Diregularity of almost Moore Digraphs

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ABSTRACT

An almost Moore digraph is a digraph of diameter $k \geq 2$, maximum out-degree $d \geq 2$ and order $n = d + d^2 + \ldots + d^k$, that is, one less than the Moore bound. In this paper, we present a study of the out-regularity of almost Moore digraphs, and then we prove that the in-degree of an almost Moore digraph is constant (= d), that is, every almost Moore digraph is diregular of degreed.

Keywords : diregularity, almost moore digraph, moore bound.

INTRODUCTION

Basic concept

A *directed graph* or *digraph G* is a pair of sets (V,A) where V is a finite nonempty set of distinct elements called *vertices*; and A is a set of ordered pairs (u,v) of distinct vertices $u,v \in A$ called *arcs*.

The *order n* of a digraph G is the number of vertices in G, that is, $n = |V|$. An *in-neighbour* (respectively, *out-neighbour*) of a vertex v in G is a vertex u (respectively, w) such that $(u,v) \in A$ (respectively, $(u,v) \in A$). The set of all in-neighbours (respectively, out-neighbours) of a vertex v is denoted by $N^+(v)$ (respectively, $N^-(v)$). The *in-degree* (respectively *out-degree*) of a vertex v is the number of its in-neighbours (respectively out-neighbours). We denote by $d^-(v)$ the in-degree of v in G. If the in-degree equals the out-degree (= d, say) for every vertex in G, then G is called a *diregular* digraph of degree d.

A *walk* of length l in a digraph G is an alternating sequence $v_0a_1v_1a_2\ldots a_lv_l$ of vertices and arcs in G such that $a_i = (v_{i-1},v_i)$ for each $1 \leq i \leq l$. A walk is closed if $v_0 = v_l$. If all the vertices of a v_0 - v_l walk are distinct, then such a walk is called a *path*. A *cycle* is a closed path. The *distance* from vertex u to v, denoted by $\delta(u,v)$, is the length of the shortest path from vertex u to vertex v. Note that in general $\delta(u,v)$ is not necessarily equal to $\delta(v,u)$. The *diameter* of digraph G is the longest distance between any two vertices in G.

The well known *degree/diameter problem* for digraphs is to determine the largest possible order $n_{d,k}$ of a digraph, given out-degree at most $d \geq 1$ and diameter $k \geq 1$. There is a natural upper bound on the order of digraphs given out-degree at most d and diameter k. For any given vertex v of a digraph G, we can count the number of vertices at a particular distance from that vertex. Let n_i, for $0 \leq i \leq k$, be the number of vertices at distance i from v. Then $n_i \leq d^k$, for $0 \leq i \leq k$, and consequently,

$$n_{d,k} = \sum_{i=0}^{k} n_i \leq 1 + d + d^2 + \ldots + d^k \qquad (1)$$

The right-hand side of (1), denoted by $M_{d,k}$, is called the *Moore bound*. If the equality sign holds in (1) then the digraph is called a *Moore digraph*. It is well known that Moore digraphs exist only in the cases when $d = 1$ (directed cycles of length $k + 1$, C_{k+1}, for any $k \geq 1$) or $k = 1$ (complete digraphs of order $d + 1$, K_{d+1}, for any $d \geq 1$) (Plesnik and Znam, 1974) dan (Bridges and Toueg, 1980).

The non-existence of Moore digraphs for given maximum out-degree $d \geq 2$ and diameter $k \geq 2$ motivated several authors to consider digraphs of order close to the Moore bound. One research activity concerning digraphs close to the Moore bound is the study of the existence of such digraphs. For example, several authors (Baskoro, *et al*., 1995) and (Baskoro, *et al*., 1998) and (Miller and Fris, 1992) and (Miller and Siran, 2001). studied the existence of almost Moore digraphs (that is, digraphs of order one less than the Moore bound) for particular out-degree d.

Another research activity concerning digraphs with order close to the Moore bound is the study of the diregularity of such digraphs, for example see (Slamin and Mirka Miller, 2000) and (Slamin, *et al*., 2002). In this paper, we consider the diregularity of almost Moore digraphs. We start this paper with the concept of *repeat* followed by a study of the out-regularity of digraphs close to the Moore bound, and then a study of the in-regularity of almost Moore digraphs.

RESEARCH PROCEDURE

This research was conducted by the following procedure.

1. Studying the known results concerning on digraphs of order close to Moore bound.
2. Studying the out-regularity of almost Moore digraphs that is the digraph of order one less than the Moore bound.
3. Proving the diregularity of almost Moore digraphs as consequence of the out-regularity and the in-regularity of almost Moore digraphs.

RESULTS AND DISCUSSION

Repeat

Let G be an almost Moore digraph of out-degree $d \geq 2$ and diameter $k \geq 2$. We will use the following notation throughout. For each vertex u of an almost Moore digraph G and for $1 \leq s \leq k$, let $T_k^+(u)$ be the multiset of all endvertices of directed paths in G of length at most s which start at u. Similarly, by $T_k^-(u)$ we denote the multiset of all starting vertices of directed paths of length at most s in G which terminate at u. Observe that the vertex u is in both $T_k^+(u)$ and $T_k^-(u)$, as it corresponds to a path of zero length. Let $N_k^+(u)$ be the set of all endvertices of directed paths in G of length exactly s which start at u. Similarly, by $N_k^-(u)$ we denote the set of all starting vertices of directed paths of length exactly s in G which terminate at u. If $s = 1$ the sets $N_k^+(u)$ and $N_k^-(u)$ represent the out- and in-neighbourhood of the vertex u in the digraph G; we denote them simply by $N^+(u)$ and $N^-(u)$, respectively, as introduced in Section 1. We depict the notations $T_s^+(u)$ and $N_s^+(u)$, for $1 \leq s \leq k$, as shown in Figure 1.

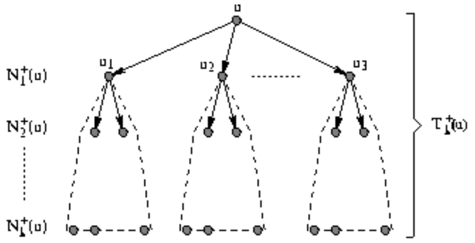
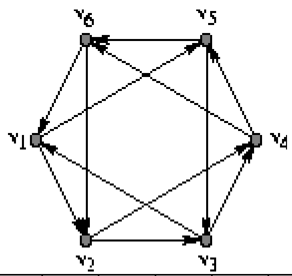


Figure 1: Multiset $T_k^+(u)$

A counting argument presented in (Baskoro, *et al*., 1995) shows that for each vertex u of G there exists exactly one vertex $r(u)$ in G with the property that there are two $u \rightarrow r(u)$ walks in G of length not exceeding k. The vertex $r(u)$ is called the *repeat* of u; this concept was introduced in (Miller and Fris, 1992). If also the maximum in-degree is d then it follows from (Baskoro, *et al*., 1995) that the mapping $v \rightarrow r(v)$ is an *automorphism* of the digraph G. Figure 2 shows an example of an almost Moore digraph of out-degree 2, diameter 2 and order 6 with its repeats.



Vertex	V_1	V_2	V_3	V_4	V_5	V_6
Repeat	V_3	V_4	V_3	V_3	V_1	V_2

Figure 2: Digraph and its repeats

For further considerations it will be useful to state a few facts (which easily follow from (Baskoro, *et al*., 1995) about repeats and the sets introduced above. Let G be an almost Moore digraph for the pair d,k and let $N^+(u) = \{u_1,u_2,\ldots,u_d\}$ for some vertex u of G. Then the union of all sets $T_{k-1}^+(u_i)$, $1 \leq i \leq d$, has cardinality either $M_{d,k}$ - 1 if $u = r(u)$, or $M_{d,k}$ - 2 if $u \neq r(u)$. In the first case the sets $T_{k-1}^+(u_i)$ are pairwise disjoint and the vertex u lies in exactly one directed cycle of length k (and in no shorter directed cycles). In the second case there are exactly two sets among the $T_{k-1}^+(u_i)$ that intersect, and their intersection consists of a single vertex, namely $r(u)$, the vertex u is then contained in no directed cycle of length at most k.

Out-regularity of digraphs of order close to the Moore bound

In this section we consider the out-regularity of digraphs of order close to the Moore bound, that is, the digraphs of order $M_{d,k} - \delta$, for $1 \le \delta < d$. The out-regularity of digraphs of order close to the Moore bound was considered by Baskoro, Miller and Plesník as mentioned in (Baskoro, *et al.*, 1998). We state their observation in the following theorem.

Theorem 1 *The digraph of maximum out-degree $d \ge 2$, diameter $k \ge 2$ and order $M_{d,k} - \delta$, for $1 \le \delta < d$, must be out-regular of out-degree d.*

Proof. We suppose that the digraph contains a vertex u with out-degree $d_1 < d$ (i.e., $d_1 \le d-1$), then considering the number of vertices in the out-bound spanning tree starting from vertex u, the order of the digraph,

$$n \le 1 + d_1 + d_1 d + ... + d_1 d^{k-1}$$
$$n = 1 + d_1(1 + d + ... + d^{k-1})$$
$$n \le 1 + (d-1)(1 + d + ... + d^{k-1})$$
$$n = (1 + d + ... + d^k) - (1 + d + ... + d^{k-1})$$
$$n < M_{d,k} - d$$

Hence the out-degree of any vertex in a digraph of order $M_{d,k} - \delta$, for $1 \le \delta < d$, must be equal to d, that is, such a digraph must be out-regular.

Establishing the diregularity or otherwise of the in-degree of digraphs of order close to the Moore bound is not easy. We shall consider the in-regularity of almost Moore digraphs in the next section.

In-regularity of almost Moore digraphs

A Moore digraph for a given pair d, k has diameter *equal* to k and the out-degree of *each* of its vertices *equal* to d. It is easy to show that all *in-degrees* in such a Moore digraph must be equal to d. These observations follow from the well known classification (Bridges and Toueg, 1980) and (Plesnik and Znam, 1974) which says that Moore digraphs exist only in the trivial cases when $d = 1$ (directed cycles of length $k + 1$, for any $k \ge 1$) or $k = 1$ (complete digraphs of order $d + 1$, for any $d \ge 1$).

In contrast with Moore digraphs, establishing the regularity or otherwise of in-degree of an *almost* Moore digraph is by no means obvious. This is best documented by the fact that there *exist* digraphs of out-degree d and diameter k whose order is just two or three less than the Moore bound and in which *not all* vertices have the same in-degree! To see this we give, for $i = 2, 3$, two examples of digraphs G_i of diameter 2, out-degree i, and order $M_{i,2} - i$ with vertices not all of the same in-degree, as shown in Figure 3.

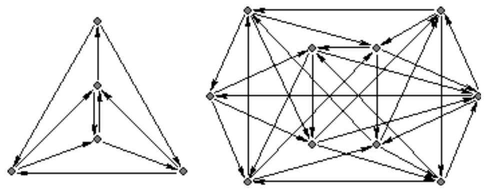


Figure 3: Non-diregular digraphs G_2 and G_3

Recall that an almost Moore digraph has $M_{d,k} - 1$ vertices. In this section we consider the regularity of the in-degree for all almost Moore digraphs. We first prove two auxiliary results concerning the possible existence of low in-degree vertices in almost Moore digraphs.

Lemma 1 *Let G be an almost Moore digraph of out-degree d and diameter k and let S be the set of all vertices of G of in-degree less than d. Then*

1. $|S| \le d$ and each vertex in S has in-degree d - 1;
2. $S \subseteq N^+(r(u))$ for each vertex u of G.

Proof. Let v be a vertex in S. Consider an arbitrary vertex $u \in V(G)$, $u \ne v$, and let $N^+(u) = \{u_1, u_2, \ldots, u_d\}$ be the out-neighbourhood of u. As the diameter of G is equal to k, the vertex v must occur in each of the sets $T_k^+(u_i)$, $1 \le i \le d$. It follows that for each i, $1 \le i \le d$, there exists a vertex $w_i \in \{u\} \cup T_{k-1}^+(u_i)$ such that $w_i v$ is a directed edge of G. Since the in-degree of v is less than d, the in-neighbours w_i of v are not all distinct; without loss of generality we may assume that $w_1 = w_2$. The case $w_1 = w_2 = u$ is clearly absurd. But then the only possible vertex which occurs in both $T_{k-1}^+(u_i)$, i = 1, 2, is $r(u)$, the repeat of u. We thus have $w_1 = w_2 = r(u)$. Invoking the comments immediately preceding the Lemma we see that no other pair of in-neighbours of v can have this property, that is, the vertices $w_2,\ldots,w_d$ are mutually distinct. Therefore the in-degree of v is d - 1, and $v \in N^+(r(u))$ for *any* vertex u of G. Realizing that $|N^+(r(u))| = d$, both statements of the Lemma follow.

As usual, if u is a vertex of a digraph G then we denote by $d^-(u)$ the in-degree of u in G. Further, for $j \geq 0$ let $N_j^-(u)$ be the set of all starting vertices of directed paths in G of length *exactly j* whose endvertex is u.

Lemma 2 *Let G be an almost Moore digraph of out-degree d ≥ 2 and diameter k ≥ 2, and, again, let S be the set of all vertices of G of in-degree less than d. Then either S is empty or |S| = d.*

Proof. Let S' be the set of vertices of G whose in-degree is larger than d. Assume that $|S| \neq \emptyset$. Let $\sigma = \sum_{w \in S'} (d^-(w) - d)$. Since every vertex of G has out-degree d, it follows that the *average* in-degree of G is d as well. Combined with Lemma 1, this gives

$$\sigma = \sum_{w \in S'} (d^-(w) - d) = \sum_{v \in S} (d - d^-(v)) = |S|$$

Take an arbitrary vertex $v \in S$; from Lemma 1 we see that $|N^+(v)| = |N_1^+(v)| = d - 1$. By the diameter assumption, the union of all the sets $N_t^-(v)$, for $0 \leq t \leq k$ is the entire vertex set $V(G)$ of G, which implies that

$$|V(G)| \leq \sum_{t=0}^{k} |N_t^-(v)| \qquad (2)$$

Our next goal is to estimate the above sum. Observe that for $2 \leq t \leq k$ we have

$$|N_t^-(v)| \leq \sum_{u \in N_{t-1}^-(v)} d^-(u) = d\,|N_{t-1}^-(v)| + \varepsilon_t \qquad (3)$$

where $\varepsilon_2 + \varepsilon_3 + \ldots + \varepsilon_k \leq \sigma$. It is not difficult to see that a safe upper bound on the sum in (2) is obtained from the recurrence inequality (3) by setting $\varepsilon_2 = \sigma = |S|$ and $\varepsilon_t = 0$, for $3 \leq t \leq k$; note that the latter is equivalent to assuming that *all* vertices from $S \setminus \{v\}$ are contained in $N_k^-(v)$ and that all vertices of S belong to $N_1^-(v)$. This way we successively obtain:

$$|V(G)| \leq 1 + |N_1^-(v)| + |N_2^-(v)| + |N_3^-(v)| + \ldots + |N_k^-(v)|$$
$$|V(G)| \leq 1 + (d-1) + (d(d-1) + |S|)(1 + d + \ldots + d^{k-2})$$ But G is
$$|V(G)| = d + d^2 + \ldots + d^k + (|S| - d)(1 + d + \ldots + d^{k-2})$$
$$|V(G)| = M_{d,k} - 1 + (|S| - d) M_{d,k-2}$$

an almost Moore digraph and so $V(G) = M_{d,k} - 1$; this together with the preceding inequality and Lemma 1 gives $|S| = d$. Thus the set S is either empty or contains exactly d vertices.

Note that it is possible for the set S to be empty as in the case of the almost Moore digraph in Figure 2.

With the two Lemmas in hand we are now in the position to prove that all almost Moore digraphs are diregular.

Theorem 2 *Let G be an almost Moore digraph of out-degree d ≥ 2 and diameter k ≥ 2. Then the in-degree of each vertex of G is equal to d.*

Proof.
Assume the contrary and let an almost Moore digraph G for $d,k \geq 2$ contain a vertex of in-degree not equal to d, that is, the set S of the vertices of G whose in-degree is less than d is non-empty. By Lemma 2, we have $|S| = d$; let $S = \{u_1, u_2,\ldots, u_d\}$. Applying the second part of Lemma 1 to $u = u_1$ and recalling that the out-degree of each vertex is d, we see that

$$S = N^+(r(u_1)) = \{u_1, u_2, \ldots, u_d\} \qquad (4)$$

It follows that $r(u_1) = N_1^-(u_1)$. We know from Lemma 1 that the in-degree of u_1 is equal to d - 1; let $N^-(u_1) = \{z_1, z_2, \ldots, z_{d-1}\}$ where $z_1 = r(u_1)$. By the diameter assumption, each vertex of $S \setminus \{u_1\}$ must appear in one of the sets $T_{k-1}^-(z_j)$, $1 \leq j \leq d - 1$. First we notice that $S \cap T_{k-1}^-(z_1) \neq \emptyset$. Indeed, if some vertex $u_i \in S$ were in $T_{k-1}^-(z_1)$ then, according to (4), we would have in G a directed cycle through u_i of length at most k. This would imply that $r(u_i)$ =

u_i which is impossible since, by Lemma 1 applied to $u = u_i$, we have $u_i \in N^+(r(u_i))$. The pigeonhole principle now shows that there are two vertices $u_{i1}, u_{i2} \in S \cap T^-_{k-1}(z_j)$ for some j such that $2 \le j \le d - 1$. Since by (4) we also have $u_{i1}, u_{i2} \in N^+(r(u_1))$, we thus obtain two $r(u_1) \to z_j$ walks in G of length at most k and therefore the vertex z_j is the repeat of $r(u_1)$.

But invoking Lemma 1, again, the set S is a subset of the out-neighbourhood of *any* vertex which is a repeat; in particular, $u_{i1}, u_{i2} \in N^+(z_j)$. This gives rise to two distinct directed cycles of length at most k through the vertex z_j. However, the existence of two such cycles through a single vertex contradicts the facts listed in the short summary in the last paragraph of Section 2. Therefore, an almost Moore digraph is in-regular of in-degree d.

Corollary 1 *Every almost Moore digraph is diregular*.

CONCLUSION

We have proved that every almost Moore digraph (that is, the digraph of order one less than the Moore bound) is diregular. However, the diregularity of digraph of defect two (that is, the digraph of order two less than the Moore bound) for any out-degree and diameter is still open. Thus for future work direction on the diregularity of digraphs close to the Moore bound, we conclude this paper with the following open problem.

Open Problem 1
Is every digraph of order two less than the Moore bound diregular?

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BIBLIOGRAPHY

Baskoro E.T., M. Miller and J. Plesník, On the structure of diregular digraphs with order close to the Moore bound, *Graphs and Combinatorics*, to appear.

Baskoro E.T., M. Miller, J. Plesník and Š. Znám, Digraphs of degree 3 and order close to the Moore bound, *Journal of Graph Theory* **20** (1995) 339-349.

Baskoro E.T., M. Miller, J. Plesník and Š. Znám, Digraphs of degree 3 and order close to Moore bound, *J. Graph Theory* **20** (1995) 339-349.

Baskoro E.T., M. Miller and J. Plesník, On the structure of digraphs with order close to the Moore bound, *Graphs and Combinatorics* **14(2)** (1998) 109-119.

Plesník J. and Š. Znám, Strongly geodetic directed graphs, *Acta F. R. N. Univ. Comen. - Mathematica* **XXIX** (1974) 29-34.

Miller M. and I. Fris, Maximum order digraphs for diameter 2 or degree 2, *Pullman volume of Graphs and Matrices, Lecture Notes in Pure and Applied Mathematics* **139** (1992) 269-278.

Miller M. and Slamin, On the monotonicity of minimum diameter with respect to order and maximum out-degree, *Proceedings of COCOON 2000, Lecture Notes in Computer Science* **1858** (D.-Z. Du, P. Eades, V. Estivill-Castro, X. Lin (eds.)) (2000) 193-201.

Miller M. and J. Širán, Digraphs of degree two which miss the Moore bound by two, *Discrete Mathematics* **226** (2001) 269-280.

Slamin and M. Miller, Diregularity of digraphs close to Moore bound, *Prosiding Konperensi Nasional Matematika X, MIHMI* **6 (No.5)** (2000) 185-192.

Slamin, E.T. Baskoro and M. Miller, Diregularity of digraphs of out-degree three and order two less than Moore bound, *Proceedings of Twelveth Australasian Workshop on Combinatorial Algorithms* (2001) 164-173.

Bridges W.G. and S. Toueg, On the impossibility of directed Moore graphs, *J. Combinatorial Theory* Series **B29** (1980) 339-341.