



Hans Freudenthal: a mathematician on didactics and curriculum theory

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The main ideas in the work of Hans Freudenthal (1905-1990), the Dutch mathematician and mathematics educator, related to curriculum theory and didactics are described. Freudenthal's educational credo, 'mathematics as a human activity', is explored. From this pedagogical point of departure, Freudenthal's criticism of educational research and educational theories is sketched and fleshed out. Freudenthal's approaches to mathematics education, developmental research and curriculum development can be seen as alternatives to the mainstream 'Anglo-Saxon' approaches to curriculum theory.

During his professional life, Hans Freudenthal's views contradicted almost every contemporary approach to educational 'reform': the 'new' mathematics, operationalized objectives, rigid forms of assessment, standardized quantitative empirical research, a strict division of labour between curriculum research and development, or between development and implementation. Looking back from the present, it is of great interest to see how his ideas, which may at the time have seemed to embody recalcitrance for its own sake, have now become widely accepted. It would, of course, be far-fetched to suggest that this correlation implies a causal relationship, but it does indicate Hans Freudenthal's special role, not only in mathematics education, but also in the development of curriculum theory and research methodology.

Hans Freudenthal had already earned his spurs as a research mathematician when he developed an interest in mathematics education and made himself acquainted with educational and psychological traditions in Europe and the US. Today, he is probably best known as one of the most influential mathematics educators of his time.¹ In this paper, we shall try to highlight some of Freudenthal's main ideas, while acknowledging that we cannot do justice to his wide-ranging work – even if we were able to.² Our point of view will centre on curriculum theory and pedagogy, and we

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will consider the aspects of Freudenthal's work and theories that are relevant from those perspectives (Blankertz 1973, Hopmann and Riquarts 1995a, b).

We start with a consideration of Freudenthal's place within the curriculum-meets-Didaktik discussion and try to clarify the main sources of Freudenthal's theory of mathematics education. We continue by elaborating Freudenthal's philosophy of mathematics education, the cornerstone of his work. We move on to his criticism of what had become 'tradition' in educational research and educational theory. We subsequently discuss Freudenthal's proposed alternative to 'research, development and diffusion', i.e. the traditional RD&D model. His alternative incorporates a philosophy, or theory, of mathematics education, its elaboration in developmental research, and an understanding of the overarching process of educational development in which this is embedded. This will be followed by a short description and analysis of Freudenthal's influence on educational change in the Netherlands. We conclude with a discussion of curriculum theory, didactics, and 'mathematics for all'.

Didaktik, curriculum theory, and Freudenthal

Although it has been suggested that curriculum theory as developed in the US and European didactics are concerned with the same questions, there are striking differences in approach (Westbury 1995, 2000). These differences derive from more basic differences in cultural, philosophical and institutional backgrounds. In the European context, the concept of Didaktik is embedded in a pedagogical theory; thus the notion of pedagogy as a form of *Geisteswissenschaftliche* along with the phenomenological theories of *Bildung* (i.e. 'formation'), have their points of departure in the *practice* of education, i.e. in educational reality. And *Bildung* can be seen as opposed to *Ausbildung*. *Bildung* refers to the ideal of personality formation, and does not entail simply the transmission of knowledge, but also the development of the knowledge, norms and values associated with 'good' citizenship and/or a membership of the cultural and intellectual elite. *Ausbildung*, on the other hand, refers to vocational and professional education. In this context, Didaktik is primarily concerned with theories of the aims and content of education and instruction.³

In the Netherlands, Didaktik was related to the *Geisteswissenschaftliche* phenomenological pedagogy, as represented, for example, by Langeveld (1965) at the University of Utrecht. This position lost its dominance in the 1960s and 1970s and, as a consequence, the concept of a general Didaktik was (to some extent, and gradually) replaced by formal models of learning and instruction as seen in the work of US educational psychologists such as Robert Glaser, Robert de Cecco, and Benjamin Bloom. However, subject-matter Didaktik – as developed within faculties and institutes of mathematics and science education – was not totally swamped by this movement.

Although it does not show in his rare references, Freudenthal was well-acquainted with educational and psychological traditions in Europe and the

US. He made many visits, which he called '*Bildung* journeys', and was also influenced by the pedagogical ideas of his wife Suus Lutter-Freudenthal, one of the driving forces behind the reception of Peter Petersen's Jena-plan movement in the Netherlands. He was also strongly influenced by the reform-pedagogy of the Belgian educationist Ovide Decroly (Freudenthal 1973b) and was an active member of the New Education Fellowship in which Ovide Decroly participated, where he was also influenced by Pierre van Hiele and van Hiele's wife, Hieke Geldof-van Hiele (both mathematics teachers who conducted their doctoral research under Freudenthal and Langeveld).⁴ For example, Decroly's educational idea of *centre d'intérêt*, which could be elaborated in space and time, resembles Freudenthal's ideas on the learning of mathematics in 'real-life' contexts. Decroly's principle of elaboration in space (how it appears in different countries) and time (development in history), under the teacher's guidance, corresponds to Freudenthal's (and Dewey's) idea of guided reinvention.

Although Freudenthal never referred to scholars like the German Wolfgang Klafki, the basic questions that Klafki addressed were also Freudenthal's questions (see Freudenthal 1973a): What is to be taught in a school subject? for what purpose? and to whom? His credo 'mathematics as a human activity' can be seen as an expression of a *Geisteswissenschaftliche*, phenomenological theory of mathematics education which has its point of departure in the practice of education and teaching, and not in the transmission of mathematics as a pre-formed system. Some of his main ideas (such as 'reinvention') and his criticism of the 'antididactic inversion' of traditional (deductive) instruction were probably inspired by the reform-pedagogy movement, i.e. by progressive education in which the ideas of, among others, Peter Petersen and Maria Montessori were important.

As viewed by Freudenthal, curriculum theory is not a fixed, pre-stated set of theories, aims and means, contents, and methods. Rather, it is always related to processes. Understood positively, the word 'curriculum' is more often than not used in combination with change or development, for example, as in curriculum development or developmental research. For Freudenthal, curriculum theory was a practical endeavour from which new theoretical ideas might arise as a kind of scientific by-product. For him, curriculum development was not to be conducted from academic ivory towers, but in schools, in collaboration with teachers and students (Freudenthal 1973a). Similar ideas are expressed by Schwab (1970: see also Walker 1990), who, in his plea for curriculum as 'practical', eloquently challenged the mainstream RD&D curriculum theory of his time. Thus, there are similarities between some branches of the Anglo-Saxon approach to curriculum theory and Freudenthal's understanding of curriculum.

But, when the word 'curriculum' appears in the work of Freudenthal, it usually has a negative connotation. He writes about the mainstream of the Anglo-Saxon curriculum movement as a theory-driven, top-down endeavour, and referred to this approach as 'boxology'. As we have suggested, the most striking feature of Freudenthal's position is his view of curricula as processes and he proposed his own alternative to curriculum development which he called *educational development*. Whereas curriculum development

centres on the development of curriculum materials, Freudenthal wanted to go one step further: educational development should seek to foster actual change in on-going classroom teaching. Consequently, such educational development is much more than instructional design; it is an all-embracing innovation strategy, based, on the one hand, on an explicit educational philosophy and, on the other hand, incorporates developments in all sorts of educational materials as *part* of its strategy. The engine of this whole process is *developmental research*, an approach which fits the pedagogical tradition very well; it is a qualitative/interpretative research tied to teaching experiments in individual classrooms. A central role is given to *dialogue* between researchers, curriculum developers, and teachers.

Mathematics as a human activity

Freudenthal was an outspoken opponent of the 'new mathematics' of the 1960s, which took its starting point as the attainment of modern mathematics, especially set theory. With this criticism, he showed himself as an exponent of the pedagogical tradition in the sense that his criticisms were grounded in a discussion about what was to be taught, and why. Thus, he acknowledged generality and wide applicability as special characteristics of mathematics, and he also acknowledged that modern mathematics abstracted mathematics even further while at the same time enhancing flexibility. However, in his view, abstracting was the source of the pedagogical problem.

In an objective sense the most abstract mathematics is without doubt also the most flexible. But not subjectively, since it is wasted on individuals who are not able to avail themselves of this flexibility (Freudenthal 1968: 5).

Since the applicability of mathematics was also often problematic, he concluded that mathematics had to be taught in order to be *useful*. He observed that this could not be accomplished by simply teaching a 'useful mathematics'; that would inevitably result in a kind of mathematics that was useful only in a limited set of contexts. However, he also rejected the alternative: 'If this means teaching pure mathematics and afterwards showing how to apply it, I'm afraid we shall be no better off. I think this is just the wrong order' (Freudenthal 1968: 5). Instead, mathematics should be taught as *mathematizing*. This view of the task of school mathematics was not only motivated by its importance for usefulness; for Freudenthal mathematics was first and foremost an activity, a *human* activity, as he often emphasized. As a research mathematician, doing mathematics was more important to Freudenthal than mathematics as a ready-made product. In his view, the same should hold true for mathematics education: mathematics education was a process of doing mathematics that led to a result, mathematics-as-a-product. In traditional mathematics education, the result of the mathematical activities of *others* was taken as a starting point for instruction, and Freudenthal (1973b) characterized this as an *anti-didactical inversion*. Things were upside down if one started by teaching the result of an activity rather than by teaching the activity itself.⁵

[Mathematics as a human activity] is an activity of solving problems, of looking for problems, but it is also an activity of organizing a subject matter. This can be a matter from reality which has to be organized according to mathematical patterns if problems from reality have to be solved. It can also be a mathematical matter, new or old results, of your own or others, which have to be organized according to new ideas, to be better understood, in a broader context, or by an axiomatic approach (Freudenthal 1971: 413–414).

He termed this organizing activity ‘mathematizing’ in other publications and it should be emphasized that it involves both ‘matter from reality’ and ‘mathematical matter’. In other words, Freudenthal included both applied mathematics and pure mathematics in his conception of mathematizing. In this sense, his starting point differed from other mathematics educators who also emphasized mathematical activity but focused on a mathematical discourse that was modelled on the discourse of pure research mathematicians – as this was reconstructed, e.g. by Lakatos (1976).

The image of mathematical activity that Freudenthal elected as a paradigm for mathematics education differed from this in two ways. First, it included, as mentioned earlier, applied mathematics, or to be more precise, ‘mathematizing matter from reality’. Secondly, the focus was *not* on the form of the activity, but on the activity itself, as well as on its effect. Moreover, the notion of ‘discourse’ referred to a social practice, whereas the idea of mathematizing put a stronger emphasis on mental activity. Freudenthal’s broader definition of mathematics as a human activity fitted in better with a more pragmatic discourse, such as one might expect in applied mathematics. In such discourse there would be more emphasis on adequacy and efficiency, and less on goal-free conjecturing, for instance.

Freudenthal used the word ‘mathematizing’ in a broad sense: it was a form of *organizing* that also incorporated mathematical matter. By choosing the word ‘organizing’, Freudenthal also indicated that, for him, mathematizing was not just a translation into a ready-made symbol system. Instead, a way of symbolizing might emerge in the process of organizing the subject matter. It was the organizing activity itself that was central to Freudenthal’s conception.

Mathematizing literally stands for ‘making more mathematical’. To clarify what ‘more mathematical’ means, one may think of such characteristics of mathematics as generality, certainty, exactness, and brevity. To clarify what is to be understood by mathematizing we may look at the following specific strategies within these characteristics (Gravemeijer 1994; see also Treffers 1987):

- *for generality*: generalizing (looking for analogies, classifying, structuring);
- *for certainty*: reflecting, justifying, proving (using a systematic approach, elaborating and testing conjectures, etc.);
- *for exactness*: modelling, symbolizing, defining (limiting interpretations and validity); and
- *for brevity*: symbolizing and schematizing (developing standard procedures and notations).

Viewed from this angle, mathematizing subject matter from mathematics and mathematizing matter from reality share the same characteristics. And, this was fundamental for Freudenthal, since, in his view, mathematics education for young children should aim above all at mathematizing everyday reality. Young children cannot mathematize mathematical matter, since, at the beginning, there is no mathematical matter that is experientially real to them. Moreover, mathematizing subject matter from reality also familiarizes the students with a mathematical approach to everyday-life situations. We may also refer here to the mathematical activity of 'looking for problems', mentioned by Freudenthal, which implies a mathematical attitude that encompasses knowledge of the possibilities and the limitations of a mathematical approach, i.e. knowing when a mathematical approach is appropriate and when it is not.

This emphasis on 'mathematizing reality' fits in with the call for 'mathematics for all' (see Damerow and Westbury 1985, Keitel 1987). Freudenthal stressed that not all students are future mathematicians: for the majority, all the mathematics they will ever use will be to solve problems in everyday-life situations. Therefore, familiarizing students with a mathematical approach to this type of problem-solving deserved to be a highest priority in mathematics education. This goal could be combined with the objective of having students mathematize problem situations that would be experientially real to them.⁶

In this light, it will not come as a surprise that Freudenthal forcefully attacked the *transposition didactique*, espoused by the French mathematics educator Chevallard (1985), who took the expert knowledge of the mathematician as his point of departure:

The mathematics that the vast majority of our future citizens learn in school does not reflect any kind of rendering – for didactic purposes or otherwise – of philosophical or scientific insights, unless they are those of an epoch long past (Freudenthal 1986: 326; our translation).

According to Keitel (1987), the central question is to realize a 'mathematics for all' that remains 'mathematics'. Consequently, she argues, it may be necessary at times for the teacher to leave behind everyday-life problems and refer to the science of mathematics – in order to show the constellations of concepts, structures, and systems which have been invented and tested there. Elaborating Freudenthal's idea of mathematizing, Treffers (1987) made a distinction between horizontal and vertical mathematization. The former involves converting a contextual problem into a mathematical problem, the latter involves taking mathematical matter onto a higher plane. Vertical mathematization can be induced by setting problems which admit solutions on different mathematical levels.

Freudenthal (1991: 41, 42) characterized this distinction as follows:

Horizontal mathematization leads from the world of life to the world of symbols. In the world of life one lives, acts (and suffers); in the other one symbols are shaped, reshaped, and manipulated, mechanically, comprehendingly, reflectingly: this is vertical mathematization. The world of life is what is experienced as reality (in the sense I used the word before), as is a symbol

world with regard to abstraction. To be sure the frontiers of these worlds are vaguely marked. The worlds can expand and shrink – also at one another's expense.

As Freudenthal indicates, the boundaries between what is to be denoted as 'horizontal mathematization' and 'vertical mathematization' are not clear-cut. The crux lies in what is to be understood as 'reality' and he (1991: 17) provided the following elucidation: 'I prefer to apply the term reality to what common sense experiences as real at a certain stage'. Reality is understood as a mixture of interpretation and sensual experience, which implies that mathematics, too, can become part of a person's reality. Reality and what a person counts as common sense are not static but grow, and are affected by the individual's learning process. This is also how Freudenthal's (1991: 18) statement 'Mathematics starting at, and staying within, reality' must be understood.

It will be clear that, in Freudenthal's view, 'common sense' and 'reality' were construed from the viewpoint of the actor. This implies that the boundary between horizontal and vertical mathematization has to be assessed from the actor's point of view as well. Whether a certain aspect of a person's mathematical activity is to be called 'vertical' or 'horizontal' depends on the question as to whether the activity involves some extension of that person's mathematical reality. A symbolizing activity, for instance, could be a routine activity for a student. This would be a case of horizontal mathematizing. However, if the same manner of symbolizing were a new invention for another student, then this would involve vertical mathematization. Vertical mathematization is the most clearly visible if a student explicitly replaces his or her solution method by one on a higher level. This could be a shift to a solution method, or a way of describing that is more sophisticated, better organized, or, in short, more mathematical (in accordance with the characteristics we laid out earlier).

Such shifts can be induced by reflecting upon solution methods and underlying understanding. Whole-class discussions of solution methods, interpretations, and insights will enhance the likelihood of those shifts; especially if the problem at hand gives rise to a variety of solution methods on different levels.⁷ When comparing and discussing their solution methods, for instance, some students may realize that other solution methods have advantages over their current method. This crucial role of dialogue as applied to interpretations, ideas and methods once more shows that an emphasis on mathematizing does not only imply solitary activity on the part of the individual student.

But, the dialogue need not only take the form of whole-class discussions. Freudenthal also espoused group work. His first plea for learning in small groups was in 1945, during a symposium of the New Educational Fellowship. Later, he advocated mathematics education in heterogeneous groups (Freudenthal 1987, 1991). In his opinion, both weaker and stronger students would profit from collaboration. And, as Freudenthal (1987: 338) noted, on re-reading the work he had produced from 1945 onwards, he realized, to his own surprise, how consistently, and for more than 40 years,

he had been a protagonist of co-operative learning in small heterogeneous groups.⁸

Criticism of educational research

To some, Freudenthal is perhaps better known for his criticism of 'traditional' educational research than for his own ideas and theories. In the Netherlands, he was a dreaded opponent of anyone in the educational research community who used an empiricist methodology and over-sophisticated statistics. He used his powers as a mathematician to show the many flaws in the manner in which mathematics (i.e. statistics) was used in many examples of 'hard' empirical research.

Freudenthal's opposition to much educational research was related to his conviction that discontinuities in the learning process are essential. Such discontinuities may be seen as creating shortcuts, or taking different perspectives (Freudenthal 1991; see also van den Heuvel-Panhuizen 1996). It is in such discontinuity, he argued, that one can perceive whether a student has achieved a certain level of comprehension. *To be able to identify these discontinuities, students must be followed individually.* This implies that group means and the like are not particularly useful, since means wash out the individual discontinuities. Moreover, the emphasis should be on *observing* learning processes, not on testing 'objective' learning outcomes. In addition, Freudenthal believed that such 'hard' research could not answer the *educational* questions of for what purpose a subject is being taught, and to whom (Freudenthal 1973a, b, 1988).

Freudenthal directed a second set of criticisms towards the testing movement. He was skeptical of objective testing methods and condemned the negative influence of examinations and testing techniques on education. The hard core of his criticisms centred on ignorance of subject matter and the overestimation of reliability at the expense of validity (Freudenthal 1980, 1991) and he did not share the optimism of the objective testing movement.⁹

More generally, Freudenthal's criticisms of educational research focused on methodologists whose strength consisted in '... knowing everything about research, but nothing about education' (Freudenthal 1991: 151). He fiercely opposed the separation of content and form. In his view, this leads to empty models that have to be filled by content experts: 'They gladly leave to the educational researcher the responsibility of his own to fill empty vessels with educational contents, but they are unconcerned about whether these fit or not' (Freudenthal 1991: 151). He offered similar criticisms of general educational theories.

Criticisms of general education theories

Freudenthal believed that general education theories not only do not fit the situation of mathematics education, but in many cases are detrimental to

the kind of education he endorsed. We may see this in his criticisms of Bloom, Gagné, and Piaget. Thus, he judged Bloom's *Taxonomy of Educational Objectives* to be inappropriate for mathematics education. Instead of aiming for a classification (resulting in taxonomies), he proposed that the activity of reality-structuring should be looked at. It is by structuring that students get a grip on reality; the artificial character of the categories of educational goals in the *Taxonomy* have a negative effect on both schooling and test development (Freudenthal 1979). Bloom's strategy of mastery learning was also vigorously rejected by Freudenthal (1980); he accused Bloom of conceiving of learning as a process in which knowledge is poured into the heads of the students.

Robert Gagné also came under fire from Freudenthal. He found the idea of task analyses, as presented in *The Conditions of Learning* (Gagné 1977), to be completely incompatible with the idea of mathematics as an human activity. 'A feeling of loneliness seized me: is mathematics really so different? I wish that someone who profoundly understands both mathematics and psychology would show us the bridge' (Freudenthal 1973b: vi).

Gagné conceived of the learning process as a continuous process that moved from the acquisition of simple to complex structures. Freudenthal saw educational processes as discontinuous: from rich, complex structures of the world of everyday-life to the abstract structures of the world of symbols – and not the other way around. Starting points should be found in situations that 'beg to be organized' where, as Freudenthal (1991: 30) put it, categories are not pre-defined but are developed by the learners themselves, and need to be accommodated to their needs.

Freudenthal also criticized Piaget for his mathematics and his experiments. What worried Freudenthal the most, however, was that Piaget's work seduced teaching methodologists into translating its research findings into instructional settings for mathematics education:

It is a sad story to see didacticians founding their practice on theories they learned from a psychologist; what they borrow from Piaget are not the results of his experiments but the wrong, or at least misunderstood, mathematical presuppositions (Freudenthal 1973b: 193).¹⁰

Freudenthal (1991) also addressed constructivism. But, although he criticizes the constructivist epistemology from an observer's point of view, it can be argued that the way he sees mathematics from an actor's point of view is compatible with this epistemology. Thus, from the perspective of an active mathematician, he characterizes mathematics as a form of (well-developed) common sense – a notion that is strongly tied to his idea of 'expanding reality'. Moreover, his educational goal is to make sure that the students experience 'objective mathematical knowledge' as a seamless extension of their everyday-life experience. This leads us to conclude that Freudenthal stands much closer to constructivism than one might gather from his attack on it.

Realistic mathematics education

We can summarize Freudenthal's view on mathematics education as follows. Mathematics must be seen foremost as a process, a human activity. However, at the same time, this activity has to result in mathematics as a product. This leads to the (design) question of how to shape a mathematics education that integrates both goals. Freudenthal's work was based on a number of ideas about how to deal with these questions. These ideas can be discussed under the headings of 'guided reinvention', 'levels in the learning process', and 'didactical phenomenology'.

Guided reinvention

According to the reinvention principle, a route to learning along which a student is able, in principle, to find the intended mathematics by himself or herself has to be mapped out (Freudenthal 1973b). To do so, the curriculum developer starts with a thought experiment, imagining a route by which he or she could have arrived at a personal solution. Knowledge of the history of mathematics can be used as a heuristic device in this process.

Freudenthal (1991) spoke of 'guided reinvention' with an emphasis on the character of the learning process rather than on inventing as such. The idea was to allow learners to come to regard the knowledge they acquire as their own, personal knowledge, knowledge for which they themselves are responsible. On the teaching side, students should be given the opportunity to build their own mathematical knowledge-store on the basis of such a learning process.

Freudenthal acknowledged the history of mathematics as a source of his inspiration. Subsequently, it has become clear that the reinvention principle can also be inspired by informal solution procedures (Streefland 1990, Gravemeijer 1994). More often than not, students' informal strategies can be interpreted as anticipating more formal procedures. In such cases, mathematizing similar solution procedures constitutes the reinvention process.

In general, contextual problems that allow for a wide variety of solution procedures will be selected, preferably solution procedures that in themselves reflect a possible learning route. Freudenthal saw the reinvention approach as an elaboration of the Socratic method and to illustrate the Socratic method, he spoke of 'thought experiments', i.e. the thought-experiment of teachers or textbook authors who imagine they are teaching students while interacting with them and dealing with their probable reactions. One part of the thought-experiment, therefore, lies in anticipating student reactions. The other part consists in the design of a course of action that fits anticipated student reactions. More precisely, the idea is that teaching matter is re-invented by students in such interaction. Freudenthal (1991: 100–101) comments:

Though the student's own activity is a fiction in the Socratic method, the student should be left with the feeling that it [i.e. understanding and insight] arose during the teaching process; that it was born during the lesson, with the teacher only acting as midwife.

Freudenthal did not subscribe to the Socratic method as such. He gave a much more active role to students in the process of constructing their own knowledge. The similarity, however, lies in the anticipation, in the *planning* of possible learning trajectories (Simon 1995). Such conjectured learning trajectories encompass the various problems that should be posed, the anticipated mental activities of students, and the actions that should be taken to make the reinvention process possible.

Levels in the learning process

Freudenthal complemented the concept of reinvention with what Treffers (1987) called 'progressive mathematization'. What could be seen as reinvention from an observer's point of view, should be experienced by the student as 'progressive mathematization' – from an actor's point of view. Students should begin by mathematizing subject matter from reality. Next, they should switch to analysing their own mathematical activity. This latter procedure is essential since it contains a vertical component, which Freudenthal (1971: 417), with reference to Van Hiele, described in the following manner: 'The activity on one level is subjected to analysis on the next, the operational matter on one level becomes subject matter on the next level'.

This shift from 'operational' to 'subject matter' relates to the shift from procedures to objects, which Sfard (1995) observed in the history of mathematics. It also relates to what Ernest (1991: 78) has called 'reification'. Freudenthal's level-theory shaped the RME-view on educational models. Instead of ready-made models, RME looks for models that emerge first as operational models of situated solution procedures, and then gradually evolve into entities of their own to function as models for formal mathematical reasoning (Gravemeijer 1999).

Didactical phenomenology

Freudenthal emphasized the importance of a phenomenological embedding of mathematical objects. In opposition to the concept-attainment approach, which implies the embodiment of concepts in concrete materials, Freudenthal proposed the use of phenomenologically rich situations: situations that are begging to be organized. In such a didactical phenomenology (Freudenthal 1983), situations should be selected in such a way that they can be organized by the mathematical objects which the students are supposed to construct. The objective is to figure out how the 'thought-matter' (*nooumenon*) describes and analyses the 'phenomenon'. How it would make the phenomenon accessible for calculation and thinking

activity. Such phenomenological analysis lays the basis for a didactical phenomenology which also incorporates a discussion of what phenomenological analysis meant from an educational perspective. For example, to construct length as a mathematical object students should be confronted with situations where phenomena have to be organized by length.

Within the framework of a didactical phenomenology, situations where a given mathematical topic is applied are to be investigated in order to assess their suitability as points of impact for a process of progressive mathematization. If we viewed mathematics as having evolved historically from practical problem solving, it would be reasonable to expect to find the problems which gave rise to this process in present-day applications. Next we could imagine that formal mathematics came into being in a process of generalizing and formalizing situation-specific problem solving procedures and concepts about a variety of situations. The goal of a phenomenological investigation is, therefore, to find problem-situations from which situation-specific approaches can be generalized, and to find situations that can evoke paradigmatic solution-procedures as the basis for vertical mathematization. To find phenomena which can be mathematized, we can seek to understand how they were invented.

Research for the sake of educational change

When Freudenthal (1991) thought about research he typically asked himself 'What is the use of it?' and he invariably gave the answer 'Change'. Education had to be constantly adapted to a changing society. Therefore, 'change' as a concept was to be preferred over the notion of 'improvement', inasmuch as what is regarded as a better education is dependent on the needs and priorities of society at a given moment in time – and as society changes, education would need to change also. In this light, an important task of the researcher is trail-blazing. Freudenthal saw the chain from research to the classroom as too long in traditional research: trail-blazing should not start in armchairs or the laboratory, but in the classroom. It was this philosophy of the aims and function of research that guided the approach to research adopted by the Institute for Development of Mathematics Education (IOWO), of which Freudenthal became the director.

At the time IOWO was founded, the RD&D model was in vogue within the Dutch educational research community. In this model, curriculum development, and what was called 'implementation', were completely separate and it was in opposition to this approach that Freudenthal (1991) set out his concept of 'educational development'. This concept meant more than just curriculum development but also contained the end-goal of changing educational practice. Moreover, educational development not only implied that the implementation of the curriculum was anticipated from the outset, it also implied choice of a broad change approach, comprising teacher education, counselling, test-development, and opinion-shaping – all based on the same educational philosophy. In contrast to the curriculum movement, Freudenthal integrated research,

development, implementation, and dissemination. As a consequence of his orientation on educational practice, he proposed to involve all participants from the start under the slogan, 'educational development in dialogue with the field'.

The type of change Freudenthal pursued was guided by his (1973b) idea of mathematics as a human activity. However, when IOWO was launched little research was available about this kind of mathematics education. Consequently, questions of how to develop instruction had to be answered during the process of development itself.

Developmental research

Freudenthal was initially reluctant to call to what was done at IOWO research. 'At our institute we regard ourselves not as researchers but engineers'. In addition, he (1973a) regarded theory as a by-product of educational development. Later, however, he (1988) argued that this metaphor separates research from educational development, and, thus, cannot do justice to the intertwined character of development and research in 'developmental research'.¹¹ New knowledge had to be legitimated by the process by which it is gained: to bring the developmental process to consciousness and to explicate it was the essence of developmental research.

Experiencing the cyclic process of development and research so consciously, and reporting on it so candidly that it justifies itself, and that this experience can be transmitted to others to become like their own experience (Freudenthal 1991: 161).

To put it differently, the aim of developmental research to Freudenthal was to create the opportunity for outsiders, e.g. teachers, to retrace the learning process of the researcher, what Smaling (1987) called 'trackability'. To ensure such trackability, Freudenthal demanded a constant awareness of the developmental process. And, if its result were to become credible and transferable, as much as possible of this reflection needed to be reported.

At the heart of this reflection lay the 'thought experiments' of the researcher. The developer would envision how teaching-learning processes proceed and would subsequently try to find evidence in a teaching experiment to show whether these expectations were right or wrong. The feedback from practical experience to (new) thought-experiments would induce an iteration of development and research: What was invented behind the desk would be put into practice immediately; what happened in the classroom would be analysed in a consistent manner and the results used to continue the developmental work. This process of deliberating and testing would result in a product that was theoretically and empirically founded, well-considered, and well-tried.

In this view, developmental research can offer teachers a frame of reference which can provide a basis for their own decisions. Against the backdrop of this framework, teachers can develop hypothetical learning trajectories (Simon 1995) that take into account both the actual situation of

their classroom and their own goals and values. Teachers are given arguments and guidelines to enable them to shape their own instruction, a starting point that is firmly built into the European Didaktik tradition.

Developmental research has a double output-channel: one at the level of theories, and one at the level of curriculum products. Thus, the developmental research in RME, carried out in and outside the IOWO and its successor, OWandOC (the present Freudenthal Institute), has resulted in a wealth of prototypical instructional sequences and other practical publications. In the Netherlands, these publications have had a strong influence on mathematics in schools. Over time, ~80% of Dutch primary schools voluntarily switched to so-called 'realistic' textbooks. At the secondary level, curriculum changes initiated by the government gave the Freudenthal Institute several commissions for the development of new curricula. As a consequence, all curricula in the Netherlands were, or are being, exchanged for curricula based on the RME philosophy.

National educational assessment studies have shown that, in the last year of primary school, Dutch students working with modern textbooks were in general more successful than students working with traditional textbooks – with the exception of the topics of written algorithms and measurement (Bokhove *et al.* 1996).¹² It seems reasonable to attribute this success to the innovation strategy – 'educational development in dialogue with the field' – used in the introduction of these curricula and texts into Dutch schools. In empirical and retrospective studies on innovation in mathematics in primary and secondary schools the following agents of success have been identified (Gravemeijer and Ruinaard 1995, Vermeulen *et al.* 1997):

- a powerful and inspiring philosophy of mathematics education;
- the development of examples, and prototypical instructional sequences;
- professionalization activities;
- the constitution of a mathematics-educational community as a mediating infrastructure;
- teacher enhancement via in-service teacher training and journals;
- textbook review;
- revision of examinations; and
- developmental research as the engine of innovation.

Developmental research lies at the heart of the innovation strategy. This work produces the prototypes and the theories that inform teacher-trainers, textbook authors, and school counsellors. These, in turn, function as mediators between the developers and the teachers. In accordance with the concept of educational development in dialogue with the practitioners, these information streams are bi-directional. To put it differently, the core idea of the Institute was the concept of 'educational development in dialogue with practitioners'; rather than innovations being developed in ivory towers, practitioners, such as teacher-trainers, counsellors, textbook-authors, researchers, test-constructors, and the teachers themselves, were involved in research and development from early on.¹³

Conclusions and discussion

To place the work of Freudenthal in the contexts of didactics and curriculum studies is not an easy task, due to his eclectic style of writing which is, for example, almost without references to the authors by whom he was inspired. In this final section, we shall focus on three main aspects of his work: didactics, curriculum theory, and 'mathematics for all'.

Didactics

Freudenthal often used the term didactics to mean correct teaching and learning processes, starting with, and staying within, 'reality'. He referred to the converse as 'the anti-didactical conversion' (the deductive approach), which he fiercely rejected. According to Freudenthal, didactics should be concerned with processes. So, there is a commonality and a striking difference between Freudenthal's didactics and Klafki's use of the term. Both are influenced by the phenomenological theory of *Bildung* and reform-pedagogy. Both take their point of departure from the practice of education (educational reality) and endeavoured at certain points in their professional lives to overcome the exalted and élitist aspects of the *Bildung*-theory. Both stressed the practical side of education and advocated the comprehensive school as a necessary educational reform. Klafki, however, mainly focused on lesson-planning or the preparation of lessons where the process of learning is not real. Klafki's fundamental questions primarily concerned the content of *Bildung*, while more or less ignoring teaching methods and processes.

Curriculum theory

Freudenthal used the word 'curriculum', but less often than 'didactics'. In his views on curriculum development and the role of theory, there is a striking similarity with the curriculum work of Joseph Schwab, who has occupied a special position in curriculum theory in the US. Along the same lines as, but independently from, Schwab, Freudenthal stressed the practical character of curriculum work and the process of dialogue between curriculum specialists and teachers. Freudenthal was against any fixed curricular system and he fiercely opposed content being bottled and funnelled into schemes and structures. This was a remarkable position at a time when curriculum theory was dominated by behaviouristic orientations and RD&D approaches, and was seen in Germany and the Netherlands as the long-expected cure-all for everything (Hopmann and Riquarts 1995a, b). Freudenthal proposed, instead, mathematics as a human activity and as guided reinvention. It was this humanistic, practical, process-oriented, phenomenological, and reform-pedagogical credo, elaborated in the context of curriculum development, that made Freudenthal's position unique among most of his fellow mathematics educators of his time. His credo inevitably brought Freudenthal into conflict with, for example,

behaviouristically-oriented psychologists such as Bloom and proponents of the 'new math' movement who advocated the development of a curriculum for mathematics as an abstract deductive system.

Mathematics for all

We finally turn to Freudenthal's position in the debate concerning 'mathematics for all' and the common curriculum (see Damerow and Westbury 1985, Keitel 1987, Dekker 1991). Although Freudenthal was educated in, and influenced by, the traditional German *Bildung* tradition in a dual school system, he rejected an exclusive form of *Bildung* for an élite as separate from schooling for the masses. He strongly advocated 'mathematics for all' and tried to make mathematics accessible to everybody. He condemned all forms of streaming and setting by referring to the inevitable 'Matthew effects'. Freudenthal (1973a) was convinced that students from different ability levels in the first years of secondary education (which in the Dutch context concerned the 12–15-year-olds) should not only stay in the same classroom, but should also follow a common curriculum. His plea for heterogeneous learning groups built on the other main aspects of his pedagogical credo.

Several aspects of Freudenthal's ideas are still under discussion. Thus, there is a strong movement *against* educational theories of his kind from psychologists, who look at learning from an information-processing point of view (Anderson *et al.* 1996). But, also there is sometimes opposition from inside the mathematics and the mathematics-education communities to the basic idea that students should proceed from the real world to the mathematical world. The main criticism of the RME approach is that it is often impossible to proceed from experientially real situations to 'mathematics'. Reinvention, in this view, is a waste of time (Verstappen 1991, Keune 1998).

These criticisms have to be mentioned, but it must also be noted that the opponents of Freudenthal's ideas are short on empirical evidence for their point of view. While various teaching experiments have shown the value of the RME approach (de Lange 1987, Nelissen 1987, van den Brink 1989, Streefland 1990), the outcomes of several research studies into the effects of mathematics curricula inspired by the ideas of Freudenthal, show clearly that learning mathematics in real-life contexts in heterogeneous learning groups is feasible and effective (Terwel 1990, 1999, Dekker 1991, Terwel *et al.* 1994, Perrenet 1995, Hoek *et al.* 1997, 1999, Hoek 1998, Roelofs and Terwel 1999). Also, there is still broad support for RME among practitioners, educators, theorists and curriculum developers. Almost all Dutch mathematics textbooks show the impact of Freudenthal's ideas. But, although there is empirical and practical evidence for the feasibility and effectiveness of RME, Freudenthal's most convincing argument for RME is that not all students are future mathematicians but, rather that, for the majority, the mathematics that they will use will be to solve problems in everyday-life situations.

Notes

1. He was President of the International Committee on Mathematics Instruction (ICMI), founding editor of *Educational Studies in Mathematics*, and one of the founders of the International Group for Psychology and Mathematics Education (PME) established to overcome the dominating behaviourist approach in educational psychology. He was also founder and president of the *Commission Internationale pour l'Étude et l'Amélioration de l'Enseignement des Mathématiques* (CIEAEM), which celebrates more than 50 years of activity. In the Netherlands, Freudenthal was the founder and director of the former institute for the development of mathematics education, IOWO, later called OW&OC, and now named the Freudenthal Institute.
2. In this context, we want to draw the attention to *The Legacy of Hans Freudenthal* (Streefland 1993).
3. The main representatives of this concept of a Didaktik are Erich Weniger and Wolfgang Klafki (Blankertz 1973). Klafki (1995, 2000) sought to elaborate the philosophical notion of *Bildung* into a more practical theory of lesson-planning; at the heart of his Didaktik is a set of basic questions concerning, for example, the selection of content, structuring, meaningfulness, and methods.
4. This international fellowship, which was founded in 1920 with sections in many countries, was more or less the practical counterpart of the *Geisteswissenschaftliche* theory of pedagogy and Didaktik. Freudenthal also explored eastern European socio-cultural theory. He had a long-lasting debate with Van Parreren, who, in collaboration with Carpay, introduced Vygotsky's ideas into the Netherlands.
5. In this respect, we may observe an affinity with Vygotsky's (1978: 64) point of departure: '... we need to concentrate not on the product of development, but on the very process...'
6. We stress that the point of departure is not that everyday-life problems will, by definition, be experientially-real for the students, nor that experientially-real problem situations necessarily have to deal with real-life situations. This is a common misunderstanding, evoked by the term 'realistic mathematics education'. Here, realistic is to be interpreted as referring to experientially-real, not to everyday-life reality.
7. We may note that the quality of this discussion largely depends on what is called the 'didactical contract'. However, Freudenthal's elaborations of mathematizing take on a psychological perspective at the expense of a social perspective. Although his arguments for working in heterogeneous groups clearly reflect his acknowledgement of learning as a social process.
8. Furthermore, he did not confine himself to theorizing about co-operative learning, but was highly involved and supportive as a supervisor of the research project 'Mixed-ability grouping in mathematics for 12–16 year-olds' (Terwel 1984, 1990, Freudenthal 1987).
9. As an example of his stance, we may take his attack on the studies of the International Association for the Evaluation of Educational Achievement (IEA): his criticism of IEA focused, among other things, on the validity of the instruments used in those studies and the lack of fit between the national curricula and the testing instruments. He observed that the only actual check carried out concerned only one country, and even in that case the correlation was not good (Freudenthal 1975: 134). Freudenthal observed that the lack of correspondence between the national curricula and the test items resurfaced in the IEA report under the variable 'opportunity-to-learn'. Here, according to Freudenthal, things were turned upside-down; the lack of concurrence of the test items and the curricula was now presented as an important explanatory variable, which, in turn, was explained by a lack of implementation of the official curriculum plans. Freudenthal attributed the differences between countries to a lack of correspondence of the national curricula and the IEA instruments.
10. Freudenthal also criticized Piaget's experiments for their lack of validity. He pointed to two interrelated flaws: the artificial character of the questions and the pre-determined interpretations of students' reactions. In an appendix to his book *Mathematics as an Educational Task*, Freudenthal (1973b, 662–677) supports his critique on Piaget's mathematics, with extensive translations from Piaget's work.

11. He set himself against the ideal of educational research as modelled on research in the natural sciences. In the natural sciences, he argued, it is easy to present new knowledge as the result of experiments, since experiments are easily replicated. In education, replication is impossible in the strict sense of the word. An educational experiment can never be repeated in an identical manner, under identical conditions.
12. The most recent research of the IEA Third International Mathematics and Science Study (TIMSS) on 13–14 year olds has shown that Dutch students do very well in international competitions, although the TIMSS test items only poorly fit the Dutch curriculum and probably would not withstand Freudenthal's (1975) critique.
13. Admittedly, this dialogue was split into two levels: the dialogue between the researchers and the teacher-trainers, etc., on the one hand, and the dialogue between this group and (a number of) teachers on the other. And, in practice, commercial textbooks were the main carriers of innovation, so that the actual innovation assumed the character of a curriculum-as-a-document to a much greater extent than ever intended. The lack of the financial resources required for greater involvement by teachers played a major role in the strategies that were adopted. Not surprisingly, research has shown that the instructional practices in school have differed significantly from the form of practice envisioned by the innovators. In an empirical study of primary grades 1–3, the characteristics of realistic mathematics education were only partly found (Gravemeijer *et al.* 1993). And, in an empirical study by Kuiper (1994), it was concluded that Freudenthal's ideas are far from being enacted in everyday classroom practices in secondary education.

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