Filling an N-sided hole using combined subdivision schemes

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Abstract

A new algorithm is presented for filling n-sided holes. The algorithm generates a subdivision surface which connects smoothly (G^1) with the surface around the hole. The data to the algorithm consists of n boundary curves and cross boundary derivatives, given in any parametric form.

The algorithm is based on Sabin's variant of Catmull-Clark scheme [1], which operates in the surface interior. Near the boundary we introduce new subdivision rules that involve the given boundary curves and cross-boundary derivatives. The generated subdivision surface is G^2-continuous except at one extraordinary point. In the neighborhood of this point the surface curvature is bounded.

The proposed subdivision scheme falls into the category of *combined subdivision schemes*, which combine operations on control points with operations on functions. The theory of *combined subdivision schemes* is used to derive the scheme and to establish its properties.

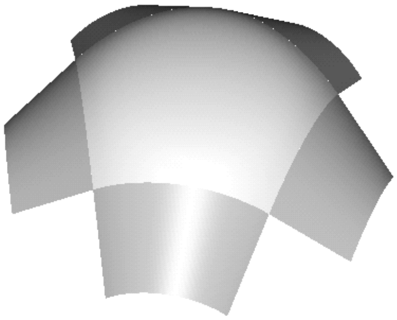

Figure 1: A 5-sided surface.

1 Background

The problem of constructing N-sided surface patches occurs frequently in computer-aided geometric design. The N-sided patch is required to connect smoothly to given surfaces surrounding a polygonal hole, as shown in figure 1.

Referring to [10, 25, 26], n-sided patches can be generated basically in two ways. Either the polygonal domain, which is to be mapped into 3D, is subdivided in the parametric plane, or one uniform equation is used as a combination of 3D constituents. In the former case triangular or rectangular elements are put together [2, 6, 12, 20, 23] or recursive subdivision methods are applied [5, 8, 24]. In the latter case either the known control-point based methods are generalized or a weighted sum of 3D interpolants gives the surface equation [1, 3, 4, 22, 26].

This paper presents a subdivision schemes specially designed for the task of filling n-sided holes, which belongs to the former case. Subdivision schemes provide efficient algorithms for the design, representation and processing of smooth surfaces of arbitrary topological type. Their simplicity and their multiresolution structure make them attractive for applications in 3D surface modeling, and in computer graphics [7, 9, 11, 13, 19, 27, 28].

The subdivision scheme presented in this paper falls into the category of *combined subdivision schemes* [14, 15, 17, 18], where the underlying surface is represented not only by a control net, but also by the given boundary

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conditions. The scheme repeatedly applies a subdivision operator to the control net, which becomes more and more dense. In the limit, the vertices of the control net converge to a smooth surface. Samples of the boundary conditions participate in every iteration of the subdivision, and as a result the limit surface satisfies the given conditions, regardless of their representation. Thus, our scheme performs so-called transfinite interpolation.

The motivation behind the specific subdivision rules, and the smoothness analysis of the scheme are presented in [16]. In the following sections, we describe Catmull-Clark's scheme, and we present the details of our scheme.

2 Catmull-Clark's scheme

A net $N = (V, E)$ consists of a set of vertices V and the topological information of the net E, in terms of edges and faces. A net is closed when each edge is shared by exactly two faces.

Camull Clark's subdivision scheme is defined over closed nets of arbitrary topology, as an extension of the tensor product bi-cubic B-spline subdivision scheme (see [5, 8]). Variants of the original scheme were analyzed by Ball and Storry [24]. Our algorithm employs a variant of Catmull-Clark's scheme due to Sabin [21], which generates limit surfaces that are G^2-continuous everywhere except at a finite number of irregular points. In the neighborhood of those points the surface curvature is bounded. The irregular points come from vertices of the original control net that have valency other than 4, and from faces of the original control net that are not quadrilateral.

Given a net N, the vertices V' of the new net $N' = (V', E')$ are calculated by applying the following rules on N (see figure 2):

1. For each old face f, make a new face-vertex $v(f)$ as the weighted average of the old vertices of f, with weights W_n that depend on the valency n of each vertex.

2. For each old edge e, make a new edge-vertex $v(e)$ as the weighted average of the old vertices of e and the new face vertices associated with the two faces originally sharing e. The weights W_n (which are the same as the weights used in rule 1) depend on the valency n of each vertex.

3. For each old vertex v, make a new vertex-vertex $v(v)$ at the point given by the following linear combination, whose coefficients $\alpha_n, \beta_n, \gamma_n$ depend on the valency n of v:

   $\alpha_n\cdot$ (the centroid of the new edge vertices of the edges meeting at v) + $\beta_n\cdot$ (the centroid of the new face vertices of the faces sharing those edges) + $\gamma_n \cdot v$.

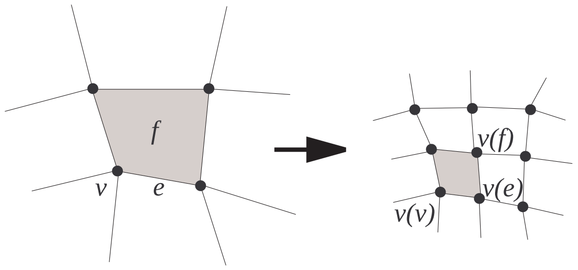

Figure 2: Catmull-Clark's scheme.

The topology E' of the new net is calculated by the following rule:

> For each old face f and for each vertex v of f, make a new quadrilateral face whose edges join $v(f)$ and $v(v)$ to the edge vertices of the edges of f sharing v (see figure 2).

We present the procedure for calculating the weights mentioned in §2, as formulated by Sabin in [21].

Let $n > 2$ denote a vertex valency. Let $k := \cos(\pi/n)$. Let x be the unique real root of

$$x^3 + (4k^2 - 3)x - 2k = 0,$$

satisfying $x > 1$. Then

$$\begin{aligned} W_n &= x^2 + 2kx - 3, \qquad (1)\\ \alpha_n &= 1,\\ \gamma_n &= \frac{kx + 2k^2 - 1}{x^2(kx+1)},\\ \beta_n &= -\gamma_n. \end{aligned}$$

The original paper by Sabin [21] contains a mistake: the formulas for the parameters α, β and γ that appear in §4 there, are $\beta := 1,\ \gamma := -\alpha$.

n	W_n	γ_n
3	1.23606797749979...	0.06524758424985...
4	1	0.25
5	0.71850240323974...	0.40198344690335...
6	0.52233339335931...	0.52342327689253...
7	0.39184256502794...	0.61703187134796...

Table 1: The weights used in Sabin's variant of Catmull-Clark's subdivision scheme

The weights W_n and γ_n for $n = 3, \ldots, 7$ are given in table 1.

3 N-sided input data

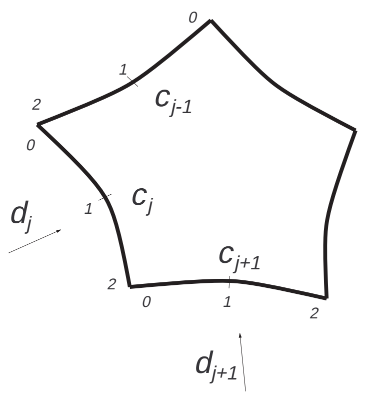

Figure 3: The data to the N-sided hole-filling problem.

The input to our scheme consists of N smooth parametric curves $c_j : [0,2] \to \mathbb{R}^3$ defined over the parametric interval $[0,2]$, and corresponding cross-boundary derivative functions $d_j : [0,2] \to \mathbb{R}^3$ (see figure 3). We say that the input data is C^0 **compatible** at the j-th corner, if

$$c_j(2) = c_{j+1}(0).$$

The conditions for C^1 **compatibility** are

$$\begin{aligned} d_j(0) &= -c'_{j-1}(2), \\ d_j(2) &= c'_{j+1}(0). \end{aligned}$$

We say that the input data is C^2 **compatible** if the curves $\{c_j\}$ have Hölder continuous second deriatives, the functions $\{d_j\}$ have Hölder continuous derivatives, and the following twist condition is satisfied:

$$d'_j(2) = -d'_{j+1}(0). \qquad (2)$$

In [16] we prove that our scheme generates surfaces that are C^2-continuous near the boundary for C^2-compatible data. However, the algorithm is applicable also for data which is not C^2-compatible, as demonstrated by the examples in §5.

4 The algorithm

In this section we describe our our algorithm for the design of an N-sided patch satisfying the boundary conditions described in §3. The key ingredients of the algorithm are two formulas for calculating the boundary vertices of the net. These formulas are given in §4.3 and §4.4.

In the following we denote by n the iteration number, where $n = 0$ corresponds to the first iteration.

4.1 Constructing an initial control net

The algorithm starts by constructing an initial control net, whose faces are all quadrilateral, with $2N$ boundary vertices and one middle vertex, as shown in figure 4. The boundary vertices are placed at the parameter values $0, 1, 2$ on the given curves. The middle vertex can be arbitrarily chosen by the designer, and it controls the shape of the resulting surface.

4.2 An iteration of subdivision

In the n-th iteration we perform three steps:

1. Relocate the boundary vertices according to the rules given below in §4.3 - §4.4.

2. Apply Sabin's variant of Catmull-Clark's scheme to calculate the new net topology and the position of the new **internal** vertices. For the purpose of choosing appropriate weights in the averaging process, we consider the boundary vertices as if the all have valency 4. This makes up for the fact that the net is not closed.

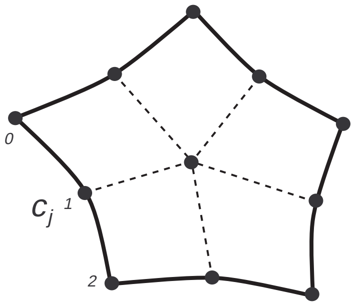

Figure 4: The initial control net.

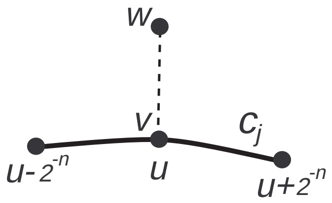

Figure 5: A smooth boundary rule.

3. Sample the boundary vertices from the given curves at uniformly spaced parameter values with interval length $2^{-(n+1)}$.

4.3 A smooth boundary rule

Let v denote a boundary vertex corresponding to the parameter $0 < u < 2$ on the curve c_j. Let w denote the unique internal vertex which shares an edge with v (see figure 5). At the first step of the n-th iteration, we calculate the position of the v by the following formula:

$$\begin{aligned} v &= 2c_j(u) - \frac{1}{2}w - \\ &\quad \frac{1}{4}\left(c_j\left(u+2^{-n}\right) + c_j\left(u-2^{-n}\right)\right) - \\ &\quad 2^{-n}\frac{1}{12}\left(d_j\left(u+2^{-n}\right) + d_j\left(u-2^{-n}\right)\right) + \\ &\quad 2^{-n}\frac{2}{3}d_j(u). \end{aligned}$$

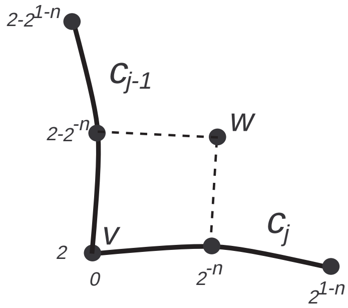

Figure 6: A corner rule.

4.4 A corner rule

Let v denote a boundary vertex corresponding to the point $c_{j-1}(2) = c_j(0)$. Let w be the unique internal vertex sharing a face with v (see figure 6). we calculate the position of the v by the following formula:

$$\begin{aligned} v &= \frac{5}{2}c_j(0) + \frac{1}{4}w - \left(c_j(2^{-n}) + c_{j-1}(2-2^{-n})\right) + \\ &\quad \frac{1}{8}\left(c_j(2^{1-n}) + c_{j-1}(2-2^{1-n})\right) + \\ &\quad 2^{-n}\frac{29}{48}\left(d_j(0) + d_{j-1}(2)\right) - \\ &\quad 2^{-n}\frac{1}{12}\left(d_j(2^{-n}) + d_{j-1}(2-2^{-n})\right) - \\ &\quad 2^{-n}\frac{1}{48}\left(d_j(2^{1-n}) + d_{j-1}(2-2^{1-n})\right). \end{aligned}$$

5 Properties of our scheme

In [16] we prove that the vertices generated by the above procedure converge to a surface which is C^2-continuous almost everywhere, for C^2-compatible data (as defined in §3). The only point where the surface is not C^2-continuous is a middle-point (corresponding to the middle vertex, which has valency N), where the surface is only G^1-continuous. In the neighborhood of this extraordinary point, the surface curvature is bounded.

The limit surface interpolates the given curves, for C^0-compatible data. For C^1-compatible data, the tangent plane of the limit surface at the point $c_j(u)$ is spanned

by the vectors $c'_j(u)$ and $d_j(u)$, thus the surface satisfies C^1-boundary conditions.

Due to the locality of this scheme, we have that the limit surface is C^2 near the boundaries except at points where the C^2-compatibility condition is not satisfied. For the examples in figures 7 - 10 we used data which is only C^1-compatible since it does not satisfy the twist condition (2) at the corners. Thus, the limit surfaces are C^2-continuous near the boundary except at the corners.

Figures 8-10 demonstrate that the limit surface behaves mederately even in the presence of wavy boundary conditions.

Figure 11 shows the limit surface resulting from repeated application of our scheme to data which is not C^1-compatible. The surface is C^2-continuous on its boundary except at two of its corners, where the tangent-plane is not defined.

In figure 12 we show that the surfaces generated by our algorithm can be easily deformed by editing internal vertices of the net at some level of the subdivision. Smale-scale deformations can be done by moving internal vertices in an advanced iteration of subdivision, while large-scale deformation corresponds to moving vertices in one of the first subdivision iterations. Thus, our approach allows free-form multiresolution editing, which is natural to subdivision schemes [28].

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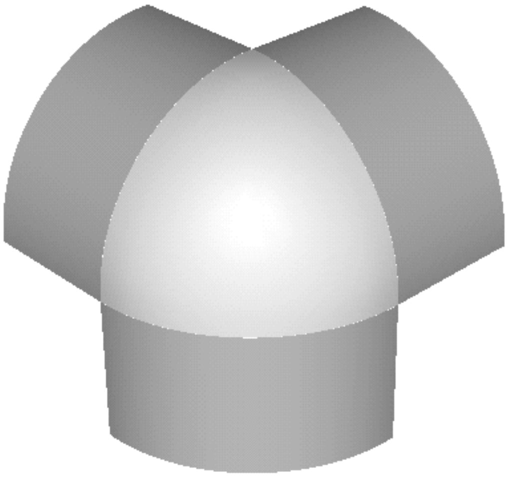

Figure 7: A 3-sided surface.

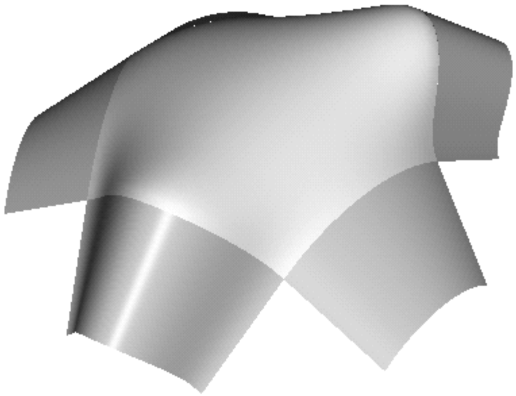

Figure 9: A 5-sided surface.

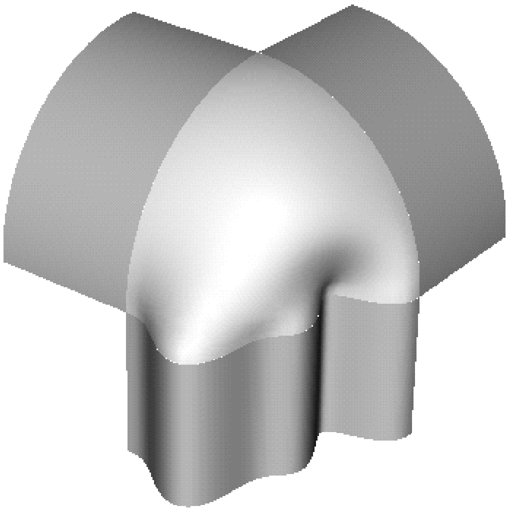

Figure 8: A 3-sided surface with wavy boundary conditions.

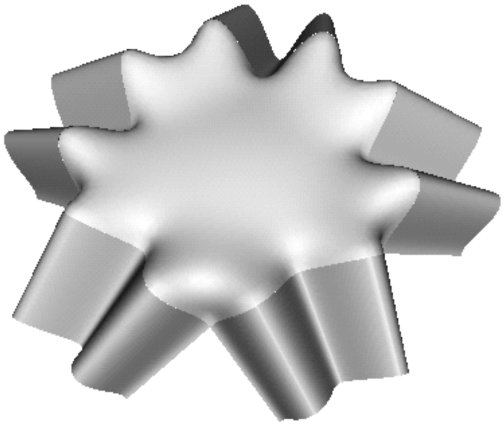

Figure 10: A 5-sided surface with wavy boundary conditions.

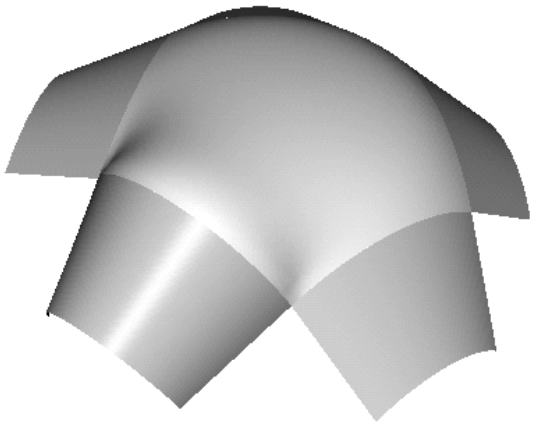

Figure 11: A 5-sided surface with boundary conditions that are not C^1 compatible.

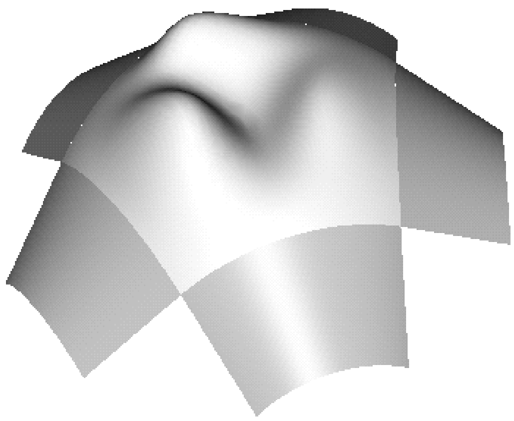

Figure 12: A 5-sided surface deformed by modifying vertices after the first iteration of subdivision.