at math as they get older.

The aforementioned theories suggest that there should be a positive association between age and performance on equivalence problems because, in general, they assume that children's difficulties with mathematics are due to something that children lack (e.g., advanced logical structures, a mature working memory system, proficiency with basic arithmetic facts). According to this view, children perform poorly on equivalence problems because they lack the cognitive structures or functions necessary for solving the problems correctly, and performance should improve over childhood as those structures or functions develop.

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In contrast to theories that focus on what children lack, change-resistance accounts (e.g., Luchins & Luchins, 1950; McNeil & Alibali, 2005b) suggest that children's difficulties are due, at least in part, to something children have—existing knowledge. Change-resistance accounts are rooted in classic "top-down" approaches to cognition and problem solving (e.g., Bruner, 1957; Luchins, 1942; Rumelhart, 1980; Tolman, 1948), as well as in contemporary theories that focus on the role of domain-general statistical learning mechanisms in development (e.g., Rogers, Rakison, & McClelland, 2004; Saffran, Aslin, & Newport, 1996; Zevin & Seidenberg, 2002). On the whole, these theories tend to focus on the specific representations that learners construct through domain experience and practice, as well as learners' tendencies to persist in the use of those representations, even when they are not appropriate. The general theoretical claim is that early learning constrains later learning. According to this view, the patterns with which children initially gain experience within a domain become entrenched, and learning difficulties arise when to-be-learned information overlaps with, but does not map directly onto, entrenched patterns.

In the United States, children's early mathematics experience is weighted heavily toward arithmetic operations such as addition, subtraction, multiplication, and division. This is because mathematics is considered to be a domain in which early competence with "basic skills" is necessary for advanced thinking and problem solving. Children learn arithmetic in a very procedural fashion for many years before they learn to reason about equations as expressions of mathematical equivalence. Indeed, data from the Third International Mathematics and Science Study (Beaton et al., 1996) show that, unlike children from higher achieving countries, children in the United States spend substantial amounts of class time reviewing basic arithmetic skills throughout the elementary and middle school years (National Science Board, 2002; see also Valverde & Schmidt, 1997).

Research has shown that children pick up on at least three recurrent, arithmetic-specific patterns. First, children learn that the equal sign and answer blank come together at the end of the problem (e.g., $3 + 4 = \_\_$, McNeil & Alibali, 2004; Seo & Ginsburg, 2003). Second, children learn to interpret the equal sign as an operator (like $+$ or $-$) that means "calculate the total" (Baroody & Ginsburg, 1983; Behr, Erlwanger, & Nichols, 1980; Kieran, 1981; McNeil & Alibali, 2005a; Rittle-Johnson & Alibali, 1999). Third, children learn to solve math problems by performing all given operations on all given numbers (McNeil & Alibali, 2000, 2004, 2005b; Perry, 1991; Perry et al., 1988). These three patterns have been called "operational patterns" in prior work (McNeil & Alibali, 2005b).

Although children's representations of the operational patterns likely facilitate performance on traditional arithmetic problems (e.g., $3 + 4 = \_\_$), they are unlikely to help (and may even hinder) performance on equivalence problems, which do not adhere to the traditional form (McNeil & Alibali, 2004). To solve an equivalence problem (e.g., $3 + 4 + 5 = 3 + \_\_$) correctly, children must (a) notice that the equal sign is not at the end of the problem, (b) understand that the equal sign denotes an equivalence relation between the two sides of the equation, and (c) manipulate the numbers and operations to arrive at an answer that makes both sides of the equation have the same value. To satisfy these conditions, children must either ignore or override their long-term memory representations of the operational patterns.

According to a change-resistance account, children's long-term memory representations of the operational patterns interfere with performance on equivalence problems (McNeil & Alibali, 2005b). If this is the case, then the development of children's ability to solve equivalence problems should be tied directly to children's susceptibility to interference from their representations. Over the course of the elementary school years, this should give rise to a U-shaped association between age and performance on equivalence problems, as described next.

First- and second-grade children (ages 6–8) have just started learning arithmetic in a formal school setting, so they may not have strong representations of the operational patterns yet. Although they have known how to add and subtract for some time, they likely have not had enough practice with school-based arithmetic to extract the operational patterns from their experience. Even if children have started to extract some of the operational patterns, these representations are newly developing and are unlikely to be robust. Thus, even though children in this age range have difficulties inhibiting their representations of information that is associated with but irrelevant to the task at hand (Dempster, 1992; Ridderinkhof, van der Molen, Band, & Bashore, 1997), their performance on equivalence problems should be relatively good because their representations of the operational patterns are not yet strong enough to interfere (cf. Munakata, 1998; Shinskey & Munakata, 2005). As children continue to receive practice with arithmetic procedures in school in third and fourth grade (ages 8–10), their representations of the operational patterns gain strength. Because children in this age range continue to have difficulties inhibiting their representations of information that is associated with but irrelevant to the task at hand (Dempster, 1992; Lorsbach, Katz, & Cupak, 1998), performance on equivalence problems should decrease during this period. Once children reach fifth grade and beyond (ages 10–14), they start to become familiar with several math topics that contradict their knowledge of the operational patterns (e.g., equivalent fractions, equalities/inequalities, pre-algebra, algebra). As a result, the operational patterns lose their predictive power, and children's representations decrease in strength. Because children in this age range also improve in their ability to inhibit their representations of information that is associated with but irrelevant to the task at hand (Bjorklund & Harnishfeger, 1990; Dempster, 1992), performance on equivalence problems should improve dramatically. Following this logic, there should be a U-shaped association between age and performance on equivalence problems over the elementary school years. This U-shaped function would also be consistent with at least two other current theories: the hierarchical competing systems account (Marcovitch & Zelazo, 1999, 2006) and the representation shift account (Church, Kelly, & Lynch, 2000). We consider the specifics of these theories in the discussion.

Does children's performance on equivalence problems improve with age, as many prevailing theories would predict, or is the association U-shaped? The answer has yet to be determined. As mentioned previously, many studies have shown that elementary school children (ages 8–10) do not solve equivalence problems correctly. There also has been some speculation that children's performance improves dramatically sometime between the ages of 9 and 12 (Perry et al., 1988). However, to date there have not been any systematic investigations into the association between age and performance on equivalence problems. Most studies have examined children within a narrow age range (Alibali & Goldin-Meadow, 1993), have not analyzed age effects (McNeil & Alibali, 2000; Perry, 1991), or have shown no evidence of an association between age and performance (Carpenter, Levi, & Farnsworth, 2000; McNeil & Alibali, 2005b; Rittle-Johnson & Alibali, 1999). The closest exception is a study by Alibali (1999), which reported that the likelihood of learning a correct strategy after instruction on equivalence problems was greater for children tested early in the school year than for those tested late in the school year. However, this was an unexpected effect, and it was an effect of time of year, rather than age per se.

As a side note, it should be mentioned that the time-of-year effect observed by Alibali (1999) would be readily predicted by accounts that focus on change resistance. Assuming that most children do not practice their arithmetic skills over summer break, it is likely that children's representations of the operational patterns are weaker early in the school year than they are late in the school year after months of practice with arithmetic operations. According to a change-resistance account, children's performance on equivalence problems should get worse over the course of a school year as children's representations of the operational patterns gain strength.

The present article reports two studies designed to investigate the association between age and performance on equivalence problems. In both studies, children of various ages were tested on their ability to solve equivalence problems, such as $7 + 4 + 5 = 7 +$ __. Many prevailing theories predict a positive association between age and performance on equivalence problems, whereas change-resistance accounts predict a U-shaped association. The goal was to test these differing predictions.

Study 1

Method

Participants. Eighty-seven children (48 boys, 39 girls; 18 first graders, 19 second graders, 23 third graders, 27 fourth graders) participated. Children ranged in age from 6 years 9 months to 10 years 8 months (M = 8 years 10 months). The study was conducted at a public school in a suburb of Pittsburgh, PA during the last month of the school year. The racial/ethnic makeup of the school was 4% African American, 1% Hispanic, and 95% White. Approximately 38% of children received free or reduced-price lunch.

Procedure. Children completed a paper-and-pencil problem-solving test consisting of 12 equivalence problems (e.g., $9 + 4 + 3 = 9 +$ __, $3 + 5 + 7 =$ __ $+ 7$). Children were tested in their classroom settings. The average number of children per classroom during testing was 22. Children's classroom teachers handed out the tests at the beginning of their regular math periods. After handing out the tests, teachers read the following instructions aloud: "Today you are going to solve some math problems. It's not a test or anything, so you won't be graded. Just do your best. If a problem seems too hard for you to solve, then you can just give it your best guess. After you finish solving the problems, turn your paper over, and sit quietly until everyone has finished." It took children between 5–15 min to finish. Teachers collected the tests after all children had finished. Teachers were told that they were not allowed to help children with the problems until after all tests were collected. In the event that a child asked a question during the test, teachers were instructed to say, "I'm interested in what you think about the problems. You can just give it your best guess."

Coding. Children's problem-solving strategies were coded using a system developed by Perry et al. (1988). Strategies were assigned based on the solutions children wrote in the blank. As in prior work, solutions were coded as reflecting a particular strategy as long as they were within ± 1 of the solution that would be achieved with that particular strategy. Examples of common strategies for the problem "$9 + 4 + 3 = 9 +$ __" are presented in Table 1. In addition to correct strategies, we were particularly interested in use of the *add-all* strategy. The add-all strategy corresponds to one of the *operational patterns* identified by McNeil and Alibali (2005b)—the "perform all given operations on all given numbers" strategy. Commitment to the operational patterns is hypothesized to increase between the ages of 7 and 9 as children continue to gain narrow experience with traditional arithmetic in school. Research suggests that children who use the add-all strategy may be less likely than children who use other incorrect strategies to benefit from instruction on mathematical equivalence (McNeil & Alibali, 2005b, Experiment 1).

Results and Discussion

Correctness. Figure 1 presents a frequency plot of the number of correctly solved equivalence problems (out of 12). The figure displays the number of children who got zero correct, 1 correct, 2 correct, and so on, up to 12 correct. Performance was poor overall and not normally distributed, with nearly half of the children (N = 40) getting zero correct. This low level of performance is consistent with prior work (McNeil & Alibali, 2004; Perry et al., 1988). Because frequency distributions tend to be highly skewed and/or bimodal in studies of children's performance on equivalence problems, nonparametric statistics are typically used to analyze performance. Such was the case here. Children were categorized according to whether or not they solved at least one problem correctly, and logistic regression was used to analyze the data.

Logistic regression was used to predict the log of the odds of solving at least one equivalence problem correctly. Predictor variables included age linear (in months, centered), age quadratic (in months, centered), and gender (0 = girl, 1 = boy) as a control variable. The linear age coefficient was significant when controlling for the other predictors, $\hat{\beta} = 0.066$, $z = 2.75$, *Wald* $(1, N = 87) = 7.59$, $p = .006$. More importantly, as predicted by a change-resistance account, the quadratic age coefficient was significant when controlling for the other predictors, $\hat{\beta} = 0.0084$, $z = 4.00$, *Wald* $(1, N = 87) = 15.89$, $p < .001$. Gender was not

Table 1
Common Solutions for the Problem 9 + 4 + 3 = 9 + __

Solution	Strategy
7	Correct
25	Add all
4	Carry
16	Add to equal sign

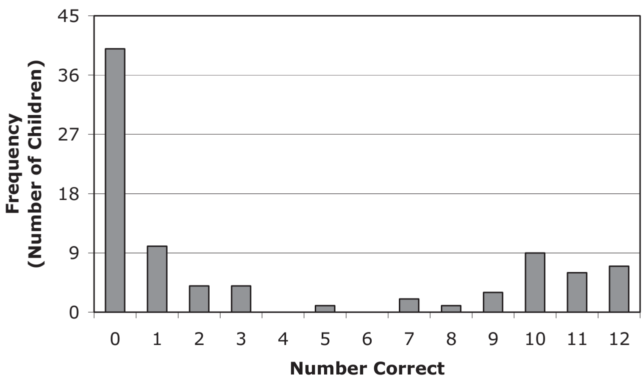

Figure 1. Frequency plot of the number of correctly solved equivalence problems (out of 12).

significant when controlling for the other predictors, $\hat{\beta} = 0.39$, $z = 0.75$, *Wald* $(1, N = 87) = 0.56$, $p = .45$. These results are summarized in Table 2.

Figure 2 presents the proportion of children who solved at least one equivalence problem correctly as a function of age (grouped into bins for illustrative purposes). The predicted U-shaped function can be seen in the figure. The youngest and oldest children in the sample were far more likely than children in the middle age range to solve at least one equivalence problem correctly.

It should be noted that the U-shaped pattern is robust. It does not depend on the criterion of "at least one correct." The quadratic age coefficient remains significant if the criterion is changed to "at least two correct," $\hat{\beta} = 0.0071$, $z = 3.74$, *Wald* $(1, N = 87) = 14.61$, $p < .001$, or "at least three correct," $\hat{\beta} = 0.0058$, $z = 3.05$, *Wald* $(1, N = 87) = 9.33$, $p = .002$. It also remains significant in a linear regression analysis when number correct (out of 12) is used as a continuous outcome variable, $\hat{\beta} = 0.011$, $t(83) = 3.99$, $p < .001$. Equally important, the findings hold up to different ways of representing the predictor variable, age. For example, the quadratic coefficient remains significant even if grade level is used as the predictor variable instead of age in months, $\hat{\beta} = 1.175$, $z = 4.47$, *Wald* $(1, N = 87) = 19.98$, $p < .001$. Thus, there is solid evidence for a U-shaped association between age and use of correct strategies on equivalence problems in this sample of children.

Use of the add-all strategy. According to change-resistance accounts, children's performance on equivalence problems worsens between ages 7 and 9 because children become more committed to the operational patterns they encounter in traditional arithmetic. One of these operational patterns is the "perform all given operations on all given numbers" strategy (McNeil & Alibali, 2005b). For example, when presented with a typical addition problem such as $3 + 4 + 5 + 6 =$ __, children learn to add up all the numbers and put the total, 18, in the blank. Children who are committed to this strategy will solve equivalence problems by adding up all the numbers (i.e., the add-all strategy). Use of this add-all strategy should decline in fifth grade and beyond as children start to become familiar with several math topics that contradict their knowledge of the operational patterns (e.g., equivalent fractions, equalities/inequalities, pre-algebra, algebra). Thus, children should increase in their use of the add-all strategy between the ages of 7 and 9, and then decrease in their use of the add-all strategy thereafter. This pattern corresponds to an inverted U-shaped association between age and children's use of the add-all strategy on equivalence problems.

There are at least two ways to analyze children's use of the add-all strategy. One possibility is to consider only incorrect strategies and then examine the proportion of incorrect strategies that were add-all. We did this using linear regression. Seven children were excluded from this analysis because they did not use any incorrect strategies. Predictor variables included age linear (in months, centered), age quadratic (in months, centered), and gender (0 = girl, 1 = boy) as a control variable. The linear age coefficient was significant when controlling for the other predictors, $pr = -.40$, $t(76) = -3.83$, $p < .001$. More importantly, as predicted by a change-resistance account, the quadratic age coefficient was significant when controlling for the other predictors, $pr = -.26$, $t(76) = -2.32$, $p = .02$. Gender was not significant when controlling for the other predictors, $pr = .12$, $t(76) = 1.01$, $p = .31$.

Another way to examine use of the add-all strategy is to examine the total number of problems solved using the strategy (out of 12). We did this using linear regression. Predictor variables were the same as in the previous analysis. The linear age coefficient was significant when controlling for the other predictors, $pr = -.45$, $t(83) = -4.56$, $p < .001$. And once again, the quadratic age coefficient was significant when controlling for the other predictors, $pr = -.30$, $t(83) = -2.88$, $p = .005$. Gender was not significant when controlling for the other predictors, $pr = .065$, $t(83) = 0.59$, $p = .55$. Thus, both ways of analyzing use of the add-all strategy yielded similar results. Findings correspond to the predicted inverted U-shaped pattern.

Is everything U-shaped in this sample? Thus far, we have shown a U-shaped association between age and correctness, and an inverted U-shaped association between age and use of the add-all strategy. Both of these findings support a change-resistance account. However, it is difficult to believe that 7-year-olds perform better than 9-year-olds on the target problems, which are essentially pre-algebra problems. Given the unintuitive nature of the findings, one might suspect that there is something peculiar about the children who participated in the study. The 7-year-olds may have been unusually accelerated, or the 9-year-olds may have been unusually delayed, or both. It would be problematic if the association between age and performance on all math tasks were U-shaped in this sample.

To alleviate this concern, it would help to show that there is not a U-shaped association between age and performance on another math task (e.g., calculating arithmetic facts, telling time, making

Table 2
Logistic Regression Analysis of Children's (Ages 7–11) Performance on Equivalence Problems

Predictor	$\hat{\beta}$	z	*Wald*	p
Age linear (in months, centered)	0.066	2.75	7.59	.006
Age quadratic (in months, centered)	8.4×10^{-3}	4.00	15.89	< .001
Gender (girl = 0, boy = 1)	0.39	0.75	0.56	.45

Note. Model predicts the log of the odds of solving at least one problem correctly.

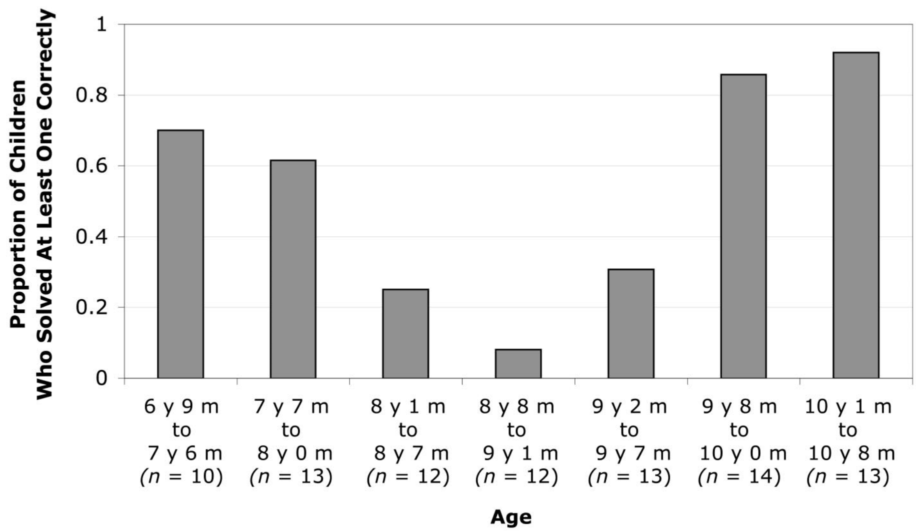

Figure 2. Proportion of children who solved at least one equivalence problem correctly as a function of age. y = years; m = months.

change, etc.). Although we did not have the foresight to test children's performance on several different math tasks, it was possible for us to get a rough estimate of children's addition skills, which should not show a U-shaped pattern. To get a rough estimate of children's addition skills, we examined whether children made arithmetic errors when carrying out the add-all or *add-to-equal* strategies (e.g., writing 24 or 17 in the blank for the problem 9 + 4 + 3 = 9 + __). We considered all children who used the add-all or add-to-equal strategy on at least one problem (N = 63). On average, these children used the add-all or add-to-equal strategy on 55% of the problems they solved. Of these children, 42 (67%) did not make any arithmetic errors when carrying out the strategies; 17 (27%) made one arithmetic error; 2 (3%) made two errors, and 2 (3%) made three errors. None of the children made greater than three arithmetic errors. We used logistic regression to predict the log of the odds of making an arithmetic error. Predictor variables included age linear (in months, centered), age quadratic (in months, centered), and gender (0 = girl, 1 = boy) as a control variable. None of the predictors were significant; if anything, there was a slight trend for the likelihood of making an arithmetic error to decrease linearly with age: linear age coefficient, $\hat{\beta} = -0.043$, $z = -1.59$, *Wald* $(1, N = 63) = 2.46$, $p = .12$; quadratic age coefficient, $\hat{\beta} = 2.7 \times 10^{-3}$, $z = 0.15$, *Wald* $(1, N = 63) = 0.022$, $p = .88$; gender, $\hat{\beta} = 0.37$, $z = 0.066$, *Wald* $(1, N = 63) = 4.3 \times 10^{-3}$, $p = .95$.

Thus, the U-shaped pattern does not hold for all measures of math performance in this sample. This finding helps to alleviate concerns about the participating children. However, some readers may still not be convinced. It is still possible that decrements in performance on equivalence problems between the ages of 7 and 9 are specific to these particular children. Thus, a second study was conducted with another sample of children. The goal of the second study was to see if the decrements in performance between the ages of 7 and 9 could be replicated with a different group of children under a different set of conditions. Few theories would predict performance decrements in math between the ages of 7 and 9. In contrast, most prevailing theories would predict improvements between the ages of 9 and 11. Thus, the second study focused on children between the ages of 7 and 9, not the entire 7–11 age range.

Study 2

Method

Participants. Thirty-five children (16 boys, 19 girls; 7 second graders, 28 third graders) participated. Children ranged in age from 7 years 0 months to 8 years 11 months (M = 8 years 4 months). The study was conducted at a public school in a suburb of Raleigh, NC in the second month of the school year. The racial/ethnic makeup of the school was 24% African American, 1% Asian, 6% Hispanic, and 69% White. Approximately 37% of children received free or reduced-price lunch.

Procedure. The procedure was identical to that of Study 1, with one exception. Teachers in Study 2 gave children two brief lessons about mathematical equivalence prior to the paper-and-pencil problem-solving test. This supplement helped to make the children across the age range more comparable in terms of background knowledge about the equal sign before they solved the equivalence problems. The lessons were modeled after the lessons in McNeil (2004). During each lesson, teachers presented children with a correctly solved equation (e.g., 15 + 13 = 28, 1 foot = 12 inches) and told them to "notice that whatever is on one side of the equal sign has to be the same amount as whatever is on the other side of the equal sign." Each child received this instruction in the context of four different equations, but none received instruction in the context of a mathematical equivalence problem. Previous research has shown that conceptual instruction on the meaning of the equal sign improves children's performance on equivalence problems (Rittle-Johnson & Alibali, 1999). Thus, the instruction should improve children's overall performance on the equivalence problems.

Coding. Coding was identical to that in Study 1.

Results and Discussion

Even with the addition of the brief lessons on the concept of mathematical equivalence prior to the problem-solving test, performance was still very poor overall, with over half of the children ($N = 19$) getting zero correct. This poor performance after brief classroom-based equal sign instruction is consistent with prior work (e.g., McNeil, 2004). Logistic regression was used to examine the association between age (in months, centered) and the likelihood of solving at least one equivalence problem correctly. The likelihood of solving at least one problem correctly decreased with age, $\hat{\beta} = -0.17$, $z = -2.34$, $Wald\ (1, N = 35) = 5.48$, $p = .02$. Figure 3 presents the proportion of children who solved at least one equivalence problem correctly as a function of age (grouped into bins for illustrative purposes). The predicted negative linear association can be seen in the figure. Specifically, the proportion of children who solved at least one problem correctly decreased from age 7 to 9. This result replicates the decrements in performance between the ages of 7 and 9 that were shown in Study 1. Taken together with results of Study 1, this finding suggests that performance on mathematical equivalence problems declines between the ages of 7 and 9 before it starts to improve thereafter.

General Discussion

Despite decades of research on children's (mis)understanding of mathematical equivalence, the current studies were the first to investigate the association between age (7–11 years) and performance on equivalence problems. The absence of previous studies may be due to the fact that most people assume that the association is positive (and thus, trivial). However, the two studies presented here revealed that the association is U-shaped. In the first study, performance on equivalence problems declined between the ages of 7 and 9 and improved between the ages of 9 and 11. The decrements in performance between ages 7 and 9 were replicated in a second study. These findings challenge intuition and several prevailing theories, but they support change-resistance accounts.

A number of prevailing theories of learning and development in the domain of mathematics attribute children's difficulties with math to something that children lack (e.g., advanced logical structures, a mature working memory system, proficiency with basic arithmetic facts). In contrast, change-resistance accounts suggest that difficulties are due, at least in part, to the knowledge that children have. According to this view, early and prolonged practice with arithmetic procedures hinders performance on equivalence problems because equivalence problems do not map onto to the operational patterns learned in arithmetic.

Children focus on arithmetic procedures in the early school years, and they encounter the same operational patterns repeatedly, virtually without exception (Seo & Ginsburg, 2003). Through this experience, children construct long-term memory representations of the operational patterns. Eventually, these internal representations become strong enough to interfere with information that is actually present in an external problem (Bruner & Postman, 1949; Gray & Fu, 2001; McNeil & Alibali, 2004; Munakata, 1998). Because equivalence problems do not map onto children's representations of the operational patterns, children must ignore or override their representations in order to solve equivalence problems correctly.

Based on the developmental and experiential changes that take place across the elementary school years, children's susceptibility to interference from their representations of the operational patterns should be greatest around age 9. According to change-resistance accounts, this is why performance on equivalence problems is at a minimum at this age. However, there are at least three alternative explanations for the observed poor performance on equivalence problems at age 9. First, arithmetic proficiency could

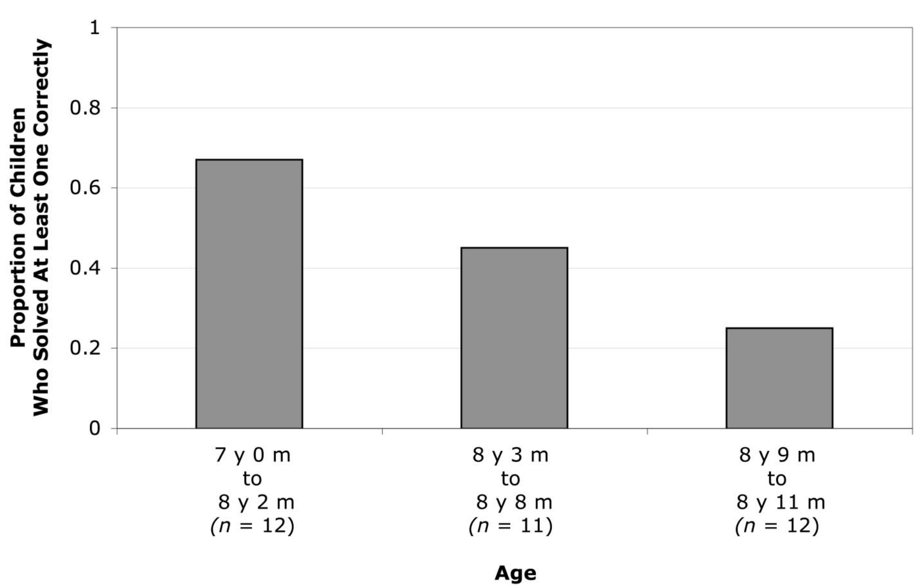

Figure 3. Proportion of children who solved at least one equivalence problem correctly as a function of age. y = years; m = months.

drop substantially at this age. There is some evidence for this claim. For example, Miller and Paredes (1990) suggested that proficiency with addition may decrease when children start learning multiplication between the ages of 8 and 10. However, it is unclear how to interpret this finding because several other studies (e.g., Barrouillet & Lepine, 2005; Geary et al., 1996; Hecht et al., 2001; Svenson & Sjoberg, 1983) have shown increases in arithmetic proficiency over the course of the early elementary school years.

The second possibility is that the association between arithmetic proficiency and performance on equivalence problems could, itself, be U-shaped. In this case, arithmetic proficiency would increase over the course of the elementary school years, but performance on equivalence problems would suffer at intermediate levels of arithmetic proficiency. This hypothesis is consistent with Marcovitch and Zelazo's (1999, 2006) view that children's behavior is controlled by two hierarchical competing systems: a response-based system and a conscious representational system. Although Marcovitch and Zelazo's hierarchical competing systems model was developed to explain performance on the A-not-B search task, it can be applied to performance on equivalence problems. Specifically, it suggests that as arithmetic proficiency increases, the response-based system increases in its tendency to apply the add-all strategy until an asymptote is reached. At the same time, cognitive resources are "freed" and made available for conscious reflection. As opportunities for conscious reflection increase, the likelihood of understanding the problem also increases, and eventually, the incorrect add-all response is overridden by the conscious representational system (cf. Crowley, Shrager, & Siegler, 1997). This account suggests that intermediate levels of arithmetic proficiency may correspond to what Hatano (1988) has called "routine" expertise (see also Dowker, Flood, Griffiths, Harriss, & Hook, 1996). Routine expertise is associated with inflexible application of knowledge. Thus, children who have intermediate levels of arithmetic proficiency may apply their knowledge of arithmetic procedures inflexibly when they encounter an equivalence problem. As children increase their proficiency beyond the intermediate level, however, they may eventually develop "adaptive" expertise and exhibit the flexibility of true experts (Dowker, 1992; Hatano, 1988). Future work will need to specify the association between arithmetic proficiency and performance on equations. In this spirit, my colleagues and I are currently manipulating arithmetic proficiency by having individuals practice arithmetic facts (or not) before they solve equivalence problems. Results will inform our understanding of the effects of arithmetic proficiency on equation-solving performance.

The third possibility is that there is a more general developmental factor behind the U-shaped association between age and performance on equivalence problems. Specifically, children may go through a developmental shift in their representational skills around the age of 9 and 10 years that causes them to rely rigidly on customs and conventions. Consider a study by Church, Kelly, and Lynch (2000). They performed two experiments to examine how participants of various ages process messages that contain mismatching speech and gesture. In their first experiment, they presented participants with a videotape of children speaking and gesturing about a concept. Then, they asked participants to recall what they saw. Nine- and 10-year-old participants focused exclusively on the information contained in speech, whereas younger and older participants attended to both channels. According to the authors, one reason the 9- and 10-year-old children focused on speech was because it is the customary channel of communication. According to this rationale, the 9-year-old children in the present study may have been more likely to add up all the numbers in the equivalence problems because adding up all the numbers is the customary way to solve a math problem. This is an intriguing hypothesis that will need to be addressed in future work.

Taken together with other studies that document U-shaped developmental patterns (e.g., Church, Kelly, & Lynch, 2000; Namy, Campbell, & Tomasello, 2004), the current study suggests that performance does not always improve with age, as many theories (including intuition) would suggest. Rather, performance sometimes gets worse before it gets better. As a general rule, U-shaped developmental functions (like the one revealed here) provide the field with insights about the factors that may or may not drive development (Siegler, 2004). They indicate that cognitive development may not always be driven by simple, monotonic increases in some attribute of the cognitive system (e.g., working memory capacity or arithmetic skill). Instead, development is driven by complex interactions among cognitive structures (e.g., long-term memory representations), cognitive processes (e.g., resistance to interference), and features of the external environment (e.g., problem type). Detailed investigations of these interactions will lead to a deeper understanding of the mechanisms that drive (and constrain) cognitive development across the life span.

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