Curry–Howard terms for Linear Logic

David Albrecht[1,2,4] Frank A. Bäuerle[3,4] John N. Crossley[1,2] and John S. Jeavons[1]

Abstract

In this paper we 1. provide a natural deduction system for full first-order linear logic, 2. prove its equivalence to Girard's original system, 3. introduce Curry–Howard–style terms for this version of linear logic, 4. extend the notion of substitution of Curry-Howard terms for term variables and 5. prove strong normalization for the full system of linear logic using a development of Girard's candidates for reducibility, thereby providing an alternative to Girard's proof using proof–nets.

Contents

[1]Department of Mathematics, Monash University, Melbourne, Australia.
[2]Department of Computer Science, Monash University, Melbourne, Australia.
[3]Department of Mathematics, University of California, San Diego, CA, USA.
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1 Introduction

In recent years connexions between computer science and logic have become much more obvious and explicit. The basis of one such connexion is the so-called Curry–Howard isomorphism between term formation rules in lambda calculus and rules of inference in propositional logic. A number of people have extended this to other logics, in particular Crossley–Shepherdson [4] presents an extension to intuitionistic logic and then Heyting arithmetic by giving a typed lambda calculus notation for proofs in a natural deduction proof system for intuitionistic logic. The terms, which we call Curry–Howard terms, give an explicit representation of the whole proof which can be readily handled by the machine in contast to the traditional representations by mathematicians (see e.g. Gentzen's work in [5]) where, in discussing the manipulation of proofs, large parts are simply represented by $\cdots$, and it is left to the reader to determine precisely what the $\cdots$ means.

In the traditional situation the manipulations could be effected without too much difficulty. In the case of Linear Logic, as we shall see below, the substitution of a proof for a sub–proof has repercussions all the way up the proof.

The algorithmic content of the Curry–Howard–style notation together with strong normalization allows one to extract explicit computer programs from proofs. Abramsky [1] began an extension of this process to a fragment of linear logic. In the present paper we extend the Curry–Howard process (as we prefer to call it) to full first order linear logic and prove a strong normalization theorem for the Curry–Howard terms so obtained. This theorem, because of the Curry–Howard correspondence (isomorphism is perhaps too strong a word unless one is very pedantic in the definitions and statement of the theorem) implies strong normalization for the logical calculus. The proof is an alternative to Girard's which uses proof-nets (see Girard[7]) and its technology is a development of Girard's candidates for reducibility ([6]). In a sequel we plan to extend this work to a linear logic version of arithmetic and extract programs from the (Curry–Howard terms for the) proofs.

The innovations in this paper include contributions to the following.

1. The logical system.
2. The Curry–Howard terms.
3. Replacement of proof-nets by Curry-Howard terms.
4. Substitution of terms within terms.

1. In our view, Curry–Howard terms provide an explicit notation for proofs. They are explicit in the sense that, given a Curry–Howard term, one can write down the corresponding proof and *vice versa*. It then follows that manipulations of proofs as in the work of Gentzen [5] and his successors can be mirrored by manipulations of Curry–Howard terms. Thus creating a Curry–Howard term may be regarded as the "implementation issue" of entering descriptions of proofs into a machine (computer).

   In order for the manipulations to be made mechanical – as opposed to merely "obvious" (or "clear") as in most logic books where the manifold occurrences of ... hide a great deal of mechanical detail – the logical system has to have the nice property that replacing one proof by another can be formally notated. Such a manipulation is made easier by putting the logical system into a natural deduction style.

The system we use has been developed from the ones in Troelstra [13] and Benton *et al* [2]. It is a system for full first order linear logic which includes (representability of) all Girard's connectives.

2. The Curry–Howard terms themselves have new operators and reduction rules in addition to the customary λ operator of lambda calculus and the various pairing and unpairing operators (see e.g. Crossley–Shepherdson [4]). Nevertheless we feel that the style of term construction and reduction rules is the same as in previous works.

3. Since we prove that Curry–Howard terms and proofs correspond we do not have the difficulties Girard faced in [7] where, in order to prove his strong normalization theorem, he had to restrict the networks that he constructed to those which were actually "proof-nets" according to his definition. All our Curry–Howard terms automatically correspond to proof-nets.

4. In linear logic substituting one sub-proof for another may have repercussions on the premisses of a rule which are reflected back up the proof as well as being transmitted down the proof as in Curry–Howard terms for traditional intuitionistic logic (see Howard [10] or Crossley–Shepherdson [4]). This means we have to give a careful (and long) definition for substitution in our Curry–Howard terms.

Recently Ronchi della Rocca *et al.* [11] have introduced a natural deduction system for intuitionistic linear logic. However, their system has rules introducing two different connectives at once whereas ours follow the traditional pattern of introducing or eliminating at most one kind of connective at a time.

The outline of the paper is as follows. In section 2 we define a natural deduction system for linear logic that is based on Troelstra's system as defined in [13] and modified by Benton *et al* [2] (for the multiplicative connectives only). In section 4.3 we show that this system is logically equivalent to Girard's version of linear logic [7]. We also point out how this system behaves with respect to Girard's objections in [8]. We then define the Curry–Howard terms associated with proofs in this natural deduction system in section 4. This section contains a new style of substitution in sub-section 4.4 and the appropriate reduction rules in section 4.5. We give a proof of strong normalization for this system in section 5.

2 A Natural Deduction System

In predicate calculus we usually have rules equivalent to the following two rules (given here in a Gentzen sequent calculus style). We read $\Gamma \vdash \beta$ as Γ yields β.

$$\frac{\alpha, \alpha \vdash \beta}{\alpha \vdash \beta}\ (\text{Contraction}) \qquad \frac{\alpha \vdash \beta}{\alpha, \gamma \vdash \beta}\ (\text{Weakening})$$

In linear logic instead of these general rules, weakening and contraction are only allowed when the premisses are of a special form, that is to say, formulae of the form $!\alpha$ with the new unary connective "!" (where we read "!" as "of course"). Thus, rules in linear logic achieve something similar to the following (for exact details see section 2.2)

$$\frac{\cdots \quad !\alpha, !\alpha \vdash \beta}{!\alpha \vdash \beta}\ (\text{Contraction}) \qquad \frac{\cdots \quad !\alpha \vdash \beta}{!\alpha, !\gamma \vdash \beta}\ (\text{Weakening})$$

where $\cdots$ indicates the possibility of other formulae or sequents occurring in the top sequent of the rule.

A formula of the form $!\alpha$ is to be thought of as a stored, or renewable, resource, i.e. $!\alpha$ can be duplicated without further ado. Another consequence of the absence of general weakening and contraction is that two, usually logically equivalent, versions of $\wedge$ ("and") are no longer equivalent in linear logic.

$$(\wedge, 1)\quad \frac{\Gamma \vdash \alpha \quad \Delta \vdash \beta}{\Gamma, \Delta \vdash (\alpha \wedge \beta)} \qquad\qquad (\wedge, 2)\quad \frac{\Gamma \vdash \alpha \quad \Gamma \vdash \beta}{\Gamma \vdash (\alpha \wedge \beta)}$$

This gives rise to two types of $\wedge$ in linear logic which Girard defined as *multiplicative and*, $\otimes$ (read "times"), which corresponds to $(\wedge, 1)$ and *additive and*, & (read "with"), which corresponds to $(\wedge, 2)$. There are also two types of "or".

The language we use contains the propositional constants $\top$ (read "top") and $\bot$ (read "bottom"), the binary connectives $\otimes$, $\oplus$ (read "plus"), & and $\multimap$ (read "linearly implies" or "lollipop" for short), the unary connective ! ("of course"), and the quantifiers $\forall$ ("for all") and $\exists$ ("there exists"). We abbreviate $\alpha \multimap \bot$ by $\alpha^{\bot}$, $\bot^{\bot}$ by 1 and $\top^{\bot}$ by 0. (Girard's other connectives are definable, see section 3 below.)

The formulae will be generated from atomic formulae in the usual manner, using the connectives and the first order quantifiers, and will be denoted by lower case Greek letters. Upper case Greek letters will denote multisets of formulae (which may be empty). The rules will be defined on sequents of the form $\Gamma \vdash \alpha$, where Γ will be called the *declared premisses* of the sequent, and α will be called the *conclusion* of the sequent. In a rule

$$\frac{\Gamma \vdash \alpha}{\Delta \vdash \beta}$$

we shall call $\Gamma \vdash \alpha$ the *top sequent* and $\Delta \vdash \beta$ the *bottom sequent.*

We shall not distinguish the order of the formulae in the premisses Γ. In this way we do not need an exchange rule. Nevertheless, the system could easily be modified to use ordered premisses and an exchange rule.

The rules for the multiplicative connectives, weakening and contraction are essentially those of Benton *et al.*[2]. The rule for additive $\oplus$ is analogous to that for traditional $\vee$ (or) while $⅋$ (read "par") is a connective defined in terms of $\multimap$ and $\bot$.

We now present the axioms and rules of our natural deduction system, $\mathcal{N}$.

2.1 Axioms

For any formula α, and any multiset of formulae Γ, we have the following axioms.

$$\begin{array}{llcl} (\alpha) & \alpha & \vdash & \alpha \\ (\top) & \Gamma & \vdash & \top \\ (1) & & \vdash & 1 \end{array}$$

2.2 Rules

For each connective * (except $\perp$) we have an introduction rule (*I) and an elimination rule (*E). Also, we let $\alpha(x/t)$ denote the result of substituting t for all free occurrences of x in α (see section 4.4 for more details).

$$(\otimes I)\quad \frac{\Gamma \vdash \alpha \quad \Delta \vdash \beta}{\Gamma, \Delta \vdash \alpha \otimes \beta} \qquad\qquad (\otimes E)\quad \frac{\Gamma \vdash \alpha \otimes \beta \quad \Delta, \alpha, \beta \vdash \gamma}{\Gamma, \Delta \vdash \gamma}$$

$$(\& I)\quad \frac{\Gamma \vdash \alpha \quad \Gamma \vdash \beta}{\Gamma \vdash \alpha \& \beta}$$

$$(\& E)\quad \frac{\Gamma \vdash \alpha \& \beta}{\Gamma \vdash \alpha} \qquad\qquad \frac{\Gamma \vdash \alpha \& \beta}{\Gamma \vdash \beta}$$

$$(\oplus I)\quad \frac{\Gamma \vdash \alpha}{\Gamma \vdash \alpha \oplus \beta} \qquad\qquad \frac{\Gamma \vdash \beta}{\Gamma \vdash \alpha \oplus \beta}$$

$$(\oplus E)\quad \frac{\Gamma \vdash \alpha \oplus \beta \quad \Delta, \alpha \vdash \gamma \quad \Delta, \beta \vdash \gamma}{\Gamma, \Delta \vdash \gamma}$$

$$(\perp\!\circ I)\quad \frac{\Gamma, \alpha \vdash \beta}{\Gamma \vdash \alpha \perp\!\circ \beta} \qquad\qquad (\perp\!\circ E)\quad \frac{\Gamma \vdash \alpha \perp\!\circ \beta \quad \Delta \vdash \alpha}{\Gamma, \Delta \vdash \beta}$$

$$(\perp)\quad \frac{\Gamma \vdash \alpha^{\perp\perp}}{\Gamma \vdash \alpha}$$

$$(\exists I)\quad \frac{\Gamma \vdash \alpha(x/t)}{\Gamma \vdash \exists x \alpha} \qquad\qquad (\exists E)\quad \frac{\Gamma \vdash \exists x \alpha \quad \Delta, \alpha \vdash \gamma}{\Gamma, \Delta \vdash \gamma}$$

$$(\forall I)\quad \frac{\Gamma \vdash \alpha}{\Gamma \vdash \forall x \alpha} \qquad\qquad (\forall E)\quad \frac{\Gamma \vdash \forall x \alpha}{\Gamma \vdash \alpha(x/t)}$$

$$(!I)\quad \frac{\Gamma_1 \vdash !\alpha_1 \cdots \Gamma_n \vdash !\alpha_n \quad !\alpha_1, \ldots, !\alpha_n \vdash \beta}{\Gamma_1, \ldots, \Gamma_n \vdash !\beta} \qquad\qquad (!E)\quad \frac{\Gamma \vdash !\alpha}{\Gamma \vdash \alpha}$$

$$(W)\quad \frac{\Gamma \vdash !\alpha \quad \Delta \vdash \gamma}{\Gamma, \Delta \vdash \gamma} \qquad\qquad (C)\quad \frac{\Gamma \vdash !\alpha \quad \Delta, !\alpha, !\alpha \vdash \gamma}{\Gamma, \Delta \vdash \gamma}$$

Remark 1: In the rules $(\exists E)$ and $(\forall I)$ x must not be free in Γ, γ or Δ.
Remark 2: The choice of the rules and axioms was mainly motivated by their suitability for forming Curry–Howard terms based on very simple principles.
Remark 3: The rule $(\perp)$ is somewhat of an elimination rule and one could introduce a rule $(\perp I)$ which could be regarded as a special case of $(\perp\!\circ I)$ when β is $\perp$.
Remark 4: The weakening rule (W) can be considered as an introduction rule with the contraction rule (C) as the corresponding elimination rule.
Remark 5: There is no (CUT) rule as such. However a cut

$$\frac{\Gamma, \alpha \vdash \beta \quad \Delta \vdash \alpha}{\Gamma, \Delta \vdash \beta}$$

can be represented by $(\multimap I)$ followed by $(\multimap E)$:

$$\dfrac{\dfrac{\Gamma, \alpha \vdash \beta}{\Gamma \vdash \alpha \multimap \beta}\ (\multimap I) \qquad \Delta \vdash \alpha}{\Gamma, \Delta \vdash \beta}\ (\multimap E)$$

It therefore follows that adding a (CUT) rule does not change the set of provable sequents (see also section 5).

Remark 6: In [8] §3.2 Girard lists certain limitations, $\alpha) \perp \delta)$ below, of natural deduction systems for linear logic.

"α) natural deduction is not equipped to deal with classical symmetry: several hypotheses and one (distinguished) conclusion. To cope with symmetrical systems one should be able to accept several conclusions at once... But then one immediately loses the tree–like structure of natural deductions, with its obvious advantage: a well–determined last rule. Hence natural deduction cannot answer the question. However it is still a serious candidate for an intuitionistic version of linear logic; we shall ... only discuss the fragment $(\otimes, \multimap)$, for which there is an obvious natural deduction system."

"γ) due to discharge, the introduction rule for $\multimap$ (and the elimination rule for $\otimes$) does not apply to a formula, but a whole proof. This *gobal* character of the rule is quite a serious defect."

"δ) last but not least, the elimination rule for $\otimes$ mentions an extraneous formula C which has nothing to do with $A \otimes B$. In intuitionistic natural deduction, we have the same problem with the rules for disjuction and existence which mention an extraneous formula C; the theory of normalisation ("commutative conversions") then becomes extremely complex and algorithmically awkward."

We comment as follows:

α) Although the symmetry between premisses and conclusions in Girard's system is lost in our natural deduction system, nevertheless any proof in Girard's system can be directly translated into a proof in ours and therefore, from a technical point of view, though not from a visual one, no symmetry is lost.

On the other hand the fact that we have a well defined "last rule" is a substantial benefit for the normalization process.

β) Girard calls his remark β) "very inessential" and we omit comment on it.

γ) The fact that a premiss is discharged in introducing $\multimap$ (or $\supset$ in intuitionistic logic) does not affect our system as much as a traditional natural deduction system since we have a list of premisses in each sequent and a discharged premiss simply disappears from this list (to appear as part of the conclusion formula) when $(\multimap I)$ is used.

δ) Girard says that the elimination rule for $\otimes$ (as well as those for $\exists$ and $\oplus$) have an "extraneous" conclusion formula, γ, which has "nothing to do with" the $\alpha \otimes \beta$ from which $\otimes$ is being eliminated.

While this is true, the actual normalization process is an algorithmic modification of a proof by removing an introduction immediately followed by a elimination without changing the conclusion.

On the other hand in the proof that the reduction proceedure terminates we do have to perform a complicated induction which is driven by having to account for any possible formula as the "extraneous formula".

3 Equivalence of the Natural Deduction System to Girard's Linear Logic

In this section we demonstrate that our system, $\mathcal{N}$, introduced in section 2, is logically equivalent to Girard's linear logic as introduced in [7].

First we shall present the axioms and rules as given by Girard in [7] and then we shall show that our system is logically equivalent to Girard's. That is: for every formula provable in one system an equivalent one is provable in the other.

Girard's system, $\mathcal{G}$, was presented in a one-sided Gentzen sequent calculus style. We allow multisets and thereby are able to dispense with the exchange rule. The propositional connectives consisted of

1. multiplicative symbols $\bot, 1, \otimes, \parr$
2. additive symbols $0, \top, \oplus, \&$ and
3. exponentials $!, \Gamma$.

For formulae α, β and γ and multisets of formulae Γ and Δ we have the following axioms and rules.

$$(\alpha) \quad \vdash \quad \alpha, \alpha^{\bot}$$

$$(1) \quad \vdash \quad 1$$

$$(\top) \quad \vdash \quad \top, \Gamma$$

$$(CUT) \quad \frac{\vdash \gamma, \Gamma \quad \vdash \gamma^{\bot}, \Delta}{\vdash \Gamma, \Delta}$$

$$(\bot) \quad \frac{\vdash \Gamma}{\vdash \bot, \Gamma}$$

For each connective * we have an introduction rule (*I). We do not have elimination rules, since CUT will provide for eliminations. Again we let $\alpha(x/t)$ denote the result of substituting t for all free occurrences of x in α.

$$(\&) \quad \frac{\vdash \alpha, \Gamma \quad \vdash \beta, \Gamma}{\vdash \alpha \& \beta, \Gamma}$$

$$(\oplus_1) \quad \frac{\vdash \alpha, \Gamma}{\vdash \alpha \oplus \beta, \Gamma} \qquad (\oplus_2) \quad \frac{\vdash \beta, \Gamma}{\vdash \alpha \oplus \beta, \Gamma}$$

$$(\otimes) \quad \frac{\vdash \alpha, \Gamma \quad \vdash \beta, \Delta}{\vdash \alpha \otimes \beta, \Gamma, \Delta} \qquad (\parr) \quad \frac{\vdash \alpha, \beta, \Gamma}{\vdash \alpha \parr \beta, \Gamma}$$

$$(\mathrm{D}\Gamma) \quad \frac{\vdash \alpha, \Gamma}{\vdash \Gamma\alpha, \Gamma} \qquad (!) \quad \frac{\vdash \alpha, \Gamma\Gamma}{\vdash !\alpha, \Gamma\Gamma}$$

$$(W\Gamma)\quad \frac{\vdash \Gamma}{\vdash \Gamma\alpha, \Gamma} \qquad\qquad (C\Gamma)\quad \frac{\vdash \Gamma\alpha, \Gamma\alpha, \Gamma}{\vdash \Gamma\alpha, \Gamma}$$

$$(\forall)\quad \frac{\vdash \alpha, \Gamma}{\vdash \forall x\alpha, \Gamma} \qquad\qquad (\exists)\quad \frac{\vdash \alpha(x/t), \Gamma}{\vdash \exists x\alpha, \Gamma}$$

Remark 1: As usual, in the $\forall$ rule x must not appear free in Γ.
Remark 2: We have used the following notation. If $\Gamma = \alpha_1, \ldots, \alpha_n$ then $\Gamma\Gamma = \Gamma\alpha_1, \ldots, \Gamma\alpha_n$.
Remark 3: Since we have used multisets rather than the exchange rule, the above system $\mathcal{G}$ is not exactly the same as the system described in Girard [7] but it is clearly equivalent to it.

Linear negation is defined in the following way.

1. $1^\perp = \perp, \perp^\perp = 1, \top^\perp = 0, 0^\perp = \top$, and for α an atomic formula $\alpha^{\perp\perp} = \alpha$.
2. $(\alpha \otimes \beta)^\perp = \alpha^\perp \parr \beta^\perp, (\alpha \parr \beta)^\perp = \alpha^\perp \otimes \beta^\perp$,
3. $(\alpha \oplus \beta)^\perp = \alpha^\perp \& \beta^\perp, (\alpha \& \beta)^\perp = \alpha^\perp \oplus \beta^\perp$,
4. $(!\alpha)^\perp = \Gamma(\alpha^\perp), (\Gamma\alpha)^\perp = !(\alpha^\perp)$,
5. $(\exists x\alpha)^\perp = \forall x(\alpha^\perp), (\forall x\alpha)^\perp = \exists x(\alpha^\perp)$.

Given a formula α in $\mathcal{G}$ then it can be automatically regarded as a formula of $\mathcal{N}$ if we regard $\alpha \parr \beta$ as an abbreviation for $(\alpha^\perp \otimes \beta^\perp)^\perp$ where we recall that in $\mathcal{N}$, $\alpha^\perp$ is an abbreviation for $\alpha \perp\!\circ \perp$, and 1 and 0 are abbreviations for $\perp^\perp$ and $\top^\perp$, respectively.

Definition 1 *The* **interpretations** *of a sequent* $\vdash \alpha_1, \cdots, \alpha_{i\perp 1}, \alpha_i, \alpha_{i+1}, \cdots, \alpha_n$ *from Girard's linear logic system,* $\mathcal{G}$*, in our system,* $\mathcal{N}$*, are all sequents* $\alpha_1^\perp, \cdots, \alpha_{i\perp 1}^\perp, \alpha_{i+1}^\perp, \cdots, \alpha_n^\perp \vdash \alpha_i$ *for* $1 \leq i \leq n$ *and, in addition, the sequent* $\alpha_1^\perp, \cdots, \alpha_n^\perp \vdash \perp$.

Lemma 1 *An interpretation in* $\mathcal{N}$ *for a given sequent in* $\mathcal{G}$ *is provable in* $\mathcal{N}$ *if, and only if, all interpretations in* $\mathcal{N}$ *of that sequent are provable in* $\mathcal{N}$.

Proof : Assume that $\alpha_1^\perp, \cdots, \alpha_{i\perp 1}^\perp, \alpha_{i+1}^\perp, \cdots, \alpha_n^\perp \vdash \alpha_i$ is provable in $\mathcal{N}$. Then one application of $\perp\!\circ E$ shows that $\alpha_1^\perp, \cdots, \alpha_n^\perp \vdash \perp$ is provable in $\mathcal{N}$ since $\alpha_i^\perp \vdash \alpha_i^\perp$ is an axiom:

$$\frac{\alpha_1^\perp, \cdots, \alpha_{i\perp 1}^\perp, \alpha_{i+1}^\perp, \cdots, \alpha_n^\perp \vdash \alpha_i \quad \alpha_i^\perp \vdash \alpha_i \perp\!\circ \perp}{\alpha_1^\perp, \cdots, \alpha_n^\perp \vdash \perp} \ (\perp\!\circ E)$$

One application each of $\perp\!\circ I$ and of $\perp$ now shows the following.

$$\cfrac{\cfrac{\alpha_1^\perp, \cdots, \alpha_{j\perp 1}^\perp, \alpha_j^\perp, \alpha_{j+1}^\perp, \cdots, \alpha_n^\perp \vdash \perp}{\alpha_1^\perp, \cdots, \alpha_{j\perp 1}^\perp, \alpha_{j+1}^\perp, \cdots, \alpha_n^\perp \vdash \alpha_j^{\perp\perp}} \ (\perp\!\circ I)}{\alpha_1^\perp, \cdots, \alpha_{j\perp 1}^\perp, \alpha_{j+1}^\perp, \cdots, \alpha_n^\perp \vdash \alpha_j} \ (\perp)$$

It is now obvious that with these two proof structures one can generate a proof of any interpretation from a proof of one particular interpretation.

□

In a similar way one can also easily prove the following lemma.

Lemma 2 *In $\mathcal{N}$, $\gamma, \Delta \vdash \alpha$ if, and only if, $\gamma^{\perp\perp}, \Delta \vdash \alpha$.*

Notation. If $\Gamma = \alpha_1, \ldots, \alpha_n$, then $!\Gamma$ is an abbreviation for $!\alpha, \ldots, !\alpha_n$, and similarly for $\Gamma^\perp$, and $\Gamma\Gamma$

Theorem 1 *If a sequent is provable in $\mathcal{G}$, then every interpretation of this sequent in $\mathcal{N}$ is provable in $\mathcal{N}$.*

Proof: The proof is by induction on the complexity of the proof, in $\mathcal{G}$, of a sequent $\vdash \Gamma$. By Lemma 1, it suffices to show that one interpretation of the sequent is provable in $\mathcal{N}$.
Base Cases: Axioms:

1. Given $\vdash \alpha, \alpha^\perp$ in $\mathcal{G}$. An interpretation in $\mathcal{N}$ is $\alpha^\perp \vdash \alpha^\perp$ which is an axiom.
2. Given $\vdash \top, \Gamma$ in $\mathcal{G}$. An interpretation in $\mathcal{N}$ is $\Gamma^\perp \vdash \top$ which is an axiom.
3. Given $\vdash 1$ in $\mathcal{G}$. An interpretation in $\mathcal{N}$ is $\vdash 1$ which is an axiom.

Induction Step: Rules:
We shall omit the obvious cases $\&, \oplus_1, \oplus_2, \forall, \exists$ and present the slightly more interesting remaining cases.

(CUT) Given that the interpretations $\Gamma^\perp \vdash \gamma$ and $\Delta^\perp \vdash \gamma^\perp$ are provable in $\mathcal{N}$, it follows from one application of $(\perp\!\circ E)$ that $\Gamma^\perp, \Delta^\perp \vdash \perp$ is provable in $\mathcal{N}$:

$$\frac{\Gamma^\perp \vdash \gamma \quad \Delta^\perp \vdash \gamma \perp\!\circ \perp}{\Gamma^\perp, \Delta^\perp \vdash \perp}\ (\perp\!\circ E)$$

Therefore an interpretation of the sequent $\vdash \Gamma, \Delta$ is provable in $\mathcal{N}$. (Recall that $\gamma^\perp$ is an abbreviation for $\gamma \perp\!\circ \perp$.)

$(\perp)$ In the light of Lemma 1 there is nothing to prove since an interpretation of $\vdash \Gamma$ is $\Gamma^\perp \vdash \perp$ and an interpretation of $\vdash \perp, \Gamma$ is also $\Gamma^\perp \vdash \perp$.

$(\parr)$ Assume that the interpretation $\Gamma^\perp, \alpha^\perp \vdash \beta$ is provable in $\mathcal{N}$. From this with one application of $(\perp\!\circ I)$ the sequent $\Gamma^\perp \vdash \alpha^\perp \perp\!\circ \beta$ is provable in $\mathcal{N}$. This is an interpretation of $\vdash \alpha \parr \beta, \Gamma$.

$(D\Gamma)$ Assume the interpretation $\Gamma^\perp, \alpha^\perp \vdash \perp$ is provable in $\mathcal{N}$. Then

$$\cfrac{\dfrac{!(\alpha^\perp) \vdash !(\alpha^\perp)}{!(\alpha^\perp) \vdash \alpha^\perp}\ (!E) \qquad \dfrac{\Gamma^\perp, \alpha^\perp \vdash \perp}{\Gamma^\perp \vdash \alpha^\perp \perp\!\circ \perp}\ (\perp\!\circ I)}{\dfrac{\Gamma^\perp, !(\alpha^\perp) \vdash \perp}{(\Gamma)^\perp \vdash (!(\alpha^\perp))^\perp}\ (\perp\!\circ I)}\ (\perp\!\circ E)$$

The conclusion of this proof segment is an interpretation of $\vdash \Gamma\alpha, \Gamma$.

$(!)$ Assume that the interpretation $(!(\Gamma^\perp))^{\perp\perp} \vdash \alpha$ is provable in $\mathcal{N}$, and $\Gamma = \gamma_1, \ldots, \gamma_n$. Then, using n axioms of the form $\delta \vdash \delta$, we have

$$\frac{(!(\gamma_1^\perp))^{\perp\perp} \vdash (!(\gamma_1^\perp))^{\perp\perp} \cdots (!(\gamma_n^\perp))^{\perp\perp} \vdash (!(\gamma_n^\perp))^{\perp\perp} \qquad (!(\Gamma^\perp))^{\perp\perp} \vdash \alpha}{(!(\gamma_1^\perp))^{\perp\perp}, \cdots, (!(\gamma_n^\perp))^{\perp\perp} \vdash !(\alpha)}\ (!I)$$

The conclusion of this proof segment is an interpretation of $\vdash !\alpha, \Gamma\Gamma$.

($W\Gamma$) Assume that the interpretation $\Gamma^\perp \vdash \perp$ is provable in $\mathcal{N}$. Then

$$\dfrac{\dfrac{!(\alpha^\perp) \vdash !(\alpha^\perp) \quad \Gamma^\perp \vdash \perp}{!(\alpha^\perp), \Gamma^\perp \vdash \perp}\ (W)}{\Gamma^\perp \vdash (!(\alpha^\perp))^\perp}\ (\perp\circ I)$$

The conclusion of this proof segment is an interpretation of $\vdash \Gamma\alpha, \Gamma$.

($C\Gamma$) Assume that the interpretation $\Gamma^\perp, (!(\alpha^\perp))^{\perp\perp}, (!(\alpha^\perp))^{\perp\perp} \vdash \perp$ is provable in $\mathcal{N}$. Then

$$\dfrac{\dfrac{!(\alpha^\perp) \vdash !(\alpha^\perp) \qquad \dfrac{(\Gamma)^\perp, (!(\alpha^\perp))^{\perp\perp}, (!(\alpha^\perp))^{\perp\perp} \vdash \perp}{\Gamma^\perp, !(\alpha^\perp), !(\alpha^\perp) \vdash \perp}\ (\textit{Lemma 2})}{\Gamma^\perp, !(\alpha^\perp) \vdash \perp}\ (C)}{\Gamma^\perp \vdash (!(\alpha^\perp))^\perp}\ (\perp\circ I)$$

The conclusion of this proof segment is an interpretation of $\vdash \Gamma\alpha, \Gamma$.

□

Next we shall show the converse by induction on the construction of a given proof. The interpretation of a sequent $\Gamma \vdash \alpha$ from our system in Girard's system is the sequent $\vdash \Gamma^\perp, \alpha$. Since $\perp\circ$ is not in Girard's language we interpret $\alpha \perp\circ \beta$ as $\alpha^\perp ⅋ \beta$.

Theorem 2 *Every provable sequent in our system has an interpretation that is provable in Girard's system.*

Proof:

Base Cases: Axioms:

1. $\alpha \vdash \alpha$. The interpretation is $\vdash \alpha, \alpha^\perp$ which is an axiom.
2. $\Gamma \vdash \top$. The interpretation is $\vdash \Gamma^\perp, \top$ which is an axiom.
3. $\vdash 1$. The interpretation is $\vdash 1$ which is an axiom.

Induction Step: Rules:

Again we shall omit the obvious cases $\otimes I, \& I, \oplus I, \exists I, \forall I$ and present the remaining cases.

($\otimes E$) Assume that the interpretations of $\Gamma \vdash \alpha \otimes \beta$ and $\Delta, \alpha, \beta \vdash \gamma$ are provable. Then

$$\dfrac{\vdash \alpha \otimes \beta, \Gamma^\perp \qquad \dfrac{\vdash \gamma, \Delta^\perp, \alpha^\perp, \beta^\perp}{\vdash \gamma, \Delta^\perp, \alpha^\perp ⅋ \beta^\perp}\ (⅋)}{\vdash \Gamma^\perp, \gamma, \Delta^\perp}\ (CUT)$$

Note that $\alpha^\perp ⅋ \beta^\perp = (\alpha \otimes \beta)^\perp$. The conclusion of this proof segment is the interpretation of $\Gamma, \Delta \vdash \gamma$.

($\& E$) Assume that the interpretation of $\Gamma \vdash \alpha \& \beta$ is provable. Then

$$\dfrac{\vdash \alpha \& \beta, \Gamma^\perp \qquad \dfrac{\vdash \alpha, \alpha^\perp}{\vdash \alpha, \alpha^\perp \oplus \beta^\perp}\ (\oplus)}{\vdash \alpha, \Gamma^\perp}\ (CUT)$$

In an analogous way one can also prove $\vdash \beta, \Gamma^{\perp}$. Note that $\alpha^{\perp} \oplus \beta^{\perp} = (\alpha \& \beta)^{\perp}$. The conclusion of this proof segment is the interpretation of $\Gamma \vdash \alpha$.

($\oplus E$) Assume that the interpretations of $\Gamma \vdash \alpha \oplus \beta$, $\Delta, \alpha \vdash \gamma$ and $\Delta, \beta \vdash \gamma$ are provable. Then

$$\dfrac{\vdash \alpha \oplus \beta, \Gamma^{\perp} \qquad \dfrac{\vdash \gamma, \Delta^{\perp}, \alpha^{\perp} \quad \vdash \gamma, \Delta^{\perp}, \beta^{\perp}}{\vdash \gamma, \Delta^{\perp}, \alpha^{\perp} \& \beta^{\perp}}\ (\&)}{\vdash \gamma, \Delta^{\perp}, \Gamma^{\perp}}\ (CUT)$$

Note that $\alpha^{\perp} \& \beta^{\perp} = (\alpha \oplus \beta)^{\perp}$. The conclusion of this proof segment is the interpretation of $\Gamma, \Delta \vdash \gamma$.

($\perp\!\circ I$) Assume that the interpretation of $\Gamma, \alpha \vdash \beta$ is provable. Then

$$\dfrac{\vdash \Gamma^{\perp}, \alpha^{\perp}, \beta}{\vdash \Gamma^{\perp}, \alpha^{\perp} \invamp \beta}\ (\invamp)$$

The conclusion of this proof segment is the interpretation of $\Gamma \vdash \alpha \perp\!\circ \beta$.

($\perp\!\circ E$) Assume that the interpretations of $\Gamma \vdash \alpha \perp\!\circ \beta$ and $\Delta \vdash \alpha$ are provable. Then using axioms of the form $\vdash \gamma, \gamma^{\perp}$ we have:

$$\dfrac{\vdash \Gamma^{\perp}, \alpha^{\perp} \invamp \beta \qquad \dfrac{\vdash \beta, \beta^{\perp} \qquad \dfrac{\vdash \alpha, \Delta^{\perp} \quad \vdash \alpha^{\perp}, \alpha^{\perp\perp}}{\vdash \alpha^{\perp\perp}, \Delta^{\perp}}\ (CUT)}{\vdash \beta, \Delta^{\perp}, \alpha^{\perp\perp} \otimes \beta^{\perp}}\ (\otimes)}{\vdash \Gamma^{\perp}, \Delta^{\perp}, \beta}\ (CUT)$$

The conclusion of this proof segment is the interpretation of $\Gamma, \Delta \vdash \beta$.

($\perp$) Assume that the interpretation of $\Gamma \vdash \alpha^{\perp\perp}$ is provable. Then

$$\dfrac{\vdash \alpha^{\perp\perp}, \Gamma^{\perp} \quad \vdash \alpha, \alpha^{\perp}}{\vdash \alpha, \Gamma^{\perp}}\ (CUT)$$

The conclusion of this proof segment is the interpretation of $\Gamma \vdash \alpha$.

($\exists E$) Assume that the interpretations of $\Gamma \vdash \exists x \alpha$ and $\Delta, \alpha \vdash \gamma$ are provable. By our definition of the ($\exists E$)-rule we can also assume that x does not occur free in γ, Γ and Δ. Then

$$\dfrac{\vdash \Gamma^{\perp}, \exists x \alpha \qquad \dfrac{\vdash \Delta^{\perp}, \alpha^{\perp}, \gamma}{\vdash \Delta^{\perp}, \forall x \alpha^{\perp}, \gamma}\ (\forall)}{\vdash \Delta^{\perp}, \Gamma^{\perp}, \gamma}\ (CUT)$$

The conclusion of this proof segment is the interpretation of $\Gamma, \Delta \vdash \gamma$.

($\forall E$) Assume that the interpretation of $\Gamma \vdash \forall x \alpha$ is provable. Then

$$\dfrac{\vdash \Gamma^{\perp}, \forall x \alpha \qquad \dfrac{\vdash \alpha^{\perp}(x/t), \alpha(x/t)}{\vdash \exists x \alpha^{\perp}, \alpha(x/t)}\ (\exists)}{\vdash \Gamma^{\perp}, \alpha(x/t)}\ (CUT)$$

The conclusion of this proof segment is the interpretation of $\Gamma \vdash \alpha(x/t)$.

$(!I)$ Assume that the interpretations of $\Gamma_1 \vdash !\alpha_1, \cdots, \Gamma_n \vdash !\alpha_n$ and $!\alpha_1, \cdots, !\alpha_n \vdash \beta$ are provable. Then

$$\dfrac{\vdash !\alpha_1, \Gamma_1^\perp \qquad \dfrac{\vdash \beta, (!\alpha_1)^\perp, \cdots, (!\alpha_n)^\perp}{\vdash !\beta, (!\alpha_1)^\perp, \cdots, (!\alpha_n)^\perp}\,(!)}{\vdash !\beta, \Gamma_1^\perp, (!\alpha_2)^\perp, \cdots, (!\alpha_n)^\perp}\,(CUT)$$

and with $n \perp 1$ more cuts of the same form for the remaining $!\alpha_i$ one obtains $\vdash !\beta, \Gamma_1^\perp, \cdots, \Gamma_n^\perp$ which is the interpretation of $\Gamma_1, \cdots, \Gamma_n \vdash !\beta$.

$(!E)$ Assume that the interpretation of $\Gamma \vdash !\alpha$ is provable. Then

$$\dfrac{\vdash \Gamma^\perp, !\alpha \qquad \dfrac{\vdash \alpha, \alpha^\perp}{\vdash \alpha, \Gamma(\alpha^\perp)}\,(D\Gamma)}{\vdash \alpha, \Gamma^\perp}\,(CUT)$$

The conclusion of this proof segment is the interpretation of $\Gamma \vdash \alpha$.

(W) Assume that the interpretations of $\Gamma \vdash !\alpha$ and $\Delta \vdash \gamma$ are provable. Then

$$\dfrac{\dfrac{\vdash \gamma, \Delta^\perp}{\vdash \gamma, \Delta^\perp, \Gamma(\alpha^\perp)}\,(W) \qquad \vdash \Gamma^\perp, !\alpha}{\vdash \gamma, \Gamma^\perp, \Delta^\perp}\,(CUT)$$

The the conclusion of this proof segment is the interpretation of $\Gamma, \Delta \vdash \gamma$.

(C) Assume that the interpretations of $\Gamma \vdash !\alpha$ and $\Delta, !\alpha, !\alpha \vdash \gamma$ are provable. Then

$$\dfrac{\vdash !\alpha, \Gamma^\perp \qquad \dfrac{\vdash \gamma, \Delta^\perp, \Gamma(\alpha^\perp), \Gamma(\alpha^\perp)}{\vdash \gamma, \Delta^\perp, \Gamma(\alpha^\perp)}\,(C\Gamma)}{\vdash \gamma, \Gamma^\perp, \Delta^\perp}\,(CUT)$$

The conclusion of this proof segment is the interpretation of $\Gamma, \Delta \vdash \gamma$.

□

It immediately follows that the two systems are logically equivalent.

4 Curry–Howard Terms

4.1 Preliminaries

In this section we define Curry–Howard terms for linear logic. These will be of the form

$$(Z^\Gamma; F^\alpha)$$

and such a term will correspond to a proof of a sequent $\Gamma \vdash \alpha$ in the natural deduction system introduced in section 2.

For each formula, α, we have *term variables* $X^\alpha, Y^\alpha, \cdots$, of type α. For the constants $\top$ and 1, we use $A_1^\top, A_2^\top, \ldots$ and $A_1^1, A_2^1, \ldots$ to denote *term constants* of types $\top$ and 1, respectively. If $\Gamma = \{\alpha_1, \ldots, \alpha_n\}$ is a multiset of formulae and $Z_1^{\alpha_1}, \ldots, Z_n^{\alpha_n}$ are distinct term variables, we shall abbreviate $Z_1^{\alpha_1}, \ldots, Z_n^{\alpha_n}$ by Z^Γ, which we shall call a *vector of term variables of type* Γ. Moreover, if $Z^\Gamma = Z_1^{\alpha_1}, \ldots, Z_n^{\alpha_n}$, then $Z^{!\Gamma} = Z_1^{!\alpha_1}, \ldots, Z_n^{!\alpha_n}$. In $T = (Z^\Gamma; F^\alpha)$ the term variables in the term vector Z^Γ are called the *declared premisses* of T, expression F is called the *conclusion* of T, and α is called the *conclusion type* of T and also the *type* of F.

In the following constructions, the free individual variables and term variables of the new term constructed from the terms $(Z^\Gamma; F^\alpha)$ and $(W^\Delta; G^\beta)$, etc., are the free variables of F^α, G^β, etc., unless otherwise stated. Because linear logic is a kind of resource logic it is necessary to ensure that each occurrence of a premiss is separately recorded in the Curry–Howard term. We therefore make the following

Premiss Convention. In the following, Z^Γ, W^Δ, etc. will denote vectors of term variables, X^α, Y^β, etc. will denote term variables, and all the term variables in the term vectors together with the other term variables *will be assumed to be distinct.*

Thus when we write e.g. Z^Γ, X^α, W^Δ we shall be assuming, in particular, that X^α does not occur in Z^Γ or W^Δ. In particular, if Z^Γ and W^Δ are such vectors then the vector Z^Γ, W^Δ will *only* be used if all term variables occurring are distinct. At this stage of the paper we shall ensure that our terms will encode sufficient information about the proof to recover it fully. To this end we need to store not only the types of the formulae involved so that we can recover the last inference rule used, but also the declared premisses at each stage of the term building. In writing Curry–Howard terms in this way we shall write rather repetitious terms — the same symbol may occur several times. It is possible, indeed probable, that a tighter notation could be devised perhaps using ideas as in de Bruijn [3]. However, our approach guarantees that *all* the information as to what the previous sequents (and indeed what the proof[s] up to this stage) were is present in the term.

Thus, for example, in the case of the $(\top)$-axiom, which allows us to introduce an arbitrary (finite) number of declared premisses simultaneously, we record all these new premisses. Thus if we had two sequents arising from the use of the $(\top)$-axiom joined by an application of, say, a $\otimes$-introduction rule, then without exact knowledge about where the declared premisses were introduced we would not be able to decide what was the original proof. Of course we can search all the (finitely many) possible proofs for a correct one, but there may be more than one correct one.

The main ideas and reasons underlying the given construction of the Curry–Howard terms are the following:

1. Elimination of declared premisses corresponds to abstraction on the associated term variables. The intended meaning of this binding is to prepare a "socket" for a "plug" of the same type that will be applied to it with an application *cf.* Girard [7]. There are two types of abstraction (see section 4.4 for substitution of Curry–Howard terms into term variables for more detail):

   (a) λ-abstraction is used in three settings but behaves essentially as usual.

   (b) κ-abstraction, is used in connexion with the contraction rule.

2. Application in general will correspond to plugging a Curry–Howard term T of type $\Gamma \vdash \alpha$ into a socket of type α (namely a term variable X^α) in a term S, thereby replacing a

bound term variable X^α by a (term T of a) proof, while also adding the premisses of the term T to the declared premisses of the term S.

As an example of this consider the implication rules : $\multimap$-Introduction corresponds to λ-abstraction (since we bind on a premiss), and $\multimap$-Elimination corresponds to application (we "apply" a proof of α to a proof of $\alpha \multimap \beta$ to obtain a proof of β).

3. The symbols τ, ω, and ι serve as markers that distinguish uses of a $\perp$ rule, (W)-rule, or $(!E)$-rule respectively. *Note.* We use ι (iota) as a mnemonic for !, κ for copying, ω for weakening and τ for twice, *viz.* double [negation].

4. Introductions of $\otimes, \&$, and $\oplus$ will correspond to pairings all of which are denoted by (,) where the type superscript will encode which rule of inference was used. The declared premisses of the bottom sequent will be formed from the declared premisses of the top sequents in the same way as in other terms.

5. The projections π_1, π_2 in the $\otimes, \&$, and $\oplus$–Elimination rules mainly serve the purpose of selecting the correct branch of the proof that gave evidence for the introduction rule. To understand the function of the projections fully one needs to look ahead to the reduction rules in section 4.5.

4.2 Definition of Curry–Howard terms

Now here are the Curry–Howard term formation rules, labelled by the corresponding axiom or rule name:

(α) If X^α is a term variable, then the following is a Curry–Howard term.

$$(X^\alpha; X^\alpha)$$

The free individual variables of this term are the free individual variables of α, and the (unique) free term variable of this term is X^α.

($\top$) If Z^Γ is a term vector of distinct term variables, and $A^\top$ is a term constant, then the following is a term.

$$(Z^\Gamma; A^\top)$$

The free variables of this term are the term variables in Z^Γ (corresponding to the declared premisses) and the free individual variables are the free individual variables of the declared premisses.

(1) If A^1 is a term constant, then the following is a term.

$$(; A^1)$$

This term has no free term variables and no free individual variables.

Note 1. In the following term construction all term variables in the terms corresponding to the premisses are assumed distinct according to our Premiss Convention above.
Note 2. Occurrences of variables are free or bound as determined in the premisses of the rule except where noted explicity.

($\otimes I$) If $T_\alpha = (Z^\Gamma; F^\alpha)$ and $T_\beta = (W^\Delta; G^\beta)$ are terms, then the following is a term.

$$(Z^\Gamma, W^\Delta; (T_\alpha, T_\beta)^{\alpha\otimes\beta})$$

($\otimes E$) If $T_{\alpha\otimes\beta} = (Z^\Gamma; F^{\alpha\otimes\beta})$ and $T_\gamma = (W^\Delta, X^\alpha, Y^\beta; G^\gamma)$ are terms, and $L^{\alpha\otimes\beta}$ is a term variable, then the following is a term. The term variables X^α, Y^β, and $L^{\alpha\otimes\beta}$ are bound (at all occurrences) and do not occur in Z^Γ or W^Δ by our Premiss Convention.

$$(Z^\Gamma, W^\Delta; ((\lambda L^{\alpha\otimes\beta}.(\lambda X^\alpha Y^\beta.T_\gamma)(\pi_1 L^{\alpha\otimes\beta})(\pi_2 L^{\alpha\otimes\beta}))(T_{\alpha\otimes\beta}))^\gamma)$$

($\& I$) If $T_\alpha = (Z^\Gamma; F^\alpha)$ and $T_\beta = (Z^\Gamma; G^\beta)$ are terms, then the following is a term.

$$(Z^\Gamma; (T_\alpha, T_\beta)^{\alpha\&\beta})$$

($\& E$) If $T_{\alpha\&\beta} = (Z^\Gamma; F^{\alpha\&\beta})$ is a term, then the following are terms.

$$(Z^\Gamma; (\pi_1 T_{\alpha\&\beta})^\alpha) \text{ and } (Z^\Gamma; (\pi_2 T_{\alpha\&\beta})^\beta)$$

($\oplus I$) If $T_\alpha = (Z^\Gamma; F^\alpha)$ is a term, then $(Z^\Gamma; (\pi_1, T_\alpha)^{\alpha\oplus\beta})$ is a term. Similarly, if $T_\beta = (Z^\Gamma; G^\beta)$ is a term, then $(Z^\Gamma; (\pi_2, T_\beta)^{\alpha\oplus\beta})$ is a term.

($\oplus E$) If $T_{\alpha\oplus\beta} = (Z^\Gamma; F^{\alpha\oplus\beta})$, $S_\gamma = (W^\Delta, X^\alpha; G^\gamma)$ and $R_\gamma = (W^\Delta, Y^\beta; H^\gamma)$ are terms, and $L^{\alpha\oplus\beta}$ is a term variable, then the following is a term.

$$(Z^\Gamma, W^\Delta; ((\lambda L^{\alpha\oplus\beta}.(\pi_1 L^{\alpha\oplus\beta})((\lambda X^\alpha.S_\gamma), (\lambda Y^\beta.R_\gamma))(\pi_2 L^{\alpha\oplus\beta}))(T_{\alpha\oplus\beta}))^\gamma)$$

The occurrences of $L^{\alpha\oplus\beta}$, of X^α in S_γ and of Y^β in R_γ are bound in this term.

($\perp\!\circ I$) If $T_\beta = (Z^\Gamma, X^\alpha; F^\beta)$ is a term, then the following is a term, which has the term variable X^α bound.

$$(Z^\Gamma; (\lambda X^\alpha.T_\beta)^{\alpha\perp\circ\beta})$$

($\perp\!\circ E$) If $T_\alpha = (W^\Delta; F^\alpha)$ and $T_{\alpha\perp\circ\beta} = (Z^\Gamma; G^{\alpha\perp\circ\beta})$ are terms, then the following is a term.

$$(Z^\Gamma, W^\Delta; (T_{\alpha\perp\circ\beta}(T_\alpha))^\beta)$$

($\perp$) If $T_{\alpha^{\perp\perp}} = (Z^\Gamma; F^{\alpha^{\perp\perp}})$ is a term, then the following is a term.

$$(Z^\Gamma; (\tau T_{\alpha^{\perp\perp}})^\alpha)$$

($\exists I$) If $T_{\alpha(x/t)} = (Z^\Gamma; F^{\alpha(x/t)})$ is a term and t is an individual term, then the following is a term.

$$(Z^\Gamma; (t, T_{\alpha(x/t)})^{\exists x\alpha})$$

($\exists E$) If $T_{\exists x\alpha} = (Z^\Gamma; F^{\exists x\alpha})$ and $T_\gamma = (W^\Delta, X^\alpha; G^\gamma)$ are terms, $L^{\exists x\alpha}$ is a term variable, and x is not free in γ nor in any types of term variables of W^Δ, then the following is a term.

$$(Z^\Gamma, W^\Delta; ((\lambda L^{\exists x\alpha}.(\lambda x X^\alpha.T_\gamma)(\pi_1 L^{\exists x\alpha})(\pi_2 L^{\exists x\alpha}))(T_{\exists x\alpha}))^\gamma)$$

The occurrences of $L^{\exists x\alpha}$, the occurrences of X^α in T_γ, and the occurrences of the individual term x in T_γ and α, are bound.

($\forall I$) If $T_\alpha = (Z^\Gamma; F^\alpha)$ is a term and x does not occur free in the type of any term variable of Z^Γ, then the following is a term.

$$(Z^\Gamma; (\lambda x.T_\alpha)^{\forall x\alpha})$$

The occurrences of x in this term, including all occurrences of x in all types of all subterms of T_α, are bound.

($\forall E$) If $T_{\forall x\alpha} = (Z^\Gamma; F^{\forall x\alpha})$ is a term and t is an individual term, then the following is a term.

$$(Z^\Gamma; (T_{\forall x\alpha}(t))^{\alpha(x/t)})$$

($!I$) If $T_{!\alpha_i} = (Z_i^{\Gamma_i}; F_i^{!\alpha_i})$ are terms, for $i = 1, \ldots, n$, and $T_\beta = (X_1^{!\alpha_1}, \ldots, X_n^{!\alpha_n}; G^\beta)$ is a term, then the following is a term.

$$(Z_1^{\Gamma_1}, \ldots, Z_n^{\Gamma_n}; ((\lambda X_1^{!\alpha_1} \ldots X_n^{!\alpha_n}.T_\beta), T_{!\alpha_1}, \ldots, T_{!\alpha_n})^{!\beta})$$

Note the comma before $T_{!\alpha_1}$.

($!E$) If $T_{!\alpha} = (Z^\Gamma; F^{!\alpha})$ is a term, then the following is a term.

$$(Z^\Gamma; (\iota T_{!\alpha})^\alpha)$$

(W) If $T_{!\alpha} = (Z^\Gamma; F^{!\alpha})$ and $S_\gamma = (W^\Delta; G^\gamma)$ are terms, then the following is a term.

$$(Z^\Gamma, W^\Delta; (\omega(T_{!\alpha}, S_\gamma))^\gamma)$$

(C) If $T_{!\alpha} = (Z^\Gamma; F^{!\alpha})$ and $T_\gamma = (W^\Delta, X^{!\alpha}, Y^{!\alpha}; G^\gamma)$ are terms, then the following is a term, which has the occurrences of the term variables $X^{!\alpha}$ and $Y^{!\alpha}$ bound in T_γ.

$$(Z^\Gamma, W^\Delta; ((\kappa X^{!\alpha} Y^{!\alpha}.T_\gamma)(T_{!\alpha}))^\gamma)$$

Note 1. We are assuming throughout that the vectors such as Z^Γ, W^Δ or W^Δ, Y^β have no term variables in common by our premiss convention.

Note 2. When we say “x is bound” we mean “bound at the obvious occurrence(s) by the newly introduced operator”. For a more explicit description see Crossley–Shepherdson [4].

Here is an example of a proof and the associated Curry–Howard terms:

$$\dfrac{\beta\oplus\gamma\vdash\beta\oplus\gamma \qquad \dfrac{\dfrac{\alpha\vdash\alpha \quad \beta\vdash\beta}{\alpha,\beta\vdash\alpha\otimes\beta}}{\alpha,\beta\vdash\alpha\otimes\beta\oplus\alpha\otimes\gamma} \qquad \dfrac{\dfrac{\alpha\vdash\alpha \quad \gamma\vdash\gamma}{\alpha,\gamma\vdash\alpha\otimes\gamma}}{\alpha,\gamma\vdash\alpha\otimes\beta\oplus\alpha\otimes\gamma}}{\alpha,\beta\oplus\gamma\vdash\alpha\otimes\beta\oplus\alpha\otimes\gamma}$$

The associated Curry–Howard terms are as follows:

$$\dfrac{(V^{\beta\oplus\gamma};V^{\beta\oplus\gamma}) \qquad \dfrac{\dfrac{(X^\alpha;X^\alpha) \quad (Y^\beta;Y^\beta)}{(X^\alpha,Y^\beta;(X^\alpha,Y^\beta)^{\alpha\otimes\beta})}}{(X^\alpha,Y^\beta;(\pi_1,(X^\alpha,Y^\beta;(X^\alpha,Y^\beta)^{\alpha\otimes\beta}))^{\alpha\otimes\beta\oplus\alpha\otimes\gamma})} \qquad \dfrac{\dfrac{(X^\alpha;X^\alpha) \quad (W^\gamma;W^\gamma)}{(X^\alpha,W^\gamma;(X^\alpha,W^\gamma)^{\alpha\otimes\gamma})}}{(X^\alpha,W^\gamma;(\pi_2,(X^\alpha,W^\gamma;(X^\alpha,W^\gamma)^{\alpha\otimes\gamma}))^{\alpha\otimes\beta\oplus\alpha\otimes\gamma})}}{\begin{array}{l}(V^{\beta\oplus\gamma},X^\alpha;(\ (\lambda L^{\beta\oplus\gamma}.(\pi_1 L^{\beta\oplus\gamma})((\lambda Y^\beta.(X^\alpha,Y^\beta;(\pi_1,(X^\alpha,Y^\beta;(X^\alpha,Y^\beta)^{\alpha\otimes\beta}))^{\alpha\otimes\beta\oplus\alpha\otimes\gamma})),\\ \quad (\lambda W^\gamma.(X^\alpha,W^\gamma;(\pi_2,(X^\alpha,W^\gamma;(X^\alpha,W^\gamma)^{\alpha\otimes\gamma}))^{\alpha\otimes\beta\oplus\alpha\otimes\gamma})))(\pi_2 L^{\beta\oplus\gamma}))((V^{\beta\oplus\gamma};V^{\beta\oplus\gamma})))^{\alpha\otimes\beta\oplus\alpha\otimes\gamma})\end{array}}$$

Theorem 3 *a) For every proof of a sequent $\Gamma \vdash \beta$ in the natural deduction system for linear logic we can construct a Curry–Howard term $(Z^\Gamma; F^\beta)$ unique (up to renaming of variables and reordering of premisses).*
b) Given a Curry–Howard term $(Z^\Gamma; F^\beta)$ we can construct from it a unique proof of the sequent $\Gamma \vdash \beta$ in the natural deduction system for linear logic.

Proof: a) This follows easily by induction on the complexity of a proof of a sequent $\Gamma \vdash \beta$. For an axiom there is a unique (up to renaming of term variables) Curry–Howard term, and given a rule there is a unique way to construct the Curry–Howard term for the bottom sequent from the Curry–Howard term(s) for the top sequent(s).

b) This again follows easily by induction on the complexity of the given Curry Howard term. In general the type of T can be obtained by reading off the superscripts of the declared premisses and of the conclusion term of T. It is then just a matter of checking whether the conclusion term contains sufficient information to recover the previous sequent or sequents (if any). More precisely we proceed as follows. Let T be the given Curry–Howard term.

Base Case (Axioms): If $T = (X^\alpha; X^\alpha)$, then T codes a sequent of type $\alpha \vdash \alpha$ which is a (unique) proof. If $T = (Z^\Gamma, A^\top)$, then T codes a sequent of type $\Gamma \vdash \top$ which is a (unique) proof. If $T = (; A^1)$, then T codes a sequent of type $\vdash 1$ which is a proof unique up to the order of the premisses.

Induction Step (Rules): It suffices to check whether $(Z^\Gamma; F^\alpha)$ contains sufficient information to uniquely recover the (Curry–Howard) term(s) T' that gave rise to T , i.e. we need to be able to recover which was the last rule of inference and what were the formulae involved. We do this for a couple of rules and leave the rest of the cases to the reader.

($\otimes I$) If $T = (Z^\Gamma, W^\Delta; (T_1, T_2)^{\alpha\otimes\beta})$, where $T_1 = (Z^\Gamma; F^\alpha)$ and $T_2 = (W^\Delta; G^\beta)$ are Curry–Howard terms, then Z^Γ and W^Δ are uniquely determined by T_1, T_2, respectively and contain no common term variable. By the induction hypothesis T_1, T_2 code up unique proofs P_1, P_2 of $\Gamma \vdash \alpha$ and $\Delta \vdash \beta$, respectively. Hence

$$\frac{P_1 \qquad P_2}{\Gamma, \Delta \vdash \alpha \otimes \beta}$$

is the unique proof required.

($\otimes E$) If

$$T = (Z^\Gamma, W^\Delta; ((\lambda L^{\alpha\otimes\beta}.(\lambda X^\alpha Y^\beta.T_2)(\pi_1 L^{\alpha\otimes\beta})(\pi_2 L^{\alpha\otimes\beta}))(T_1))^\gamma),$$

where $T_1 = (Z^\Gamma; F^{\alpha\otimes\beta})$ and $T_2 = (W^\Delta, X^\alpha, Y^\beta; G^\gamma)$ are Curry–Howard terms, then Z^Γ and W^Δ are uniquely determined and moreover X^α, Y^β do not occur in Z^Γ nor in W^Δ.

By the induction hypothesis T_1, T_2 code up unique proofs P_1, P_2 of $\Gamma \vdash \alpha \otimes \beta$ and $\Delta, \alpha, \beta \vdash \gamma$, respectively. Hence

$$\frac{P_1 \quad P_2}{\Gamma, \Delta \vdash \gamma}$$

is the unique proof required.

□

4.3 Equivalence

When performing a substitution we must ensure that free individual variables do not become bound. Following Crossley–Shepherdson [4] we introduce the concepts of equivalent first order formula, equivalent Curry–Howard types, and equivalent Curry–Howard terms.

Let α be a first order formula, x an individual variable, and t an individual term. Then, as usual, we say *t is free for x in α* if no free occurrence of x in α is within the scope of a quantifier $\forall y$ or $\exists y$ for any variable y which occurs in t. If $\forall x\beta$ or $\exists x\beta$ is a sub-formula of α, then a change of x to y in the quantifier, and at free occurrences of x within β, is called *legitimate* if y does not occur free in β and y is free for x in β. If a formula α' can be obtained from α by a sequence of zero or more legitimate changes of bound variables we say α' is *equivalent* to α, write $\alpha \equiv \alpha'$, and denote the class of first order formulae equivalent to α by $[\alpha]$. We say term vectors X^{α} and $Y^{\alpha'}$ are *equivalent* if $\alpha \equiv \alpha'$. We say the term vectors $X^{\gamma_1}, \ldots, X^{\gamma_n}$ and $Y^{\gamma'_1}, \ldots, Y^{\gamma'_n}$ are *equivalent* if each $\gamma_i \equiv \gamma'_i$; and we say the sequents $\Gamma \vdash \alpha$ and $\Gamma' \vdash \alpha'$ are *equivalent*, where $\Gamma = \gamma_1, \ldots, \gamma_n$ and $\Gamma' = \gamma'_1, \ldots, \gamma'_n$, if each $\gamma'_i \equiv \gamma_i$ and $\alpha' \equiv \alpha$.

Now, suppose that x is a bound individual variable in a Curry–Howard term, C, that y does not occur in C, and that, as a result of a replacement, every bound occurrence of x in C is changed to y, and the type of the term variable, X^{α} in C, was changed, to say β, when x was changed to y. We now choose a term variable of type β, say Y^{β}, which does not occur in C and change every occurrence of X^{α} in C to Y^{β}. It then follows from the rules of Curry–Howard term formation that the resulting expression is a Curry–Howard term, say C', which has the same basic structure as C. We call a change from C to C', as described, a *legitimate change of a bound individual variable* in C. Next, suppose that X^{α} is a bound term variable of type α in C, and that Y^{α} is term variable of type α which does not occur in C. Then we say that a change of every occurrence of X^{α}, in C, to Y^{α} is a *legitimate change of a bound term variable* in C. Then, if a Curry–Howard term C' can be obtained from C by a sequence of zero or more changes, each of which is either a legitimate change of bound individual variable or a legitimate change of a bound term variable, we say C' is *equivalent* to C.

4.4 Substitution

In this section we successively define substitution for first order formulae, substitution for term vectors, and substitution for Curry–Howard terms. Finally we show that the collection of Curry–Howard terms is closed under substitution. We refer the reader to Crossley–Shepherdson [4] for further clarification.

Let α be a first order formula, $x = x_1, \ldots, x_n$ be a list of distinct individual variables, and $t = t_1, \ldots, t_n$ be a list of individual terms. We let $\alpha(x_1/t_1, \ldots, x_n/t_n)$ (or $\alpha(x/t)$ in vector notation) denote the result of simultaneous substitution of $t_1, \ldots, t_n$ for all free occurences of $x_1, \ldots, x_n$, respectively, in α, after changing bound variables in α so that if a variable y occurs in a quantifier $\forall y$ or $\exists y$ whose scope includes a free occurrence of some x_i, then y is distinct from all the variables in t_i. We now define $\alpha(x/t)$ inductively as follows:

0. If α is $\top$ or $\bot$, then $\alpha(x/t)$ is $\top$ or $\bot$ respectively.

1. If α is atomic, then $\alpha(x/t)$ denotes the simultaneous replacement of each occurrence of x_i in α by t_i.

2. If $\alpha = \beta \star \gamma$, where $\star$ is one of the connectives $\oplus, \otimes, \&, \multimap$, then $\alpha(x/t) = \beta(x/t) \star \gamma(x/t)$.

3. If $\alpha = !\beta$, then $\alpha(x/t) = !\beta(x/t)$.

4. If $\alpha = \forall y\beta$, then $\alpha(x/t) = \forall z(\beta(y/z)(x^*/t^*))$, where x^* is the list of those x_i in x which occur free in α, t^* is the corresponding sublist of t, and z is any variable which is not free in α, is free for y in β and is not in t^*. Similarly for $\alpha = \exists y\beta$.

We now extend this notation of substitution. We use the same notation (x/t) but we are in fact introducing new definitions since hitherto we have only applied (x/t) to a formula.

Because we have a restriction in the formation of Curry–Howard terms that each premiss be represented by a distinct term variable (so that two occurrences of a premiss α will be represented by distinct term variables, say X^α, Y^α) we have to ensure that this condition is preserved when we make substitutions. For example, we could clearly make substitutions in certain formulae $\alpha(x)$ and $\alpha(y)$ which would make them coincide. In this case a naive substitution would make the term variables $X^{\alpha(x)}$ and $X^{\alpha(y)}$ become identical. To avoid this, and yet guarantee that every time we substitute, say, t for x, in a term variable, X^α, we substitute the same term variable of type $\alpha(x/t)$ for X^α, we introduce a new strategy.

Let TV be a list of tuples (X^α, x, t) for all term variables X^α, all lists of individual variables $x = x_1, \ldots, x_n$, and all list of individual terms $t = t_1, \ldots, t_n$. We associate a unique term variable of type $\alpha(x/t)$ with each such tuple. Given (X^α, x, t) we denote the associated unique term variable by $X^\alpha(x/t)$. The term variable $X^\alpha(x/t)$ has the following properties:

1. If x^* is a sublist of x consisting of those x_i which are free in α, and t^* is the corresponding sublist of t, then $X^\alpha(x/t) = X^\alpha(x^*/t^*)$. In particular, if each x_i is not free in α, then $X^\alpha(x/t) = X^\alpha$.

2. If every x_i and y_i is free in α, then $X^\alpha(x/t) = Y^\beta(y/s)$ if, and only if, $X^\alpha = Y^\beta$, $x = y$, and $t = s$ (where here $=$ denotes syntactic identity).

Let $Z^\Gamma = Y_1^{\gamma_1}, \ldots, Y_n^{\gamma_n}$ be a vector of term variables, let x be a list of individual variables, and let t be a list of individual terms. Then $Z^\Gamma(x/t) = Y_1^{\gamma_1}(x/t), \ldots, Y_n^{\gamma_n}(x/t)$. Note that since all the term variables in Z^Γ are distinct so are all those in $Z^\Gamma(x/t)$.

Let $W = W_1^{\Delta_1}, \ldots, W_m^{\Delta_m}$ be a list of term vectors which do not contain any term variable which also occurs in Z^Γ, and let $X = X_1^{\alpha_1}, \ldots, X_m^{\alpha_m}$ be a list of term variables. Then we define $Z^\Gamma(X_1/W_1, \ldots, X_n/W_n)$ (or $Z^\Gamma(X/W)$ in vector notation) to denote the result of simultaneously substituting $W_1^{\Delta_1}, \ldots, W_m^{\Delta_m}$ for all occurrences of $X_1^{\alpha_1}, \ldots, X_m^{\alpha_m}$ in Z^Γ.

Let C be a Curry–Howard term, let $x = x_1, \ldots, x_n$ be a list of individual variables, let $t = t_1, \ldots, t_n$ be a list of individual terms, let $X = X_1^{\alpha_1}, \ldots, X_n^{\alpha_n}$ be a list of term variables, and let $T = T_{\alpha_1}, \ldots, T_{\alpha_n}$ be a list of Curry–Howard terms, where $T_{\alpha_k} = (W_k^{\Delta_k}; G_k^{\alpha_k})$. Then $C(x_1/t_1, \ldots, x_n/t_n)$ and $C(X_1^{\alpha_1}/T_{\alpha_1}, \ldots, X_n^{\alpha_n}/T_{\alpha_n})$ (or $C(x/t)$ and $C(X/T)$ in vector notation) will denote,respectively, the simultaneous substitution of $t_1, \ldots, t_n$ for $x_1, \ldots, x_n$ in C and the simultaneous substitution of $T_{\alpha_1}, \ldots, T_{\alpha_n}$ for $X_1^{\alpha_1}, \ldots, X_n^{\alpha_n}$ in C. We will define $C(x/t)$ to be $C'(x^*/t^*)$, where x^* is the sublist of x of all the x_k which occur free in C, t^* is the corresponding sublist of t, and C' is an equivalent Curry–Howard term to C, obtained after completing legitimate changes in bound individual variables in C so that if an individual variable y is bound in C' then it is distinct from all the variables in t.

Let X^* be the sublist of X of all the $X_k^{\alpha_k}$ which occur free in C, and let T^* be the corresponding sublist of T. Then we define $C(X/T)$ to be $C'(Y^*/T^*)$, where C' is an equivalent Curry–Howard term to C, obtained after completing legitimate changes in bound term variables so that every bound term variable in C' is distinct from the term variables in T, and Y^* is a list of term variables, which the term variables in X^* changed to when C changed to C'. We can now assume each x_k is free in C and each $X_k^{\alpha_k}$ is free in C, and define $C(x/t)$ and $C(X/T)$ inductively as follows.

Recall $T = T_{\alpha_1}, \ldots, T_{\alpha_n}$ is a list of Curry–Howard terms, where $T_{\alpha_k} = (W_k^{\Delta_k}; G_k^{\alpha_k})$. Let W denote the list $W_1^{\Delta_1}, \ldots, W_n^{\Delta_n}$ of term vectors. If Y^β and V^β are term variables, let (Y/V) be an abbreviation for $(Y/(V;V))$.

(α) Suppose $C = (X^\alpha; X^\alpha)$, $X = X^\alpha$ and $T = T_\alpha$. Then

$$C(x/t) = (X^\alpha(x/t); X^\alpha(x/t)),$$

and

$$C(X/T) = T_\alpha.$$

($\top$) Suppose $C = (Z^\Gamma; A^\top)$, where Z^Γ is a term vector. Then $C(x/t) = (Z^\Gamma(x/t); A^\top)$ and $C(X/T) = (Z^\Gamma(X/W); A^\top)$.

(1) Suppose $C = (; A^1)$. Then $C(x/t) = C$ and $C(X/T) = C$.

($\otimes I$) Suppose $C = (Z^\Gamma; (S_1, S_2)^{\gamma_1 \otimes \gamma_2})$, where Z^Γ is a term vector, and S_1 and S_2 are Curry–Howard terms. Then

$$C(x/t) = (Z^\Gamma(x/t); (S_1(x/t), S_2(x/t))^{(\gamma_1 \otimes \gamma)(x/t)})$$

and

$$C(X/T) = (Z^\Gamma(X/W); (S_1(X/T), S_2(X/T))^{\gamma_1 \otimes \gamma_2}).$$

($\otimes E$) Suppose

$$C = (Z^\Gamma; ((\lambda L^{\gamma_1 \otimes \gamma_2}.(\lambda Y_1^{\gamma_1} Y_2^{\gamma_2}.S)(\pi_1 L^{\gamma_1 \otimes \gamma_2})(\pi_2 L^{\gamma_1 \otimes \gamma_2}))(R)^\delta),$$

where Z^Γ is a term vector; $Y_1^{\gamma_1}$, $Y_2^{\gamma_2}$ and $L^{\gamma_1 \otimes \gamma_2}$ are term variables; and S and R are Curry–Howard terms. Then

$$C(x/t) = (Z^\Gamma(x/t); ((\lambda L'.(\lambda Y_1' Y_2'.S(x/t))(\pi_1 L')(\pi_2 L'))(R(x/t))^{\delta(x/t)}),$$

where L', Y_1' and Y_2' are, respectively, $L^{\gamma_1 \otimes \gamma_2}(x/t)$, $Y_1^{\gamma_1}(x/t)$, and $Y_2^{\gamma_2}(x/t)$; and

$$C(X/T) = (Z^\Gamma(X/W); ((\lambda L^{\gamma_1 \otimes \gamma_2}.(\lambda V_1^{\gamma_1} V_2^{\gamma_2}.S')(\pi_1 L^{\gamma_1 \otimes \gamma_2})(\pi_2 L^{\gamma_1 \otimes \gamma_2}))(R(X/T))))^\delta),$$

where S' is $S(Y_1/V_1)(Y_2/V_2)(X/T)$ and V_1 and V_2 are term variables of type γ_1 and γ_2, respectively, which are not free in C, are not bound in S, and are not in T.

($\&I$) Suppose $C = (Z^\Gamma; (S_1, S_2)^{\gamma_1 \& \gamma_2})$, where Z^Γ is a term vector, and S_1 and S_2 are Curry–Howard terms. Then

$$C(x/t) = (Z^\Gamma(x/t); (S_1(x/t), S_2(x/t))^{(\gamma_1 \& \gamma)(x/t)})$$

and

$$C(X/T) = (Z^\Gamma(X/W); (S_1(X/T), S_2(X/T))^{\gamma_1 \& \gamma_2}).$$

($\&E$) Suppose $C = (Z^\Gamma; (\pi_1 S)^\gamma)$, where Z^Γ is a term vector, and S is a Curry–Howard term. Then

$$C(x/t) = (Z^\Gamma(x/t); (\pi_1 S(x/t))^{\gamma(x/t)}) \quad \text{and} \quad C(X/T) = (Z^\Gamma(X/W); (\pi_1 S(X/T))^\gamma).$$

Similarly, if $C = (Z^\Gamma; (\pi_2 S)^\gamma)$ then

$$C(x/t) = (Z^\Gamma(x/t); (\pi_2 S(x/t))^{\gamma(x/t)}) \quad \text{and} \quad C(X/T) = (Z^\Gamma(X/W); (\pi_2 S(X/T))^\gamma).$$

($\oplus I$) Suppose $C = (Z^\Gamma; (\pi_1, S)^{\gamma_1 \oplus \gamma_2})$, where Z^Γ is a term vector, and S is a Curry–Howard term. Then

$$C(x/t) = (Z^\Gamma(x/t); (\pi_1, S(x/t))^{(\gamma_1 \oplus \gamma_2)(x/t)})$$

and

$$C(X/T) = (Z^\Gamma(X/W); (\pi_1, S(X/T))^{\gamma_1 \oplus \gamma_2}).$$

Similarly, if $C = (Z^\Gamma; (\pi_2, S)^{\gamma_1 \oplus \gamma_2})$ then

$$C(x/t) = (Z^\Gamma(x/t); (\pi_2, S(x/t))^{(\gamma_1 \oplus \gamma_2)(x/t)})$$

and

$$C(X/T) = (Z^\Gamma(X/W); (\pi_2, S(X/T))^{\gamma_1 \oplus \gamma_2}).$$

($\oplus E$) Suppose

$$C = (Z^\Gamma; ((\lambda L^{\gamma_1 \oplus \gamma_2}.(\pi_1 L^{\gamma_1 \oplus \gamma_2})((\lambda Y_1^{\gamma_1}.S_1), (\lambda Y_2^{\gamma_2}.S_2))(\pi_2 L^{\gamma_1 \oplus \gamma_2}))(S_3))^\delta),$$

where Z^Γ is a term vector; Y_1, Y_2 and $L^{\gamma_1 \oplus \gamma_2}$ are term variables; and S_1, S_2, and S_3 are Curry–Howard terms. Then

$$C(x/t) = (Z^\Gamma(x/t); ((\lambda L'.(\pi_1 L')((\lambda Y_1'.S_1(x/t)), (\lambda Y_2'.S_2(x/t)))(\pi_2 L'))(S_3(x/t)))^{\delta(x/t)}),$$

where L', Y_1' and Y_2' are, respectively, $L^{\gamma_1 \oplus \gamma_2}(x/t)$, $Y_1^{\gamma_1}(x/t)$, and $Y_2^{\gamma_2}(x/t)$; and

$$C(X/T) = (Z^\Gamma(X/W); ((\lambda L^{\gamma_1 \oplus \gamma_2}.(\pi_1 L^{\gamma_1 \oplus \gamma_2})((\lambda V_1.S_1'), (\lambda V_2.S_2'))(\pi_2 L^{\gamma_1 \oplus \gamma_2}))(S_3'))^\delta),$$

where S_1',S_2' and S_3' are, respectively, $S_1(Y_1/V_1)(X/T)$, $S_2(Y_2/V_2)(X/T)$ and $S_3(X/T)$; and V_1 and V_2 are term variables of type γ_1 and γ_2, respectively, which are not free in C, are not bound in S_1 and S_2, respectively, and are not in T.

($\perp\!\circ I$) Suppose $C = (Z^\Gamma; (\lambda Y^{\gamma_1}.S)^{\gamma_1 \perp\!\circ \gamma_2})$, where Z^Γ is term vector, Y^{γ_1} is term variable, and S is a Curry–Howard term. Then

$$C(x/t) = (Z^\Gamma(x/t); (\lambda Y'.S(x/t))^{(\gamma_1 \perp\!\circ \gamma_2)(x/t)}),$$

where Y' is $Y^{\gamma_1}(x/t)$; and

$$C(X/T) = (Z^\Gamma(X/W); (\lambda V.(S(Y/V)(X/T)))^{\gamma_1 \perp\!\circ \gamma_2}),$$

where V is a term variable of type γ_1 which is not free in C, is not bound in S, and is not in T.

($\perp\!\circ E$) Suppose $C = (Z^\Gamma; (S_1(S_2))^\delta)$, where Z^Γ is a term vector; and S_1 and S_2 are Curry–Howard terms. Then

$$C(x/t) = (Z^\Gamma(x/t); (S_1'(S_2(x/t)))^{\delta(x/t)}),$$

where $S_1' = S_1(x/t)$.

$$C(X/T) = (Z^\Gamma(X/W); (S_1''(S_2(X/T)))^\delta),$$

where $S_1'' = S_1(X/T)$.

($\perp$) Suppose $C = (Z^\Gamma; (\tau S)^\gamma)$, where Z^Γ is term vector, and S is a Curry–Howard term. Then

$$C(x/t) = (Z^\Gamma(x/t); (\tau(S(x/t)))^{\gamma(x/t)}) \quad \text{and} \quad C(X/T) = (Z^\Gamma(X/W); (\tau(S(X/T)))^\gamma).$$

($\exists I$) Suppose $C = (Z^\Gamma; (s, S)^{\exists y \gamma})$, where Z^Γ is a term vector, s is an individual term, and S is a Curry–Howard term. Then

$$C(x/t) = (Z^\Gamma(x/t); (s(x/t), S(x/t))^{(\exists y \gamma)(x/t)}),$$

and

$$C(X/T) = (Z^\Gamma(X/W); (s, S(X/T))^{\exists y \gamma}).$$

($\exists E$) Suppose

$$C = (Z^\Gamma; ((\lambda L^{\exists y\gamma}.(\lambda y Y^\gamma.S_1)(\pi_1 L^{\exists y\gamma})(\pi_2 L^{\exists y\gamma}))(S_2))^\delta),$$

where Z^Γ is a term vector; y is an individual variable; Y^γ is a term variable; and S_1 and S_2 are Curry–Howard terms. Then

$$C(x/t) = (Z^\Gamma(x/t); ((\lambda L'.(\lambda v Y'.S_1')(\pi_1 L')(\pi_2 L'))(S_2'))^{\delta(x/t)}),$$

where L', Y', S_1', and S_2' are, respectively, $L^{\exists y\gamma}(y/v)(x/t)$, $Y^\gamma(y/v)(x/t)$, $S_1(y/v)(x/t)$, and $S_2(y/v)(x/t)$; and v is an individual variable which is not free in C, is free for y in S_1, and is not in t.

$$C(X/T) = (Z^\Gamma(X/W); ((\lambda L^{\exists y\gamma}.(\lambda y V^\gamma.(S_1')(\pi_1 L^{\exists y\gamma})(\pi_2 L^{\exists y\gamma})))(S_2(X/T)))^\delta),$$

where S_1' is $S_1(Y/V)(X/T)$; and V is a term variable of type γ which is not free in C, is not bound in S_1, and is not in T.

($\forall I$) Suppose $C = (Z^\Gamma; (\lambda y.S)^{\forall y\gamma})$, where Z^Γ is a term vector, y is an individual variable, and S is a Curry–Howard term. Then

$$C(x/t) = (Z^\Gamma(x/t); (\lambda v.S(y/v)(x/t))^{\forall v(\gamma(y/v)(x/t))}),$$

where v is an individual variable which is not free in C, is free for y in S, and is not in T, and

$$C(X/T) = (Z^\Gamma(X/W); (\lambda y.(S(X/T)))^{(\forall y\gamma)}).$$

($!I$) Suppose

$$C = (Z^\Gamma; ((\lambda Y_1^{\gamma_1} \cdots Y_n^{\gamma_n}.S_0), S_1, \ldots, S_n)^{!\gamma}),$$

where Z^Γ is a term vector; $Y_1, \ldots, Y_n$ are term variables; and $S_0, \ldots, S_n$ are Curry–Howard terms. Then

$$C(x/t) = (Z^\Gamma(x/t); ((\lambda Y_1' \cdots Y_n'.S_0(x/t)), S_1(x/t), \ldots, S_n(x/t))^{!\gamma(x/t)}),$$

where each Y_k' is $Y_k(x/t)$; and

$$C(X/T) = (Z^\Gamma(X/W); ((\lambda V_1 \cdots V_n.(S_0'), S_1(X/T), \ldots, S_n(X/T)))^{!\gamma}),$$

where S_0' is $S_0(Y_1/V_1) \cdots (Y_n/V_n)(X/T)$; and $V_1, \ldots, V_n$ are term variables of type $\gamma_1, \ldots, \gamma_n$, respectively, and are not free in C, not bound in S_0, and are not in T.

($!E$) Suppose $C = (Z^\Gamma; (\iota S)^\delta)$, where Z^Γ is a term vector and S is a Curry–Howard term. Then

$$C(x/t) = (Z^\Gamma(x/t); (\iota S(x/t))^{\delta(x/t)}) \quad \text{and} \quad C(X/T) = (Z^\Gamma(X/W); (\iota S(X/T))^\delta).$$

(W) Suppose $C = (Z^\Gamma; (\omega(S,R))^\delta)$, where Z^Γ is a term vector, and S and R are Curry–Howard terms. Then

$$C(x/t) = (Z^\Gamma(x/t); (\omega(S(x/t), R(x/t)))^{\delta(x/t)}),$$

and

$$C(X/T) = (Z^\Gamma(X/W); (\omega(S(X/T), R(X/T)))^\delta).$$

(C) Suppose $C = (Z^\Gamma; ((\kappa Y_1^{!\gamma} Y_2^{!\gamma}.S_1)(S_2))^\delta)$, where Z^Γ is a term vector; Y_1 and Y_2 are term variables; and S_1 and S_2 are Curry–Howard terms. Then

$$C(x/t) = (Z^\Gamma(x/t); (\kappa Y_1' Y_2'.S_1(x/t))(S_2(x/t)))^{\delta(x/t)}),$$

where Y_1' and Y_2' are, respectively, $Y_1(x/t)$ and $Y_2(x/t)$; and

$$C(X/T) = (Z^\Gamma(X/W); ((\kappa V_1^{!\gamma_1} V_2^{!\gamma_2}.(S_1(Y_1/V_1)(Y_2/V_2)))(S_2))^\delta),$$

where V_1 and V_2 are term variables of type $!\gamma_1$ and $!\gamma_2$, respectively, which are not free in C, are not bound in S_1, and are not in T.

Lemma 3 *If C is a Curry–Howard term of type $\Gamma \vdash \gamma$ then, $C(x/t)$ and $C(X/T)$ are Curry–Howard terms whose types are equivalent, respectively, to $\Gamma(x/t) \vdash \gamma(x/t)$ and $\Gamma' \vdash \gamma$ (where Γ' is the sequence of types in the term vector $Z^\Gamma(X/W)$).*

Proof: Follows immediately from the definitions above, by induction on the structure of C.

□

4.5 The Reduction Rules

As we mentioned earlier, a Curry–Howard term is just another notation for a (given) proof of a theorem, with the property that we can fully recover the original proof from the Curry–Howard term. What makes this particular notation nice and very useful is its algorithmic content, i.e. Curry–Howard terms are given in a lambda-calculus-style notation. This makes it very easy to remove unnecessary portions of the given proof, namely those portions where an introduction of a connective *via* one of the introduction rules of a given type is immediately followed by the elimination of that connective *via* the elimination rule of the same type (with a slight difference in the case of the $\perp$-Reduction rule). Such a portion can be removed from the proof without any effect on the validity of the given proof, provided we make the necessary adjustments to the premisses. The lambda notation allows this to be handled very neatly, and to make this precise we shall give the reduction rules for our Curry–Howard terms. Of course such reduced Curry–Howard terms will not contain sufficient information to fully recover the original proof, but they will nonetheless encode a valid proof of the same theorem, i.e. we can construct from them a valid proof. We say that a term E is *reducible* to a term F, denoted by $E \succ F$, if F can be obtained from E by a finite sequence of applications of one of the reduction rules given below. If it takes only one application of a reduction rule to obtain F from E we say that F is *immediately reducible* to E and denote this with $E \succ^i F$. The term reductions are based on the following simple operations, where T, S, and R (possibly with subscripts) are Curry–Howard terms and Z^{Γ} and W^{Δ} are vectors of term variables.

$$
\begin{array}{ll}
(Z^{\Gamma}; \pi_1(T,S)) & \succ T \\
(Z^{\Gamma}; \pi_2(T,S)) & \succ S \\
(Z^{\Gamma}; (\lambda x.T)(t)) & \succ T(x/t) \\
(Z^{\Gamma}; (\lambda X.T)(S))) & \succ T(X/S) \\
(Z^{\Gamma}; (\iota(Z^{\Gamma}; (\lambda X_1 \ldots X_n.T), S_1, \ldots, S_n))) & \succ T(X_1/S_1)\cdots(X_n/S_n) \\
(Z^{\Gamma}, W^{\Delta}; (\kappa XY.(X,Y,W^{\Delta}; (\omega(R,T)))(S))) & \succ T(X/R(Y/S))
\end{array}
$$

We omit type superscripts where no confusion is possible. Moreover, in order to increase legibility we shall often denote a term by an upper case letter with a type *subscript* indicating the type of the *conclusion* of the last applied rule.

Note: The reader can reconstruct the changes in proofs which correspond to these reductions (using, in fact, Theorem 3).

($\otimes$ Intro, $\otimes$ Elim) If $T_{\alpha} = (Z^{\Gamma}; F^{\alpha})$, $T_{\beta} = (W^{\Delta}; G^{\beta})$, $T_{\alpha\otimes\beta} = (Z^{\Gamma}, W^{\Delta}; (T_{\alpha}, T_{\beta})^{\alpha\otimes\beta})$, and $T_{\gamma} = (U^{\Theta}, X^{\alpha}, Y^{\beta}; H^{\gamma})$ are terms, then

$$(Z^{\Gamma}, W^{\Delta}, U^{\Theta}; ((\lambda L^{\alpha\otimes\beta}.(\lambda X^{\alpha} Y^{\beta}.T_{\gamma})(\pi_1 L^{\alpha\otimes\beta})(\pi_2 L^{\alpha\otimes\beta}))(T_{\alpha\otimes\beta}))^{\gamma}) \succ^{i} S_{\gamma},$$

where $S_{\gamma} = (Z^{\Gamma}, W^{\Delta}, U^{\Theta}; K^{\gamma})$ is equivalent to $T_{\gamma}(X^{\alpha}/T_{\alpha})(Y^{\beta}/T_{\beta})$.

(& Intro, & Elim) If $T_{\alpha} = (Z^{\Gamma}; F^{\alpha})$, $T_{\beta} = (Z^{\Gamma}; G^{\beta})$, and $T_{\alpha\&\beta} = (Z^{\Gamma}; (T_{\alpha}, T_{\beta})^{\alpha\&\beta})$ are terms, then

$$
\begin{array}{l}
(Z^{\Gamma}; (\pi_1 T_{\alpha\&\beta})^{\alpha}) \succ^{i} T_{\alpha} \\
(Z^{\Gamma}; (\pi_2 T_{\alpha\&\beta})^{\beta}) \succ^{i} T_{\beta}
\end{array}
$$

($\oplus$ Intro, $\oplus$ Elim) If $T_{\alpha} = (Z^{\Gamma}; F^{\alpha})$, $T_{\alpha\oplus\beta} = (Z^{\Gamma}; (\pi_1, T_{\alpha})^{\alpha\oplus\beta})$, $T_{\gamma} = (W^{\Delta}, X^{\alpha}; G^{\gamma})$ and $S_{\gamma} = (W^{\Delta}, Y^{\beta}; H^{\gamma})$ are terms and $L^{\alpha\oplus\beta}$ is a term variable, then

$$(Z^{\Gamma}, W^{\Delta}; ((\lambda L^{\alpha\oplus\beta}.(\pi_1 L^{\alpha\oplus\beta})((\lambda X^{\alpha}.T_{\gamma}), (\lambda Y^{\beta}.S_{\gamma}))(\pi_2 L^{\alpha\oplus\beta}))(T_{\alpha\oplus\beta}))^{\gamma}) \succ^{i} R_{\gamma},$$

where $R_{\gamma} = (Z^{\Gamma}, W^{\Delta}; K^{\gamma})$ is equivalent to $T_{\gamma}(X^{\alpha}/T_{\alpha})$.

Also, if $S_{\beta} = (Z^{\Gamma}; F^{\beta})$, $S_{\alpha\oplus\beta} = (Z^{\Gamma}; (\pi_2, S_{\beta})^{\alpha\oplus\beta})$, $T_{\gamma} = (W^{\Delta}, X^{\alpha}; G^{\gamma})$ and $S_{\gamma} = (W^{\Delta}, Y^{\beta}; H^{\gamma})$ are terms and $L^{\alpha\oplus\beta}$ is a term variable, then

$$(Z^{\Gamma}, W^{\Delta}; ((\lambda L^{\alpha\oplus\beta}.(\pi_1 L^{\alpha\oplus\beta})((\lambda X^{\alpha}.T_{\gamma}), (\lambda Y^{\beta}.S_{\gamma}))(\pi_2 L^{\alpha\oplus\beta}))(S_{\alpha\oplus\beta}))^{\gamma}) \succ^{i} Q_{\gamma},$$

where $Q_{\gamma} = (Z^{\Gamma}, W^{\Delta}; P^{\gamma})$ is equivalent to $S_{\gamma}(Y^{\beta}/S_{\beta})$.

($\bot\!\circ$ Intro, $\bot\!\circ$ Elim) If $T_\beta = (Z^\Gamma, X^\alpha; F^\beta)$, $T_{\alpha\bot\circ\beta} = (Z^\Gamma; (\lambda X^\alpha.T_\beta)^{\alpha\bot\circ\beta})$ and $T_\alpha = (W^\Delta; G^\alpha)$ are terms, then

$$(Z^\Gamma, W^\Delta; (T_{\alpha\bot\circ\beta}(T_\alpha))^\beta) \succ^i S_\beta,$$

where $S_\beta = (Z^\Gamma, W^\Delta; H^\beta)$ is equivalent to $T_\beta(X^\alpha/T_\alpha)$.

($\exists$ Intro, $\exists$ Elim) If $T_{\alpha(x/t)} = (Z^\Gamma; F^{\alpha(x/t)})$, $T_{\exists x\alpha} = (Z^\Gamma; (t, T_{\alpha(x/t)})^{\exists x\alpha})$, and $T_\gamma = (W^\Delta, X^\alpha; G^\gamma)$ are terms, and x is not free in Γ, Δ, or γ, then

$$(Z^\Gamma, W^\Delta; ((\lambda L^{\exists x\alpha}.(\lambda x X^\alpha.T_\gamma)(\pi_1 L^{\exists x\alpha})(\pi_2 L^{\exists x\alpha}))(T_{\exists x\alpha}))^\gamma) \succ^i S_\gamma,$$

where $S_\gamma = (Z^\Gamma, W^\Delta; H^\gamma)$ is equivalent to $T_\gamma(x/t)(X^\alpha(x/t)/T_{\alpha(x/t)})$.

Note: Since x is not free in Γ, Δ, or γ, it follows that $Z^\Gamma(x/t) = Z^\Gamma$, $W^\Delta(x/t) = W^\Delta$, and $T_\gamma(x/t)(X^\alpha(x/t)/T_{\alpha(x/t)})$ has a type equivalent to $\Gamma, \Delta \vdash \gamma$. Therefore, it is always possible to find an $S_\gamma = (Z^\Gamma, W^\Delta; H^\gamma)$ which is equivalent to $T_\gamma(x/t)(X^\alpha(x/t)/T_{\alpha(x/t)})$.

($\forall$ Intro, $\forall$ Elim) If $T_\alpha = (Z^\Gamma; F^\alpha)$, and $T_{\forall x\alpha} = (Z^\Gamma; (\lambda x.T_\alpha)^{\forall x\alpha})$ are terms, and x is not free in Γ, then

$$(Z^\Gamma; (T_{\forall x\alpha}(t))^{\alpha(x/t)}) \succ^i S_{\alpha(x/t)},$$

where $S_{\alpha(x/t)} = (Z^\Gamma; G^{\alpha(x/t)})$ is equivalent to $T_\alpha(x/t)$.

Note: Since x is not free in Γ, it follows that $Z^\Gamma(x/t) = Z^\Gamma$, and $T_\alpha(x/t)$ has a type equivalent to $\Gamma \vdash \alpha(x/t)$. Therefore, it is always possible to find an $S_{\alpha(x/t)} = (Z^\Gamma; G^{\alpha(x/t)})$ which is equivalent to $T_\alpha(x/t)$.

(! Intro, ! Elim) If $T_{!\alpha_i} = (Z_i^{\Gamma_i}; F_i^{!\alpha_i})$, for $i = 1, \ldots, n$, $T_\beta = (X_1^{!\alpha_1}, \ldots, X_n^{!\alpha_n}; G^\beta)$, and $T_{!\beta} = (Z_1^{\Gamma_1}, \ldots, Z_n^{\Gamma_n}; ((\lambda X_1^{!\alpha_1} \ldots X_n^{!\alpha_n}.T_\beta), T_{!\alpha_1}, \ldots, T_{!\alpha_n})^{!\beta})$ are terms, then

$$(Z_1^{\Gamma_1}, \ldots, Z_n^{\Gamma_n}; (\iota T_{!\beta})^\beta) \succ^i S_\beta,$$

where $S_\beta = (Z_1^{\Gamma_1}, \ldots, Z_n^{\Gamma_n}; H^\beta)$ is equivalent to $T_\beta(X_1^{!\alpha_1}/T_{!\alpha_1}) \cdots (X_n^{!\alpha_n}/T_{!\alpha_n})$.

(W Intro, C Elim) If $T_\gamma = (W_1^{\Delta_1}, X^{!\alpha}; G^\gamma)$, $T_{!\alpha} = (W_2^{\Delta_2}, Y^{!\alpha}; F^{!\alpha})$,

$$S_\gamma = (W_1^{\Delta_1}, W_2^{\Delta_2}, X^{!\alpha}, Y^{!\alpha}; (\omega(T_{!\alpha}, T_\gamma))^\gamma),$$

and $S_{!\alpha} = (Z^\Gamma; H^{!\alpha})$ are terms, then

$$(W_1^{\Delta_1}, W_2^{\Delta_2}, Z^\Gamma; ((\kappa X^{!\alpha} Y^{!\alpha}.S_\gamma)(S_{!\alpha}))^\gamma) \succ^i R_\gamma,$$

where $R_\gamma = (W_1^{\Delta_1}, W_2^{\Delta_2}, Z^\Gamma; H^\gamma)$ is equivalent to $T_\gamma(X^{!\alpha}/T_{!\alpha}(Y^{!\alpha}/S_{!\alpha}))$.

Lemma 4 *Suppose S is a Curry–Howard term, $S \succ^i S'$, $x = x_1, \ldots, x_n$ is a list of individual variables, $t = t_1, \ldots, t_n$ is a list of individual terms, $X = X_1^{\alpha_1}, \ldots, X_m^{\alpha_m}$ is a list of term variables, and $T = T_{\alpha_1}, \ldots, T_{\alpha_m}$ is a list of Curry–Howard terms, where $T_{\alpha_k} = (W_k^{\Delta_k}; F_k^{\alpha_k})$. Then S' is a Curry–Howard term, the type of S' equals the type of S, and $S(x/t)(X/T) \succ^i S'(x/t)(X/T)$.*

Proof: Immediate from the defintions of immediate reducts and the definition of $S(x/t)(X/T)$, by induction on the structure of S.

□

5 Strong Normalization

In this section we prove strong normalization for our natural deduction system for linear logic. The normalization of a proof is achieved through applications of reduction rules. As proved earlier an application of a reduction rule sends a Curry–Howard term of type $\Gamma \vdash \alpha$ to another Curry–Howard term whose type is equal to $\Gamma \vdash \alpha$. The effect of this reduction is the removal of an unnecessary pair of rule applications, namely an application of an introduction rule immediately followed by an application of an elimination rule (this is our analogue of cut-elimination). We cannot expect to be able to eliminate all applications of an elimination rule, for instance the ones immediately following an axiom cannot be eliminated. There are two obvious questions related to reductions:

1. Is there an order of applying reductions that will lead to an irreducible Curry–Howard term after a finite number of stepsΓ
2. Is this process finite, i.e. are we guaranteed to end up with an irreducible Curry–Howard term after finitely many reductions Γ

If the answer to question 1 is "Yes", then we have Weak Normalization. If the answer to question 2 (and therefore also to 1) is "Yes" then we have Strong Normalization, and this is what we shall prove. This will be done by extending the technique used by Crossley–Shepherdson in [4]. This technique is, in turn, an adaptation of Girard's proof for his system F, (see [6]), and utilizes the concepts of "candidate for reducibility" (an extension of "R-term" originally due to Tait[12]) and "neutral terms". These concepts together with the stronger induction hypothesis, namely that a term is not only strongly normalizable but is also an element of a candidate for reducibilty (defined below), allow us to prove strong normalization.

5.1 Candidates for Reducibility

First we need to introduce some (standard) terminology. We say a term is *normal* if it is irreducible and that it is *strongly normalizable* if all its reduction sequences are finite.

Let $N(T)$ denote the least upper bound of the lengths of the reduction sequences for a term T. Then it follows that if $N(T)$ is finite for a term T, then T is strongly normalizable. Conversely, from König's lemma it follows that if T is strongly normalizable, then $N(T)$ is finite.

The proof of strong normalization heavily utilizes the following concept of a *neutral* term, which intuitively refers to Curry–Howard terms that correspond to axioms or encode the bottom sequent of an elimination rule.

Definition 2 *A Curry–Howard term is* **neutral** *if it is an axiom , or its conclusion term is of one of the following forms:*

1. $(\lambda L^{\alpha\otimes\beta}.(\lambda X^{\alpha} Y^{\beta}.R)(\pi_1 L^{\alpha\otimes\beta})(\pi_2 L^{\alpha\otimes\beta}))(S)$ *(which corresponds to the* $(\otimes E)$ *rule);*
2. $\pi_1 S$ *and* $\pi_2 S$ *(which correspond to the two forms in the* $(\& E)$ *rule);*

3. $(\lambda L^{\alpha\oplus\beta}.(\pi_1 L^{\alpha\oplus\beta})((\lambda X^\alpha.R),(\lambda Y^\beta.U))(\pi_2 L^{\alpha\oplus\beta}))(S)$ *(which corresponds to the* $(\oplus E)$ *rule);*

4. $S(R)$ *(which corresponds to the* $(\bot\!\circ E)$ *rule);*

5. $(\lambda L^{\exists x\alpha}.(\lambda x X^\alpha.R)(\pi_1 L^{\exists x\alpha})(\pi_2 L^{\exists x\alpha}))(S)$ *(which corresponds to the* $(\exists E)$ *rule);*

6. $S(t)$ *(which corresponds to the* $(\forall E)$ *rule);*

7. ιS *(which corresponds to the* $(!E)$ *rule);*

8. τS *(which corresponds to the* $(\bot)$ *rule);*

9. $(\kappa X^{!\alpha}Y^{!\alpha}.R)(S)$ *(which corresponds to the* (C) *rule);*

where S, R *and* U *are Curry–Howard terms,* $X^{!\alpha}$, $Y^{!\alpha}$, X^α, *and* Y^β *are distinct term variables,* x *is an individual variable and* t *is an individual term.*

One immediate consequence of this definition is the following lemma:

Lemma 5 *With the notation of the definition above: Let* T *be a neutral term and let* S *be the corresponding subterm of* T *as in the corresponding definition above. If* S *is also neutral, then all immediate reducts of* T *are obtained by reducing either* S*, or the corresponding subterm* R *of* T *(if it occurs) or the corresponding subterm* U *of* T *(if it occurs).*

Proof : If a term T is neutral it corresponds to the conclusion of an elimination rule (amongst which we include the (C) rule and the $(\bot)$ rule), say, the rule for * elimination. If the subterm S is also neutral then it too corresponds to the conclusion of an elimination rule. However for the reduction not to be inside S we should have to have an introduction rule for * .

□

Definition 3 *A set of terms,* C_α*, whose conclusion types are equivalent to* α*, will be called a* **candidate for reducibility (CR) of type** $[\alpha]$*, if it satisfies the following conditions:*

CR0: *(closure under equivalence) If* T *is in* C_α *and* T' *is equivalent to* T*, then* T' *is in* C_α*.*

CR1: *If* T *is in* C_α*, then* T *is strongly normalizable.*

CR2: *(closure under* $\succ^i$*) If* T *is in* C_α *and* $T \succ^i T'$*, then* T' *is in* C_α*.*

CR3: *If* T *is neutral and all immediate reducts of* T *are in* C_α*, then* T *is in* C_α*.*

5.2 Operations on CR

The candidates for reducibility will be defined inductively in the style of Crossley–Shepherdson [4].

Suppose C_α and C_β are CR of types $[\alpha]$ and $[\beta]$, respectively. Then define

$C_\alpha \otimes C_\beta$ is the set of terms equivalent to some $T = (Z^\Gamma; F^{\alpha\otimes\beta})$ for some Γ such that, for all types $[\gamma]$, all CR C_γ of type $[\gamma]$, all terms $S = (W^\Delta, X^\alpha, Y^\beta; G^\gamma)$ in C_γ, for which $S(X^\alpha/U_\alpha, Y^\beta/U_\beta)$ is in C_γ for every term U_α in C_α and term U_β in C_β, the term

$$(Z^\Gamma, W^\Delta; ((\lambda L^{\alpha\otimes\beta}.(\lambda X^\alpha Y^\beta.S)(\pi_1 L^{\alpha\otimes\beta})(\pi_2 L^{\alpha\otimes\beta}))(T))^\gamma)$$

is in C_γ,

$C_\alpha \& C_\beta$ is the set of terms equivalent to some $T = (Z^\Gamma; F^{\alpha\&\beta})$ for some Γ such that $(Z^\Gamma; (\pi_1 T)^\alpha)$ is in C_α and $(Z^\Gamma; (\pi_2 T)^\beta)$ is in C_β,

$C_\alpha \oplus C_\beta$ is the set of terms equivalent to some $T = (Z^\Gamma; F^{\alpha\oplus\beta})$ for some Γ such that, for all types $[\gamma]$, all CR C_γ of type $[\gamma]$, all terms $S = (W^\Delta, X^\alpha; G^\gamma)$ and $R = (W^\Delta, Y^\beta; H^\gamma)$ in C_γ, for which $S(X^\alpha/U_\alpha)$ and $R(Y^\beta/U_\beta)$ are in C_γ for every term U_α in C_α and term U_β in C_β, the term

$$(Z^\Gamma, W^\Delta; ((\lambda L^{\alpha\oplus\beta}.((\pi_1 L^{\alpha\oplus\beta})((\lambda X^\alpha.S), (\lambda Y^\beta.R))(\pi_2 L^{\alpha\oplus\beta})))(T))^\gamma)$$

is in C_γ,

$C_\alpha \multimap\!\!\!\!\perp C_\beta$ is the set of terms equivalent to some $T = (Z^\Gamma; F^{\alpha \perp\!\circ \beta})$ for some Γ such that, for all terms $S = (W^\Delta; G^\alpha)$ in C_α, the term $(Z^\Gamma, W^\Delta; (T(S))^\beta)$ is in C_β.

$!C_\alpha$ is the set of terms equivalent to some $T = (Z^\Gamma; F^{!\alpha})$ for some Γ such that $(Z^\Gamma; (\iota T)^\alpha)$ is in C_α.

$\perp C_\alpha$ is the set of terms equivalent to some $T = (Z^\Gamma; F^{\alpha^{\perp\perp}})$ for some Γ such that $(Z^\Gamma; (\tau T)^\alpha)$ is in C_α.

$\exists x C_\alpha$ is the set of terms equivalent to some $T = (Z^\Gamma; F^{\exists x\alpha})$ for some Γ such that, x is not free in Γ, for all types $[\gamma]$ where x is not free in γ, for all CR C_γ of type $[\gamma]$, all terms $S = (W^\Delta, X^\alpha; G^\gamma)$, for which x is not free in Δ and $S(x/t)(X^\alpha(x/t)/U_{\alpha(x/t)})$ is in $C_{\gamma(x/t)}$ for every individual term t and term $U_{\alpha(x/t)}$ in $C_{\alpha(x/t)}$, the term

$$(Z^\Gamma, W^\Delta; ((\lambda L^{\exists x\alpha}.(\lambda x X^\alpha.S)(\pi_1 L^{\exists x\alpha})(\pi_2 L^{\exists x\alpha}))(T))^\gamma)$$

is in C_γ.

$\forall x C_\alpha$ is the set of terms equivalent to some $T = (Z^\Gamma; F^{\forall x\alpha})$ for some Γ such that, for all individual terms t, the term $(Z^\Gamma; (T(t))^{\alpha(x/t)})$ is in $C_{\alpha(x/t)}$.

Lemma 6 *All the above sets of terms are CR.*

Proof: Suppose C_α and C_β are CR of type $[\alpha]$ and $[\beta]$ respectively. For each of the above sets we must verify CR0, CR1, CR2, and CR3. To verify CR0 is trivial in all the cases, and hence we omit it. We do the case of $C_\alpha \otimes C_\beta$ as an example and relegate the others to the appendix.

$C_\alpha \otimes C_\beta$**:** Let $T = (Z^\Gamma; F^{\alpha\otimes\beta})$.

CR1: Suppose T is in $C_\alpha \otimes C_\beta$. Take any CR, $C_\top$, of type $[\top]$. Since $\alpha, \beta \vdash \top$ is an axiom, by CR3 for $C_\top$, it follows that $S = (X^\alpha, Y^\beta; A^\top)$ is in $C_\top$. Take any $U_\alpha = (M^{\Theta_\alpha}; Q^\alpha)$ in C_α and $U_\beta = (N^{\Theta_\beta}; P^\beta)$ in C_β. Then $S(X^\alpha/U_\alpha, Y^\beta/U_\beta)$ corresponds to the axiom $\Theta_\alpha, \Theta_\beta \vdash \top$, and hence is in $C_\top$. Let

$$\mathcal{C} = (Z^\Gamma; ((\lambda L^{\alpha\otimes\beta}.(\lambda X^\alpha Y^\beta.S)(\pi_1 L^{\alpha\otimes\beta})(\pi_2 L^{\alpha\otimes\beta}))(T))^\top)$$

Then by the definition of $C_\alpha \otimes C_\beta$, $\mathcal{C}$ is in $C_\top$. Therefore $\mathcal{C}$ is strongly normalizable and thus $N(\mathcal{C}) < \infty$. Since T is a proper subterm of $\mathcal{C}$ this implies that $N(T) < \infty$ and hence T is strongly normalizable.

CR2: Suppose T is in $C_\alpha \otimes C_\beta$, and let $T \succ^i T'$. Take any $S = (W^\Delta, X^\alpha, Y^\beta; G^\gamma)$ such that S is in a CR C_γ of type $[\gamma]$, and such that $S(X^\alpha/U_\alpha, Y^\beta/U_\beta)$ is in C_γ for every term U_α in C_α and term U_β in C_β, and let

$$\mathcal{C}(T) = (Z^\Gamma, W^\Delta; ((\lambda L^{\alpha\otimes\beta}.(\lambda X^\alpha Y^\beta.S)(\pi_1 L^{\alpha\otimes\beta})(\pi_2 L^{\alpha\otimes\beta}))(T))^\gamma)$$

Then $C(T)$ is in C_γ and $\mathcal{C}(T) \succ^i \mathcal{C}(T')$. Therefore, by CR2 for C_γ, $\mathcal{C}(T')$ is in C_γ and hence T' is in $C_\alpha \otimes C_\beta$.

CR3: Suppose T is neutral and all its immediate reducts are in $C_\alpha \otimes C_\beta$. Take any $S = (W^\Delta, X^\alpha, Y^\beta; G^\gamma)$ such that S is in a CR C_γ of type $[\gamma]$, and such that $S(X^\alpha/U_\alpha, Y^\beta/U_\beta)$ is in C_γ for every term U_α in C_α and term U_β in C_β, and let

$$\mathcal{C}(T, S) = (Z^\Gamma, W^\Delta; ((\lambda L^{\alpha\otimes\beta}.(\lambda X^\alpha Y^\beta.S)(\pi_1 L^{\alpha\otimes\beta})(\pi_2 L^{\alpha\otimes\beta}))(T))^\gamma)$$

Since $\mathcal{C}(T, S)$ is neutral, by CR3 for C_γ, we only need to show all the immediate reducts of $\mathcal{C}(T, S)$ are in C_γ. We will prove this by induction on $N(S)$. By Lemma 5, since T and $\mathcal{C}(T, S)$ are neutral it follows that the immediate reducts of $\mathcal{C}(T, S)$ are either $\mathcal{C}(T', S)$, where $T \succ^i T'$, in which case T' is in $C_\alpha \otimes C_\beta$, so $\mathcal{C}(T', S)$ is in C_γ, or else $\mathcal{C}(T, S')$, where $S \succ^i S'$; in this case $N(S') < N(S)$ so the result follows by Lemma 4 and the induction hypothesis.

□

5.3 Definition of canonical CR

For each first order formula α we define a canonical CR, C_α, by induction on α.

0: If α is an atomic formula (including the cases of $\top$ and $\bot$), then C_α is the CR of all strongly normalizable terms whose conclusion type is equivalent to α.

$\otimes$: $C_{\alpha\otimes\beta} = C_\alpha \otimes C_\beta$.

&: $C_{\alpha\&\beta} = C_\alpha \& C_\beta$.

$\oplus$: $C_{\alpha\oplus\beta} = C_\alpha \oplus C_\beta$.

$\bot\circ$: $C_{\alpha\bot\circ\beta} = C_\alpha \bot\circ C_\beta$.

!: $C_{!\alpha} = !C_\alpha$.

$\exists x\alpha$: $C_{\exists x\alpha} = \exists x C_\alpha$.

$\forall x$: $C_{\forall x\alpha} = \forall x C_\alpha$.

It is easily checked that the C_α for α atomic are indeed CRs and the other cases follow from lemma 6 above. (note that $C_{\alpha^\bot}$ is a special case of $C_{\alpha\bot\circ\beta}$ when β is $\bot$.)

5.4 The Strong Normalization Theorem

Now we are ready to prove the main theorem. We shall show that every Curry–Howard term of type $\Gamma \vdash \alpha$ belongs to $C_{[\Gamma\vdash\alpha]}$, and is therefore strongly normalizable.

Remarks on the structure of the proof of strong normalization.

1. The cases for (W) and (C) are dealt with here in a manner which is different from, but apparently equivalent to, that in Benton *et al* [2]. Benton *et al.* may be regarded as using global commutation properties for (W) and (C) whereas we are only looking at a one stage movement of the rule. However it would appear that the two methods are equivalent for the cases considered. In our case we intuitively view the weakening rule as less fixed in the temporal sequence of the proof. Thus when we have a proof $\Gamma \vdash \gamma$ whose last rule was (W) we consider, not the extra hypothesis introduced into Γ, but the structure of γ, in particular its last connective ($\star$, say) if there is one. If there is one we treat the term for $\Gamma \vdash \gamma$ in the same way as if $(\star I)$ were its last rule. Otherwise, if there is no connective then, in the case of (W), no reduction is possible on this last stage so any reduction must have taken place earlier.

In the case of (C) we have to consider two initial possiblilites. The first is where the (C) immediately follows a (W) which permits a reduction. This is dealt with straightforwardly, i.e. like other reductions. The second is when there is no reduction. In this case we proceed as above by induction on the construction of the conclusion formula.

2. The extra inductions involved in the cases of (W) and (C) (where we use the construction of the conclusion formula) clearly substantially increase the ordinal associated with the reductions, just as one would expect.

3. Because the cases for the connectives follow the traditional pattern we have relegated most of the details to the appendix. However, we found it so difficult to get the details exactly right that we felt it essential to include such details.

Theorem 4 *Each Curry–Howard term $T = (Z^\Gamma; F^\alpha)$ of type $\Gamma \vdash \alpha$ belongs to the canonical CR, C_α, of type $[\alpha]$.*

Proof : First we define some notation (for more details on this notation, see section 4.4). Let $y = y_1, \ldots, y_n$ be a list of distinct individual variables which are free in T, let $s = s_1, \ldots, s_n$ be a list of individual terms, let $Y = Y_1^{\beta_1}, \ldots, Y_m^{\beta_m}$ be a list of distinct term variables which are free in T, let $S' = S'_{\beta'_1}, \ldots, S'_{\beta'_m}$ be a list of Curry–Howard terms in $C_{\beta'_1}, \ldots, C_{\beta'_m}$, respectively, where $S'_{\beta'_k} = (W_k^\Delta; G_k^{\beta'_k})$ and $\beta'_k = \beta_k(y/s)$. Moreover, let $Y' = Y(y/s)$, let $\alpha' = \alpha(y/s)$, and let Γ' be the sequence of the types in the term vector $Z^\Gamma(y/s)(Y'/S')$. (Note Γ' is a function of Γ, y, s, Y and S'.)

The proof is by induction on the structure of T, and to make the induction go through we strengthen the hypothesis to:

"If $T = (Z^\Gamma; F^\alpha)$ is a Curry–Howard term of type $\Gamma \vdash \alpha$ then $T(y/s)(Y'/S')$ is a Curry–Howard term of type $\Gamma' \vdash \alpha'$ and is in the CR $C_{\alpha'}$."

First, if $T = (Z^\Gamma; F^\alpha)$ is a Curry–Howard term of type $\Gamma \vdash \alpha$ then it follows from the substitution lemma 3 that $T(y/s)(Y'/S')$ is a Curry–Howard term of type $\Gamma' \vdash \alpha'$. We now proceed to prove, by induction on the structure of T, that $T(y/s)(Y'/S')$ is in $C_{\alpha'}$.

Base Case: If T is an axiom then $T(y/s)(Y'/S')$ is either an axiom or one of the $S'_{\beta'_i}$'s. In either case, $T(y/s)(Y'/S')$ is in $C_{\alpha'}$.

Induction Step: We now consider, in turn, each rule of term formation, on the assumption that all proper sub-terms of the term under consideration satisfy the above hypothesis.

Also, we will write α' for $\alpha(y/s)$, γ' for $\gamma(y/s)$, α'_i for $\alpha_i(y/s)$ and Γ', Γ'_i, and Θ' for the sequence of types in the term vectors $Z^{\Gamma}(y/s)(Y'/S')$, $Z_i^{\Gamma_i}(y/s)(Y'/S')$, and $W^{\Theta}(y/s)(Y'/S')$, respectively.

$(\otimes I)$ Let $T_{\alpha_1} = (Z_1^{\Gamma_1}; F_1^{\alpha_1})$ and $T_{\alpha_2} = (Z_2^{\Gamma_2}; F_2^{\alpha_2})$ satisfy the hypothesis, and let

$$T = (Z^{\Gamma_1}, Z^{\Gamma_2}; (T_{\alpha_1}, T_{\alpha_2})^{\alpha_1 \otimes \alpha_2}).$$

Then we wish to show that $R = T(y/s)(Y'/S')$ is in $C_{\alpha'_1 \otimes \alpha'_2}$.

Take any $U = (W^{\Theta}, X_1^{\alpha'_1}, X_2^{\alpha'_2}; G^{\gamma})$, such that U is in a CR $\mathcal{C}$ of type $[\gamma]$, and such that $U(X_1^{\alpha'_1}/V_{\alpha'_1}, X_2^{\alpha'_2}/V_{\alpha'_2})$ is in $\mathcal{C}$ for every term $V_{\alpha'_1}$ in $C_{\alpha'_1}$ and term $V_{\alpha'_2}$ in $C_{\alpha'_2}$. Let

$$\mathcal{H}(R,U) = (Z_1^{\Gamma'_1}, Z_2^{\Gamma'_2}, W^{\Theta}; ((\lambda L.(\lambda X_1^{\alpha'_1} X_2^{\alpha'_2}.U)(\pi_1 L)(\pi_2 L))(R))),$$

where L is a new term variable of type $\alpha'_1 \otimes \alpha'_2$. Then we need to show that $\mathcal{H}(R,U)$ is in $\mathcal{C}$. We do this by a subsidiary induction on $N(R) + N(U)$. Since $\mathcal{H}(R,U)$ is a neutral term, we only need to show that all its immediate reducts are in $\mathcal{C}$. These reducts are equivalent to the following forms:

1. $\mathcal{H}(R', U)$, where $R \succ^i R'$. Then $N(R') < N(R)$, so the term is in $\mathcal{C}$ by the subsidiary induction hypothesis.
2. $\mathcal{H}(R, U')$, where $U \succ^i U'$. Then $N(U') < N(U)$ so the term is in $\mathcal{C}$ by the subsidiary induction hypothesis.
3. $U(X_1^{\alpha'_1}/T_{\alpha_1}(y/s)(Y/S), X_2^{\alpha'_2}/T_{\alpha_2}(y/s)(Y/S))$ which is in $\mathcal{C}$ by hypothesis.

For the other cases involving connective rules see the appendix. We now deal with $(\perp)$, (W) and (C).

$(\perp)$ Let $T_{\alpha^{\perp\perp}} = (Z^{\Gamma}; F^{\alpha^{\perp\perp}})$ satisfy the hypothesis, and let

$$T = (Z^{\Gamma}; (\tau T_{\alpha^{\perp\perp}})^{\alpha}).$$

Then we wish to show that $R = T(y/s)(Y'/S')$ is in $C_{\alpha'}$.

We consider the following cases: α is atomic, $\alpha = \alpha_1 \otimes \alpha_2$, $\alpha = \alpha_1 \& \alpha_2$, $\alpha = \alpha_1 \oplus \alpha_2$, $\alpha = \alpha_1 \perp\!\circ \alpha_2$, $\alpha = !\gamma$, $\alpha = \gamma^{\perp\perp}$, $\alpha = \exists x \gamma$, or $\alpha = \forall x \gamma$.

Assume α is atomic. Then we wish to show that R is in $C_{\alpha'}$. To do this we need to show that R is strongly normalizable. However,

$$R = (Z^{\Gamma'}; (\tau T_{\alpha^{\perp\perp}}(y/s)(Y'/S'))^{\alpha'}).$$

Therefore $N(R) = N(T_{\alpha^{\perp\perp}}(y/s)(Y'/S'))$ since there is no reduction for τ; and hence R is strongly normalizable.

Now, assume $\alpha = \alpha_1 \otimes \alpha_2$. Then we wish to show that R is in $C_{\alpha'_1 \otimes \alpha'_2}$.

Take $U = (W^{\Theta}, X_1^{\alpha'_1}, X_2^{\alpha'_2}; G^{\delta})$ in $\mathcal{C}$, such that U is a CR $\mathcal{C}$ of type $[\gamma]$ and such that $U(X_1^{\alpha'_1}/V_{\alpha'_1}, X_2^{\alpha'_2}/V_{\alpha'_2})$ is in $\mathcal{C}$ for every term $V_{\alpha'_1}$ in $C_{\alpha'_1}$ and term $V_{\alpha'_2}$ in $C_{\alpha'_2}$. Let

$$\mathcal{H}(R, U) = (Z^{\Gamma'}, W^{\Theta}; ((\lambda L.(\lambda X_1^{\alpha'_1} X_2^{\alpha'_2}.U)(\pi_1 L)(\pi_2 L))(R))),$$

where L is a new term variable of type $\alpha'_1 \otimes \alpha'_2$, and let $R_1 = T_{\alpha^{\perp\perp}}(y/s)(Y'/S')$.

Then we need to show that $\mathcal{H}(R, U)$ is in $\mathcal{C}$. We do this by a subsidiary induction on $N(R_1) + N(U)$. Since $\mathcal{H}(R, U)$ is a neutral term, we only need to show that all its immediate reducts are in $\mathcal{C}$. These reducts are equivalent to the following forms:

1. $\mathcal{H}(R', U)$, where $R \succ^i R'$. In turn R' are equivalent to $(Z^{\Gamma'}; (\tau R'_1)^{\alpha'})$, where $R_1 \succ^i R'_1$. Then $N(R'_1) < N(R_1)$, so the term $\mathcal{H}(R', U)$ is in $\mathcal{C}$ by the subsidiary induction hypothesis.

2. $\mathcal{H}(R, U')$, where $U \succ^i U'$. Then $N(U') < N(U)$ so the term is in $\mathcal{C}$ by the subsidiary induction hypothesis.

The other cases of γ, when $\gamma = \alpha_1 \& \alpha_2$, etc, are handled similarly.

(W) Let $T_{!\alpha} = (Z_1^{\Gamma_1}; F_1^{!\alpha})$ and $T_{\gamma} = (Z_2^{\Gamma_2}; F_2^{\gamma})$ satisfy the hypothesis, and let

$$T = (Z_1^{\Gamma_1}, Z_2^{\Gamma_2}; (\omega(T_{!\alpha}, T_{\gamma}))^{\gamma}).$$

Then we wish to show that $R = T(y/s)(Y'/S')$ is in $C_{\gamma'}$.

We now consider the following cases: γ is atomic, $\gamma = \alpha_1 \otimes \alpha_2$, $\gamma = \alpha_1 \& \alpha_2$, $\gamma = \alpha_1 \oplus \alpha_2$, $\gamma = \alpha_1 \perp\!\circ \alpha_2$, $\gamma = !\alpha$, $\gamma = \alpha^{\perp\perp}$, $\gamma = \exists x \alpha$, or $\gamma = \forall x \alpha$.

Assume γ is atomic. Then we wish to show that R is in $C_{\gamma'}$. To do this we need to show that R is strongly normalizable. However,

$$R = (Z_1^{\Gamma'_1}, Z_2^{\Gamma'_2}; (\omega(T_{!\alpha}(y/s)(Y'/S'), T_{\gamma}(y/s)(Y'/S')))^{\gamma(y/s)}).$$

Therefore

$$N(R) \leq N(T_{!\alpha}(y/s)(Y'/S')) + N(T_{\gamma}(y/s)(Y'/S')) < \infty,$$

and hence R is strongly normalizable.

Now, assume $\gamma = \alpha_1 \otimes \alpha_2$. Then we wish to show that R is in $C_{\alpha'_1 \otimes \alpha'_2}$.

Take any $U = (W^{\Theta}, X_1^{\alpha'_1}, X_2^{\alpha'_2}; G^{\delta})$, such that U is in a CR $\mathcal{C}$ of type $[\delta]$, and such that $U(X_1^{\alpha'_1}/V_{\alpha'_1}, X_2^{\alpha'_2}/V_{\alpha'_2})$ is in $\mathcal{C}$ for every term $V_{\alpha'_1}$ in $C_{\alpha'_1}$ and term $V_{\alpha'_2}$ in $C_{\alpha'_2}$. Let

$$\mathcal{H}(R, U) = (Z_1^{\Gamma'_1}, Z_2^{\Gamma'_2}, W^{\Theta}; ((\lambda L.(\lambda X_1^{\alpha'_1} X_2^{\alpha'_2}.U)(\pi_1 L)(\pi_2 L))(R))),$$

where L is a new term variable of type $\alpha'_1 \otimes \alpha'_2$, let $R_{!\alpha} = T_{!\alpha}(y/s)(Y'/S')$, let $R_{\gamma} = T_{\gamma}(y/s)(Y'/S')$, and let

$$\mathcal{W}(R_{!\alpha}, R_{\gamma}) = (Z_1^{\Gamma'_1}, Z_2^{\Gamma'_2}; \omega(R_{!\alpha}, R_{\gamma})).$$

Then we need to show that $\mathcal{H}(R, U)$ is in $\mathcal{C}$. We do this by a subsidiary induction on $N(R_{!\alpha}) + N(R_\gamma) + N(U)$. Since $\mathcal{H}(R, U)$ is a neutral term, we only need to show that all its immediate reducts are in $\mathcal{C}$. These reducts are equivalent to the following forms:

1. $\mathcal{H}(R', U)$, where $R \succ^i R'$. In turn R' are equivalent to the following forms:
   (a) $\mathcal{W}(R'_{!\alpha}, R_\gamma)$, where $R_{!\alpha} \succ^i R'_{!\alpha}$. Then $N(R'_{!\alpha}) < N(R_{!\alpha})$, so the term $\mathcal{H}(R', U)$ is in $\mathcal{C}$ by the subsidiary induction hypothesis.
   (b) $\mathcal{W}(R_{!\alpha}, R'_\gamma)$, where $R_\gamma \succ^i R'_\gamma$. Then $N(R'_\gamma) < N(R_\gamma)$, so the term $\mathcal{H}(R', U)$ is in $\mathcal{C}$ by the subsidiary induction hypothesis.
2. $\mathcal{H}(R, U')$, where $U \succ^i U'$. Then $N(U') < N(U)$ so the term is in $\mathcal{C}$ by the subsidiary induction hypothesis.

The other cases of γ, when $\gamma = \alpha_1 \& \alpha_2$, etc, are handled similarly.

(C) Let $T_{!\alpha} = (Z_1^{\Gamma_1}; F^{!\alpha})$ and $T_\gamma = (Z_2^{\Gamma_2}, Z_3^{\Gamma_3}, X_1^{!\alpha}, X_2^{!\alpha}; G^\gamma)$ satisfy the hypothesis, and let

$$T = (Z_1^{\Gamma_1}, Z_2^{\Gamma_2}; ((\kappa X_1^{!\alpha} X_2^{!\alpha}.T_\gamma)(T_{!\alpha}))^\gamma).$$

We want to show that $R = T(y/s)(Y'/S')$ is in $C_{\gamma'}$.

There are two cases to consider; either we can apply a (W Intro, C Elim) reduction rule to T, or we cannot.

Case 1: Suppose we can apply a (W Intro C Elim) reduction rule to T. In this case we shall suppose $P_\gamma = (Z_2^{\Gamma_2}, X_2^{!\alpha}; H^\gamma)$ and $P_{!\alpha} = (Z_3^{\Gamma_3}, X_1^{!\alpha}; K^{!\alpha})$ both satisfy the induction hypothesis, and

$$T_\gamma = (Z_2^{\Gamma_2}, Z_3^{\Gamma_3}, X_1^{!\alpha}, X_2^{!\alpha}; (\omega((P_{!\alpha}, P_\gamma))^\gamma).$$

We now consider the following cases: γ is atomic, $\gamma = \alpha_1 \otimes \alpha_2$, $\gamma = \alpha_1 \& \alpha_2$, $\gamma = \alpha_1 \oplus \alpha_2$, $\gamma = \alpha_1 \perp\!\circ \alpha_2$, $\gamma = !\alpha$, $\gamma = \alpha^{\perp\perp}$, $\gamma = \exists x \alpha$, or $\gamma = \forall x \alpha$.

Let $R_{!\alpha} = T_{!\alpha}(y/s)(Y'/S')$, let $R_\gamma = T_\gamma(y/s)(Y'/S')$, and let

$$\mathcal{K}(R_{!\alpha}, R_\gamma) = (Z_1^{\Gamma_1}, Z_2^{\Gamma_2}; ((\kappa X_1^{!\alpha} X_2^{!\alpha}.R_\gamma)(R_{!\alpha}))^\gamma).$$

Assume γ is atomic. Then we wish to show that R is strongly normalizable. We do this by a subsidiary induction on $N(R_{!\alpha}) + N(R_\gamma)$. To show R is strongly normalizable it suffices to show all its immediate reducts are strongly normalizable. These reducts are equivalent to the following forms:

1. $\mathcal{K}(R'_{!\alpha}, R_\gamma)$, where $R_{!\alpha} \succ^i R'_{!\alpha}$. Then $N(R'_{!\alpha}) < N(R_{!\alpha})$, so the term is strongly normalizable by the subsidiary induction.
2. $\mathcal{K}(R_{!\alpha}, R'_\gamma)$, where $R_\gamma \succ^i R'_\gamma$. Then $N(R'_\gamma) < N(R_\gamma)$, so the term is strongly normalizable by the subsidiary induction.

3. $P_\gamma(y/s)(Y'/S', X_2(y/s)/P_{!\alpha}(y/s)(Y'/S', X_1(y/s)/R_{!\alpha}))$, which by the hypothesis is strongly normalizable.

Now, assume $\gamma = \alpha_1 \otimes \alpha_2$. Then we wish to show that R is in $C_{\alpha'_1 \otimes \alpha'_2}$.

Take $U = (W^\Theta, X_1^{\alpha'_1}, X_2^{\alpha'_2}; G^\delta)$, such that U is in a CR $\mathcal{C}$ of type $[\delta]$, and such that $U(X_1^{\alpha'_1}/V_{\alpha'_1}, X_2^{\alpha'_2}/V_{\alpha'_2})$ is in $\mathcal{C}$ for every term $V_{\alpha'_1}$ in $C_{\alpha'_1}$ and term $V_{\alpha'_2}$ in $C_{\alpha'_2}$. Let

$$\mathcal{H}(R, U) = (Z_1^{\Gamma'_1}, Z_2^{\Gamma'_2}, Z_3^{\Gamma'_3}, W^\Theta; ((\lambda L.(\lambda X_1^{\alpha'_1} X_2^{\alpha'_2}.U)(\pi_1 L)(\pi_2 L))(R))),$$

where L is a new term variable of type $\alpha'_1 \otimes \alpha'_2$.

Then we need to show that $\mathcal{H}(R, U)$ is in $\mathcal{C}$. We do this by a subsidiary induction on $N(R_{!\alpha}) + N(R_\gamma) + N(U)$. Since $\mathcal{H}(R, U)$ is a neutral term, we only need to show that all its immediate reducts are in $\mathcal{C}$. These reducts are equivalent to the following forms:

1. $\mathcal{H}(R', U)$, where $R \succ^i R'$. In turn R' are equivalent to the following forms:
   (a) $\mathcal{K}(R'_{!\alpha}, R_\gamma)$, where $R_{!\alpha} \succ^i R'_{!\alpha}$. Then $N(R'_{!\alpha}) < N(R_{!\alpha})$, so the term $\mathcal{H}(R', U)$ is in $\mathcal{C}$ by the subsidiary induction hypothesis.
   (b) $\mathcal{K}(R_{!\alpha}, R'_\gamma)$, where $R_\gamma \succ^i R'_\gamma$. Then $N(R'_\gamma) < N(R_\gamma)$, so the term $\mathcal{H}(R', U)$ is in $\mathcal{C}$ by the subsidiary induction hypothesis.
   (c) $P_\gamma(y/s)(Y'/S', X_2(y/s)/P_{!\alpha}(y/s)(Y'/S', X_1(y/s)/R_{!\alpha}))$, so by hypothesis, and the definition of $C_{\alpha'_1 \otimes \alpha'_2}$, the term $\mathcal{H}(R', U)$ is in $\mathcal{C}$.
2. $\mathcal{H}(R, U')$, where $U \succ^i U'$. Then $N(U') < N(U)$ so the term is in $\mathcal{C}$ by the subsidiary induction hypothesis.

The other cases of $\gamma = \alpha_1 \& \alpha_2$, etc, are handled by similar subsidiary inductions.

Case 2: We now suppose we cannot apply a (W Intro C Elim) reduction rule to T.

We now consider the following cases: γ is atomic, $\gamma = \alpha_1 \otimes \alpha_2$, $\gamma = \alpha_1 \& \alpha_2$, $\gamma = \alpha_1 \oplus \alpha_2$, $\gamma = \alpha_1 \multimap \alpha_2$, $\gamma = !\alpha$, $\gamma = \alpha^{\perp\perp}$, $\gamma = \exists x \alpha$, or $\gamma = \forall x \alpha$.

Let $R_{!\alpha} = T_{!\alpha}(y/s)(Y'/S')$, let $R_\gamma = T_\gamma(y/s)(Y'/S')$, and let

$$\mathcal{K}(R_{!\alpha}, R_\gamma) = (Z_1^{\Gamma_1}, Z_2^{\Gamma_2}; ((\kappa X_1^{!\alpha} X_2^{!\alpha}.R_\gamma)(R_{!\alpha}))^\gamma).$$

Assume γ is atomic. Then we wish to show that R is strongly normalizable. We do this by a subsidiary induction on $N(R_{!\alpha}) + N(R_\gamma)$. To show R is strongly normalizable it suffices to show all its immediate reducts are strongly normalizable. These reducts are equivalent to the following forms:

1. $\mathcal{K}(R'_{!\alpha}, R_\gamma)$, where $R_{!\alpha} \succ^i R'_{!\alpha}$. Then $N(R'_{!\alpha}) < N(R_{!\alpha})$, so the term is strongly normalizable by the subsidiary induction.
2. $\mathcal{K}(R_{!\alpha}, R'_\gamma)$, where $R_\gamma \succ^i R'_\gamma$. Then $N(R'_\gamma) < N(R_\gamma)$, so the term is strongly normalizable by the subsidiary induction.

Now, assume $\gamma = \alpha_1 \otimes \alpha_2$. Then we wish to show that R is in $C_{\alpha_1' \otimes \alpha_2'}$.

Take $U = (W^\Theta, X_1^{\alpha_1'}, X_2^{\alpha_2'}; G^\delta)$, such that U is in a CR $\mathcal{C}$ of type $[\delta]$, and such that $U(X_1^{\alpha_1'}/V_{\alpha_1'}, X_2^{\alpha_2'}/V_{\alpha_2'})$ is in $\mathcal{C}$ for every term $V_{\alpha_1'}$ in $C_{\alpha_1'}$ and term $V_{\alpha_2'}$ in $C_{\alpha_2'}$. Let

$$\mathcal{H}(R, U) = (Z_1^{\Gamma_1'}, Z_2^{\Gamma_2'}, W^\Theta; ((\lambda L.(\lambda X_1^{\alpha_1'} X_2^{\alpha_2'}.U)(\pi_1 L)(\pi_2 L))(R))),$$

where L is a new term variable of type $\alpha_1' \otimes \alpha_2'$.

Then we need to show that $\mathcal{H}(R, U)$ is in $\mathcal{C}$. We do this by a subsidiary induction on $N(R_{!\alpha}) + N(R_\gamma) + N(U)$. Since $\mathcal{H}(R, U)$ is a neutral term, we only need to show that all its immediate reducts are in $\mathcal{C}$. These reducts are equivalent to the following forms:

1. $\mathcal{H}(R', U)$, where $R \succ^i R'$. In turn R' are of the following form:
   (a) $\mathcal{K}(R'_{!\alpha}, R_\gamma)$, where $R_{!\alpha} \succ^i R'_{!\alpha}$. Then $N(R'_{!\alpha}) < N(R_{!\alpha})$, so the term $\mathcal{H}(R', U)$ is in $\mathcal{C}$ by the subsidiary induction hypothesis.
   (b) $\mathcal{K}(R_{!\alpha}, R'_\gamma)$, where $R_\gamma \succ^i R'_\gamma$. Then $N(R'_\gamma) < N(R_\gamma)$, so the term $\mathcal{H}(R', U)$ is in $\mathcal{C}$ by the subsidiary induction hypothesis.
2. $\mathcal{H}(R, U')$, where $U \succ^i U'$. Then $N(U') < N(U)$ so the term is in $\mathcal{C}$ by the subsidiary induction hypothesis.

The other cases of $\gamma = \alpha_1 \& \alpha_2$, etc, are handled by similar subsidiary inductions.

□

Theorem 5 *Every Curry–Howard term is strongly normalizable.*

Proof Immediate from the proof above, by CR1.

□

Corollary 1 *The logical system $\mathcal{N}$ is equivalent to the same system with (W) and (C) restricted to sequents with atomic conclusions.*

Proof Immediate from the cases of (W) and (C) in the proof above.

□

6 Future work

We believe that the system of terms introduced here forms a sound basis for the mechanical (i.e. computer) extraction of programs from proofs in linear logic. In a planned sequel we intend to extend this work to a linear logic version of Peano arithmetic where we expect to be able to give resource bounds for the programs extracted.

We are also investigating a modification of the system to accommodate Girard, Scedrov and Scott's Bounded Linear Logic [9]. This should lead to giving explicit polynomial-time bounds for the programs extracted from proofs of formulae.

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Appendix

Here we present the missing details of proofs.

Proof of lemma 6 (concluded).

$C_\alpha \& C_\beta$: Let $T = (Z^\Gamma; F^{\alpha\&\beta})$.

CR1: Suppose T is in $C_\alpha \& C_\beta$. Let $\mathcal{C} = (Z^\Gamma; (\pi_1 T)^\alpha)$. Then by definition of $C_\alpha \& C_\beta$, $\mathcal{C}$ is in C_α. Therefore $\mathcal{C}$ is strongly normalizable and hence $N(\mathcal{C}) < \infty$. Since T is a proper subterm of S this implies that $N(T) < \infty$ and hence T is strongly normalizable.

CR2: Suppose T is in $C_\alpha \& C_\beta$ and $T \succ^i T'$. Let

$$\mathcal{C}_1(T) = (Z^\Gamma; (\pi_1 T)^\alpha) \text{ and } \mathcal{C}_2(T) = (Z^\Gamma; (\pi_2 T)^\beta)$$

Then $\mathcal{C}_1(T)$ is in C_α, $\mathcal{C}_2(T)$ is in C_β, $\mathcal{C}_1(T) \succ^i \mathcal{C}_1(T')$ and $\mathcal{C}_2(T) \succ^i \mathcal{C}_2(T')$. So, by CR2 for C_α and C_β, $\mathcal{C}_1(T')$ is in C_α and $\mathcal{C}_2(T')$ is in C_β. Therefore, T' is in $C_\alpha \& C_\beta$.

CR3: Suppose T is neutral and every immediate reduct of T is in $C_\alpha \& C_\beta$. Let

$$\mathcal{C}_1(T) = (Z^\Gamma; (\pi_1 T)^\alpha) \text{ and } \mathcal{C}_2(T) = (Z^\Gamma; (\pi_2 T)^\beta)$$

Since $\mathcal{C}_1(T)$ and T are neutral, by Lemma 5, the immediate reducts of $\mathcal{C}_1(T)$ are $\mathcal{C}_1(T')$, where $T \succ^i T'$. By assumption $\mathcal{C}_1(T')$ is in C_α, and so by CR3 for C_α, $\mathcal{C}_1(T)$ is in C_α. Similarly, by Lemma 5, since $\mathcal{C}_2(T)$ and T are neutral the immediate reducts of $\mathcal{C}_2(T)$ are $\mathcal{C}_2(T')$, where $T \succ^i T'$. By assumption $\mathcal{C}_2(T')$ is in C_β, and so by CR3 for C_β, $\mathcal{C}_2(T)$ is in C_β. Hence T is in $C_\alpha \& C_\beta$.

$C_\alpha \oplus C_\beta$: Let $T = (Z^\Gamma; F^{\alpha\oplus\beta})$.

CR1: Suppose T is in $C_\alpha \oplus C_\beta$. Take any CR, $C_\top$, of type $[\top]$. Since $\alpha \vdash \top$ and $\beta \vdash \top$ are axioms, by CR3 for $C_\top$, it follows that $S = (X^\alpha; A_1^\top)$ and $R = (Y^\beta; A_2^\top)$ are in $C_\top$. Take any $U_\alpha = (M^{\Theta_\alpha}; Q^\alpha)$ in C_α and $U_\beta = (N^{\Theta_\beta}; P^\beta)$ in C_β. Then $S(X^\alpha/U_\alpha)$ and $R(Y^\beta/U_\beta)$ correspond, respectively, to the axioms $\Theta_\alpha \vdash \top$ and $\Theta_\beta \vdash \top$, and hence $S(X^\alpha/U_\alpha)$ and $R(Y^\beta/U_\beta)$ are in $C_\top$. Let

$$\mathcal{C} = (Z^\Gamma; ((\lambda L^{\alpha\oplus\beta}.(\pi_1 L^{\alpha\oplus\beta})((\lambda X^\alpha.S), (\lambda Y^\beta.R))(\pi_2 L^{\alpha\oplus\beta}))(T))^\top)$$

Then by the definition of $C_\alpha \oplus C_\beta$, $\mathcal{C}$ is in $C_\top$. Therefore $\mathcal{C}$ is strongly normalizable and thus $N(\mathcal{C}) < \infty$. Since T is a proper subterm of $\mathcal{C}$ this implies that $N(T) < \infty$ and hence T is strongly normalizable.

CR2: Suppose T is in $C_\alpha \oplus C_\beta$, and let $T \succ^i T'$. Take any type $[\gamma]$, any CR C_γ of type $[\gamma]$, and any $S = (W^\Delta, X^\alpha; G^\gamma)$ and $R = (W^\Delta, Y^\beta; H^\gamma)$ in C_γ, such that $S(X^\alpha/U_\alpha)$ and $R(Y^\beta/U_\beta)$ is in C_γ for every term U_α in C_α and term U_β in C_β, and let

$$\mathcal{C}(T) = (Z^\Gamma, W^\Delta; ((\lambda L^{\alpha\oplus\beta}.(\pi_1 L^{\alpha\oplus\beta})((\lambda X^\alpha.S), (\lambda Y^\beta.R))(\pi_2 L^{\alpha\oplus\beta}))(T))^\gamma)$$

Then $C(T)$ is in C_γ and $\mathcal{C}(T) \succ^i \mathcal{C}(T')$. Therefore, by CR2 for C_γ, $\mathcal{C}(T')$ is in C_γ and hence T' is in $C_\alpha \oplus C_\beta$.

CR3: Suppose T is neutral and all its immediate reducts are in $C_\alpha \oplus C_\beta$. Take any type $[\gamma]$, any CR C_γ of type $[\gamma]$, and any $S = (W^\Delta, X^\alpha; G^\gamma)$ and $R = (W^\Delta; Y^\beta; H^\gamma)$ in C_γ, such that $S(X^\alpha/U_\alpha)$ and $R(Y^\beta/U_\beta)$ is in C_γ for every term U_α in C_α and term U_β in C_β, and let

$$\mathcal{C}(T,S) = (Z^\Gamma, W^\Delta; ((\lambda L^{\alpha\oplus\beta}.(\pi_1 L^{\alpha\oplus\beta})((\lambda X^\alpha.S),(\lambda Y^\beta.R))(\pi_2 L^{\alpha\oplus\beta}))(T))^\gamma)$$

Since $\mathcal{C}(T,S)$ is neutral, by CR3 for C_γ, we only need to show all the immediate reducts of $\mathcal{C}(T,S)$ are in C_γ. We will prove this by induction on $N(S)$. By Lemma 5, since T and $\mathcal{C}(T,S)$ are neutral it follows that the immediate reducts of $\mathcal{C}(T,S)$ are either $\mathcal{C}(T',S)$, where $T \succ^i T'$, in which case T' is in $C_\alpha \oplus C_\beta$, so $\mathcal{C}(T',S)$ is in C_γ, or else $\mathcal{C}(T,S')$, where $S \succ^i S'$; in this case $N(S') < N(S)$ so the result follows by Lemma 4 and the induction hypothesis.

$C_\alpha \multimap C_\beta$: Let $T = (Z^\Gamma; F^{\alpha\multimap\beta})$.

CR1: Suppose T is in $C_\alpha \multimap C_\beta$. Take any term $S = (W^\Delta; G^\alpha)$ in C_α, and let $\mathcal{C} = (Z^\Gamma, W^\Delta; (T(S))^\beta)$. Then by the definition of $C_\alpha \multimap C_\beta$, $\mathcal{C}$ is in C_β. Therefore $\mathcal{C}$ is strongly normalizable and thus $N(\mathcal{C}) < \infty$. Since T is a proper subterm of $\mathcal{C}$ this implies that $N(T) < \infty$ and hence T is strongly normalizable.

CR2: Suppose T is in $C_\alpha \multimap C_\beta$ and let $T \succ^i T'$. Take any term $S = (W^\Delta; G^\alpha)$ in C_α, and let

$$\mathcal{C}(T) = (Z^\Gamma, W^\Delta; (T(S))^\beta)$$

Then $\mathcal{C}(T)$ is in C_β and $\mathcal{C}(T) \succ^i \mathcal{C}(T')$. Therefore, by CR2 for C_β, $\mathcal{C}(T')$ is in C_β and hence T' is in $C_\alpha \multimap C_\beta$.

CR3: Suppose T is neutral and all its immediate reducts are in $C_\alpha \multimap C_\beta$. Take any term $S = (W^\Delta; G^\alpha)$ in C_α, and let

$$\mathcal{C}(T,S) = (Z^\Gamma, W^\Delta; (T(S))^\beta)$$

Since $\mathcal{C}(T,S)$ is neutral, by CR3 for C_β, we only need to show that all its immediate reducts are in C_β. We will prove this by induction on $N(S)$. By Lemma 5, since T and $\mathcal{C}(T,S)$ are neutral, it follows that all the immediate reducts of $\mathcal{C}(T,S)$ are either $\mathcal{C}(T',S)$, where $T \succ^i T'$, in which case T' is in $C_\alpha \multimap C_\beta$, so $\mathcal{C}(T',S)$ is in C_β, or else $\mathcal{C}(T,S')$, where $S \succ^i S'$; in this case $N(S') < N(S)$ so the result follows by Lemma 4 and the induction hypothesis.

$!C_\alpha$ Let $T = (Z^\Gamma; F^{!\alpha})$.

CR1: Suppose T is in $!C_\alpha$. Then $\mathcal{C} = (Z^\Gamma; (\iota T)^\alpha)$ is in C_α. Therefore $\mathcal{C}$ is strongly normalizable and thus $N(\mathcal{C}) < \infty$. Since T is a proper subterm of $\mathcal{C}$ this implies that $N(T) < \infty$ and hence T is strongly normalizable.

CR2: Suppose T is in $!C_\alpha$, $T \succ^i T'$, and let $\mathcal{C}(T) = (Z^\Gamma; (\iota T)^\alpha)$. Then $\mathcal{C}(T)$ is in C_α and $\mathcal{C}(T) \succ^i \mathcal{C}(T')$. So, by CR2 for C_α, $\mathcal{C}(T')$ is in C_α and hence T' is in $!C_\alpha$.

CR3: Supoose T is neutral and all its immediate reducts are in $!C_\alpha$. Let $\mathcal{C}(T) = (Z^\Gamma; (\iota T)^\alpha)$. Since $\mathcal{C}(T)$ is neutral, by CR3 for C_α, we only need to show all its immediate reducts are in C_α. However all its immediate reducts are of the form $\mathcal{C}(T')$, where $T \succ^i T'$, in which case T' is in $!C_\alpha$, and so $\mathcal{C}(T')$ is in C_α.

$\exists x C_\alpha$ Let $T = (Z^\Gamma; F^{\exists x\alpha})$.

CR1: Suppose T is in $\exists x C_\alpha$. Take any CR, $C_\top$, of type $[\top]$. Since $\alpha \vdash \top$ is an axiom, by CR3 for $C_\top$, it follows that $S = (X^\alpha; A^\top)$ is in $C_\top$. Take any $U_{\alpha(x/t)} = (M^{\Theta_\alpha}; Q^{\alpha(x/t)})$ in $C_{\alpha(x/t)}$ and individual term t. Then $S(x/t)(X^\alpha(x/t)/U_{\alpha(x/t)})$ corresponds to the axiom $\Theta_\alpha(x/t) \vdash \top$, and hence $S(x/t)(X^\alpha(x/t)/U_{\alpha(x/t)})$ is in $C_\top$. Let

$$\mathcal{C} = (Z^\Gamma; ((\lambda L^{\exists x\alpha}.(\lambda x X^\alpha.S)(\pi_1 L^{\exists x\alpha})(\pi_2 L^{\exists x\alpha}))(T))^\top)$$

Then, by the definition of $\exists x C_\alpha$, $\mathcal{C}$ is in $C_\top$. Therefore $\mathcal{C}$ is strongly normalizable and thus $N(\mathcal{C}) < \infty$. Since T is a proper subterm of $\mathcal{C}$ this implies that $N(T) < \infty$ and hence T is strongly normalizable.

CR2: Suppose T is in $\exists x C_\alpha$, and let $T \succ^i T'$. Take any $S = (W^\Delta, X^\alpha; G^\gamma)$, such that S is in a CR C_γ of type $[\gamma]$, and x is not free in γ or Δ, and such that $S(x/t)(X^\alpha(x/t)/U_{\alpha(x/t)})$ is in $C_{\gamma(x/t)}$, for every individual term t and term $U_{\alpha(x/t)}$ in $C_{\alpha(x/t)}$. Let

$$\mathcal{C}(T) = (Z^\Gamma, W^\Delta; ((\lambda L^{\exists x\alpha}.(\lambda x X^\alpha.S)(\pi_1 L^{\exists x\alpha})(\pi_2 L^{\exists x\alpha}))(T))^\gamma)$$

Then $\mathcal{C}(T)$ is in C_γ. However $T \succ^i T'$, so $\mathcal{C}(T) \succ^i \mathcal{C}(T')$. Therefore, by CR2 for C_γ, $\mathcal{C}(T')$ is in C_γ and hence T' is $\exists x C_\alpha$.

CR3: Suppose T is neutral and all its immediate reducts are in $\exists x C_\alpha$. Take any $S = (W^\Delta, X^\alpha; G^\gamma)$, such that S is in a CR C_γ of type $[\gamma]$, and x is not free in γ or Δ, and such that $S(x/t)(X^\alpha(x/t)/U_{\alpha(x/t)})$ is in $C_{\gamma(x/t)}$, for every individual term t and term $U_{\alpha(x/t)}$ in $C_{\alpha(x/t)}$. Let

$$\mathcal{C}(T, S) = (Z^\Gamma, W^\Delta; ((\lambda L^{\exists x\alpha}.(\lambda x X^\alpha.S)(\pi_1 L^{\exists x\alpha})(\pi_2 L^{\exists x\alpha}))(T))^\gamma)$$

Since $\mathcal{C}(T, S)$ is neutral, by CR3 for C_γ, we only need to show that all its immediate reducts are in C_γ. We will prove this by induction on $N(S)$. By Lemma 5, since T and $\mathcal{C}(T, S)$ are neutral, it follows that all the immediate reducts of $\mathcal{C}(T, S)$ are either $\mathcal{C}(T', S)$, where $T \succ^i T'$, in which case T' is in $\exists x C_\alpha$, so $\mathcal{C}(T', S)$ is in C_γ, or else $\mathcal{C}(T, S')$, where $S \succ^i S'$; in this case $N(S') < N(S)$ so the result follows by Lemma 4 and the induction hypothesis.

$\forall x C_\alpha$: Let $T = (Z^\Gamma; F^{\forall x\alpha})$.

CR1: Suppose T is in $\forall x C_\alpha$. Take any individual term t, and let $\mathcal{C} = (Z^\Gamma; (T(t))^{\alpha(x/t)})$. Then $\mathcal{C}$ is in $C_{\alpha(x/t)}$. Therefore $\mathcal{C}$ is strongly normalizable and thus $N(\mathcal{C}) < \infty$. Since T is a proper subterm of $\mathcal{C}$ this implies that $N(T) < \infty$ and hence T is strongly normalizable.

CR2: Suppose T is in $\forall x C_\alpha$ and $T \succ^i T'$. Take any individual term t, and let $\mathcal{C}(T) = (Z^\Gamma; (T(t))^{\alpha(x/t)})$. Then $\mathcal{C}(T)$ is in $C_{\alpha(x/t)}$ and $\mathcal{C}(T) \succ^i \mathcal{C}(T')$. So, by CR2 for $C_{\alpha(x/t)}$, $\mathcal{C}(T')$ is in $C_{\alpha(x/t)}$ and hence T' is in $\forall x C_\alpha$.

CR3: Suppose T is neutral and all its immediate reducts are in $\forall x C_\alpha$. Take any individual term t, and let $\mathcal{C}(T) = (Z^\Gamma; (T(t))^{\alpha(x/t)})$. Then the immediate reducts of $\mathcal{C}(T)$ are $\mathcal{C}(T')$ where $T \succ^i T'$. Therefore, by assumption, T' is in $\forall x C_\alpha$ and hence $\mathcal{C}(T')$ is in $C_{\alpha(x/t)}$. So, since $\mathcal{C}(T)$ is neutral, by CR3 for $C_{\alpha(x/t)}$, $\mathcal{C}(T)$ is in $C_{\alpha(x/t)}$. Therefore, T is in $\forall x C_\alpha$.

□

Proof of theorem 4 (concluded).

($\otimes E$) Let $U = (W^\Theta, X_1^{\alpha_1}, X_2^{\alpha_2}; G^\gamma)$ and $R = (Z^\Gamma; F^{\alpha_1 \otimes \alpha_2})$ satisfy the hypothesis, and let

$$T = (Z^\Gamma, W^\Theta; ((\lambda L.(\lambda X_1^{\alpha_1} X_2^{\alpha_2}.U)(\pi_1 L)(\pi_2 L))(R))),$$

where L is a new term variable of type $\alpha_1 \otimes \alpha_2$. Then we wish to show that $T(y/s)(Y'/S')$ is in $C_{\gamma'}$. However this follows from the definition of $C_{\alpha'_1 \otimes \alpha'_2}$, and the induction hypothesis for R and U.

($\& I$) Let $T_{\alpha_1} = (Z^\Gamma; F^{\alpha_1})$ and $T_{\alpha_2} = (Z^\Gamma; F^{\alpha_2})$ satisfy the hypothesis, and let

$$T = (Z^\Gamma; (T_{\alpha_1}, T_{\alpha_2})^{\alpha_1 \& \alpha_2}).$$

Then we want to show that $R = T(y/s)(Y'/S')$ is in $C_{\alpha'_1 \& \alpha'_2}$. We do this by two subsidiary inductions on $N(R)$, which show that $(Z^{\Gamma'}; (\pi_1 R)^{\alpha'_1})$ and $(Z^{\Gamma'}; (\pi_2 R)^{\alpha'_2})$ are in, respectively, $C_{\alpha'_1}$ and $C_{\alpha'_2}$. Now, since $(Z^{\Gamma'}; (\pi_1 R)^{\alpha'_1})$ is a neutral term, we only need to show that all its immediate reducts are in $C_{\alpha'_1}$. These reducts are equivalent to the following forms:

1. $(Z^{\Gamma'}; (\pi_1 R')^{\alpha'_1})$, where $R \succ^i R'$. Then $N(R') < N(R)$, so the term is in $C_{\alpha'_1}$ by the subsidiary induction.
2. $T_{\alpha_1}(y/s)(Y'/S')$ which is in $C_{\alpha'_1}$ by hypothesis.

Similarly, since $(Z^{\Gamma'}; (\pi_2 R)^{\alpha'_2})$ is a neutral term, we only need to show that all its immediate reducts are in $C_{\alpha'_2}$. These reducts are equivalent to the following forms:

1. $(Z^{\Gamma'}; (\pi_2 R')^{\alpha'_2})$, where $R \succ^i R'$. Then $N(R') < N(R)$, so the term is in $C_{\alpha'_2}$ by the subsidiary induction.
2. $T_{\alpha_2}(y/s)(Y'/S')$ which is in $C_{\alpha'_2}$ by hypothesis.

($\&E$) Let $R = (Z^\Gamma; F^{\alpha_1 \& \alpha_2})$ satisfy the hypothesis, and let

$$T = (Z^\Gamma; (\pi_1 R)^{\alpha_1}) \quad \text{or} \quad T = (Z^\Gamma; (\pi_2 R)^{\alpha_2}).$$

Then we want to show that $T(y/s)(Y'/S')$ is in either $C_{\alpha'_1}$ or $C_{\alpha'_2}$, respectively. However, this follows from the definition of $C_{\alpha'_1 \& \alpha'_2}$, and the induction hypothesis for R.

($\oplus I$) Suppose $T_{\alpha_1} = (Z^\Gamma; F^{\alpha_1})$ satisfies the hypothesis, and let $T = (Z^\Gamma; (\pi_1, T_{\alpha_1})^{\alpha_1 \oplus \alpha_2}))$. Then we need to show that $R = T(y/s)(Y'/S')$ is in $C_{\alpha'_1 \oplus \alpha'_2}$.

Take any $U_{\alpha'_1} = (X_1^{\alpha'_1}, W^\Theta; G_1^\gamma)$ and $U_{\alpha'_2} = (X_2^{\alpha'_2}, W^\Theta; G_2^\gamma)$ such that $U_{\alpha'_1}$ and $U_{\alpha'_2}$ are in a CR $\mathcal{C}$ of type $[\gamma]$ and such that $U_{\alpha'_1}(X_1^{\alpha'_1}/V_{\alpha'_1})$ and $U_{\alpha'_2}(X_2^{\alpha'_2}/V_{\alpha'_2})$ are in $\mathcal{C}$, for every term $V_{\alpha'_1}$ in $C_{\alpha'_1}$ and term $V_{\alpha'_2}$ in $C_{\alpha'_2}$. Let

$$\mathcal{H}(R, U_{\alpha'_1}, U_{\alpha'_2}) = (Z^{\Gamma'}, W^\Theta; (\lambda L.((\pi_1 L)((\lambda X_1^{\alpha'_1}.U_{\alpha'_1})(\lambda X_2^{\alpha'_2}.U_{\alpha'_2}))(\pi_2 L)))(R)),$$

where L is a new term variable of type $\alpha'_1 \oplus \alpha'_2$. Then we need to show that $\mathcal{H}(R, U_{\alpha'_1}, U_{\alpha'_2})$ is in $\mathcal{C}$. We do this by subsidiary induction on $N(R) + N(U_{\alpha'_1}) + N(U_{\alpha'_2})$. Now, since $\mathcal{H}(R, U_{\alpha'_1}, U_{\alpha'_2})$ is a neutral term, we only need to show that all its immediate reducts are in $\mathcal{C}$. These reducts are euivalent to the following forms:

1. $\mathcal{H}(R', U_{\alpha'_1}, U_{\alpha'_2})$, where $R \succ^i R'$. Then $N(R') < N(R)$, so by the subsidiary induction the term is in $\mathcal{C}$.
2. $\mathcal{H}(R, U'_{\alpha'_1}, U_{\alpha'_2})$, where $U_{\alpha'_1} \succ^i U'_{\alpha'_1}$. Then $N(U'_{\alpha'_1}) < N(U_{\alpha'_1})$, so by the subsidiary induction, the term is in $\mathcal{C}$.
3. $\mathcal{H}(R, U_{\alpha'_1}, U_{\alpha'_2})$, where $U_{\alpha'_2} \succ^i U'_{\alpha'_2}$. Then $N(U'_{\alpha'_2}) < N(U_{\alpha'_2})$, so by the subsidiary induction, the term is in $\mathcal{C}$.
4. $U_{\alpha'_1}(X^{\alpha'_1}/T_{\alpha_1}(y/s)(Y/S))$ which is in $\mathcal{C}$ by hypothesis.

Similarly, if $T_{\alpha_2} = (Z^\Gamma; F^{\alpha_2})$ satisfies the hypothesis, and we let $T = (Z^\Gamma; (\pi_2, T_{\alpha_2})^{\alpha_1 \oplus \alpha_2}))$. Then we can show that $R = T(y/s)(Y'/S')$ is in $C_{\alpha'_1 \oplus \alpha'_2}$.

($\oplus E$) let $U_{\alpha_1} = (W^\Theta, X_1^{\alpha_1}; G^\gamma)$, $U_{\alpha_2} = (W^\Theta, X_2^{\alpha_2}; G^\gamma)$, and $R = (Z^\Gamma; F^{\alpha_1 \oplus \alpha_2})$ satisfy the hypothesis, and let

$$T = (Z^\Gamma, W^\Theta; (\lambda L.((\pi_1 L)((\lambda X_1^{\alpha_1}.U_{\alpha_1})(\lambda X_2^{\alpha_2}.U_{\alpha_2}))(\pi_2 L)))(R)),$$

where L is a distinct term variable of type $\alpha_1 \oplus \alpha_2$. Then we wish to show that $T(y/s)(Y'/S')$ is in $C_{\gamma'}$. However, this follows from the definition of $C_{\alpha'_1 \oplus \alpha'_2}$ and the induction hypothesis for R, U_{α_1} and U_{α_2}.

$(\perp\!\circ I)$ Let $T_{\alpha_2} = (Z^\Gamma, X^{\alpha_1}; F^{\alpha_2})$ satisfy the hypothesis, and let $T = (Z^\Gamma; (\lambda X^{\alpha_1}.T_{\alpha_2})^{\alpha_1 \perp\circ \alpha_2})$. Then we wish to show that $R = T(y/s)(Y'/S')$ is in $C_{\alpha'_1 \perp\circ \alpha'_2}$. Take any $U = (W^\Theta; G^{\alpha'_1})$ in $C_{\alpha'_1}$. Let

$$\mathcal{H}(R, U) = (Z^{\Gamma'}, W^\Theta; R(U)).$$

Then we need to show that $\mathcal{H}(R, U)$ is in $C_{\alpha'_2}$. We do this by a subsidiary induction on $N(R)+N(U)$. Now, since $\mathcal{H}(R, U)$ is a neutral term, we only need to show that all its immediate reducts are in $C_{\alpha'_2}$. These reducts are equivalent to the following forms:

1. $\mathcal{H}(R', U)$, where $R \succ^i R'$. Then $N(R') < N(R)$, so by the subsidiary induction, the term is in $C_{\alpha'_2}$.

2. $\mathcal{H}(R, U')$, where $U \succ^i U'$. Then $N(U') < N(U)$, so by the subsidiary induction, the term is in $C_{\alpha'_2}$.

3. $T_{\alpha_2}(y/s)(Y'/S', X^{\alpha'_1}/U)$ which by hypothesis is in $C_{\alpha'_2}$.

$(\perp\!\circ E)$ Let $U = (W^\Theta; G^{\alpha_1})$ and $R = (Z^\Gamma; F^{\alpha_1 \perp\circ \alpha_2})$ satisfy the hypothesis, and let

$$T = (W^\Theta, Z^\Gamma; R(U)).$$

Then we wish to that $T(y/s)(Y'/S')$ is in $C_{\alpha'_2}$. However this follows from the definition of $C_{\alpha'_1 \perp\circ \alpha'_2}$ and the induction hypothesis for R and U.

$(\forall I)$ Let $T_\alpha = (Z^\Gamma; F^\alpha)$ satisfy the hypothesis. Suppose x does not occur free in Γ, and let

$$T = (Z^\Gamma; (\lambda x.T_\alpha)^{\forall x \alpha}).$$

Then we wish to show that $R = T(y/s)(Y'/S')$ is in $C_{\forall x \alpha'}$. Take any term t. Then we need to show that $\mathcal{H}(R) = (Z^{\Gamma'}; R(t))$ is in $C_{\alpha'(x/t)}$. We do this by a subsidiary induction on $N(R)$. Now since $\mathcal{H}(R)$ is a neutral term, we only need to show that all its immediate reducts are in $C_{\alpha'(x/t)}$. These reducts are equivalent to the following forms:

1. $\mathcal{H}(R')$, where $R \succ^i R'$. Then $N(R') < N(R)$, so the term is in $C_{\alpha'(x/t)}$ by the subsidiary induction hypothesis.

2. $T_\alpha(y/s, x/t)(Y'/S')$, which is in $C_{\alpha'(x/t)}$ by hypothesis.

$(\forall E)$ Let $R = (Z^\Gamma; F^{\forall x\alpha})$ satisfy the hypothesis, let t be a term, and let $T = (Z^\Gamma; R(t))$. Then we wish to show that $T(y/s)(Y'/S')$ is in $C_{\alpha'(x/t)}$. However this follows from the definition of $C_{\forall x\alpha'}$ and the induction hypothesis for R.

$(\exists I)$ Let t be a term, let $T_{\alpha(x/t)} = (Z^\Gamma; F^{\alpha(x/t)})$, and let

$$T = (Z^\Gamma; (t, T_{\alpha(x/t)})^{\exists x\alpha}).$$

Then we wish to show that $R = T(y/s)(Y'/S')$ is in the CR $C_{\exists x\alpha'}$.

Take any $U = (W^\Theta, X^{\alpha'}; G^\gamma)$ such that U is in a CR $\mathcal{C}$ of type $[\gamma]$ such that x is not free in γ or Θ, and such that $U(x/t)(X^{\alpha'}(x/t)/V_{\alpha'(x/t)})$ is in $\mathcal{C}$, for every individual term t and term $V_{\alpha'(x/t)}$ in $C_{\alpha'(x/t)}$. Let

$$\mathcal{H}(R, U) = (Z^{\Gamma'}, W^\Theta; (\lambda L.((\lambda x X^{\alpha'}.U)(\pi_1 L)(\pi_2 L))(R))),$$

where L is a new term variable of type $\exists x\alpha'$. Then we need to show that $\mathcal{H}(R, U)$ is in $\mathcal{C}$. We do this by a subsidiary induction on $N(R) + N(U)$. Now since $\mathcal{H}(R, U)$ is a neutral term, we only need to show that all its immediate reducts are in $\mathcal{C}$. These reducts are equivalent to the following forms:

1. $\mathcal{H}(R', U)$, where $R \succ^i R'$. Then $N(R') < N(R)$, so the term is in $\mathcal{C}$ by the subsidiary induction hypothesis.
2. $\mathcal{H}(R, U')$, where $U \succ^i U'$. Then $N(U') < N(U)$, so the term is in $\mathcal{C}$ by the subsidiary induction hypothesis.
3. $U(x/t)(X^{\alpha'(x/t)}/T_{\alpha(x/t)}(y/s)(Y/S))$ which is in $\mathcal{C}$ by hypothesis.

$(\exists E)$ Let $U = (W^\Theta, X^\alpha; G^\gamma)$ and $R = (Z^\Gamma; F^{\exists x\alpha})$ satisfy the hypothesis. Suppose x is not free in γ, Θ, or Γ, and let

$$T = (Z^\Gamma, W^\Theta; (\lambda L.((\lambda x X^\alpha.U)(\pi_1 L)(\pi_2 L))(R))),$$

where L is a new term variable of type $\exists x\alpha$. Then we need to show $T(y/s)(Y'/S')$ is in $C_{\gamma'}$. However this follows from the definition of $C_{\exists x\alpha'}$ and the induction hypothesis for R and U.

$(!I)$ Let $T_{!\alpha_i} = (Z_i^{\Gamma_i}; F_i^{!\alpha_i})$, for $i = 1, \ldots, n$, and $T_\gamma = (X_1^{!\alpha_1}, \ldots, X_n^{!\alpha_n}; G^\gamma)$ satisfy the hypothesis, and let

$$T = (Z_1^{\Gamma_1}, \ldots, Z_n^{\Gamma_n}; ((\lambda X_1^{!\alpha_1} \cdots X_n^{!\alpha_n}.T_\gamma), T_{!\alpha_1} \ldots, T_{\alpha_n})^{!\gamma}).$$

Then we want to show that $R = T(y/s)(Y'/S')$ is in $C_{!\gamma}$. Let

$$\mathcal{H}(R) = (Z_1^{\Gamma'_1}, \ldots, Z_n^{\Gamma'_n}; (\iota R)).$$

Then we need to show that $\mathcal{H}(R)$ is in $C_{\gamma'}$. We do this by a subsidiary induction on $N(R)$. Now since $\mathcal{H}(R)$ is a neutral term, we only need to show that all its immediate reducts are in $C_{\gamma'}$. These reducts are equivalent to the following forms:

1. $\mathcal{H}(R')$, where $R \succ^i R'$. Then $N(R') < N(R)$, so the term is in $C_{\gamma'}$ by the subsidiary induction hypothesis.

2. $T_\gamma(y/s)(Y/S, X_1^{!\alpha'_1}/T_{!\alpha_1}(y/s)(Y/S), \cdots, X_1^{!\alpha'_n}/T_{!\alpha_n}(y/s)(Y/S))$ which is in $C_{\gamma'}$ by hypothesis.

$(!E)$ Let $R = (Z^\Gamma; F^{!\alpha})$ satisfy the hypothesis, and let $T = (Z^\Gamma; (\iota R))$. Then we want to show that $T(y/s)(Y'/S')$ is in $C_{\alpha'}$. However this follows from the definition of $C_{!\alpha'}$ and the induction hypothesis for R.

$\square$