DYNAMICAL BEHAVIOR OF A ROTOR UNDER ROTATIONAL RANDOM BASE EXCITATION

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ABSTRACT
This paper investigates the dynamical behavior of a rotor under a random rotational base excitation, which is assumed to be a stationary and ergodic truncated Gaussian white noise. As the base motion is a rotation, the equations of motion present both internal and external random excitations. The stability of the rotor is then studied by computing the largest Lyapunov exponent with an iterative formula. Then, the power spectrum density of the stationary forced response is obtained from a Monte Carlo simulation. Finally, we perform a comparative analysis on the influence of the required number of modal shapes to describe accurately the response.

INTRODUCTION
Rotating machines are used in many industrial domains like automotive, aeronautics or nuclear. In these applications the rotor base is frequently submitted to external motions. In the case where this motion is a rotation $\alpha$, Duchemin [1] shows that the equations of motion present time-varying coefficients as follows:

$$M\ddot{x} + C\dot{x} + Kx + K_p\dot{\alpha}^2x = f_1\dot{\alpha} + f_2\ddot{\alpha}$$  \hspace{1cm} (1)

where $M$, $K$ and $C$ are the mass, stiffness, damping and gyroscopic matrices of the rotor respectively. The vectors $f_1$ and $f_2$ weight the speed $\dot{\alpha}$ and the acceleration $\ddot{\alpha}$ of the base rotation and represent the external excitation. In Eqn. (1), the base rotation introduces parametric stiffness terms given by the matrix $K_p$ linked to $\dot{\alpha}^2$.

If $\alpha$ is an harmonic function, Eqn. 1 is periodic and many papers [1-3] have already been devoted to study the stability and the forced response of such systems. If $\alpha$ is a random process, the equations of motion become stochastic. Consequently, all tools used for the deterministic applications are not adapted. This kind of stochastic differential equations has been widely investigated in the literature and three large families of methods can be distinguished for studying the stability issue.

First, the stochastic moments of the Ito’s equation, due to the Stratonovitch and Khasminskii theory, are obtained by stochastic averaging. Subramanian [4] and Sri Namachchivaya [5] employ this method to define the stochastic stability boundaries of gyroscopic systems parametrically excited by stationary processes. The almost-sure stability of systems perturbed by stationary real-noise processes is defined by computing of the largest Lyapunov exponent which describes the exponential growth or decay of the response. This approach could be interpreted as a generaliza-
tion of the Floquet theory. Semi-analytical expressions of this exponent are proposed in [6] from a perturbation method and in [7] by combining the stochastic averaging and the Khasminskii's procedure. But these expressions are not suitable to all stochastic systems. Then numerical computations have to be envisaged: Talay [8] and Zentner and Poirion [9] suggested to compute it by an iterative formulation.

To complete this approach, the computation of the moment Lyapunov exponents can be envisaged. Xie [10] and Namachchivaya and co-workers [11] apply perturbation methods to obtain semi-analytical expressions of these moments for single or two degrees of freedom (DOF) systems under parametric excitation. Exact solutions of the response power spectrum density function of the response ([12]). To and Orisamolu [13] computed the two first response stochastic moments of a stationary and ergodic random process. This process is modeled by a Gaussian truncated white noise. However these formulations can not be generalized to all stochastic multi DOF linear systems. Regarding computations of the forced response, many features of it could be estimated. First, one can consider the probability density function of the response ([12]). To and Orisamolu [13] computed the two first response stochastic moments of a two-DOF system subjected to a parametric non-stationary random excitation. Exact solutions of the response power spectrum density (PDS) are defined by Dimenber [14] and Iourtchenko [15] for single DOF systems under Gaussian and non-Gaussian parametric noises respectively. The response PSD of multi DOF systems can be computed by a Monte Carlo simulation.

In this paper, we investigate the dynamical behavior of a rotor whose base rotation \( \alpha \) is assumed to be a second order stationary random process. This process is modeled by a Gaussian truncated white noise and is defined by its power spectrum density (PSD):

\[
S_{\alpha}(\omega) = |H(\omega)|^2 S_0
\]  

where \( S_0 \) is the PSD of the white noise and \( H(\omega) \) is the transfer function of a 2n order Butterworth filter. The almost-sure stability is first studied from the largest Lyapunov exponent computed with the Talay’s iterative formulation. By assuming that the response is stationary, its PSD is obtained by a Monte Carlo simulation. Indeed, even in this simple case (a stationary and ergodic random process of given input PSD), there is no semi-analytical expression for the output PSD response. At last a parametric study regarding the main features of the excitation and modal truncation is conducted. It provides helpful results to designers in case of such issue.

**Numerical Tools to Study the Dynamical Behavior**

In this section, we present the numerical tools used to investigate the dynamical behavior of the rotor.

**Generation of time samples of the random process**

In industrial applications, a random excitation is defined from its PSD. As the base motion is a rotation modeled by the angle \( \alpha \), the known PSD is about the process \( \{\alpha\} \). In Eqn. (1), time samples of the rotational speed \( \dot{\alpha} \) and acceleration \( \ddot{\alpha} \) are required. These samples can not be generated directly from the time derivatives of the trajectories of \( \{\alpha\} \). They have to be independent and \( \{\dot{\alpha}\} \) and \( \{\ddot{\alpha}\} \) are two distinguished processes. However, as \( \{\alpha\} \) is a stationary process, the PSD of the derivative process \( S_{\dot{\alpha}} \) is defined as:

\[
S_{\dot{\alpha}}(\omega) = \omega^2 S_{\alpha}(\omega)
\]  

So, the time samples generation of both processes \( \{\dot{\alpha}\} \) and \( \{\ddot{\alpha}\} \) is achieved from the PSD \( S_{\alpha} \). Among the different methods of simulation, the Markovianization is used to build time histories of a stationary Gaussian process [16]. This approach is based on a rational decomposition of the process PSD. Particularly the transfer function \( H(\omega) \) is the one of a Butterworth filter decomposed of \( n \) second order filters linked in cascade. \( H(\omega) \) is rewritten in the following form:

\[
H(\omega) = \prod_{k=1}^{n} H_k(j\omega) = \prod_{k=1}^{n} \frac{\Omega_k^2}{\Omega_k^2 + j\omega \beta_k \Omega_k - \omega^2}
\]  

where \( H_k(j\omega) \) is the transfer function of the kth second order filter and \( \Omega_k \) is the cutting frequency. By introducing (2) and (4) into (3), it is possible to define \( n \) intermediary processes \( z_k \) \( (k = 1..n) \) corresponding to each second order filtering. Their PSD are:

\[
S_{z_1} = \omega^2 |H_1(j\omega)|^2 S_0
\]
\[
S_{z_2} = |H_2(j\omega)|^2 S_{z_1}
\]
\[
\vdots
\]
\[
S_{z_n} = |H_n(j\omega)|^2 S_{z_{n-1}}
\]

By considering a process \( z_{10} \) such that \( z_{1} = z_{10} \) and \( S_{z_1} = \omega^2 S_{z_{10}} \), the first expression of Eqn. (5) is replaced by:

\[
S_{z_{10}} = |H_1(j\omega)|^2 S_0
\]

Then, the differential equations corresponding to each process are:

\[
\ddot{z}_{10} + \beta_1 \Omega_c \dot{z}_{10} + \Omega_c^2 z_{10} = \Omega_c^2 N_r
\]
\[
\ddot{z}_2 + \beta_2 \Omega_c \dot{z}_2 + \Omega_c^2 z_2 = \Omega_c^2 z_{10}
\]
\[
\vdots
\]
\[
\ddot{z}_n + \beta_n \Omega_c \dot{z}_n + \Omega_c^2 z_n = \Omega_c^2 z_{n-1}
\]
where \( N_t \) is the white noise. By writing Eqn. (7) in a state space form, a differential Ito’s equation is then built by:

\[
dt \begin{bmatrix} z_n \\ z_n’ \\ \vdots \\ z_2 \\ z_2’ \\ z_{10} \\ z_{10}’ \end{bmatrix} = A \begin{bmatrix} z_n \\ z_n’ \\ \vdots \\ z_2 \\ z_2’ \\ z_{10} \\ z_{10}’ \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \Omega_c^2 \end{bmatrix} dW_t
\]

with the matrix \( A \) defined by:

\[
A = \begin{bmatrix}
0 & 1 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\Omega_c^2 - \Omega_c \beta & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & \cdots & 0 & 1 & 0 & \cdots & \cdots \\
0 & 0 & \cdots & -\Omega_c^2 - \Omega_c \beta & \cdots & 0 & \Omega_c^2 \\
0 & 1 & \cdots & 0 & 0 & 0 & 1 \\
0 & 0 & \cdots & 0 & 0 & -\Omega_c^2 - \Omega_c \beta & 1
\end{bmatrix}
\]

In this expression, \( W_t \) is the Wiener process. It is linked to the white noise by: \( N_t = dW_t/\sqrt{dt} \). The time histories of the random process are then simulated by solving Eqn. (8) with a Newmark integration scheme. For each time step, \( dW_t \) is an independent sample event taken from a random Gaussian variable with a zero mean value and a standard deviation equal to \( \sqrt{dt} \).

Finally the time histories of each random process present in the equations of motion (1) are provided by:

\[
\alpha = z_n, \quad \alpha^2 = z_n^2, \quad \ddot{\alpha} = \ddot{z}_n
\]

**Stability analysis**

Before computing the forced response, the almost-sure stability of the system has to be studied. To this end, the largest Lyapunov exponent \( \lambda \) of Eqn. (1) without right hand side is considered and is defined by the formula:

\[
\lambda = \lim_{t \to \infty} \frac{1}{t} \ln \| y \| \tag{10}
\]

where \( y \) is the response of the homogeneous equations of motion written in the state form:

\[
\dot{y} = A_t y
\]

with

\[
y = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}, \quad A_t = \begin{bmatrix} 0 & \mathbf{I}_d \\ -M^{-1} (K + K_0 \alpha^2) & -M^{-1} C \end{bmatrix}
\]

It is proved [17] that relation (10) converges almost surely toward a deterministic number, which is independent of the initial value of problem (11): it is the maximum Lyapunov exponent. Following the sign of \( \lambda \), one can conclude on the stability of the system. If \( \lambda \leq 0 \), the system is stable almost surely. As the Lyapunov exponent is a deterministic number, only one time history of the process \( \{ \alpha^2 \} \) is used to compute it. In practice, because there is no any analytical expression, \( \lambda \) is computed from a very long time history of \( \alpha^2 \) until the convergence is reached. To avoid problems of numerical stability, Talay suggested to implement an iterative formula to compute \( \lambda \). Then, at the time \( t_{j+1} \), we have:

\[
\lambda_{j+1} = \lambda_j \left( 1 - \frac{1}{j+1} \right) + \frac{\ln \| y_j + 1 \|}{(j+1)dt} \tag{12}
\]

where \( y_{j+1} \) is the state response of (11) computed at the time \( t_{j+1} \) from the initial condition:

\[
y_j = \frac{\dot{y}_j}{\| y_j \|}
\]

This iterative formula (12) is applied until \( \lambda \) tends towards a constant value.

**Random steady state response**

Once the stability of the system is confirmed by the sign of the largest Lyapunov exponent, the dynamical forced response has to be computed. To this end, the Monte Carlo simulation has been employed in this work. This procedure is considered as the only numerical “true” method for solving this kind of stochastic differential problem.

We recall here briefly the principal steps of this numerical process. Beforehand, a large number of truncated time samples of the random processes \( \alpha, \alpha^2, \ddot{\alpha} \) is generated following the procedure described in the previous section. Then, each trajectory is applied to the system and the corresponding response is computed using a Newmark scheme. Afterward a PSD estimator corresponding to each truncated response is determined. Finally, the PSD of the global response is obtained from the average of all the response PSD estimators. Of course, the larger the number of time samples, the better the approximation of the PSD response.

**APPLICATION TO A ROTOR**

Let us consider the rotor shown in Fig. 1(a). This device is composed by a flexible steel shaft whose length is \( L = 0.4 \) m and radius is \( r = 0.01 \) m. A rigid steel disk of radius \( R = 0.15 \) m is mounted at the third of the shaft length. The rotor is supported by two bearings located at both ends of the shaft and characterized by linear stiffnesses and dampings. Table 1
between the rotor and the support \( \Omega \) provides the stiffness and damping coefficients given in N.m\(^{-1}\) and in N.s.m\(^{-1}\) respectively. As the rotor base is assumed to be rigid, it is not modeled in this application.

### Equations of motion

We briefly recall how to obtain the equations of motion in the form of Eqn. (1). The rotor motion is modeled by introducing three reference frames which are fixed to the rotor \( (R_R) \), the rotor base \( (R_b) \) and the ground \( (R_0) \) (Fig. 1). Only the base rotation \( \alpha \) around the \( x_0 \) direction is modeled. The angular velocity vectors between the rotor and the support \( \Omega^S_R \) and between the support and the ground \( \Omega^G_0 \) are then given by:

\[
\Omega^S_R = \begin{bmatrix}
\hat{\theta} \cos \phi - \psi \cos \theta \sin \phi \\
\hat{\phi} + \psi \sin \theta \\
\hat{\theta} \sin \phi + \psi \cos \theta \cos \phi
\end{bmatrix} \quad \Omega^G_0 = \begin{bmatrix}
\alpha \\
0 \\
0
\end{bmatrix}
\tag{13}
\]

where \( \psi, \theta \) and \( \phi \) are the Euler angles displayed in Fig. 1(b). The rotor rotation speed \( \Omega \) is assumed to be constant. Then, the angular displacement \( \phi \) is equal to \( \Omega t \). In order to apply the Lagrange's equations, the kinetic and strain energies of both disk and shaft and the generalized forces due to the bearings are computed.

The Rayleigh-Ritz method has been chosen by considering four modal shapes to describe the rotor displacement. They are both rigid body modes and both first bending strain shapes of a beam simply supported at both ends. These functions are given by:

\[
f_1(y) = 1 \quad f_2(y) = Ay + B \quad f_3(y) = \frac{\sin \frac{\pi y}{L}}{\frac{\pi y}{L}} \\
f_4(y) = a_0 y^6 + a_3 y^5 + a_4 y^4 + a_3 y^3 + a_2 y^2 + a_1 y + a_0
\tag{14}
\]

where \( f_1 \) and \( f_2 \) are the cylindrical and conical rigid body modes. The functions \( f_3 \) and \( f_4 \) correspond to the first and second beam bending mode shapes respectively, which are obtained by a finite element model of the rotor supported at both ends. The \( f_4 \) function is interpolated by a 6 order polynomial whose coefficients are \( a_i (i = 0, 6) \). Then, the rotor displacements \( u(y, t) \) and \( w(y, t) \) in the \( x \) and \( z \) directions respectively are:

\[
u(y, t) = f_1(y)q_1(t) + f_2(y)q_3(t) + f_3(y)q_5(t) + f_4(y)q_7(t) \\
w(y, t) = f_1(y)q_2(t) + f_2(y)q_4(t) + f_3(y)q_6(t) + f_4(y)q_8(t)
\tag{15}
\]

After introducing Eqn. (15) into the energies and the generalized forces, the Lagrange’s equations are applied and provide:

\[
M\ddot{q} + C\dot{q} + (K + K_p \alpha^2)q = f_s \dot{\alpha} + f_p \alpha
\tag{16}
\]

where \( M \) is the symmetric mass matrix:

\[
M = \begin{bmatrix}
m_{11} & 0 & m_{13} & 0 & m_{15} & 0 & m_{17} & 0 \\
m_{13} & m_{11} & 0 & m_{15} & 0 & m_{17} & 0 & 0 \\
m_{15} & 0 & m_{13} & m_{11} & 0 & m_{17} & 0 & 0 \\
m_{17} & 0 & 0 & m_{15} & m_{13} & m_{11} & 0 & 0 \\
m_{23} & 0 & 0 & 0 & m_{55} & 0 & m_{57} & 0 \\
m_{25} & 0 & 0 & 0 & m_{57} & m_{55} & 0 & m_{53} \\
m_{27} & 0 & 0 & 0 & 0 & m_{53} & m_{77} & 0 \\
m_{53} & 0 & 0 & 0 & m_{57} & 0 & m_{77} & m_{55} \\
\end{bmatrix}
\]
\( K \) is the symmetric stiffness matrix:

\[
K = \begin{bmatrix}
  c_{b_11} & 0 & c_{b_13} & 0 & 0 & 0 & 0 & 0 \\
  0 & c_{b_11} & 0 & c_{b_13} & 0 & 0 & 0 & 0 \\
  c_{b_13} & 0 & c_{b_{33}} & -g_{34} & 0 & -g_{36} & 0 & -g_{38} \\
  0 & c_{b_{13}} & 0 & c_{b_{33}} & g_{34} & 0 & g_{36} & 0 \\
  0 & 0 & -g_{34} & 0 & g_{36} & 0 & -g_{38} & 0 \\
  0 & 0 & 0 & 0 & 0 & g_{38} & 0 & g_{38} \\
  0 & 0 & -g_{38} & 0 & -g_{38} & 0 & -g_{78} & 0 \\
  0 & 0 & -g_{38} & 0 & g_{38} & 0 & g_{78} & 0
\end{bmatrix}
\]

In this matrix, the \( g_{ij} \) terms correspond to the gyroscopic effects and have a skew-symmetric distribution. The symmetric distribution of \( c_{b_{ij}} \) terms represent the damping due to the bearings.

\( K \) is the symmetric stiffness matrix:

\[
K = \begin{bmatrix}
  k_{b_{11}} & 0 & k_{b_{13}} & 0 & 0 & 0 & 0 & 0 \\
  0 & k_{b_{11}} & 0 & k_{b_{13}} & 0 & 0 & 0 & 0 \\
  k_{b_{13}} & 0 & k_{b_{33}} & 0 & 0 & 0 & 0 & 0 \\
  0 & k_{b_{13}} & 0 & k_{b_{33}} & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & k_{34} & 0 & k_{36} & 0 \\
  0 & 0 & 0 & 0 & 0 & k_{36} & 0 & k_{38} \\
  0 & 0 & 0 & 0 & k_{38} & 0 & k_{58} & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & k_{58} & 0
\end{bmatrix}
\]

In this matrix, the \( k_{ij} \) and the \( k_{b_{ij}} \) terms are due to the shaft bending and to the bearing respectively. One can note that, in the damping and stiffness matrices, the terms corresponding to the bearings act only on the degrees of freedom (DOF) of the rigid body modes. Indeed, the bending modes have been computed by considering the rotor simply supported at its ends. Then, the displacements and velocities at the location (shaft ends) of the real bearings (not rigid) are equal to zero for these two bending modes. Moreover, we can also observe that the stiffness terms due to the shaft bending do not act on the DOF of the rigid body modes.

\( K_p \) is the symmetric matrix of parametric stiffness terms (16):

\[
K_p = \begin{bmatrix}
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  k_{p_{22}} & 0 & k_{p_{24}} & 0 & k_{p_{26}} & 0 & k_{p_{28}} & 0 \\
  k_{p_{33}} & 0 & k_{p_{35}} & 0 & k_{p_{37}} & 0 & k_{p_{39}} & 0 \\
  k_{p_{44}} & 0 & k_{p_{46}} & 0 & k_{p_{48}} & 0 & k_{p_{50}} & 0 \\
  k_{p_{55}} & 0 & k_{p_{57}} & 0 & k_{p_{59}} & 0 & k_{p_{61}} & 0 \\
  k_{p_{66}} & 0 & k_{p_{68}} & 0 & k_{p_{70}} & 0 & k_{p_{72}} & 0 \\
  k_{p_{77}} & 0 & k_{p_{79}} & 0 & k_{p_{81}} & 0 & k_{p_{83}} & 0 \\
  SYM
\end{bmatrix}
\]

The vectors of the generalized forces are given by:

\[
f_x = \begin{bmatrix} f_{x1} \\ f_{x3} \\ 0 \\ f_{x5} \\ 0 \\ f_{x7} \\ 0 \\ f_{x9} \end{bmatrix} \quad f_z = \begin{bmatrix} 0 \\ f_{z2} \\ f_{z4} \\ f_{z6} \\ 0 \end{bmatrix}
\]

They expressions of all the terms \( m_{ij}, g_{ij}, c_{b_{ij}}, k_{ij}, k_{b_{ij}}, f_{xi} \) and \( f_{zi} \) are given in appendix.

**Results**

Now we present the results of the dynamical response of the rotor described in the previous section.

**Campbell diagram** The natural frequencies of the rotor are computed from the eigenvalues of Eqn. (16) putting the terms due to the rotor rotation to zero. The Campbell diagram (Fig. 2) is then obtained by plotting these frequencies with respect to the operating speed \( \Omega \). We can remark that the range of the frequencies is very large: the last natural frequency is sixty times larger than the first one. Then, to respect the Shannon’s theorem the time discretization used in the Newmark scheme has to be defined by: \( dt = \pi / \Omega_{max} \) where \( \Omega_{max} = 20 \omega_1 \). If the duration \( T = Ndt \) of the simulations is not enough long, the frequency discretization \( d\omega = 2\pi / T \) is not enough small to describe all frequencies of the rotor. In other words, simulations have to be realized by considering a large number \( N \) of time discretisations.
Stability Analysis  As observed in Eqn. (16), the base rotation reveals parametric stiffness terms which can lead to instabilities. In our application, this rotation is assumed to be a stationary and ergodic random process. It is modeled by a Gaussian white noise whose PSD is given by \( S_\alpha(\omega) = A_0 \) where \( A_0 \) is a constant value. This white noise is truncated by a 20th order Butterworth filter at the cutting frequency \( \Omega_c \). In Eqn. (16), the stiffness matrix \( K_p \) is multiplied by the base rotation velocity \( \dot{\alpha}^2 \) and we have seen in Eqn. (3) that the PSD of the derivative process \( \{ \dot{\alpha} \} \) was: \( S_\alpha(\omega) = \omega^2 S_\alpha(\omega) \). In other words, \( S_\alpha \) has a parabolic distribution and its maximal magnitude increases with the cutting frequency \( \Omega_c \). Figure 3 displays this relationship. Then, two parameters seem to influence the rotor stability: \( A_0 \) and \( \Omega_c \).

In order to ensure that \( \lambda \) tends towards a constant value, this one has been computed from a time history of \( \dot{\alpha}^2 \) simulated over \( N = 2^{18} \) points. Figure 4(a) illustrates how the convergence is reached by plotting the evolution of \( \lambda \) with respect to the time for \( A_0 = 10^{-6} \) rad.s and \( \Omega_c = 2000 \) rad.s\(^{-1} \) when the operating speed is \( \Omega = 5000 \) rpm.

Finally, the stability has been studied by computing the largest Lyapunov exponent \( \lambda \) for \( A_0 \) varying from \( 10^{-6} \) to \( 10^{-4} \) rad.s and \( \Omega_c \) from 50 to 6500 rad.s\(^{-1} \). Figure 4(b) presents \( \lambda \) with respect to these two parameters. As expected, we observe that the value of \( \lambda \) increases when \( A_0 \) and \( \Omega_c \) increase. From the sign of \( \lambda \), a stability card has been established on Fig. 4(c) where the dots represent the unstable zone for which \( \lambda > 0 \). This representation confirms that the rotor becomes unstable when \( A_0 \) and \( \Omega_c \) are increasing. To confirm the results provided by Fig. 4(c), we have plotted the time responses of the rotor submitted to a sample of the base rotation for \( \Omega_c = 2000 \) rad.s\(^{-1} \) and two values of \( A_0 \). Following Fig. 4(c), for \( A_0 = 10^{-6} \) and \( A_0 = 2.5 \times 10^{-5} \) rad.s, the rotor should be stable and unstable respectively. The time responses plotted on Fig. 4(d) and 4(e) confirm these affirmations. Indeed, the response for \( A_0 = 10^{-6} \) rad.s (Fig. 4(d)) seems to be bounded when the time is increasing. But, if \( A_0 = 2.5 \times 10^{-5} \) rad.s, the magnitude of the response increases as an unbounded function. We cannot say if this evolution is exponential because the response cannot be written as the product between a random function and an envelop in \( e^{\lambda t} \). To know which statistic moment is unbounded, the moment Lyapunov exponents should be computed.

Steady state forced response  In practical applications, a rotor is designed to withstand a defined excituation level. In this section, we consider that the rotation \( \alpha \) of the rotor base is a white noise of magnitude \( A_0 = 10^{-6} \) rad.s truncated at \( \Omega_c = 2000 \) rad.s\(^{-1} \). These values have obviously been chosen into the stable domain defined in Fig. 4(c). The response is computed by a Monte Carlo simulation from 1000 time samples.

![Figure 3. PSD OF THE PROCESSES {\( \alpha \)} and {\( \dot{\alpha} \)} (solid line: \( \Omega = \Omega_{\text{max}}/8 \); dotted line: \( \Omega = \Omega_{\text{max}}/4 \); dashed line: \( \Omega = \Omega_{\text{max}}/2 \))](image)

![Figure 4. STABILITY AND LYAPUNOV EXPONENTS](image)
samples of the random processes modeled over \( N = 2^{15} \) points. Then the time and frequency discretisations are \( dt = 2.8 \times 10^{-5} \) s and \( d\omega = 6.85 \) rad.s\(^{-1}\) respectively. Figure 5 plots the response PSD \( S_{uu}, S_{ww} \) and \( S_{uw} \), in the \( x \) and \( z \) directions and in the cross direction \( xz \) respectively when the operating speed is \( \Omega = 5000 \) rpm. All the peaks correspond to the natural frequencies displayed on the Campbell diagram. Obviously, the second bending mode is not excited because the cutting frequency is lesser than \( \omega_8 \). We observe also that the rigid body modes exhibit higher response levels than those observed for the first bending mode. This is a classical result in the rotor dynamic field.

**Influence of the number of the modal shapes used**

The aim of this section is to study how the modeling of the rotor influences the results on the dynamical response. In particular, we search how many modal shapes have to be considered to provide goods results in terms of frequency localization and response magnitude. This study is very useful for finite element investigations because it provides a practical information in view of a modal truncature of the model.

To this end, we consider three different modelings: the case called A in which only the 2 rigid body modes have been taken into account (i.e.: \( f_1 \) and \( f_2 \) functions), the case B including the 2 rigid body modes and the first bending modal shape (i.e.: \( f_1, f_2 \) and \( f_3 \) functions) and the case C corresponding to the entire system (four modal shapes).

**Influence on the Campbell diagram**

Here we compare the Campbell diagrams obtained from the three modeling cases. Figure 6(a) shows the frequencies of the two rigid body modes for the cases A and B. Obviously, a shift frequency phenomena is observed. We can first conclude that only the rigid body modes are not sufficient to describe accurately the first critical speeds. In Fig. 6(b), the frequencies of the three first modes (two rigid body and first bending mode) of the cases B and C are compared. The good agreement between the results leads us to guess that the second bending mode is not necessary to define the system.

**Influence on the forced response**

To verify if the second bending mode is necessary to describe the rotor behavior, we compare here the PSD of the forced response for the cases B and C. As before, the operating speed \( \Omega \) is equal to 5000 rpm, the magnitude of the white noise is \( A_0 = 10^{-6} \) rad.s and the cutting frequency is \( \Omega_c = 2000 \) rad.s\(^{-1}\). The responses are obtained from a Monte Carlo simulation over 1000 time samples of \( N = 2^{15} \) points. Figure 7 plots the response PSD \( S_{uu} \) in the \( x \) direction corresponding to the cases B and C. One more time, we observe that the results match very well. It can then be concluded that only the two rigid body and the first bending modes are sufficient to have a good description of this rotor.

**CONCLUSION**

In this paper, the dynamical behavior of a rotor whose base is subjected to a stationary and ergodic random rotational motion
has been investigated. Because it seems difficult to obtain analytical formulations, some numerical tools to model the random process and to study the rotor behavior have been presented. Finally, an application has proved that the rotor behavior is very well described by considering only both rigid body and the first bending modes. This information is then very useful for finite element studies because the system size can considerably be decreased. Then the computation time, which can be very large for Monte Carlo simulations, is too decreased. Works are in development to focus on the case of a non-stationary random base rotation as earthquake application.

References


Appendix A: Coefficients of the rotor matrices

The coefficients of the mass matrix $M$ are given by:

$$
m_{11} = M_d f_1(L_d)^2 + \rho S \int_0^L f_1(y)^2 dy$$
$$m_{13} = M_d f_1(L_d) f_2(L_d) + \rho S \int_0^L f_1(y) f_2(y) dy$$
$$m_{15} = M_d f_1(L_d) f_3(L_d) + \rho S \int_0^L f_1(y) f_3(y) dy$$
$$m_{17} = M_d f_1(L_d) f_4(L_d) + \rho S \int_0^L f_1(y) f_4(y) dy$$
$$m_{33} = M_d f_2(L_d)^2 + I_{dmg2}(L_d)^2 + \rho \int_0^L [S f_2(y)^2 + I_{amg2}(y)^2] dy$$
$$m_{35} = M_d f_2(L_d) f_3(L_d) + I_{dmg2}(L_d) g_3(L_d) + \rho \int_0^L [S f_2(y) f_3(y) + I_{amg2}(y) g_3(y)] dy$$
$$m_{37} = M_d f_2(L_d) f_4(L_d) + I_{dmg2}(L_d) g_4(L_d) + \rho \int_0^L [S f_2(y) f_4(y) + I_{amg2}(y) g_4(y)] dy$$
$$m_{55} = M_d f_3(L_d)^2 + I_{dmg3}(L_d)^2 + \rho \int_0^L [S f_3(y)^2 + I_{amg3}(y)^2] dy$$
$$m_{57} = M_d f_3(L_d) f_4(L_d) + I_{dmg3}(L_d) g_4(L_d) + \rho \int_0^L [S f_3(y) f_4(y) + I_{amg3}(y) g_4(y)] dy$$
$$m_{77} = M_d f_4(L_d)^2 + I_{dmg4}(L_d)^2 + \rho \int_0^L [S f_4(y)^2 + I_{amg4}(y)^2] dy$$

The gyroscopic terms are:

$$g_{34} = \Omega_d y_2(L_d)^2 + 2\Omega \rho \int_0^L g_2(y)^2 dy$$
$$g_{36} = \Omega_d y_2(L_d) g_3(L_d) + 2\Omega \rho \int_0^L g_2(y) g_3(y) dy$$
$$g_{38} = \Omega_d y_2(L_d) g_4(L_d) + 2\Omega \rho \int_0^L g_2(y) g_4(y) dy$$
$$g_{56} = \Omega_d y_3(L_d)^2 + 2\Omega \rho \int_0^L g_3(y)^2 dy$$
$$g_{58} = \Omega_d y_3(L_d) g_4(L_d) + 2\Omega \rho \int_0^L g_3(y) g_4(y) dy$$
$$g_{78} = \Omega_d y_4(L_d)^2 + 2\Omega \rho \int_0^L g_4(y)^2 dy$$

The damping and stiffness terms due to the bearings are:

$$c_{b11} = c_{c11} f_1(0)^2 + c_{c21} f_1(L)^2$$
$$c_{b13} = c_{c11} f_1(0) f_2(0) + c_{c21} f_1(L) f_2(L)$$
$$c_{b33} = c_{c11} f_2(0)^2 + c_{c22} f_2(L)^2$$
$$k_{b11} = k_{c11} f_1(0)^2 + k_{c21} f_1(L)^2$$
$$k_{b13} = k_{c11} f_1(0) f_2(0) + k_{c21} f_1(L) f_2(L)$$
$$k_{b33} = k_{c11} f_2(0)^2 + k_{c22} f_2(L)^2$$

The stiffness terms due to the bending of the shaft are:

$$k_{55} = E I_{am} \int_0^L h_3(y)^2 dy$$
$$k_{57} = E I_{am} \int_0^L h_3(y) h_4(y) dy$$
$$k_{77} = E I_{am} \int_0^L h_4(y)^2 dy$$

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The parametrical stiffness terms due to the rotation of the rotor base are:

\[
\begin{align*}
    k_{p22} &= -M_d f_1(L_d)^2 - \rho S \int_0^L f_1(y)^2 \, dy \\
    k_{p24} &= -M_d f_1(L_d) f_2(L_d) - \rho S \int_0^L f_1(y) f_2(y) \, dy \\
    k_{p26} &= -M_d f_1(L_d) f_3(L_d) - \rho S \int_0^L f_1(y) f_3(y) \, dy \\
    k_{p28} &= -M_d f_1(L_d) f_4(L_d) - \rho S \int_0^L f_1(y) f_4(y) \, dy \\
    k_{p33} &= (I_{dm} - I_{dy}) g_2(L_d)^2 - \rho L_{am} \int_0^L g_2(y)^2 \, dy \\
    k_{p35} &= (I_{dm} - I_{dy}) g_2(L_d) g_3(L_d) - \rho L_{am} \int_0^L g_2(y) g_3(y) \, dy \\
    k_{p37} &= (I_{dm} - I_{dy}) g_2(L_d) g_4(L_d) - \rho L_{am} \int_0^L g_2(y) g_4(y) \, dy \\
    k_{p44} &= -M_d f_2(L_d)^2 - \rho S \int_0^L f_2(y)^2 \, dy \\
    k_{p46} &= -M_d f_2(L_d) f_3(L_d) - \rho S \int_0^L f_2(y) f_3(y) \, dy \\
    k_{p48} &= -M_d f_2(L_d) f_4(L_d) - \rho S \int_0^L f_2(y) f_4(y) \, dy \\
    k_{p55} &= (I_{dm} - I_{dy}) g_3(L_d)^2 - \rho L_{am} \int_0^L g_3(y)^2 \, dy \\
    k_{p57} &= (I_{dm} - I_{dy}) g_3(L_d) g_4(L_d) - \rho L_{am} \int_0^L g_3(y) g_4(y) \, dy \\
    k_{p66} &= -M_d f_3(L_d)^2 - \rho S \int_0^L f_3(y)^2 \, dy \\
    k_{p68} &= -M_d f_3(L_d) f_4(L_d) - \rho S \int_0^L f_3(y) f_4(y) \, dy \\
    k_{p77} &= (I_{dm} - I_{dy}) g_4(L_d)^2 - \rho L_{am} \int_0^L g_4(y)^2 \, dy \\
    k_{p88} &= -M_d f_4(L_d)^2 - \rho S \int_0^L f_4(y)^2 \, dy
\end{align*}
\]

where \( E \) and \( \rho \) are the Young modulus and the volumic mass of the rotor. \( L_d, M_d, I_{dm} \) and \( I_{dy} \) are the position, the mass, the diametral and polar inertias of the disk. \( S, L_{am} \) are the section surface and the inertia of the shaft. The functions \( g_i \) and \( h_i \) \((i = 1 \ldots 4)\) are the first and second derivatives of the \( f_i \) functions with respect to \( y \) respectively. The stiffness and damping terms indexed by 1 and by 2 are due to the bearings located at \( y = 0 \) and \( y = L \) respectively.