Monotonicity Analysis in Optimum Design of Marine Risers

The optimal design of marine risers used for drilling and production of oil in offshore operations is studied. The optimization problem is formulated on a two-dimensional model for bending of circular tubular beams under tension and internal and external static pressure. A general polynomial expression describes the external hydrodynamic loads. Monotonicity analysis is used to identify active constraints, determine design rules, and reduce the size of the problem.

Linearized Static Riser Model

The general nonlinear, large deflection, three-dimensional model, derived in [5], which describes the dynamic behavior of a riser including its extensional oscillations, can be simplified to yield the model used in the present feasibility and optimization design analysis.

The linearized differential equations of equilibrium of a riser subject to small deflections and slopes in a vertical plane are:

\[ \frac{dM}{dz} - P_e \frac{dU}{dz} + Q = 0 \]  \hspace{1cm} (1)

Equilibrium of forces in the x direction

\[ \frac{dQ}{dz} = f_x \]  \hspace{1cm} (2)

Equilibrium of forces in the z direction

\[ \frac{dP_e}{dz} = W_e \]  \hspace{1cm} (3)

where

\[ W_e = W_h + W_m + W_m - B \]  \hspace{1cm} (4)

\[ P_e(z) = T(z) + \gamma_w \frac{\pi D_o^2}{4} (h_w - z) - \gamma_m \frac{\pi D_i^2}{4} (h_m - z) \]  \hspace{1cm} (5)

\[ U \] is the deflection of the riser in the x direction,

\[ T(z) \] is the actual tension in the riser,

\[ M(z) \] is the bending moment in the y direction,

\[ Q(z) \] is the shear force in the x direction,

\[ f_x(z) \] is the external force per unit length exerted on the riser.

In this problem \( f_x \) is considered time invariant. In addition,

\[ W_R = \gamma_R \frac{\pi}{4} (D_o^2 - D_i^2) \]  \hspace{1cm} (6)

is the weight of the riser per unit length,

\[ W_m = \gamma_m \frac{\pi}{4} D_i^2 \]  \hspace{1cm} (7)

is the weight of the drilling mud per unit length,
is the buoyancy of the buoyancy modules per unit length and

\[ B = \gamma_w \frac{\pi}{4} D_b^2 \]

is the buoyancy of the riser and modules per unit length.

The linearized constitutive relation of bending is

\[ M(z) = EI \left( \frac{d^2 U}{dz^2} \right) \tag{9} \]

Finally the boundary conditions are:

\[ U(0) = 0 \tag{11} \]

\[ U(L) = \Delta \tag{12} \]

\[ \frac{d^2 U}{dz^2}(0) = 0 \tag{13} \]

\[ \frac{d^2 U}{dz^2}(L) = 0 \tag{14} \]

The static external hydrodynamic force, \( f_x(z) \), is proportional to the square of the relative fluid velocity. Along the riser (depthwise) we can approximate the force profile by a

\[ f_x(z) = \frac{1}{2} \rho_0 C_D V^2 D_b \left[ \alpha_0 + \alpha_1 \left( \frac{z}{L} \right) + \alpha_2 \left( \frac{z}{L} \right)^2 + \ldots + \alpha_n \left( \frac{z}{L} \right)^n \right] \tag{15} \]

where \( V \) is a characteristic velocity and \( C_D \) is the drag coefficient. Equations (1), (2), (3), and (9) can be combined to yield the fourth order differential equation with variable coefficients which describes the riser’s response to external static loads:

\[ \frac{d^2}{dz^2} \left[ EI \frac{d^2 U}{dz^2} \right] - \frac{d}{dz} \left( W(z) + P_0(z) \right) \frac{dU}{dz} = f_x(z) \tag{16} \]

A better understanding of some of the terms involved in equations (1) to (16) is required for proper formulation of the optimization problem. Specifically, \( W_e(z) \) the effective weight of the riser per unit length and \( P_e(z) \) the effective tension defined by equations (4) and (5) are two key variables of the design problem [5].

The effective weight of the riser per unit length, \( W_e(z) \), is equal to the weight of the riser \( W_R \) plus the weight of the mud, plus the weight of the buoyancy modules \( W_B \), minus the total buoyancy \( B \) per unit length. That is, \( W_e(z) \) is the weight of riser and contents in water per unit length.

In equation (3) describing the force equilibrium in the \( z \) direction it appears as if the force exerted on the differential element \( dz \) (Fig. 1) is equal to \( W(z) \cdot dz \). This is not correct since \( dz \) is not a closed body fully wetted by water. For this reason the left-hand side of equation (3) is not the actual tension \( T(z) \) in the riser but \( P_e(z) \), the effective tension, i.e. the actual tension \( T(z) \) modified due to the internal mud static pressure and the external hydrostatic pressure [5].

In this paper \( W_e(z) \) is considered constant and consequently \( P_e(z) \) is a linear function of \( z \). As a result, the design parameters in equation (10) are \( W_e \) and \( P_0(0) \). This conclusion is particularly important in the derivation of the buckling constraint. The buckling loads are functions of the effective tension at the riser’s lower end \( P_e(0) \) [7] and not \( T(0) \). In reference [7] it is proved that a riser may actually buckle as a Euler column even if the actual tension in the riser \( T(z) \) is positive for all values of \( z \) between 0 and \( L \).

**Nomenclature**

- \( D \) = average diameter of riser
- \( D_b \) = outer diameter of buoyancy modules
- \( D_{b,max} \) = maximum allowable \( D_b \)
- \( D_i, D_o \) = inner and outer riser diameters
- \( E \) = Young’s Modulus
- \( I \) = riser cross-sectional area moment of inertia
- \( L \) = riser’s length
- \( M \) = bending moment
- \( P_e \) = effective tension
- \( Q \) = shear force
- \( R_i \) = constraint no. \( i \)
- \( S_y \) = riser’s material yield strength
- \( T(0) \) = actual tension at the riser’s lower end
- \( TTR \) = tension at the top of the riser
- \( U \) = riser’s lateral displacement
- \( W_e \) = effective weight of riser per unit length
- \( W_m \) = weight of drilling mud per unit length

\[ W_R = \text{weight of riser per unit length} \]
\[ h_m = \text{total height of mud column} \]
\[ h_w = \text{total water depth} \]
\[ f_x = \text{external hydrodynamic load per unit length} \]
\[ p = \text{dimensionless vertical coordinate along the riser} \]
\[ P_l, P_0, P_t = \text{position of maximum total stress} \]
\[ \beta = \text{dimensionless effective riser weight} \]
\[ \gamma_B, \gamma_B, \gamma_m = \text{specific weight of buoyancy modules; material, riser wall material, water} \]
\[ \gamma_m = \text{specific weight of circulating mud} \]
\[ \gamma_{m_1}, \gamma_{m_2} = \text{lower and upper limits for } \gamma_m \]
\[ \Delta = \text{static offset of drilling vessel} \]
\[ \sigma, \sigma_1, \sigma_2 = \text{principal stresses in } t, r, z \text{ directions} \]
\[ \sigma_{b}, \sigma_{e} = \text{bending and effective stresses} \]
\[ \sigma_T = \text{combined Tresca stress} \]
\[ \tau = \text{dimensionless effective tension at riser’s lower end} \]
\[ \sigma_{t}, \sigma_{z}, \sigma_{s} = \text{shear stress components} \]
Prediction of the external hydrodynamic loads exerted on circular cylinders moving in viscous fluids is very difficult and at the present state of the art can be achieved only for a very limited number of flows. For flows other than those studied experimentally the hydrodynamic loads can be predicted only crudely. Consequently, only a parametric formulation of the hydrodynamic loads in terms of a relatively large number of parameters can be realistic. This method is adopted in equation (15).

In dimensionless form equation (16) becomes
\[
\frac{d^4 U}{dp^4} - \frac{d}{dp} \left[ (\beta p + \tau) \frac{dU}{dp} \right] = -\frac{L^4}{EI} f_s(pL) = \frac{L^4D}{EI} C_h(p) \tag{17}
\]
where
\[
\beta = \frac{W_e L^3}{EI} \tag{18}
\]
\[
\tau = \frac{P_e(0) L^3}{EI} \tag{19}
\]
\[
C_h = \frac{1}{2} \rho \omega C_D V^2 \tag{20}
\]
\[
p = \frac{z}{L} \tag{21}
\]
and
\[
h(p) = a_0 + a_1 p + a_2 p^2 + \ldots + a_n p^n \tag{22}
\]
Equation (17) is subject to the following boundary conditions which are imposed by the fixed lower end of the riser, the static excursion of the drill ship, \( \Delta \), and the upper and lower ball joints.
\[
U(0) = 0 \tag{23}
\]
\[
U(1) = \Delta \tag{24}
\]
\[
\frac{d^2 U}{dp^2}(0) = 0 \tag{25}
\]
\[
\frac{d^2 U}{dp^2}(1) = 0 \tag{26}
\]
In conclusion, equation (17) subject to the boundary conditions (23) to (26) models the riser response to the static load defined by equation (15).

Approximate Solution of the Boundary Value Problem

The exact solution to the static riser boundary value problem, defined by the differential equation (17) subject to the boundary conditions (23) to (26), has been derived earlier [3]. This solution can be used to evaluate the bending stress in the riser for the formulation of the design constraints in the optimization problem. However, the purpose of this paper is to derive an approximate, simplified, approximate solution is developed for expressing the equivalent stresses. This will allow expression of the stress constraints in an explicit closed form.

For long risers the bending rigidity is significant only near the upper (\( p = 1 \)) and lower (\( p = 0 \)) ends of the riser. Away from the riser ends the first term of the differential equation (17) can be dropped. The resulting differential equation models the response of a cable under generalized static load. Near the riser ends, the cable approximation may not be satisfactory, depending on the tensions applied at the top of the riser (TTR) and the external static loads. Consequently the cable approximation should be corrected near the ends of the riser.

The linearized static cable problem is defined by the differential equation (27).
\[
-\frac{d}{dp} \left[ (\beta p + \tau) \frac{dU}{dp} \right] = L^4 D_p C_h(p) = a h(p) \tag{27}
\]
subject to the boundary conditions
\[
U(0) = 0 \tag{28}
\]
and
\[
U(1) = \Delta \tag{29}
\]
The solution to this boundary value problem is
\[
U(p) = -\int_0^p \frac{H(\xi) d\xi}{\beta \xi + \tau} + C_1 \frac{1}{\beta} \ln \left( \frac{\beta}{\tau} p + 1 \right) \tag{30}
\]
where
\[
C_1 = \frac{1}{\beta} \ln \left( \frac{\beta}{\tau} + 1 \right) \tag{31}
\]
and
\[
H(p) = \int_0^p h(\xi) d\xi \tag{32}
\]
Further, the cable approximation to the curvature of the riser
\[
\frac{d^2 U}{dp^2} = -\frac{a h(p)}{\beta p + \tau} \left[ C_1 - a H(p) \right] \tag{33}
\]
can be used to evaluate the bending stress. This approximation may not be satisfactory near the riser ends. For example Hapel [4] has proved that for external static loads, exponentially decaying with the water depth, the cable curvature overestimates the riser curvature, near the upper end, and cannot be used for design purposes. Hapel has approximated the riser curvature near the upper end by solving the problem of a beam under constant tension and near the lower end by assuming constant external loads.

An alternative method for correcting the cable approximation to the curvature, as given by equation (33), is based on the following rationale: The riser response is subject to the boundary conditions (23) and (26). Both conditions are, in general, violated by the cable approximation since a cable has no bending rigidity and cannot sustain bending moments. Concentrated bending moments exerted at the riser ends may be applied in order to correct the cable approximation to the curvature and make it comply with these boundary conditions. Near the end points, the riser is a beam under tension, \( P_e \), given by equation (5) for \( p = 0 \) and \( p = 1 \).

For \( p = 1 \) the influence function is [2]
\[
f_{p,1} = \frac{-\sinh(\sqrt{\beta} + \tau p)}{\sinh(\sqrt{\beta} + \tau)} \tag{34}
\]
For \( \sqrt{\beta} + \tau \gtrless 3/2 \) which is the prevailing case for risers
\[
f_{p,1} = e^{-\sqrt{\beta} + \tau t (1 - p)} \tag{35}
\]
Similarly, for \( p = 0 \) the influence function is
\[
f_{p,0} = \frac{-\sinh(\sqrt{\tau} (1 - p))}{\sinh(\sqrt{\tau})} \tag{36}
\]
and for \( \sqrt{\tau} \gtrless 3/2 \)
\[
f_{p,0} = e^{-\sqrt{\tau} p} \tag{37}
\]
Consequently, near the ends the curvature may be approximated by
\[
\frac{d^2 U(p)}{dp^2} = \frac{1}{\beta(p + \frac{\tau}{\beta})^2} \left[ a h(p) \left( p + \frac{\tau}{\beta} \right) - aH(p) + C_1 \right]
\]
\[-\frac{1}{\beta(1 + \frac{\tau}{\beta})^2} \left[ a h(1) \left( 1 + \frac{\tau}{\beta} \right) - aH(1) + C_1 \right] e^{-\sqrt{\beta} \pi(1 - p)}
\]
\[-\frac{1}{\beta(\frac{\tau}{\beta})^2} \left[ a h(0) \frac{\tau}{\beta} + C_1 \right] e^{-\sqrt{\beta} p}
\]

in which the second and third terms at the right-hand side are the corrections introduced to satisfy the boundary conditions (26) and (25), respectively, decaying exponentially with the distance from the lower and upper end, respectively.

The position of local maximum bending stress, \( p_m \), can be estimated satisfactorily from equation (38). The advantage of using this expression instead of the exact solution [3] is that the position of \( p_m \) can be evaluated directly in terms of the various design variables, parameters, and constants.

**Combined Stresses in the Riser**

Consider the local coordinate system (\( z, r, t \)) in the riser where \( r \) and \( t \) are the local radial and tangential directions, and let \( \sigma_r, \sigma_r, \sigma_z \) be the local principal stresses and \( \tau_{rz}, \tau_{rt}, \tau_{tr} \), be the local shear stresses.

Since the riser tubes can be considered as thin cylindrical shells the shear stresses are

\[
\tau_{rz} = \frac{Q}{I} \frac{D \delta}{4} \cos \phi \equiv 0
\]
\[\tau_{tr} = 0 \quad \text{and} \quad \tau_{rt} = 0
\]

and the normal stresses are

\[
\sigma_r = \sigma_r + \frac{\bar{\rho}_t A_l - \bar{\rho}_t A_0}{A} \sigma_r
\]
\[\sigma_r = \sigma_r - \bar{\tau}
\]
\[\sigma_z = \sigma_z + \bar{\tau}
\]

where

\[
\sigma_r = \frac{\bar{\rho}_t D_z^2 - \bar{\rho}_0 D_0^2}{(D_z^2 - D_0^2)}
\]
\[\bar{\tau} = \frac{(\bar{\rho}_t - \bar{\rho}_0) D_2^2 D_z^2}{(D_z^2 - D_0^2)^2 a r^2}
\]
\[\sigma_z = \frac{P_z}{A}
\]
\[\sigma_r = \frac{E d}{L^2} \frac{d^2 U}{dp^2}
\]

\( d \) is the distance from the neutral axis with a maximum value of \( D_o/2 \)

\[
\bar{\rho}_t = \rho_m (h_m - z)
\]
\[\bar{\rho}_0 = \rho_w (h_w - z)
\]

and \( A \) is the cross-sectional area of the riser

\[A = A_0 - A_1 = \frac{\pi D_o^2}{4} - \frac{\pi D_i^2}{4}
\]

The limit stress in the riser can be derived based on the Tresca criterion.

\[
\sigma_T = \max \left\{ \frac{1}{2} \left| \sigma_r - \sigma_r \right|, \frac{1}{2} \left| \sigma_r - \sigma_z \right|, \frac{1}{2} \left| \sigma_z - \sigma_r \right| \right\}
\]

or

\[
\sigma_T = \max \left\{ \frac{1}{2} \left| \sigma_r - \sigma_r + \bar{\tau} \right|, \frac{1}{2} \left| \sigma_r - \sigma_r - \bar{\tau} \right|, \frac{1}{2} \left| \sigma_z - \sigma_r \right| \right\}
\]

and the maximum local value of the stress as function of \( p \) can be found from

\[
\sigma_T = \max \left\{ \frac{1}{2} \left| \frac{E f}{L^2} \frac{d^2 U}{dp^2} + \frac{\bar{\rho}_t + \tau}{\beta} \right| \right\}
\]

Modeling of Optimization Problem

Several design criteria may serve as objective functions, such as riser weight, overall cost or required tension at the top of the riser (a factor influencing the size of the drilling vessel). In the present analysis the objective chosen is minimizing the riser weight \( W_R \). The constraints are developed as follows.

**Yielding.** The combined stress given by the Tresca criterion usually exhibits two local maxima along the riser [1] that should not exceed the material yield strength. These maxima occur near the upper and lower ends.

Near the upper end the stress constraint is derived using (53), (38), and (27).

Near the lower end either term at the right-hand side of equation (53) could be maximum depending on the values of the design variables. Therefore, the stress design requirements result in two constraints.

\[
R_1: \quad \frac{D_o D_r L^2 P_0(\beta, \tau)}{2I} + \frac{4E f (\beta)}{\beta} \left( \frac{p_0 + \tau}{\beta} \right) + \left( \frac{D_o^2}{D_0^2 - D_i^2} \right)^2 \leq \frac{P_s}{N_t}
\]

where \( P_s \) is the yield stress in tension, \( N_t \) is a safety factor, \( p_0 \) is the value of \( p \) at the upper end \((p_0 \equiv 1)\) depending only on \( \beta \) and \( \tau \) for sufficiently small \( \Delta \) and

\[
P_1(\beta, \tau) = \frac{1}{a} \frac{d^2 U(p_0)}{dp^2}
\]

Near the lower end either term at the right-hand side of equation (53) could be maximum depending on the values of the design variables. Therefore, the stress design requirements result in two constraints.

\[
R_2: \quad \frac{D_o D_r L^2 P_0(\beta, \tau)}{2I} + \frac{4E f (\beta)}{\beta} \left( \frac{p_0 + \tau}{\beta} \right) + \left( \frac{D_o^2}{D_0^2 - D_i^2} \right)^2 \leq \frac{P_s}{N_t}
\]

where \( N_0 \) and \( \rho_0 \) are defined in the same way as \( N_t \) and \( p_0 \), and

\[
P_0(\beta, \tau) = \frac{1}{a} \frac{d^2 U(p_0)}{dp^2}
\]
Buckling. As discussed in the section on the riser model, buckling will not occur if
\[ \tau \geq -f(\beta)/N_2 = -\tau_{\text{crit}}/N_2 \] (59)
where the buckling load is expressed in terms of \( \tau \), the non-dimensional effective stress at the lower end, and the critical buckling load is expressed as a function of \( \beta \) with a safety factor of \( N_2 \) [7].

In practice, \( T(0) \) is sufficiently greater than zero so that \( P_e(0) \) defining the buckling load is positive, thus making \( \tau \) also positive. Moreover in the cable approximation buckling will occur even when \( \tau = 0 \). Thus in the present model the buckling constraint (58) is reduced to \( \tau > 0 \).

**Equality Constraints.** The following equality constraints are derived from the relations developed in the riser model section.

Equations (4), (6), (7), (8a), (8b), (10) and (18) yield:
\[ E\beta L^3 = W_R + \left( \pi \gamma_w/4 \right) D_2^2 - \left( \pi \gamma_w/4 \right) D_B^2 \]
\[ + \left( \pi \gamma_w/4 \right) (D^2_2 - D^2_B) \]  
\[ W_R = \left( \pi \gamma_w/4 \right) (D^2_2 - D^2_B) \]  
\[ I = \left( \pi /64 \right) (D^4_B - D^4_2) \]  
(60)

Elimination of \( P_e(z) \) from (3) and (5) yields an expression for \( T(z) \). For \( z = L \), \( T(z) \) is equal to the tension at the top of the riser (TTR) given by
\[ TTR = W_R + L (0) - \left( \pi \gamma_w/4 \right) (D^2_B - D^2_2) L \]  
(63)
Expressing (63) in terms of the nondimensional variables yields
\[ (\tau + \beta) E\beta L^2 = TTR + \left( \pi \gamma_w/4 \right) (D^2_B - D^2_2) \]  
(64)

**Practical Constraints.** Drilling mud circulates in the drill string and between the drill string and the riser. Its function is to facilitate drilling, carry away cuttings and cool the drill string. For these reasons a minimum mud circulation is required which depends primarily on the drilling depth. The foregoing requirements impose the following constraint on the internal riser diameter.
\[ R_4: D_2 \geq D_{\text{min}} \]  
(65)
The density of the drilling mud \( \gamma_m \) depends primarily on the drilling depth below the sea bed and also on other factors like lubrication, protection of the riser from the drill string, and prevention of blow out. These define a rather small acceptable range of values for \( \gamma_m \), expressed by constraints \( R_6 \) and \( R_7 \).
\[ R_6, R_7: \gamma_{m1} \leq \gamma_m \leq \gamma_{m2} \]  
(66)
To keep the riser up, prevent buckling or collapse, and provide additional strength to lateral loads, a tension is applied at the top of the riser (TTR). This force, provided by the tensioning system installed in the drilling vessel, is very high particularly for long risers. To alleviate high tensile stresses imposed at the upper end, buoyancy modules are distributed along the riser thus reducing the effective riser weight in water. Part of the required upward force is provided by the tensioning system implying the constraint
\[ R_{11}: TTR > 0 \]  
(67)
Additional upward force is provided by the buoyancy modules. This partial buoyancy implies the constraint
\[ R_5: W_R - \left( \pi \gamma_w/4 \right) (D^2_B - D^2_2) \geq 0 \]  
(68)
where \( \gamma_w = \gamma_w - \gamma_B \). The equality holds for a neutrally buoyed riser. The buoyancy modules increase the hydrodynamic loads exerted on the riser. To avoid excessive external loads and high construction costs the size of the buoyancy modules is limited by
\[ R_5: D_B \leq D_{\text{max}} \]  
(69)
Finally note that all design variables are strictly positive. An obvious geometric constraint is
\[ R_{10}: D_B > D_o \]  
(70)

**Monotonicity Analysis**

Monotonicity rules will be used to identify inequality constraints that are active at the optimum [8, 10]. To apply these rules the problem is cast into normalized form and equality constraints are eliminated from the model along with an equal number of design variables. Furthermore, in order to expose monotonic structure of the functions involved the following transformations of variables are used.
\[ D_2 = D_o + x_1 \]
\[ D_2 = D_o + x_2 \]
Note that \( D_2 = D_o + 2x_1 \), where the thickness of the riser tube is very small compared to either \( D_o \) or \( D_2 \). Thus, \( D_2 \geq D_o \) with a maximum error of 5 percent and this result is used to simplify some expressions. The resulting optimization problem in normalized form is the following.

**Problem PI**
\[ \min W_R = \left( \pi \gamma_w/4 \right) D_B^2 \]
subject to
\[ R_1: C_h \left( 32L^2/(2\pi) P_h(\beta, \tau)D_B^2 \right) + \frac{E\beta D_B^2 (p_1 + \tau / \beta)}{(S_f/N_f)16L^2} \]
\[ + \frac{\left( \gamma_m - \gamma_w \right) L D_A^2 (1 - p_1)}{(S_f/N_f)2D_B^2} \leq 1 \]
\[ R_2: C_h \left( 32L^2/(2\pi) P_h(\beta, \tau) D_B^2 \right) + \frac{E\beta D_B^2 (p_2 + \tau / \beta)}{(S_f/N_f)16L^2} \]
\[ + \frac{\left( \gamma_m - \gamma_w \right) L D_A^2 (1 - p_2)}{(S_f/N_f)2D_B^2} \leq 1 \]
\[ R_3: \left( \gamma_m - \gamma_w \right) L D_A^2 (1 - p_3) \leq 1 \]
\[ R_4: 2D^2_{\text{min}} - D_A^2 \leq 1 \]
\[ R_5: D_B/D_{\text{max}} \leq 1 \]
\[ R_6: \gamma_{m1} \leq \gamma_m \leq \gamma_{m2} \]
\[ R_7: \gamma_{m1} \leq \gamma_m \leq \gamma_{m2} \]
\[ R_8: 2\left( D_B^2/D_A^2 \right) - 2 \left( \gamma_B/\gamma_w \right) \left( D_B^2/D_A^2 \right) \leq 1 \]
\[ R_9: \frac{2D_B^2}{D_A^2} - \frac{\gamma_m}{\gamma_B} = \frac{2}{\gamma_w} + 2 \frac{\gamma_B}{\gamma_w} \leq 1 \]
\[ R_{10}: D_A^2/2D_B^2 \leq 1 \]
\[ R_{11}: \left( 2D^2_{\text{min}}/D_A^2 \right) - 8(\tau + \beta) E\beta D_A^2/\gamma_A L^3 \leq 1 \]
and
\[ \tau_B D_B R_B D_B \gamma_m > 0 \]
Note that \( R_3 \) is derived from (60) by elimination of \( I \) and \( W_R \). Also, constraints \( R_{10}, R_{11} \) are assumed inactive, i.e. only the case with buoyancy modules and a riser under tension will be examined.
Consider the variable $\tau$ which appears only in constraints $R_1$ and $R_3$ and not in the objective. Then, either both $R_1$ and $R_3$ are active, or they are both inactive. In fact, if either of them were inactive, the other would have to be also inactive, since otherwise $\tau$ would appear in a single constraint and its value at the optimum would be determined directly from the optimal values of the remaining variables. This argument holds whenever the single constraint does not imply another modified constraint on the remaining variables, as for example in the case of asymptotic substitution [10]. Next, assume that $R_1$ and $R_3$ are both inactive. Constraint $R_9$ shows that $D_B$ decreases with respect to $\beta$. Implicit elimination of $D_B$ from the model with the aid of $R_9$ results in

**Problem P2**

$$\min W_R(D_{R^+})$$

subject to

$$R_1(\gamma_{m^+}, D_{A^+}, D_{R^-}) \leq 1 \quad R_8(\gamma_{m^-}) \leq 1$$

$$R_4(D_{A^-}) \leq 1 \quad R_7(\gamma_{m^+}) \leq 1$$

$$R_5(\gamma_{m^+}, D_{R}, D_{A^-}) \leq 1 \quad R_9(\gamma_{m^+}, D_{R}, D_{A^-}) \leq 1$$

The superscript plus (minus) on a design variable denotes that the objective or constraint functions are increasing (decreasing) with respect to that variable. No superscript means that monotonicity does not exist or is unknown. Noting that only $R_3$ and $R_8$ contain $\beta$, both decreasing with respect to $\beta$, it follows that either the problem is unbounded or both $R_3$ and $R_8$ are inactive. In the latter case the optimum is given by simultaneous activity of constraints $R_1$, $R_4$, and $R_5$. Thus a local optimum is found from

$$W_R^* = \left( \frac{\pi R}{4} \right) D_{\min}^2 \lambda(1 - \lambda)$$

(73)

where

$$\lambda = \frac{(\gamma_{m^-} - \gamma_{m^-})(1 - \rho_0) L}{(3\sqrt{2}/2N_b)}$$

(74)

The remaining case is when $R_1$ and $R_2$ are both active. Constraint $R_8$ shows $D_B$ to be increasing with respect to $\gamma_{m^-}$. Substitution of $D_B$ throughout problem P1 results in $R_8$ being the only constraint decreasing with respect to $\gamma_{m^-}$ while the active constraint $R_2$ is increasing with respect to $\gamma_{m^-}$. Thus, by monotonicity, $R_8$ must be active, i.e. $\gamma_{m^-}$ must be set to its lower bound. Problem P1 is now reduced to two degrees of freedom. At this point it can be assumed that constraint $R_9$ is inactive. Physically this means that the riser is actually partially buoyed, which is in agreement with the requirement that there is a positive tension exerted at the top of the riser ($TTR > 0$). Thus the case of both $R_1$ and $R_2$ being active is described by

**Problem P3**

$$\min W_R(D_{R^+})$$

subject to

$$R_1(\beta, \gamma_{m^+}, D_{A^+}, D_{R^-}) \leq 1$$

$$R_2(\beta, \gamma_{m^+}, D_{A^+}, D_{R^-}) \leq 1$$

$$R_4(D_{A^+}) \leq 1$$

$$R_8(D_{R^-}) \leq 1$$

$$R_9(\beta, D_{R}, D_{A^-}, D_{R}) = 1$$

This problem cannot be reduced further unless the external loads are specified in order to obtain exact relations in $R_1$ and $R_2$ for $P_1(\beta, \gamma_{m^+}, P_0(\beta, \gamma_{m^+}), \rho_0$).

**Discussion**

The original optimization problem was reduced to problem P1 with five degrees of freedom and eleven constraints. This was further reduced to problems P2 and P3. Problem P2 yields a constraint-bound solution in closed form which is a local optimum for problem P1. Problem P3 having two degrees of freedom can be solved numerically once the design parameters and constants for a particular application are known. Any of the local optima obtained should not violate any of the constraints assumed inactive.

From the design point of view it is found that the mud density must be set always at its lower limit. In addition, at least one stress constraint must be active. In particular, when the internal loads are predominant, i.e. when the mud static pressure induces higher loads than those induced by the external hydrodynamic forces and the tension along the riser, constraint $R_1$ is active (problem P2). A closed form solution is thus obtained which does not depend on the external loads. Furthermore, when the external loads become significant, the combined stresses both at the top and the bottom of the riser will be critical (problem P3).

**Conclusions**

The original exact solution to the riser model was adequately approximated so that explicit expressions for the stresses were obtained. This allowed an algebraic formulation of the minimum weight design problem. Subsequent application of monotonicity analysis provided one local optimum and design rules for the investigation of other possible local optima. Numerical solutions can be obtained in future research following the procedure described in this paper and establishing the values of riser parameters, and the expression of the hydrodynamic loads. Parametric analysis can then generate design charts for practical use.

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