Poisson Structures for Geometric Curve Flows in Semi-simple Homogeneous Spaces

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Abstract. We apply the equivariant method of moving frames to investigate the existence of Poisson structures for geometric curve flows in semi-simple homogeneous spaces. We derive explicit compatibility conditions that ensure that a geometric flow induces a Hamiltonian evolution of the associated differential invariants. Our results are illustrated by several examples.

1. Introduction.

In 1972, Hasimoto, [**6**], showed how the evolution of curvature and torsion of space curves under the vortex filament flow is governed by the completely integrable nonlinear Schrödinger equation. Since then, a large variety of integrable soliton equations, including all of the most familiar examples (Korteweg–deVries, modified Korteweg–deVries, Sawada–Kotera, etc.), have arisen in connection with invariant geometric curve flows in various Klein geometries, [**1**, **2**, **3**, **10**, **15**, **17**]. However, the underlying reasons for the surprisingly frequent appearance of integrability remain mysterious.

The basic geometric construction begins with a Lie group G acting on a m-dimensional manifold M — typically a homogeneous space. A G-invariant evolution equation for curves

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in M is said to define a *geometric curve flow*. Given a geometric curve flow, our focus will be on the induced evolution of the differential invariants associated with the group action. In favorable situations, they evolve according to a completely integrable bi-Hamiltonian system, cf. [**18**]. A complete understanding of this phenomenon remains obscure, and so our more modest aim is investigate the Hamiltonian structure of such differential invariant evolutions. We will apply the powerful equivariant moving frame and invariant variational bicomplex machinery developed in [**5**, **9**, **20**] to analyze the Poisson reduction of one of the canonical Poisson structures appearing on the loop space of the Lie algebra dual, denoted $\mathcal{L}\,\mathfrak{g}^*$, to the evolution of the Maurer–Cartan differential invariants associated with the moving frame. The geometric description of the resulting reduced Poisson structure was first described in [**12**, **14**].

In this paper, we restrict our attention to curve flows in homogeneous spaces $M = G/N$ in which G is a semisimple Lie group and $N \subset G$ a closed subgroup. Modulo certain extra complications, our results can be extended to curves in affine geometry; details of the latter construction will appear elsewhere. We will distinguish curves with different parametrizations, meaning that the action of G does not affect the parametrization. The case of unparametrized curves (i.e., working modulo the reparametrization pseudo-group) will be the subject of future investigations, building on the fact that the moving frame and invariant variational bicomplex constructions are equally valid in this context.

We begin by exhibiting a class of Poisson structures for curves in a Euclidean space that are prescribed by first order differential operators of a particular form. All such operators arise on the loop space $\mathcal{L}\,\mathfrak{g}^*$ based on the dual to some Lie algebra $\mathfrak{g}$, where they can be viewed as central extensions of the classic Lie–Poisson structure on $\mathfrak{g}^*$, [**8**, **18**]. Since the Maurer–Cartan differential invariants naturally take their values in the Lie algebra $\mathfrak{g}$, we first translate the Poisson evolution to the loop space $\mathcal{L}\,\mathfrak{g}$. In section 3, building on earlier work [**12**, **14**], we analyze the Poisson reduction, [**16**], of such structures to the quotient loop space $\mathcal{L}\,\mathfrak{g}/\mathcal{L}\,N$.

Our approach to the geometry of invariant curve flows will be based on the equivariant method of moving frames introduced in [**5**]. General formulas for the evolution of differential invariants under invariant geometric flows were found in [**20**], using certain invariant differential operators that arise naturally in the invariant variational bicomplex of [**9**]. In section 4, we establish a remarkable factorization of the invariant linearization operator associated with the Maurer–Cartan invariants, whose first factor can be identified with the pull back, via the Maurer–Cartan map, of the Lie algebra Poisson operator found in the first section, while the second factor, dubbed the Maurer–Cartan operator, also appears naturally in the context of the invariant variational bicomplex.

In section 5 we describe how reduction works in the differential invariant setting, and in section 6 we investigate the algebraic description of compatibility conditions found in [**14**] that are required for the reduction of a Hamiltonian flow on $\mathcal{L}\,\mathfrak{g}^*$ to coincide with the differential invariant evolution induced by a geometric curve flow. We then establish the existence of a suitable reduction of the loop space Poisson structure governing compatible flows. We will need to impose certain restrictions on the structure of the Maurer–Cartan invariants that occur in the examples of interest illustrating our results. Extensions of these methods to more general settings will be the subject of future investigations.

2. Lie Algebraic Poisson Structures.

We begin by stating a basic classification theorem for first order Poisson differential operators of a particular form, defined on the loop space of a Euclidean space. The resulting Poisson structure leads to an identification of the underlying Euclidean space with the dual vector space to a Lie algebra.

Recall that the *loop space* of a manifold M is, by definition, the space of smooth maps from the unit circle into M, denoted $\mathcal{L}\, M = \mathrm{C}^\infty(S^1, M)$. Suppose $M = \mathbb{R}^r$, with coordinates $L = (L_1, \ldots, L_r)$. Consider a first order $r \times r$ matrix differential operator $\widehat{\mathcal{P}} = \widehat{\mathcal{P}}[L]$ on the loop space $\mathcal{L}\, M$, whose entries are of the form

$$\widehat{\mathcal{P}}_{ij} = b_{ij} D_x - \sum_{k=1}^{r} c_{ij}^k L_k, \tag{2.1}$$

with b_{ij}, c_{ij}^k are constants. An easy computation based on the methods of [**18**; Section 7.2] leads to the following characterization of when such a differential operator defines a Poisson structure.

Theorem 2.1. *The first order matrix differential operator $\widehat{\mathcal{P}}$ with entries* (2.1) *is Poisson if and only if*

(*a*) c_{ij}^k *are the structure constants for an r-dimensional Lie algebra $\mathfrak{g}$ relative to a basis $\mathbf{v}_1, \ldots, \mathbf{v}_r$, and*

(*b*) $B = (b_{ij})$ *is a symmetric $r \times r$ matrix, with $\beta = \sum_{i,j=1}^r b_{ij} \mu^i \odot \mu^j$ an* ad^**–invariant symmetric 2 tensor on the Lie algebra dual $\mathfrak{g}^*$, in which $\mu^1, \ldots, \mu^r$ form the dual basis of $\mathfrak{g}^*$, i.e., the* Maurer–Cartan forms.

The ad^*–invariance of β requires that

$$\sum_{l=1}^{r} (\, c_{ij}^l b_{lk} + c_{ik}^l b_{lj}\,) = 0 \qquad \text{for all} \qquad i, j, k. \tag{2.2}$$

In view of this condition, we will refer to β as a *cocycle*.

Remark: The Poisson operator (2.1) is *not* of hydrodynamic type, [**4**], since it doesn't involve the derivative of the L_i in the second term.

We can thus identify $\mathbb{R}^r \simeq \mathfrak{g}^*$, with $L = (L_1, \ldots, L_r)$ representing the coordinates of a point

$$L = \sum_{i=1}^{r} L_i \mu^i \in \mathfrak{g}^*.$$

With this convention, the differential operator (2.1) defines a Poisson structure on the loop space $\mathcal{L}\, \mathfrak{g}^*$. The Poisson operator

$$\widehat{\mathcal{P}}[L] H = B\, D_x H + \mathrm{ad}_H^*(L), \qquad L \in \mathfrak{g}^*, \qquad H \in \mathcal{L}\, \mathfrak{g} \tag{2.3}$$

maps $\mathcal{L}\mathfrak{g} \to \mathcal{L}\mathfrak{g}^*$. Note that Theorem 2.1 does not require that the cocycle β be of maximal rank, and so the Poisson structure can be degenerate. In particular, if $\beta = 0$, then $\widehat{\mathcal{P}}[L]$ reduces to the usual Lie–Poisson structure on $\mathfrak{g}^*$, [**18**].

If $\mathfrak{g}$ is semi-simple, then β will be a multiple of the Killing metric tensor; to avoid degenerating back to the Lie–Poisson structure, we assume the multiple is non-zero. Under this restriction, B is an invertible matrix, and the Poisson structure (2.3) is non-degenerate. From here on, for brevity, we will restrict out attention to the semi-simple case, although many of our results can be extended to affine Lie algebras, and some to completely general Lie algebras. An ongoing project is to determine how far these methods can be pushed.

Let $\widehat{H}[L]$ be a Hamiltonian functional defined on $\mathcal{L}\mathfrak{g}^*$. For the Poisson structure defined by (2.3), the associated *Hamiltonian flow* is

$$\frac{\partial L}{\partial t} = \widehat{\mathcal{P}}[L]\, \frac{\delta \widehat{H}}{\delta L} = B\, D_x\, \delta\widehat{H} + \mathrm{ad}^*_{\delta\widehat{H}}(L). \tag{2.4}$$

where, on occasion, we abbreviate the variational derivative as $\delta\widehat{H} = \delta\widehat{H}/\delta L$. In coordinates,

$$\frac{\partial L_i}{\partial t} = \sum_{j=1}^{r} \left(b_{ij} D_x \frac{\delta H}{\delta L_j} \;-\; \sum_{k=1}^{r} c_{ij}^{k} L_k \frac{\delta \widehat{H}}{\delta L_j} \right).$$

Let us introduce dual variables $K = (K^1, \ldots, K^r)$, which define coordinates on the Lie algebra $\mathfrak{g}$. The cocycle β defines an invertible linear map $\beta : \mathfrak{g} \to \mathfrak{g}^*$, given by

$$L = B\, K. \tag{2.5}$$

Given a Hamiltonian functional $H[L]$ on $\mathcal{L}\mathfrak{g}^*$, let

$$\widetilde{H}[K] = \widehat{H}[B\, K] = \beta^*\big(\widehat{H}[L]\big) \tag{2.6}$$

be the corresponding Hamiltonian functional on $\mathcal{L}\mathfrak{g}$. Clearly, their variational derivatives are related by

$$\frac{\delta \widetilde{H}}{\delta K}[K] = B\, \frac{\delta \widehat{H}}{\delta L}[B\, K]. \tag{2.7}$$

We now use the cocycle map β to pull back the Hamiltonian system (2.4). Substituting (2.5, 7), and then using (2.2, 3), we find

$$\begin{aligned} \sum_{j=1}^{r} b_{ij} \frac{\partial K^j}{\partial t} &= \sum_{j=1}^{r} b_{ij} D_x \frac{\delta \widehat{H}}{\delta L_j} - \sum_{j,k=1}^{r} c_{ij}^{k} b_{kl} K^l \frac{\delta \widehat{H}}{\delta L_j} \\ &= \sum_{j=1}^{r} b_{ij} D_x \frac{\delta \widehat{H}}{\delta L_j} - \sum_{j,k=1}^{r} b_{ik} c_{jl}^{k} K^l \frac{\delta \widehat{H}}{\delta L_j}. \end{aligned} \tag{2.8}$$

Since B is invertible, in view of (2.7), the Poisson evolution on $\mathcal{L}\mathfrak{g}$ takes the form

$$\frac{\partial K}{\partial t} = \widetilde{\mathcal{Q}}[K]\, B^{-1}\, \frac{\delta \widehat{H}}{\delta K} = \widetilde{\mathcal{P}}[K]\, \frac{\delta \widetilde{H}}{\delta K}\;, \tag{2.9}$$

where $\widetilde{\mathcal{Q}}[K] = \big(\widetilde{\mathcal{Q}}_i^k \big)$ is the matrix differential operator with entries

$$\widetilde{\mathcal{Q}}_i^k = \delta_i^k D_x - \sum_{j=1}^{r} c_{ij}^k K^j, \tag{2.10}$$

and δ_i^k is the usual Kronecker symbol. Note that

$$\widetilde{\mathcal{Q}}[K](J) = \big(D_x + \mathrm{ad}_K \big) J = D_x J + [\, K, J\,] \qquad \text{for any} \qquad J \in \mathcal{L}\,\mathfrak{g}, \tag{2.11}$$

and satisfies

$$\widehat{\mathcal{P}}[L] = B\, \widetilde{\mathcal{Q}}[K] \qquad \text{when} \qquad L = B\,K. \tag{2.12}$$

On the other hand,

$$\widetilde{\mathcal{P}}[K] = \widetilde{\mathcal{Q}}[K]\, B^{-1} = B^{-1} \widehat{\mathcal{P}}[B\,K]\, B^{-1} \tag{2.13}$$

is the pull-back of the Poisson operator (2.3) under the map (2.5). As a result, $\widetilde{\mathcal{P}}$ automatically defines a Poisson structure on $\mathcal{L}\,\mathfrak{g}$.

3. Poisson Reduction.

Let G be an r dimensional Lie group, acting on a $q = r - s$ dimensional homogeneous space $M = G/N$ for some s dimensional closed subgroup $N \subset G$. The Lie algebras of N and G are denoted, respectively, by $\mathfrak{n} \subset \mathfrak{g}$, while $\mathcal{L}\,\mathfrak{n} \subset \mathcal{L}\,\mathfrak{g}$ are the associated loop spaces.

In preparation for our study of Poisson structures on invariants, we investigate the reduction of the Poisson structure (2.13) on $\mathcal{L}\,\mathfrak{g}$ to the quotient loop space $\pi\colon \mathcal{L}\,\mathfrak{g} \to \mathcal{L}\,\mathfrak{g}/\mathcal{L}\,N$. Here $\mathcal{L}\,N$ acts on $\mathcal{L}\,\mathfrak{g}$ via the loop extension of the adjoint action

$$A(N) \cdot K = N^{-1} N_x - \mathrm{Ad}_N(K) \qquad \text{for} \qquad N \in \mathcal{L}\,N. \tag{3.1}$$

The infinitesimal action is given by

$$a(n) \cdot K = n_x - \mathrm{ad}_n(K) = n_x - [\, n, K\,] = n_x + \mathrm{ad}_K(n) = \widetilde{\mathcal{Q}}[K]\, n \qquad \text{for} \qquad n \in \mathcal{L}\,\mathfrak{n}. \tag{3.2}$$

These actions are the dual to the actions on $\mathcal{L}\,\mathfrak{g}^*$ described in [**8**], and induced by a central extension of the algebra of loops. We refer the reader to [**16**] for the general theory of Poisson reduction, and [**12, 14**] for Poisson reductions related to moving frames and differential invariants.

Let $\widetilde{h}[k]$ be a functional defined on the quotient, so $k \in \mathcal{L}\,\mathfrak{g}/\mathcal{L}\,N$. Let $\widetilde{H}[K] = \widetilde{h}[\pi(K)]$ denote the corresponding functional on $\mathcal{L}\,\mathfrak{g}$, which will be constant along the orbits of $\mathcal{L}\,N$ under the action (3.1). This, infinitesimally, requires

$$0 = \langle \delta \widetilde{H}, \widetilde{\mathcal{Q}}[K]\, n \rangle = \langle \delta \widetilde{H}, B^{-1}\, \widehat{\mathcal{P}}[B\,K]\, n \rangle = -\, \langle \widehat{\mathcal{P}}[B\,K]\, B^{-1}\, \delta \widetilde{H}, n \rangle \quad \text{for} \quad n \in \mathfrak{n}. \tag{3.3}$$

Thus, $\widehat{\mathcal{P}}[B\,K]\, B^{-1}\, \delta \widetilde{H} \in \mathfrak{n}_0^*$, where $\mathfrak{n}_0^* = \mathfrak{n}^\perp \subset \mathfrak{g}^*$ is the annihilator of $\mathfrak{n}$. (This notation is atypical; other authors use $\mathfrak{n}_0 \subset \mathfrak{g}^*$ to denote the annihilator. Our notation here follows the convention of consistently denoting elements in the dual with a *.) Equivalently, using (2.12),

$$\widetilde{\mathcal{Q}}[K] B^{-1} \delta \widetilde{H} \in \mathfrak{n}_0, \tag{3.4}$$

where $\mathfrak{n}_0 = B^{-1}\mathfrak{n}_0^* \subset \mathfrak{g}$ is the corresponding subspace of the Lie algebra.

Let $\mathcal{L}\,\mathfrak{a} \subset \mathcal{L}\,\mathfrak{g}$ denote a cross-section to the $\mathcal{L}\,N$ orbits. For simplicity, we will assume from now on that $\mathcal{L}\,\mathfrak{a}$ is an affine subspace of $\mathcal{L}\,\mathfrak{g}$ (such is the case in all our examples). This condition is not actually necessary until we arrive to the compatibility condition (see [**14**]), but it serves to greatly simplify our arguments.

We can thus identify $\varphi\colon \mathcal{L}\,\mathfrak{g}/\mathcal{L}\,N \xrightarrow{\sim} \mathcal{L}\,\mathfrak{a} \subset \mathcal{L}\,\mathfrak{g}$. Then $\widetilde{h}$ will coincide with the extended Hamiltonian $\widetilde{H}$ along the affine subspace:

$$\widetilde{H}[\,\varphi(k)\,] = \widetilde{h}[\,k\,].$$

We claim that this implies the existence of a differential operator $\widetilde{\mathcal{R}}[\,k\,]$ acting on $\mathcal{L}\,\mathfrak{g}^*$ such that

$$\frac{\delta \widetilde{H}}{\delta K}\,[\,\varphi(k)\,] = B\,\widetilde{\mathcal{R}}[\,k\,]\,\frac{\delta \widetilde{h}}{\delta k}\,[\,k\,]. \tag{3.5}$$

(The B factor is added for later convenience.) In particular, $\widetilde{\mathcal{R}}$ restricted to the dual subspace $\mathcal{L}\,\mathfrak{a}^* \subset \mathcal{L}\,\mathfrak{g}^*$ is the identity. Under these conditions, the following proposition is valid:

Proposition 3.1. *The differential operator* $\widetilde{\mathcal{R}}[\,k\,]$ *is uniquely determined by the condition*

$$\widetilde{\mathcal{Q}}[\,\varphi(k)\,]\,\widetilde{\mathcal{R}}[\,k\,]\,\delta\widetilde{h}[\,k\,] \in \mathfrak{n}_0, \qquad \text{for all} \qquad k \in \mathcal{L}\,\mathfrak{a}. \tag{3.6}$$

Proof: Instead of proving that $\widetilde{\mathcal{R}}$ is uniquely determined, we will prove that $\delta\widetilde{H}[\,\varphi(k)\,]$ is uniquely determined by $\delta\widetilde{h}[\,k\,]$ and k. The existence of $\widetilde{\mathcal{R}}$ will thus follow.

On the one hand, since $\widetilde{H}$ coincides with $\widetilde{h}$ on $\mathcal{L}\,\mathfrak{a}$, its variational derivative $\delta\widetilde{H} \in \mathcal{L}\,\mathfrak{g}^*$ is determined in the direction of $\mathcal{L}\,\mathfrak{a}^*$. In other words, the operator $\widetilde{\mathcal{R}}$ restricted to a section of $\mathcal{L}\,\mathfrak{a}^*$ is the identity. On the other hand, condition (3.3) also determines the value of $\delta\widetilde{H}$ on $\widetilde{\mathcal{Q}}(\mathcal{L}\,\mathfrak{n})$ to be zero. But (3.2) implies that $\widetilde{\mathcal{Q}}(\mathcal{L}\,\mathfrak{n})$ is the space tangent to the orbits of $\mathcal{L}\,N$ on $\mathcal{L}\,\mathfrak{g}$. Thus, δH is prescribed on a complement to $\mathcal{L}\,\mathfrak{a}^*$, and so completely determined. *Q.E.D.*

Once $\widetilde{\mathcal{R}}$ has been determined, the reduced Poisson bracket evaluated at $\varphi(k) \in \mathcal{L}\,\mathfrak{a}$ is described by

$$\widetilde{\mathcal{P}}_R[\,k\,] = \widetilde{\mathcal{R}}^*[\,k\,]\,\widetilde{\mathcal{Q}}[\,\varphi(k)\,]\,\widetilde{\mathcal{R}}[\,k\,] = \widetilde{\mathcal{R}}^*[\,k\,]\,\widehat{\mathcal{P}}[\,B\,\varphi(k)\,]\,\widetilde{\mathcal{R}}[\,k\,]. \tag{3.7}$$

This satisfies the conditions to be a Poisson operator on $\mathcal{L}\,\mathfrak{a} \simeq \mathcal{L}\,\mathfrak{g}/\mathcal{L}\,N$ as a consequence of the general theory in [**16**], which relies on the fact that N is a closed subgroup, and so G/N is a smooth manifold.

4. Moving Frames for Parametrized Curves and Poisson Flows.

We are interested in the induced action of G on *parametrized curves* $u\colon X = \mathbb{R} \to M = G/N$ belonging to our homogeneous space. "Parametrized" means that we do not identify curves that have different parametrizations.

Let $\mathrm{J}^n = \mathrm{J}^n(\mathbb{R}, M)$ be the curve jet bundle of order n. Let[†] $\rho\colon \mathrm{J}^n \to G$ be a left moving frame — that is a left-equivariant map: $\rho(g \cdot u^{(n)}) = g \cdot \rho(u^{(n)})$. Existence of such a moving frame requires that n be sufficiently large in order that G act freely and regularly on an open subset of J^n, [**5**]. The moving frame is uniquely prescribed by the choice of a cross-section $K^n \subset \mathrm{J}^n$ to the group orbits through the requirement $\rho(u^{(n)})^{-1} \cdot u^{(n)} \in K^n$ for suitable jets $u^{(n)} \in \mathrm{J}^n$ — namely, those belonging to orbits that intersect the cross-section.

Once a moving frame is prescribed, there is an induced *invariantization map* ι that takes differential functions $F\colon \mathrm{J}^k \to \mathbb{R}$ to differential invariants $I = \iota(F)$ and, more generally, differential forms to invariant differential forms. Specifically, given a differential form ω on J^n, its invariantization $\iota(\omega)$ is the unique invariant differential form that agrees on the cross-section: $\iota(\omega) \mid K^n = \omega \mid K^n$. In particular, invariant forms ϖ are unaffected by invariantization: $\iota(\varpi) = \varpi$. Consequently, invariantization defines an algebra morphism that projects the exterior algebra of differential forms on J^n to the exterior algebra of invariant differential forms.

Since G does not act on the parameter space, the basic invariant horizontal one-form is just $\varpi = \iota(dx) = dx$, with $\mathcal{D} = D_x$ the dual invariant differential operator that maps differential invariants to differential invariants. Let

$$\theta^\alpha_k = du^\alpha_k - u^\alpha_{k+1}, \qquad \alpha = 1, \ldots, q, \quad k \geq 0, \tag{4.1}$$

denote the basis contact forms on the curve jet space J^n. Let

$$\vartheta = (\, \vartheta^1, \ldots, \vartheta^q \,)^T = (\, \iota(\theta^1), \ldots, \iota(\theta^q) \,)^T \tag{4.2}$$

denote the column vector of invariantized zero$^{\text{th}}$ order contact forms. The recurrence relations of [**9**] imply that one can write all other invariant contact forms as a linear combination of derivatives of the ϑ^α. In particular, the invariantized basis contact forms can be written as

$$\vartheta^\alpha_i = \iota(\theta^\alpha_i) = \mathcal{B}^\alpha_i(\vartheta) \tag{4.3}$$

for certain invariant differential operators $\mathcal{B}^\alpha_i$ that can be explicitly determined using the moving frame recurrence formulae. Examples of this process can be found below.

The induced *invariant variational bicomplex*, [**9**], is constructed by decomposing the invariant differential forms into their horizontal and contact components. A differential k form is of type (i, j), for $i + j = k$, if it is a linear combination of terms containing wedge products of i invariant horizontal forms (for curves, $0 \leq i \leq 1$) and j invariant contact forms. The isinvariant variational bicomplex is obtained by splitting $d = d_{\mathcal{H}} + d_{\mathcal{V}}$ into invariant horizontal and vertical components, which relies on the fact that the group acts projectably on the total space $\mathbb{R} \times M$, cf. [**9**].

For our purposes, the key objects of interest are the pull-backs of the Maurer–Cartan forms under the moving frame map:

$$\nu = (\, \nu^1, \ldots, \nu^r \,)^T = (\, \rho^* \mu^1, \ldots, \rho^* \mu^r \,)^T. \tag{4.4}$$

† Our notational conventions allow ρ to only be defined on an open subset of J^n.

We split them into invariant horizontal and invariant contact components,

$$\nu = \rho^* \mu = \boldsymbol{\kappa}\, \varpi + \mathcal{C}(\vartheta), \tag{4.5}$$

where $\boldsymbol{\kappa} = (\,\kappa^1, \ldots, \kappa^r\,)^T$ are known as the *Maurer–Cartan differential invariants*, while $\mathcal{C} = \big(\,\mathcal{C}^k_\alpha\,\big)$ is a $r \times q$ matrix of invariant differential operators, which we call the *Maurer–Cartan operator*. We can identify the Maurer–Cartan invariants with the components of the *Maurer–Cartan map*

$$\boldsymbol{\kappa}(u^{(n)}) = \rho(u^{(n)})^{-1} D_x \rho(u^{(n)}) \colon \mathrm{J}^n \longrightarrow \mathfrak{g}. \tag{4.6}$$

According to a theorem of Hubert, [**7**], the Maurer–Cartan invariants $\boldsymbol{\kappa}$ contain a generating set consisting of exactly q independent differential invariants $\varkappa = (\,\varkappa^1, \ldots, \varkappa^q\,)^T$, meaning that every other differential invariant is a function of the $\varkappa$ and their derivatives. Moreover, according to [**19**], if we employ a moving frame of minimal order, then the generating differential invariants can be taken to be $\varkappa^\alpha = \iota\big(\,u^\alpha_{k_\alpha+1}\,\big)$, $\alpha = 1, \ldots, q$, where k_α is the maximal order derivative of u^α that appears in the cross-section normalization equations. The explicit formulae for the Maurer–Cartan invariants $\boldsymbol{\kappa}$ and the Maurer–Cartan operator $\mathcal{C}$ in terms of the generating invariants can be deduced, using only linear differential algebraic calculation, from the moving frame recurrence formulae; see below for how this is done in practice.

Let us compute the differentials of the pulled-back Maurer–Cartan forms:

$$d\nu = d\boldsymbol{\kappa} \wedge \varpi + \boldsymbol{\kappa}\, d\varpi + d[\mathcal{C}(\vartheta)] = \, d_{\mathcal{V}}\, \boldsymbol{\kappa} \wedge \varpi + \boldsymbol{\kappa}\, d\varpi + d[\mathcal{C}(\vartheta)], \tag{4.7}$$

since $d_{\mathcal{H}}\, \boldsymbol{\kappa} = \mathcal{D}(\boldsymbol{\kappa})\, \varpi$. Let's evaluate each of the terms on the right hand side. First,

$$d\varpi = 0, \tag{4.8}$$

since $\varpi = dx$ in the parametrized case being considered here. Second,

$$d_{\mathcal{V}}\, \boldsymbol{\kappa} = \mathcal{A}(\vartheta), \tag{4.9}$$

where $\mathcal{A} = \mathcal{A}_{\boldsymbol{\kappa}}$ is an $r \times q$ matrix of invariant differential operators called the *invariant linearization operator* associated with the Maurer–Cartan invariants, [**20**]. Finally, the $(1,1)$ component of the last term of (4.7) is, in view of [**9**; eq. (5.34)],

$$d_{\mathcal{H}}\, [\mathcal{C}(\vartheta)] = \varpi \wedge \mathcal{D}[\mathcal{C}(\vartheta)]. \tag{4.10}$$

On the other hand, using the moving frame map to pull back the Maurer–Cartan structure equations

$$d\mu^i = -\,\tfrac{1}{2} \sum_{j,k=1}^{r} \, c^i_{jk}\, \mu^j \wedge \mu^k, \tag{4.11}$$

we see that the left hand side of (4.7) is given by

$$d\nu^i = -\,\tfrac{1}{2} \sum_{j,k=1}^{r} \, c^i_{jk}\, \nu^j \wedge \nu^k, \tag{4.12}$$

where c_{jk}^i are the structure constants associated with our choice of basis of $\mathfrak{g}$. Substituting (4.5), we find the the $(1,1)$ component of (4.12) is

$$- \sum_{j,k=1}^{r} c_{jk}^i \, \kappa^j \varpi \wedge \mathcal{C}^k(\vartheta), \tag{4.13}$$

where $\mathcal{C}^k$ denotes the k^{th} row of the matrix differential operator $\mathcal{C}$.

Plugging equations (4.8, 9, 10, 13) into (4.7) and canceling the common factor ϖ, we arrive at the following intriguing factorization:

$$\mathcal{A} = \mathcal{Q} \cdot \mathcal{C}, \tag{4.14}$$

with $\mathcal{C}$ the Maurer–Cartan operator (4.5), while

$$\mathcal{Q} = \widetilde{\mathcal{Q}}[\boldsymbol{\kappa}] \qquad \text{has entries} \qquad \mathcal{Q}_i^k = \delta_i^k D_x - \sum_{j=1}^{r} c_{ij}^k \, \kappa^j. \tag{4.15}$$

Observe that the operator $\widetilde{\mathcal{Q}}$ is obtained by pulling back the Lie algebra operator (2.11) via the Maurer–Cartan map (4.6), i.e., replacing the Lie algebra coordinates K by the Maurer–Cartan invariants $\boldsymbol{\kappa}$. This striking factorization plays a key role in our analysis of geometric flows.

5. Poisson Brackets Associated to a Moving Frame.

The process of replacing coordinates on $\mathfrak{g}$ by the corresponding Maurer–Cartan invariants can be described as a quotient, first established in [**11**, **13**] in special cases, and then in [**14**] for the general semi-simple homogeneous case. This quotient description will allow us to reduce the Poisson structure (2.13) on $\mathcal{L}\,\mathfrak{g}$ to a reduced Poisson structure involving the Maurer–Cartan invariants $\boldsymbol{\kappa}$.

As above, we assume G is a semisimple Lie group and $\pi\colon G \to M = G/N$ an associated homogeneous space. We can identify $N = G_{e_0}$ as the isotropy subgroup of $e_0 = \pi(e)$. We will need to assume that the curves $u\colon \mathbb{R} \to M$ have *monodromy*, meaning that there exists $T > 0$ such that

$$u(x+T) = g \cdot u(x), \qquad \text{for some } g \in G \text{ and for all } x. \tag{5.1}$$

The monodromy condition implies that the differential invariants of such a curve are periodic in x with period T. We assume that the moving frame $\rho\colon \mathrm{J}^n \to G$ is based on a cross-section $K^n \subset \mathrm{J}^n$ sitting over e_0, that is, $\rho(u^{(n)}(x)) \cdot e_0 = u(x)$. A curve (with monodromy) will be called *regular* if its jet lies in the domain of the moving frame map. The following description can be found in [**14**].

Lemma 5.1. *Under the above assumptions, the image of the Maurer-Cartan map* $\boldsymbol{\kappa}\colon \mathrm{J}^n \to \mathcal{L}\,\mathfrak{g}$ *evaluated on regular curves forms an open subset of the quotient space* $\mathcal{L}\,\mathfrak{g}/\mathcal{L}\,N \simeq \mathcal{L}\,\mathfrak{a} \subset \mathcal{L}\,\mathfrak{g}$.

The identification of Maurer–Cartan invariants is based on the following observations. First, let u be a generic curve in $M \to G/N$, and let ρ be a left moving frame whose defining cross-section sits over e_0. Given a set of Maurer–Cartan invariants $\boldsymbol{\kappa}(u^{(n)})$, let $K_\epsilon(x) \in \mathcal{L}\,\mathfrak{g}$ be a nearby loop. We can integrate K_ϵ to the group, $g_\epsilon \in \mathrm{C}^\infty(\mathbb{R}, G)$, by solving locally the system

$$g_\epsilon^{-1} D_x g_\epsilon = K_\epsilon. \tag{5.2}$$

Let $u_\epsilon(x) = g_\epsilon(x) \cdot e_0$ be the corresponding curve in M, which clearly satisfies the monodromy condition (5.1). Then, clearly,

$$g_\epsilon(x)^{-1} \cdot \rho(u_\epsilon^{(n)}(x)) \in \mathcal{L}\,\mathfrak{n}.$$

Now, the action (3.1) is induced by the action $g \mapsto g \cdot n$ of $\mathcal{L}\,N$ on solutions of (5.2), and so the algebra element representing Maurer-Cartan invariants for u_ϵ is in the same $\mathcal{L}\,N$-orbit as K_ϵ.

Keeping Lemma 5.1 and (4.15) in mind, we pull back the operators appearing in Section 3 using the Maurer–Cartan map (4.6) by replacing the coordinates on $\mathfrak{g}$ by the Maurer–Cartan invariants: $K = \boldsymbol{\kappa}(u^{(n)})$. Similarly, the coordinates on $\mathcal{L}\,\mathfrak{g}/\mathcal{L}\,N \simeq \mathcal{L}\,\mathfrak{a}$ are replaced by the generating differential invariants: $k = \varkappa(u^{(n)})$, with $\boldsymbol{\kappa} = \varphi(\varkappa)$. The resulting operators will be denoted by the same symbols without tildes, so $\mathcal{Q} = \widetilde{\mathcal{Q}}[\boldsymbol{\kappa}] = \widetilde{\mathcal{Q}}[\varphi(\varkappa)]$ and so on. In addition, the pulled back Poisson operator is

$$\mathcal{P} = \widetilde{\mathcal{P}}[\boldsymbol{\kappa}] = \widetilde{\mathcal{Q}}[\boldsymbol{\kappa}]\, B^{-1}, \qquad \text{where} \qquad \boldsymbol{\kappa} = \varphi(\varkappa). \tag{5.3}$$

The resulting operator is skew adjoint, but, without further reduction, is not of the correct shape to be Poisson on the space of differential invariants.

Indeed, the reduced Poisson operator $\mathcal{P}_R$ on the space of generating differential invariants can be found explicitly following two steps: we first apply the Maurer–Cartan invariant version of (3.6) to construct the operator $\mathcal{R}$:

$$\mathcal{Q}\,\mathcal{R}\,\delta\widetilde{h} \in \mathfrak{n}_0. \tag{5.4}$$

The reduced Poisson operator is then described by the Maurer–Cartan pull-back of our earlier reduction formula (3.7):

$$\mathcal{P}_R = \mathcal{R}^*\,\mathcal{P}\,\mathcal{R}. \tag{5.5}$$

6. Invariant Curve Flows and Reduced Hamiltonian Systems.

In general, an *invariant curve flow* on M takes the form

$$\frac{\partial C}{\partial t} = J \cdot \mathbf{n} = \sum_{\alpha=1}^{q} J^\alpha \mathbf{n}_\alpha, \tag{6.1}$$

where $J = (J^1, \ldots, J^q)$ is a vector of differential invariants, and $\mathbf{n}_1, \ldots, \mathbf{n}_q$ are the group-invariant normal directions on the curve that are dual to the order zero invariant contact forms (4.2):

$$\langle\, \vartheta^\alpha\,; \mathbf{n}_\beta \,\rangle = \delta^\alpha_\beta, \qquad \alpha, \beta = 1, \ldots, q. \tag{6.2}$$

According to [**20**] and (4.14), the induced evolution of the Maurer–Cartan invariants is then given by

$$\frac{\partial \boldsymbol{\kappa}}{\partial t} = \mathcal{A}(J) = \mathcal{Q}\,\mathcal{C}(J), \tag{6.3}$$

where $\mathcal{A} = \mathcal{A}_{\boldsymbol{\kappa}}$ is the associated invariant linearization operator (4.9). We will call (6.3) the *Maurer–Cartan flow* induced by the invariant curve flow (6.1). The Maurer–Cartan flow can clearly be reduced to a flow on the generating invariants $\varkappa$.

Our goal is to characterize in our setting those geometric flows on M that produce a Hamiltonian flow on the generating differential invariants under the reduced Poisson structure defined by (5.5). Using a direct approach, we can choose a Hamiltonian functional $\widetilde{H}[K]$ that induces the Poisson flow (2.9) on $\mathcal{L}\,\mathfrak{g}$. Identifying $K = \boldsymbol{\kappa}(u^{(n)})$, the corresponding evolution of Maurer–Cartan invariants is

$$\frac{\partial \boldsymbol{\kappa}}{\partial t} = \mathcal{P}[\boldsymbol{\kappa}]\,\frac{\delta H}{\delta K}\,[\boldsymbol{\kappa}] = \mathcal{Q}[\boldsymbol{\kappa}]\,B^{-1}\,\frac{\delta H}{\delta K}\,[\boldsymbol{\kappa}]. \tag{6.4}$$

This coincides with the Maurer–Cartan flow (6.3) provided

$$\mathcal{A}(J) = \mathcal{Q}\,\mathcal{C}(J) = \mathcal{Q}\,B^{-1}\,\frac{\delta H}{\delta K}\,. \tag{6.5}$$

In particular, this holds if $\delta H/\delta K = B\,\mathcal{C}(J)$. However, examples show that this condition is too restrictive, and a more intricate analysis is required.

We can apply the results of Section 3 to conclude that the Poisson structure on $\mathcal{L}\,\mathfrak{g}$ defined by (2.13) produces a reduced Poisson bracket on the generating differential invariants $\varkappa$. To simplify the construction, we will make the additional assumption that the generating invariants $\varkappa$ occur linearly, algebraically in the full set of Maurer–Cartan invariants $\boldsymbol{\kappa}$; in other words, we can write $\boldsymbol{\kappa} = A\,\varkappa + b$ for some constant $r \times q$ matrix A and constant vector $b \in \mathbb{R}^q$. This condition depends on an appropriate choice of cross-section. In simple examples, this can always be arranged, but, so far, we do not know general conditions on the group action that guarantee such a cross-section exists. Under this assumption, the Maurer–Cartan map (4.6) traces out an open subset of the affine subspace

$$\mathfrak{a} = \{\, \boldsymbol{\kappa} = \varphi(\varkappa) = A\,\varkappa + b \mid \varkappa \in \mathbb{R}^q \,\} \subset \mathfrak{g}. \tag{6.6}$$

We use this subspace to effect the identification $\mathcal{L}\,\mathfrak{a} \simeq \mathcal{L}\,\mathfrak{g}/\mathcal{L}\,N$.

The main result that allows us to relate reduced Hamiltonian evolutions on the affine subspace $\mathcal{L}\,\mathfrak{a}$ traced out by the Maurer–Cartan invariants directly to invariant curve flows was proved in [**14**] using a somewhat different approach. Notice that the curve evolution in [**14**] can be written as

$$\frac{\partial C}{\partial t} = \frac{\partial \Phi}{\partial u}(\rho(u^{(n)}), u)J,$$

where $\Phi\colon G \times M \to M$ is the group action, $\Phi(g,u) = g \cdot u$, and its partial is regarded as a linear transformation, or an element of $\mathrm{GL}(n)$. On the other hand, the invariantized contact frame (4.1) can also be written as $\vartheta = \dfrac{\partial \Phi}{\partial u}(\rho(u^{(n)}), u)\theta$ and so the evolution in [**14**] and (6.1) are identically formulated.

Theorem 6.1. *Under the above assumptions, the Hamiltonian flow*

$$\frac{\partial \varkappa}{\partial t} = \mathcal{P}_R \, \frac{\delta h}{\delta \varkappa}\,, \tag{6.7}$$

on $\mathcal{L}\,\mathfrak{a}$ with Hamiltonian functional $h[\,\varkappa\,]$ with respect to the reduced Poisson structure (5.5) *is induced by the Maurer–Cartan flow* (6.3) *with*

$$\mathcal{C}(J) \;\equiv\; \mathcal{R}\,\delta h \quad \mathrm{mod}\ \mathfrak{n}. \tag{6.8}$$

Condition (6.8) is called the *compatibility condition*, and requires that $\mathcal{C}(J) = \mathcal{R}\,\delta h + n$ for some $n \in \mathfrak{n}$. We remark that the left hand side, when computed modulo $\mathfrak{n}$, just amounts to an invertible matrix applied to J, and hence the compatibility condition uniquely determines the curve flow invariants J in terms of δh. On the other hand, $\mathcal{R}$ is typically not invertible, and hence only special curve flows are of Hamiltonian form.

Let us investigate how all this works in some basic examples.

Example 6.2. Let $G = \mathrm{PSL}(2)$ be the projective group acting on $M = \mathbb{R}$. We use

$$\mathbf{v}_1 = \partial_u, \qquad \mathbf{v}_2 = u\,\partial_u, \qquad \mathbf{v}_3 = u^2\,\partial_u,$$

as a basis for $\mathfrak{g} = \mathfrak{sl}(2)$. Using the commutation relations, we find that every ad^*–invariant symmetric 2 tensor is a scalar multiple of the Killing form:

$$\beta = \tfrac12\,(\mu^2)^2 - 2\,\mu^1\mu^3, \qquad \text{so that} \qquad B = \begin{pmatrix} 0 & 0 & -1 \\ 0 & \frac12 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Thus, the cocycle relation (2.5) requires that

$$L_1 = -\,K^3, \qquad L_2 = \tfrac12\,K^2, \qquad L_3 = -\,K^1. \tag{6.9}$$

The Poisson operator on $\mathcal{L}\,\mathfrak{g}^*$ is

$$\widehat{\mathcal{P}}[\,L\,] = \begin{pmatrix} 0 & 0 & -1 \\ 0 & \frac12 & 0 \\ -1 & 0 & 0 \end{pmatrix} D_x + \begin{pmatrix} 0 & -L_1 & -2\,L_2 \\ L_1 & 0 & -\,L_1 \\ 2\,L_2 & L_3 & 0 \end{pmatrix},$$

and hence

$$\widetilde{\mathcal{Q}}[\,K\,] = B^{-1}\,\widehat{\mathcal{P}}[\,B\,K\,] = \mathcal{D} + \begin{pmatrix} -\,K^2 & K^1 & 0 \\ -2\,K^3 & 0 & 2\,K^1 \\ 0 & -\,K^3 & K^2 \end{pmatrix}. \tag{6.10}$$

We adopt the standard normalizations

$$u \;\longmapsto\; 0, \qquad u_x \;\longmapsto\; 1, \qquad u_{xx} \;\longmapsto\; 0,$$

to define a left equivariant moving frame $\rho\colon \mathrm{J}^2 \to G$. The generating differential invariant is the *Schwarzian derivative*

$$\kappa = \iota(u_{xxx}) = \frac{2\,u_x u_{xxx} - 3\,u_{xx}^2}{u_x^2}\,. \tag{6.11}$$

Applying the invariant variational bicomplex machinery [**9**] to determine the recurrence formulae, the pulled back Maurer–Cartan forms are

$$
\begin{aligned}
\nu^1 &= \rho^*(\mu^1) = -\varpi - \vartheta, \\
\nu^2 &= \rho^*(\mu^2) = -\vartheta_x, \\
\nu^3 &= \rho^*(\mu^3) = -\tfrac{1}{2}\kappa\varpi - \tfrac{1}{2}\vartheta_{xx} - \tfrac{1}{2}\kappa\vartheta,
\end{aligned}
\tag{6.12}
$$

from which we deduce the Maurer–Cartan invariants and operator:

$$
\boldsymbol{\kappa} = \begin{pmatrix} -1 \\ 0 \\ -\frac{1}{2}\kappa \end{pmatrix}, \qquad \mathcal{C} = \begin{pmatrix} -1 \\ -\mathcal{D} \\ -\frac{1}{2}\mathcal{D}^2 - \frac{1}{2}\kappa \end{pmatrix}. \tag{6.13}
$$

The operator (6.10) rewritten in terms of the Maurer–Cartan invariants is

$$
\mathcal{Q} = \begin{pmatrix} \mathcal{D} & -1 & 0 \\ \kappa & \mathcal{D} & -2 \\ 0 & \frac{1}{2}\kappa & \mathcal{D} \end{pmatrix}.
$$

while the Poisson operator (2.13) is obtained by right multiplication by B^{-1}, and so

$$
\mathcal{P} = \mathcal{Q}\,B^{-1} = \begin{pmatrix} 0 & -2 & \mathcal{D} \\ 2 & 2\mathcal{D} & -\kappa \\ \mathcal{D} & \kappa & 0 \end{pmatrix}.
$$

Note that $\mathcal{P}$ is clearly skew-adjoint, but needs to be reduced in order to define a Poisson structure. Using the recurrence formulae, we discover that the invariant vertical derivatives of the Maurer–Cartan invariants are

$$
d_{\mathcal{V}}(-1) = 0, \qquad d_{\mathcal{V}}(0) = 0, \qquad d_{\mathcal{V}}(-\kappa) = -\tfrac{1}{2}\vartheta_{xxx} - \kappa\vartheta_x - \tfrac{1}{2}\kappa_x\vartheta,
$$

and hence the invariant linearization operator factors as in (4.14),

$$
\mathcal{A} = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{2}\left(\mathcal{D}^3 + 2\kappa\mathcal{D} + \kappa_x\right) \end{pmatrix} = \mathcal{Q}\cdot\mathcal{C},
$$

as can be checked by hand. Notice that $\mathcal{A}$ describes the evolution of the Maurer-Cartan invariants, and so this implies that

$$
-\tfrac{1}{2}\kappa_t = -\tfrac{1}{2}\left(\mathcal{D}^3 + 2\kappa\mathcal{D} + \kappa_x\right)J. \tag{6.14}
$$

To obtain the reduced Hamiltonian and the compatibility condition, we need to find the operator $\mathcal{R}$ satisfying (5.4). In this case, the isotropy subalgebra nd its annihilator have the form

$$
\mathfrak{n} = \begin{pmatrix} 0 \\ * \\ * \end{pmatrix}, \qquad \mathfrak{n}_0^* = \begin{pmatrix} * \\ 0 \\ 0 \end{pmatrix}, \qquad \mathfrak{n}_0 = B^{-1}\mathfrak{n}_0^* = \begin{pmatrix} 0 \\ 0 \\ * \end{pmatrix}.
$$

Recall that an extension H will coincide with h on $\boldsymbol{\kappa}$. If we want to find the reduced Hamiltonian evolution of κ itself, then $\mathcal{R} = (\,2, X, Y\,)^T$, where the constant entry is placed in the dual position to that of the generating invariant κ in $\boldsymbol{\kappa}$, with value 2 since the third entry of $\boldsymbol{\kappa}$ is $-\frac{1}{2}\,\kappa$. The condition

$$\mathcal{Q}\,\mathcal{R}\,\delta h = \mathcal{Q}\begin{pmatrix} 2 \\ X \\ Y \end{pmatrix}\delta h = \begin{pmatrix} 2\mathcal{D} - X \\ \mathcal{D}X + 2\,\kappa - 2Y \\ \mathcal{D}Y + \frac{1}{2}\,\kappa X \end{pmatrix}\delta h = \begin{pmatrix} 0 \\ 0 \\ * \end{pmatrix} \in \mathfrak{n}_0$$

implies that $X = 2\mathcal{D}$ and $Y = \mathcal{D}^2 + \kappa$. Thus, in this particular example, $\mathcal{R} = -2\,\mathcal{C}$. We conclude that the reduced Poisson operator is given by

$$\mathcal{P}_R = \mathcal{R}^* \cdot \mathcal{P} \cdot \mathcal{R} = -2\,(\mathcal{D}^3 + 2\,\kappa\,\mathcal{D} + \kappa_x),$$

which is the well-known second Hamiltonian structure of the Korteweg–deVries equation, [**18**]. The reduced Hamiltonian evolution of κ is given by

$$\kappa_t = \mathcal{P}_R\,\delta h = -2\left(\mathcal{D}^3 + 2\,\kappa\,\mathcal{D} + \kappa_x\right)\frac{\delta h}{\delta \kappa}\,. \tag{6.15}$$

Owing to the form of $\mathfrak{n}$, the compatibility condition (6.8) requires that the first entry of $\mathcal{R}\,\delta h$ coincides with the first entry of $\mathcal{C}\,J$, which requires $J = -2\,\delta h$. This condition clearly makes both flows (6.14, 15) coincide.

Example 6.3. *Centro-equi-affine plane curves*: Consider the standard linear representation of $G = \mathrm{SL}(2)$ on $M = \mathbb{R}^2$. The infinitesimal generators are

$$\mathbf{v}_1 = -\,u\,\partial_u + v\,\partial_v, \qquad \mathbf{v}_2 = v\,\partial_u, \qquad \mathbf{v}_3 = u\,\partial_v.$$

The cocycle is the Killing form:

$$\beta = 2\,(\mu^1)^2 - 2\,\mu^2\mu^3, \qquad \text{so that} \qquad B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Adopting the following moving frame normalizations

$$u \longmapsto 0, \qquad v \longmapsto 1, \qquad v_x \longmapsto 0,$$

the generating differential invariants are

$$\kappa = \iota(u_x), \qquad \tau\,\kappa = \iota(v_{xx}).$$

The Maurer–Cartan invariants and operator are found using the invariant variational bicomplex constructions:

$$\boldsymbol{\kappa} = \begin{pmatrix} 0 \\ -\,\kappa \\ -\,\tau \end{pmatrix}, \qquad \mathcal{C} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \\ -\,\frac{\tau}{\kappa} & -\,\frac{1}{\kappa}\,\mathcal{D} \end{pmatrix}.$$

The pulled back Poisson operator and its counterpart are

$$\mathcal{P} = \begin{pmatrix} 2\,\mathcal{D} & 2\,\tau & -2\,\kappa \\ -2\,\tau & 0 & \mathcal{D} \\ 2\,\kappa & \mathcal{D} & 0 \end{pmatrix}, \qquad \mathcal{Q} = \begin{pmatrix} \mathcal{D} & \tau & -\kappa \\ 2\,\kappa & \mathcal{D} & 0 \\ -2\,\tau & 0 & \mathcal{D} \end{pmatrix}.$$

Thus, in accordance with (4.14), the invariant linearization operator factorizes as

$$\mathcal{A} = \begin{pmatrix} 0 & 0 \\ -\mathcal{D} & -2\,\kappa \\ -\mathcal{D}\,\frac{\tau}{\kappa} & -\mathcal{D}\,\frac{1}{\kappa}\,\mathcal{D} + 2\,\tau \end{pmatrix} = \mathcal{Q}\cdot\mathcal{C},$$

and the Maurer–Cartan flow $\boldsymbol{\kappa}_t = \mathcal{A}\,(J)$ takes the explicit form

$$\kappa_t = \mathcal{D}J_1 + 2\,\kappa\,J_2, \qquad \tau_t = \mathcal{D}\left(\frac{\tau}{\kappa}\,J_1\right) + \left(\mathcal{D}\,\frac{1}{\kappa}\,\mathcal{D} - 2\,\tau\right) J_2. \tag{6.16}$$

The isotropy subalgebra and its annihilator are

$$\mathfrak{n} = \begin{pmatrix} 0 \\ 0 \\ * \end{pmatrix}, \qquad \mathfrak{n}_0^* = \begin{pmatrix} * \\ * \\ 0 \end{pmatrix}, \qquad \mathfrak{n}_0 = B^{-1}\mathfrak{n}_0^* = \begin{pmatrix} * \\ 0 \\ * \end{pmatrix}.$$

The entries dual to the position of $-\kappa$ and $-\tau$ in $\boldsymbol{\kappa}$ are third and second entries, respectively, and therefore

$$\mathcal{R} = \begin{pmatrix} X & Y \\ 0 & -1 \\ -1 & 0 \end{pmatrix}, \qquad \mathcal{Q}\cdot\mathcal{R} = \begin{pmatrix} \mathcal{D}X - \kappa & \mathcal{D}Y - \tau \\ 2\,\kappa\,X & 2\,\kappa\,Y - \mathcal{D} \\ -2\,\tau\,X - \mathcal{D} & -2\,\tau\,Y \end{pmatrix}.$$

Condition (5.4) requires $X = 0,\ Y = (2\,\kappa)^{-1}\,\mathcal{D}$, and thus

$$\mathcal{R} = \begin{pmatrix} 0 & \frac{1}{2\,\kappa}\,\mathcal{D} \\ 0 & -1 \\ -1 & 0 \end{pmatrix}, \qquad \mathcal{P}_R = \mathcal{R}^*\cdot\mathcal{P}\cdot\mathcal{R} = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{2}\,\mathcal{D}\,\frac{1}{\kappa}\,\mathcal{D}\,\frac{1}{\kappa}\,\mathcal{D} + \mathcal{D}\,\frac{\tau}{\kappa} + \frac{\tau}{\kappa}\,\mathcal{D} \end{pmatrix}.$$

Thus, the reduced Hamiltonian flow $\boldsymbol{\kappa}_t = \mathcal{P}_R\,\delta h$ becomes

$$\kappa_t = 0, \qquad \tau_t = \left(-\,\frac{1}{2}\,\mathcal{D}\,\frac{1}{\kappa}\,\mathcal{D}\,\frac{1}{\kappa}\,\mathcal{D} + \mathcal{D}\,\frac{\tau}{\kappa} + \frac{\tau}{\kappa}\,\mathcal{D}\right)\frac{\delta h}{\delta\tau}\,. \tag{6.17}$$

and so, for fixed κ, the differential invariant In this example, this implies that *any* reduced Hamiltonian evolution will fix the value of the "arc length" invariant κ, while the evolution of τ has the second Korteweg–deVries Hamiltonian structure. Finally, since a complement to $\mathfrak{n}$ in $\mathfrak{g}$ is given by the first two entries, the compatibility condition (6.8) says

$$J_1 = h_\tau, \qquad J_2 = -\frac{1}{2\kappa}\,\mathcal{D}h_\tau = -\,\frac{1}{2\,\kappa}\,\mathcal{D}J_1.$$

Example 6.4. *A parabolic* SL(3) *action.* Let $(u, v, w) \in M \simeq \mathbb{R}^3$ be identified with upper triangular 3×3 matrices with unit diagonal:

$$U = \begin{pmatrix} 1 & u & v \\ 0 & 1 & w \\ 0 & 0 & 1 \end{pmatrix}.$$

Consider the action of $G = \mathrm{SL}(3)$ on M that corresponds to the parabolic manifold associated to the finest gradation of $\mathfrak{sl}(3) = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$, with

$$\mathfrak{g}_{-2} = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathfrak{g}_{-1} = \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathfrak{g}_0 = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix},$$
$$\mathfrak{g}_1 = \begin{pmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ 0 & * & 0 \end{pmatrix}, \qquad \mathfrak{g}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & 0 & 0 \end{pmatrix}.$$

Namely, $A \in \mathrm{SL}(3)$ acts via

$$A : U \longmapsto \widehat{U}, \qquad \text{where} \qquad A\,U = \widehat{U}\,C \tag{6.18}$$

for some lower triangular unimodular matrix C. The infinitesimal generators of this action can be determined by standard methods:

$$\begin{aligned}
&\mathbf{v}_1 = \partial_v, && \mathbf{v}_2 = \partial_w, && \mathbf{v}_3 = \partial_u + w\,\partial_v, \\
&\mathbf{v}_4 = u\,\partial_u - w\,\partial_w, && \mathbf{v}_5 = u\,\partial_u + 2\,v\,\partial_v + w\,\partial_w, && \mathbf{v}_6 = -\,u^2\,\partial_u + v\,\partial_w, \\
&\mathbf{v}_7 = (u\,w - v)\,\partial_u - v\,w\,\partial_v - w^2\,\partial_w, && \mathbf{v}_8 = u\,(u\,w - v)\,\partial_u - v^2\,\partial_v - v\,w\,\partial_w. &&
\end{aligned}$$

The cocycle is the Killing form:

$$\beta = \tfrac{2}{3}\,(\mu^4)^2 + 2\,(\mu^5)^2 + 2\,\mu^1\mu^8 + 2\,\mu^2\mu^7 + 2\,\mu^3\mu^6,$$

so that

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The prolonged action is locally free on a dense open subset of J^2. We choose a cross-section $K^2 \subset \mathrm{J}^2$ that specifies the following moving frame normalizations:

$$u \to 0, \quad v \to 0, \quad w \to 0, \quad u_x \to 0, \quad v_x \to 1, \quad w_x \to 0, \quad u_{xx} \to 1, \quad v_{xx} \to 0.$$

The resulting generating differential invariants are

$$\kappa = \iota(w_{xx}), \qquad \tau = \iota(u_{xxx}), \qquad \rho = \iota(v_{xxx}).$$

Using the invariant variational bicomplex construction, the Maurer–Cartan invariants and operator are found to be

$$\boldsymbol{\kappa} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ -\tau \\ 0 \\ -\kappa \\ 1 \\ \frac{1}{2}\rho \end{pmatrix}, \qquad \mathcal{C} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \\ -\mathcal{D}^2 - 2\tau\,\mathcal{D} - \tau_x - \tau^2 - \frac{1}{2}\rho & -\frac{3}{2}\mathcal{D} - \tau & 0 \\ 0 & -\frac{1}{2}\mathcal{D} & 0 \\ 0 & -\kappa & -\mathcal{D} + \tau \\ \mathcal{D} + \tau & 1 & 0 \\ -\frac{1}{2}\kappa & \frac{1}{2}\mathcal{D}^2 + \frac{1}{2}\rho & \frac{1}{2} \end{pmatrix}.$$

On $\mathcal{L}\,\mathfrak{g}^*$, the Poisson operator is

$$\mathcal{P} = \begin{pmatrix} 0 & 0 & 0 & 0 & -\rho & -1 & -\kappa & \mathcal{D} \\ 0 & 0 & -\frac{1}{2}\rho & 1 & -1 & 0 & \mathcal{D}+\tau & 0 \\ 0 & \frac{1}{2}\rho & 0 & \kappa & \kappa & \mathcal{D}-\tau & 0 & 0 \\ 0 & -1 & -\kappa & \frac{2}{3}\mathcal{D} & 0 & 0 & 0 & 0 \\ \rho & 1 & -\kappa & 0 & 2\mathcal{D} & 0 & 0 & 2 \\ 1 & 0 & \mathcal{D}+\tau & 0 & 0 & 0 & 1 & 0 \\ \kappa & \mathcal{D}-\tau & 0 & 0 & 0 & -1 & 0 & 0 \\ \mathcal{D} & 0 & 0 & 0 & -2 & 0 & 0 & 0 \end{pmatrix},$$

whereas

$$\mathcal{Q} = \begin{pmatrix} \mathcal{D} & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ \kappa & \mathcal{D}-\tau & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & \mathcal{D}+\tau & 0 & 0 & 0 & 1 & 0 \\ 0 & -\frac{3}{2} & -\frac{3}{2}\kappa & \mathcal{D} & 0 & 0 & 0 & 0 \\ \frac{1}{2}\rho & \frac{1}{2} & -\frac{1}{2}\kappa & 0 & \mathcal{D} & 0 & 0 & 1 \\ 0 & \frac{1}{2}\rho & 0 & \kappa & \kappa & \mathcal{D}-\tau & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\rho & 1 & -1 & 0 & \mathcal{D}+\tau & 0 \\ 0 & 0 & 0 & 0 & -\rho & -1 & -\kappa & \mathcal{D} \end{pmatrix}.$$

The invariant linearization operator $\mathcal{A} = \mathcal{Q} \cdot \mathcal{C}$ is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \begin{matrix} -\mathcal{D}^3 - 2\tau\mathcal{D}^2 - \left(3\tau_x + \frac{1}{2}\rho + \tau^2\right)\mathcal{D} - \\ -\tau_{xx} - 2\tau\tau_x - \frac{1}{2}\rho_x + \frac{3}{2}\kappa \end{matrix} & -\frac{3}{2}\mathcal{D}^2 - \tau\mathcal{D} - \tau_x & \frac{3}{2} \\ 0 & 0 & 0 \\ -\kappa\mathcal{D}^2 - 2\kappa\tau\mathcal{D} - \kappa\tau_x - \kappa\tau^2 - \frac{1}{2}\kappa\rho & -3\kappa\mathcal{D} - \kappa_x & -\mathcal{D}^2 + 2\tau\mathcal{D} + \tau_x - \tau^2 - \frac{1}{2}\rho \\ 0 & 0 & 0 \\ -\frac{3}{2}\kappa\mathcal{D} - \kappa\tau - \frac{1}{2}\kappa_x & \frac{1}{2}\mathcal{D}^3 + \rho\mathcal{D} + \frac{1}{2}\rho_x & \frac{3}{2}\mathcal{D} - \tau \end{pmatrix}.$$

Next, the isotropy subalgebra and annihilator are

$$\mathfrak{n} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ * \\ * \\ * \\ * \\ * \end{pmatrix}, \qquad \mathfrak{n}_0^* = \begin{pmatrix} * \\ * \\ * \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathfrak{n}_0 = B\mathfrak{n}_0^* = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ * \\ * \\ * \end{pmatrix}.$$

If

$$\mathcal{R} = \begin{pmatrix} 0 & 0 & 2 \\ X_2 & Y_2 & Z_2 \\ -1 & 0 & 0 \\ 0 & -\frac{3}{2} & 0 \\ X_5 & Y_5 & Z_5 \\ X_6 & Y_6 & Z_6 \\ X_7 & Y_7 & Z_7 \\ X_8 & Y_8 & Z_8 \end{pmatrix},$$

then

$$\mathcal{Q}\mathcal{R} = \begin{pmatrix} -2X_5 & -2Y_5 & -2Z_5 + 2\mathcal{D} \\ (\mathcal{D}-\tau)X_2 - X_6 & (\mathcal{D}-\tau)Y_2 - Y_6 & (\mathcal{D}-\tau)Z_2 - Z_6 + 2\kappa \\ -(\mathcal{D}+\tau) + X_7 & Y_7 & Z_7 + 2 \\ -\frac{3}{2}X_2 + \frac{3}{2}\kappa & -\frac{3}{2}Y_2 - \frac{3}{2}\mathcal{D} & -\frac{3}{2}Z_2 \\ \frac{1}{2}X_2 + \frac{1}{2}\kappa + \mathcal{D}X_5 + X_8 & \frac{1}{2}Y_2 + \mathcal{D}Y_5 + Y_8 & \frac{1}{2}Z_2 + \rho + \mathcal{D}Z_5 + Z_8 \\ * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}$$

and so condition (5.4) results in

$$\mathcal{R} = \begin{pmatrix} 0 & 0 & 2 \\ \kappa & -\mathcal{D} & 0 \\ -1 & 0 & 0 \\ 0 & -\frac{3}{2} & 0 \\ 0 & 0 & \mathcal{D} \\ (\mathcal{D}-\tau)\kappa & -(\mathcal{D}-\tau)\mathcal{D} & 2\kappa \\ \mathcal{D}+\tau & 0 & -2 \\ -\kappa & \frac{1}{2}\mathcal{D} & -\mathcal{D}^2-\rho \end{pmatrix}.$$

From here, the reduced Poisson operator $\mathcal{P}_R$ is given by

$$\begin{pmatrix} \kappa(\mathcal{D}+\tau)^2-(\mathcal{D}-\tau)^2\kappa & \frac{1}{2}\,\rho\,\mathcal{D}+(\mathcal{D}-\tau)^2\mathcal{D} & -2\left(\kappa\mathcal{D}+(\mathcal{D}-\tau)\kappa+\kappa(\mathcal{D}+\tau)\right) \\ \frac{1}{2}\,\mathcal{D}\,\rho+\mathcal{D}(\mathcal{D}+\tau)^2 & -\frac{3}{2}\,\mathcal{D} & -3\,\mathcal{D}^2-2\,\mathcal{D}\,\tau \\ -2\left(\mathcal{D}\,\kappa+\kappa(\mathcal{D}+\tau)+(\mathcal{D}-\tau)\kappa\right) & 3\,\mathcal{D}^2-2\,\tau\,\mathcal{D} & -2(\mathcal{D}^3+\mathcal{D}\,\rho+\rho\,\mathcal{D}) \end{pmatrix}$$

Finally, the compatibility condition is given by equating the portions of $\mathcal{R}\delta h$ and $\mathcal{C}J$ that correspond to a complement of $\mathfrak{n}$ in $\mathfrak{g}$; that is, by equating the first 3 entries of each. This results in

$$-\begin{pmatrix} J_2 \\ J_3 \\ J_1 \end{pmatrix} = \begin{pmatrix} 2h_\rho \\ \kappa h_\kappa - (h_\tau)_x \\ -h_\kappa \end{pmatrix},$$

where the subscripts denote variational derivatives.

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