Geometric least-squares fitting of spheres, cylinders, cones and tori

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Abstract

This paper considers a problem arising in the reverse engineering of boundary representation solid models from three-dimensional depth maps of scanned objects. In particular, we wish to identify and fit surfaces of known type wherever these are a good fit, and we briefly outline a segmentation strategy for deciding to which surface type the depth points should be assigned. The particular contributions of this paper are methods for the least-squares fitting of spheres, cylinders, cones and tori to three-dimensional data. While plane fitting is well understood, least-squares fitting of other surfaces, even of such simple geometric type, has been much less studied; we review previous approaches to the fitting of such surfaces.

Our method has the particular advantage of being robust in the sense that as the principal curvatures of the surfaces being fitted decrease (or become more equal), the results which are returned naturally become closer and closer to the surfaces of "simpler type", *i.e.* planes, cylinders, or cones (or spheres) which best describe the data, unlike other methods which may diverge as various parameters or their combination become infinite.

Keywords. Non-linear least squares. Geometric distance. Cylinder, cone, sphere, torus, surface fitting.

1 Introduction

This paper considers the problem of least-squares fitting of spheres, cylinders, cones and tori to three-dimensional point data. The motivation for this problem lies in reverse engineering of geometric shape. A laser scanner or similar device is used to capture three-dimensional point data sampled from the surface of an object. From this we wish to construct a boundary representation solid model of the object's shape. In particular, we wish to identify and fit simple surfaces of known type to portions of the boundary wherever these are in good agreement with the point data. The problem can be decomposed into two logical steps: *segmentation*, where the data points are grouped into sets each belonging to a different surface, and *fitting*, where the best surface of an appropriate type is fitted to each set of points. The new results in this paper mainly concern the latter problem, but we first outline the method we use for solving the former, as it has a significant effect on the final model created.

While plane fitting is well understood, least-squares fitting of other surfaces, even of simple geometric type, has been relatively much less studied. We review previous approaches to the fitting of spheres, cylinders, cones and tori, and then present some new results on fitting these surfaces. Our method has the particular advantage of being robust in the sense that as the

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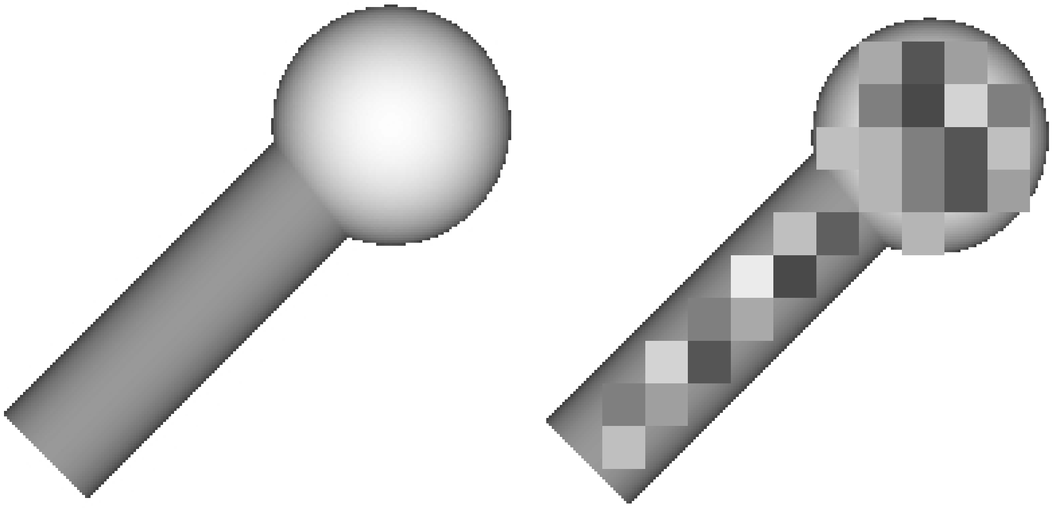

Figure 1: Initial 3D data and seed placement

principal curvatures of the surfaces being fitted decrease (or become more equal), the results which are returned naturally become closer and closer to the surfaces of "simpler type" *i.e.* planes, cylinders, or cones (or spheres) which best describe the data, unlike other methods which may diverge as various parameters or their combination become infinite.

2 Segmentation

Segmentation is the problem of grouping the points in the original dataset into subsets each of which logically belong to a single primitive surface. Various approaches exist for segmenting simple surfaces from three-dimensional data [Baj90, Bes88a, Bes88b, Bol91, Fau83, Heb82, Lio90]. Most commonly, segmentation has been viewed as a local-to-global aggregation problem with several similarity constraints employed to form a cohesive description in terms of geometric primitives. These approaches usually involve several stages, mostly applied in a sequential fashion, ranging from the estimation of local surface properties such as curvature, etc., to more complex feature clustering such as symmetry seeking. Typically, seed regions are initially small and placed at random locations in the data. The seed regions are then grown such that homogeneous regions are merged together. However, such region growing approaches tend to isolate the segmentation stage from the representation stage with the result that the data partitioning may not accord well with the given primitive types. In addition, the sensitivity of these methods to noise in the data (in particular outliers) may also lead to misclassification and, hence, poor results [Leo93]. Therefore, it is desirable for an efficient and reliable segmentation process to employ geometric knowledge of the primitive types to guide the detection and grouping processes, and to assure the coherence and consistency of models throughout the whole segmentation process [Baj90].

The approach employed to perform segmentation of the three-dimensional data in our reverse engineering project is based on the *recover and select* paradigm of Leonardis et al.[Jak96, Leo90, Leo93, Leo95]. Here an iterative process recovers and selects specific instances of the required geometric primitives (planes, spheres, cylinders, cones and tori). The basic approach partitions the data according to primitives by searching for models as regions are systematically grown such that the description is *best* in terms of global shape and error of fit. Initially seed regions are placed at arbitrary locations in the data and models of each primitive approximated (Fig. 1). Grossly mismatching models may be rejected at this stage. An iterative grow and select phase is then operated. All valid models are grown for an equal number of steps (see Fig. 2); note that the models are allowed to overlap. The resulting models are then inspected and some are selected

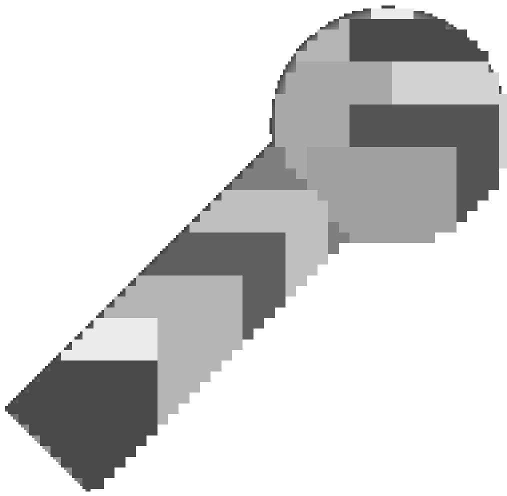

Figure 2: Grown and selected regions at an intermediate stage

for further growing. Optimal models from the overlapping sets are selected on the basis of the following criteria:

Area — the number of image elements contained in the model

Error of Fit — the maximum or average distance to the model

Quality of Fit — the number of parameters used to describe the model

Surface Type — the class of surface of the model

In the case that different models have similar goodness of fit, some types of model may be preferred to others, and the surface type is used to impose an ordering of selection in the way suggested by Besl and Jain [Bes88a, Bes88b]. In the implementation the ordering can be done in various ways although it is typically in terms of increasing surface complexity (*i.e.* plane, sphere, cylinder, cone, torus), where the simpler surface would be chosen first. A weighted sum of the above criteria is then used as a *cost-benefit* measure to select the optimal model or models.

Models that may simply have a poor error fit even if not overlapping other models are also rejected at the selection stage.

Given this segmentation framework, we thus have a need for modelling, and particularly fitting, certain primitive surface types. The rest of this paper discuss fitting methods for spheres, cylinders, cones and tori.

3 Surface fit and general non-linear least-squares

As noted in the Introduction, while plane fitting is well understood, least-squares fitting of other surfaces of simple geometric type has received much less attention. We start here by outlining some basic concepts which are generally useful, and then review previous approaches to the fitting of these surfaces.

Let us assume that each of the three-dimensional points $\mathbf{p}_i$ for $i = 1, \ldots, m$ lies close to the same member of a family of surfaces which can be parameterised by $\mathbf{s} \in G \subseteq \mathbb{R}^s$ where G is an open set. Let d be a function which is defined as the distance of the point $\mathbf{p}_i \in \mathbb{R}^3$ from that surface in the family identified by $\mathbf{s}$. Throughout, d will be called the "true" distance function of the surface (fitting methods generally rely on approximations to this distance as will be explained later).

A surface which goes through all the points can be viewed as that member of the family which corresponds to the solution of the simultaneous system of m equations:

$$d(\mathbf{s}, \mathbf{p}_i) = 0, \qquad \text{for } i = 1, \ldots, m. \tag{3.1}$$

Since the number of points (m) is usually much greater than the number of degrees of freedom (s), this system of equations is overdetermined and in general cannot be solved. However it is possible to solve it in the least-squares sense, *i.e.* to search for the surface which is the best fit to the points "on the average", minimising

$$\sum_{i=1}^{m} d(\mathbf{s}, \mathbf{p}_i)^2 \tag{3.2}$$

Sometimes we might have additional non-linear constraints like

$$H(\mathbf{s}) = \mathbf{0} \in \mathbb{R}^t \tag{3.3}$$

for some integer $t < s$. (For example, the surface form being used might describe quadric surfaces in general, but we may wish to restrict the surface being found to being a cylinder, which can by done by imposing constraints on $\mathbf{s}$.) When the principle of Lagrangian multipliers is used to include these constraints, it produces a non-linear generalized eigenvalue problem, which is not easy to solve. A simpler approach is to use Eqn. 3.3 to eliminate t unknowns, to reduce the problem to an unconstrained optimization problem in a lower dimensional space. This is the method we shall use.

Note that usually the family of surfaces is defined as points satisfying an implicit equation:

$$f(\mathbf{s}, \mathbf{x}) = 0, \qquad \text{for } \mathbf{x} \in \mathbb{R}^3, \tag{3.4}$$

where $\mathbf{s}$ is the family parameter. Although if we fix $\mathbf{s}$, f and d have the same roots in space, they may behave quite differently for points which do not lie on the surface. Thus, if instead of Expression Eqn. 3.2, one minimizes $\sum f^2$ this may give quite different results. However, this approach can be justified if *both* the function f and the constraint H in Eqn. 3.3 are of particularly simple form. If f is linear and H is quadratic in terms of the parameters then basically linear generalised eigenvalue techniques work (see *e.g.* [Fpf96]). If f is non-linear but H is still quadratic then one can readily try Taubin's generalized eigenvector fit [Tau91]. Nevertheless, the choice of form for f is important from the point view of the behaviour of the non-linear fitting algorithm and consequently the quality of the obtained solution. For fitting ellipses, Rosin [Ros96] shows that choosing f carelessly can lead to severely biassed estimates for $\mathbf{s}$. Below we shall propose 'fairly good' f functions (*i.e.* f behaves much like d near the surface) which are highly non-linear, and which have no additional constraints on the parameters $\mathbf{s}$.

4 Approximating the true distance

As far as possible one has to avoid singularities of $d(\mathbf{s}, \mathbf{p}_i)$ in the range where solutions may lie. These singularities may be places where some denominator in $d(\mathbf{s}, \mathbf{p}_i)$ vanishes, or where the distance function is not differentiable. Using Euclidean metrics such singularities arise frequently since the Euclidean distance from a given fixed point is *itself* singular in this sense. Nevertheless, most of these nonlinear singularities are only *computational*, *i.e.* unessential discontinuities in the mathematical sense, meaning that a limiting value of the distance function can still be found for the critical parameter value. Even so, the computation of the distance function (or its derivatives) can be unstable at such points, since it may require the subtraction of similar quantities etc.

Avoiding the effects of singularities can be achieved by means of various techniques. First one chooses a suitable parameterisation where the critical values do not lie on the border of G. Secondly one changes the definition of $d(\mathbf{s}, \mathbf{p}_i)$ slightly in order to get rid of singularities. We shall say that this modified definition is *faithful* to the true Euclidean distance function if, firstly, the function is zero where the true distance is zero, and secondly, at these points, the derivatives with respect to the parameters are the same for the true distance and the modified definition.

Faithful distance functions can be obtained if one approximates square roots within $d(\mathbf{s}, \mathbf{p}_i)$ in the following way. Suppose that the distance function is of the following form:

$$d(\mathbf{s}, \mathbf{p}_i) = \sqrt{g} - h, \tag{4.1}$$

where both g and h may depend on the parameter vector $\mathbf{s}$ and on the point $\mathbf{p}_i$ in three space. In order to get rid of the square root we might try to minimize $\sum(g - h^2)^2$ instead of Eqn. 3.2 since $d = 0$ when $g = h^2$. Unfortunately, the effect is now that we are searching for the surface which fits best in terms of the "average" of the *square* of the distance instead of the distance. Thus, this transformation amplifies the importance of the further points unnecessarily, and flattens the goal function in the neighbourhood of the solution. Instead, let us put

$$\tilde{d} = \frac{g - h^2}{2h}. \tag{4.2}$$

and let us minimise

$$\sum \tilde{d}^2(\mathbf{s}, \mathbf{p}_i) = \sum \frac{\left(g(\mathbf{s}, \mathbf{p}_i) - h^2(\mathbf{s}, \mathbf{p}_i)\right)^2}{4h^2(\mathbf{s}, \mathbf{p}_i)}. \tag{4.3}$$

The following simple assertion holds:

Statement 4.1. *Let us assume that a function $d : E \to \mathbb{R}$ is defined on a region $E \subset \mathbb{R}^n$ and it can be expressed by means of of two positive continuously differentiable functions g and h on E as*

$$d = \sqrt{g} - h.$$

Let us define

$$\tilde{d} = \frac{g - h^2}{2h}.$$

Then, if $\mathbf{q} \in E$, $\tilde{d}(\mathbf{q}) = 0$ holds if and only if $d(\mathbf{q}) = 0$. Moreover, in this case:

$$\frac{\partial d}{\partial q_i}(\mathbf{q}) = \frac{\partial \tilde{d}}{\partial q_i}(\mathbf{q}) \qquad \text{for } i = 1, \ldots, n.$$

More generally, if $\mathbf{q}_n \in E$ and $\mathbf{q}_n \to \mathbf{q}$, $d(\mathbf{q}_n) \to 0$, $d > 0$ and $h(\mathbf{q}_n) \to \infty$ then

$$\lim_{\mathbf{q}_n \to \mathbf{q}} \frac{d(\mathbf{q}_n)}{|\mathbf{q} - \mathbf{q}_n|} = \lim_{\mathbf{q}_n \to \mathbf{q}} \frac{\tilde{d}(\mathbf{q}_n)}{|\mathbf{q} - \mathbf{q}_n|}. \tag{4.4}$$

PROOF: Replacing g,

$$\tilde{d} = \frac{g - h^2}{2h} = d + \frac{d^2}{2h}, \tag{4.5}$$

so if $\tilde{d}$ vanishes then d must be 0 since otherwise h would have to be negative.

Moreover at a solution point ($d = 0$) the partial derivatives of the modified function $\tilde{d}$ with respect to any of the surface parameters $\mathbf{s}$ and spatial parameters $\mathbf{p}$ will be the same as those of the original function d. This is not true for the function $g - h^2$.

Note that Eqn. 4.4 is a straightforward consequence of Eqn. 4.5 and the condition $d > 0$. Note that here $\mathbf{q}$ may not be in E. ■

Remark 4.1. *In fact, approximation Eqn. 4.2 can be generalized when $h < 0$. For regions where h is negative Eqn. 4.1 should be extended to be*

$$d = \text{sign}(h)\sqrt{g} - h = \text{sign}(h) \cdot (\sqrt{g} - |h|) \tag{4.6}$$

and then Eqn. 4.2 still works. Tacitly this will be done always below.

In summary, in our approach we start with an exact expression for the distance d. This is replaced by a simplification which is easier to compute, but which still has the same zero set and derivatives at the zero set. In contrast, similar work by Taubin [Tau91] starts from a parameterised family of implicit functions $f = 0$. He notes that while f itself is not a good approximation to d, $f/|\nabla f|$ is much better, *i.e.* he replaces the original implicit function with a new one whose value is a better approximation to d. Although Taubin does not state so explicitly, it is clear that it is better because the derivatives with respect to spatial parameters of this function are the same as those of the distance function. In practice Taubin's approach can be used only if f is linear with respect to the parameters, and the system to be solved then includes a quadratic constraint. Another difference is that our approach is better behaved with respect to singularities.

5 Fitting spheres, cylinders, cones and tori

The linear least-squares fitting of second order curves and surfaces has been relatively thoroughly investigated recently (See [Pra87], [Ros93], [Ggs94], [Fpf96]). However, specific linear methods still do not exist for right cylinders and cones. The reason behind this fact is that the equations expressing the conditions for a space quadric to be a right cylinder or a cone are not quadratic. If general linear methods are used for algebraic second order surfaces the solutions found are usually not right cylinders or cones and may even be very different from the optimum surfaces of such type. In this sense, algebraic techniques which use the value of the implicit quadratic form as the "distance" from the surface approximate the true geometric distance in a rather unfaithful way.

The situation is much simpler for spheres since straightforward algebraic methods work in this case: under a suitable normalization the minimised algebraic distance will reflect the geometric distance as well. For example, the method in [Pra87] minimises

$$\sum_i \left(A(x_i^2 + y_i^2 + z_i^2) + Dx_i + Ey_i + Fz_i + G\right)^2 \tag{5.1}$$

under the condition

$$D^2 + E^2 + F^2 - 4AG = 1 \tag{5.2}$$

which is basically equivalent to our minimisation [1] in Eqn. 4.3. Nevertheless, we shall give our non-linear method for spheres in the next section as an illustration of our method, as it has certain advantages.

Nonlinear methods which take into account the true geometric distance match well with the requirements of our reverse engineering requirements. Those points belonging to the same elementary surface are selected by means of some segmentation technique which usually provides an initial approximation for the parameters of each surface, but these may be some way from being an optimal fit. Starting from these, at the expense of some computing time, one can obtain a more accurate fit. Our nonlinear methods will also work well in other applications where again some initial approximate fit for the surface is already known.

5.1 A simple formula for vector products

The following formula is used frequently in the sections below.

Lemma 5.1. *Let* $\mathbf{a}$*,* $\mathbf{b}$*,* $\mathbf{c}$ *and* $\mathbf{d}$ *be arbitrary vectors in three dimensional space. Then*

$$\langle \mathbf{a} \times \mathbf{b}, \mathbf{c} \times \mathbf{d} \rangle = \langle \mathbf{a}, \mathbf{c} \rangle \langle \mathbf{b}, \mathbf{d} \rangle - \langle \mathbf{a}, \mathbf{d} \rangle \langle \mathbf{b}, \mathbf{c} \rangle. \tag{5.3}$$

[1] Note that the most straightforward constraint, $A = 1$ may give quite unfaithful results as shown in [Pra87].

5.2 Sphere fitting

For non-linear least-squares fit the parameterisation of the sphere will be the following. Suppose that the *closest* point of the sphere (*not* its centre) to the origin is $\varrho\mathbf{n}$, where $|\mathbf{n}| = 1$ and the radius of the sphere is $1/k$. Then if $\mathbf{p}$ is an arbitrary point in space, the distance of this point from the surface of the sphere is:

$$d(\mathbf{s},\mathbf{p}) = \left|\mathbf{p} - (\varrho + \frac{1}{k})\mathbf{n}\right| - \frac{1}{k} = \sqrt{\langle \mathbf{p} - (\varrho + \frac{1}{k})\mathbf{n}, \mathbf{p} - (\varrho + \frac{1}{k})\mathbf{n}\rangle} - \frac{1}{k} \tag{5.4}$$

Since this function is of the form Eqn. 4.1 we can apply approximation Eqn. 4.2 to give

$$\tilde{d}(\mathbf{s},\mathbf{p}) = \frac{k}{2}\left(|\mathbf{p}|^2 - 2\varrho\langle \mathbf{p},\mathbf{n}\rangle + \varrho^2\right) + \varrho - \langle \mathbf{p},\mathbf{n}\rangle, \tag{5.5}$$

or if one introduces the notation

$$\widehat{\mathbf{p}} = \mathbf{p} - \varrho\mathbf{n}, \tag{5.6}$$

one obtains

$$\tilde{d}(\mathbf{s},\mathbf{p}) = \frac{k}{2}\,|\widehat{\mathbf{p}}|^2 - \langle \widehat{\mathbf{p}},\mathbf{n}\rangle. \tag{5.7}$$

Note that here $\widehat{\mathbf{p}}$ is the expression of $\mathbf{p}$ as if the origin were $\varrho\mathbf{n}$. Now let us parameterize $\mathbf{n}$ by its polar coordinates, so in the usual way

$$\mathbf{n} = (\cos\varphi\sin\vartheta, \sin\varphi\sin\vartheta, \cos\vartheta)\,, \tag{5.8}$$

where ϑ is the angle between $\mathbf{n}$ and the z axis and φ is the angle of the projection of $\mathbf{n}$ onto the plane $z = 0$ with the x axis. Obviously if one differentiates $\mathbf{n}$ with respect to φ and ϑ one obtains two partial derivative vectors which are orthogonal to each other and to $\mathbf{n}$ (upper indices denote derivatives)

$$\mathbf{n}^{\varphi} = (-\sin\varphi\sin\vartheta, \cos\varphi\sin\vartheta, 0) \tag{5.9}$$

$$\mathbf{n}^{\vartheta} = (\cos\varphi\cos\vartheta, \sin\varphi\cos\vartheta, -\sin\vartheta)\,. \tag{5.10}$$

Thus $\mathbf{n}$ and hence $\tilde{d}$ can be parameterised without constraints by $\mathbf{s} = (\varrho, \varphi, \vartheta, k)$.

Statement 5.1. *The partial derivatives of the approximate distance function Eqn. 5.5 are the following:*

$$\frac{\partial\tilde{d}}{\partial\varrho} = k\left(\varrho - \langle \mathbf{p},\mathbf{n}\rangle\right) + 1, \tag{5.11}$$

$$\frac{\partial\tilde{d}}{\partial\varphi} = (-k\varrho - 1)\langle \mathbf{p},\mathbf{n}^{\varphi}\rangle \tag{5.12}$$

$$\frac{\partial\tilde{d}}{\partial\vartheta} = (-k\varrho - 1)\langle \mathbf{p},\mathbf{n}^{\vartheta}\rangle \tag{5.13}$$

$$\frac{\partial\tilde{d}}{\partial k} = \frac{1}{2}\left(|\mathbf{p}|^2 - 2\varrho\langle \mathbf{p},\mathbf{n}\rangle + \varrho^2\right). \tag{5.14}$$

Note that unlike Eqn. 5.1, Eqn. 5.5 is nonlinear but it behaves well as the curvature of the sphere decreases, as in that case, $k \to 0$, all the terms are bounded, and Eqn. 5.5 basically reduces to the expression that would be used for least-squares plane fitting. In contrast, observe that in case of Eqn. 5.1 some of the terms will tend to infinity both in the objective function and in the constraint Eqn. 5.2.

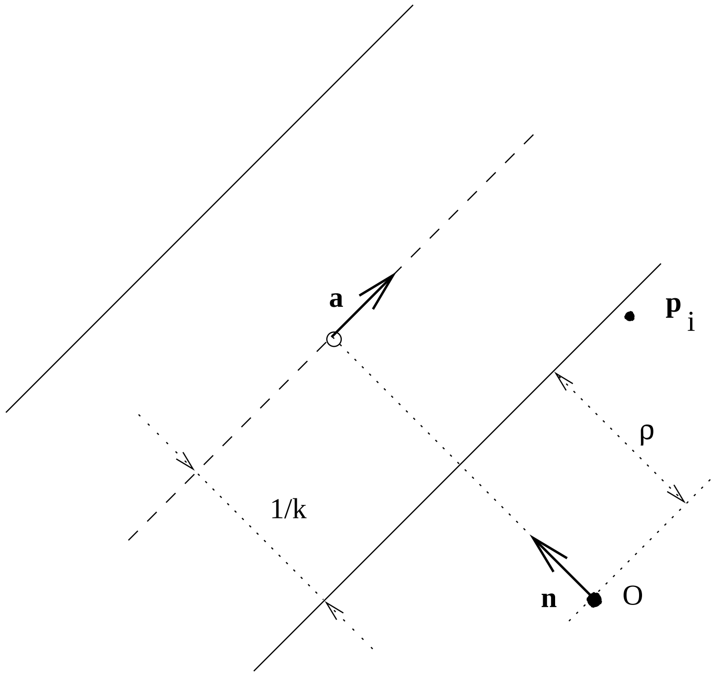

Figure 3: Parameterisation of the cylinder

5.3 Right circular cylinder fitting

The parameterisation used for the cylinder is similar to that for the sphere above. The closest point of the cylinder to the origin is $\varrho\mathbf{n}$, where $|\mathbf{n}| = 1$. Assume that the direction of the axis of the cylinder is $\mathbf{a}$ with $|\mathbf{a}| = 1$ and the radius of the cylinder is $1/k$. Note that $\langle \mathbf{n}, \mathbf{a} \rangle = 0$. Now let us suppose that $\mathbf{p}$ is an arbitrary point in space and compute the distance of this point from the surface of the cylinder. This is given by finding the distance from the symmetry axis and from it subtracting the radius of the cylinder (see Fig. 3):

$$d(\mathbf{s}, \mathbf{p}) = \left|(\mathbf{p} - (\varrho + \frac{1}{k})\mathbf{n}) \times \mathbf{a}\right| - \frac{1}{k} = \sqrt{|\mathbf{p} - (\varrho + \frac{1}{k})\mathbf{n}|^2 - \langle \mathbf{p} - (\varrho + \frac{1}{k})\mathbf{n}, \mathbf{a}\rangle^2} - \frac{1}{k} \tag{5.15}$$

Since this function is of the form Eqn. 4.1 we can apply approximation Eqn. 4.2 to obtain

$$\tilde{d}(\mathbf{s}, \mathbf{p}) = \frac{k}{2}\left(|\mathbf{p}|^2 - 2\varrho\langle \mathbf{p}, \mathbf{n}\rangle - \langle \mathbf{p}, \mathbf{a}\rangle^2 + \varrho^2\right) + \varrho - \langle \mathbf{p}, \mathbf{n}\rangle = \frac{k}{2}|\widehat{\mathbf{p}} \times \mathbf{a}|^2 - \langle \widehat{\mathbf{p}}, \mathbf{n}\rangle, \tag{5.16}$$

where $\widehat{\mathbf{p}} = \mathbf{p} - \varrho\mathbf{n}$ as in Eqn. 5.6. Using appropriate parameterisations for $\mathbf{n}$ and $\mathbf{a}$ we are going to minimise the function

$$\sum_i \tilde{d}^2(\mathbf{s}, \mathbf{p}_i).$$

Let us make some observations about formula Eqn. 5.16. Firstly, it is linear in the curvature k if all other parameters are fixed. This kind of problem is called a *separable* non-linear least squares problem (see *e.g.* [Bjö96]), and they are sometimes easier to solve than the fully non-linear case[2]. Note that Eqn. 5.16 behaves well as k gets smaller k (as ϱ is bounded within sensible limits by the geometric configuration of the scanner); compare Eqn. 5.15 which involves the subtraction of

[2] It is easy to see that we do not need an initial estimate for k if we have estimates for the other parameters: an initial value for k can be found by solving a linear least-squares problem in which all other parameters but k are fixed.

two large quantities as k becomes small. In the limit as $k \to 0$ we get $\tilde{d} = \varrho - \langle \mathbf{p}, \mathbf{n} \rangle$ which again means the problem reduces to a linear least-squares fitting of planes.

The difficult question is how to parameterise $\mathbf{n}$ and $\mathbf{a}$ in order to fulfil the relations:

$$|\mathbf{n}| = |\mathbf{a}| = 1, \quad \langle \mathbf{n}, \mathbf{a} \rangle = 0.$$

Again we use polar coordinates. The parameterisation for $\mathbf{n}$ was introduced in Eqn. 5.8, and Eqn. 5.9 and Eqn. 5.10 are the partial derivatives of $\mathbf{n}$. Thus if we put

$$\overline{\mathbf{n}^{\varphi}} = (-\sin\varphi, \cos\varphi, 0) = \frac{\mathbf{n}^{\varphi}}{\sin\vartheta}, \tag{5.17}$$

then $\mathbf{n}^{\vartheta}$, $\overline{\mathbf{n}^{\varphi}}$and $\mathbf{n}$ are all unit vectors and mutually orthogonal. Hence we can parameterise $\mathbf{a}$ as follows:

$$\mathbf{a} = \mathbf{n}^{\vartheta} \cos\alpha + \overline{\mathbf{n}^{\varphi}} \sin\alpha, \tag{5.18}$$

where α is the angle subtended between $\mathbf{a}$ and $\mathbf{n}^{\vartheta}$. Thus, $\mathbf{n}$ and $\mathbf{a}$ are parameterised through φ, ϑ and α by means of expressions Eqn. 5.8, Eqn. 5.18, Eqn. 5.10 and Eqn. 5.17.

Statement 5.2. *A non-linear distance function for right circular cylinders which is faithful up to the first derivative is Eqn. 5.16. It is parameterised in terms of ϱ, φ, ϑ, α and k using expressions Eqn. 5.8, Eqn. 5.18, Eqn. 5.10 and Eqn. 5.17. If all other parameters are fixed this function is linear in terms of the curvature k of the cylinder. The partial derivatives are the following:*

$$\frac{\partial \tilde{d}}{\partial \varrho} = k\left(\varrho - \langle \mathbf{p}, \mathbf{n} \rangle\right) + 1, \tag{5.19}$$

$$\frac{\partial \tilde{d}}{\partial \varphi} = -k\left(\varrho \langle \mathbf{p}, \mathbf{n}^{\varphi} \rangle + \langle \mathbf{p}, \mathbf{a} \rangle \langle \mathbf{p}, \mathbf{n}^{\vartheta\varphi} \cos\alpha + \overline{\mathbf{n}^{\varphi\varphi}} \sin\alpha \rangle\right) - \langle \mathbf{p}, \mathbf{n}^{\varphi} \rangle, \tag{5.20}$$

$$\frac{\partial \tilde{d}}{\partial \vartheta} = k\left(\langle \mathbf{p}, \mathbf{a} \rangle \langle \mathbf{p}, \mathbf{n} \rangle \cos\alpha - \varrho \langle \mathbf{p}, \mathbf{n}^{\vartheta} \rangle\right) - \langle \mathbf{p}, \mathbf{n}^{\vartheta} \rangle, \tag{5.21}$$

$$\frac{\partial \tilde{d}}{\partial \alpha} = k \langle \mathbf{p}, \mathbf{a} \rangle \langle \mathbf{p}, \mathbf{n}^{\vartheta} \sin\alpha - \overline{\mathbf{n}^{\varphi}} \cos\alpha \rangle, \tag{5.22}$$

$$\frac{\partial \tilde{d}}{\partial k} = \frac{1}{2}\left(|\mathbf{p}|^2 - 2\varrho \langle \mathbf{p}, \mathbf{n} \rangle - \langle \mathbf{p}, \mathbf{a} \rangle^2 + \varrho^2\right), \tag{5.23}$$

where the second derivatives of $\mathbf{n}$ *with respect to* φ *and* ϑ *are:*

$$\mathbf{n}^{\vartheta\varphi} = (-\sin\varphi\cos\vartheta, \cos\varphi\cos\vartheta, 0) \tag{5.24}$$

$$\overline{\mathbf{n}^{\varphi\varphi}} = (-\cos\varphi, \sin\varphi, 0) \tag{5.25}$$

$$\mathbf{n}^{\vartheta\vartheta} = (\cos\varphi\sin\vartheta, -\sin\varphi\sin\vartheta, -\cos\vartheta) = -\mathbf{n}. \tag{5.26}$$

PROOF: Let us compute the derivatives of $\langle \mathbf{p}, \mathbf{a} \rangle^2$:

$$\frac{\partial \langle \mathbf{p}, \mathbf{a} \rangle^2}{\partial \varphi} = 2\langle \mathbf{p}, \mathbf{a} \rangle \langle \mathbf{p}, \mathbf{n}^{\vartheta\varphi} \cos\alpha + \overline{\mathbf{n}^{\varphi\varphi}} \sin\alpha \rangle,$$

$$\frac{\partial \langle \mathbf{p}, \mathbf{a} \rangle^2}{\partial \vartheta} = -2\langle \mathbf{p}, \mathbf{a} \rangle \langle \mathbf{p}, \mathbf{n} \cos\alpha \rangle,$$

$$\frac{\partial \langle \mathbf{p}, \mathbf{a} \rangle^2}{\partial \alpha} = 2\langle \mathbf{p}, \mathbf{a} \rangle \langle \mathbf{p}, -\mathbf{n}^{\vartheta} \sin\alpha + \overline{\mathbf{n}^{\varphi}} \cos\alpha \rangle.$$

If one substitutes these formulae into the derivatives of Eqn. 5.16 one obtains Eqn. 5.19, Eqn. 5.20, Eqn. 5.21, Eqn. 5.22, Eqn. 5.23. ■

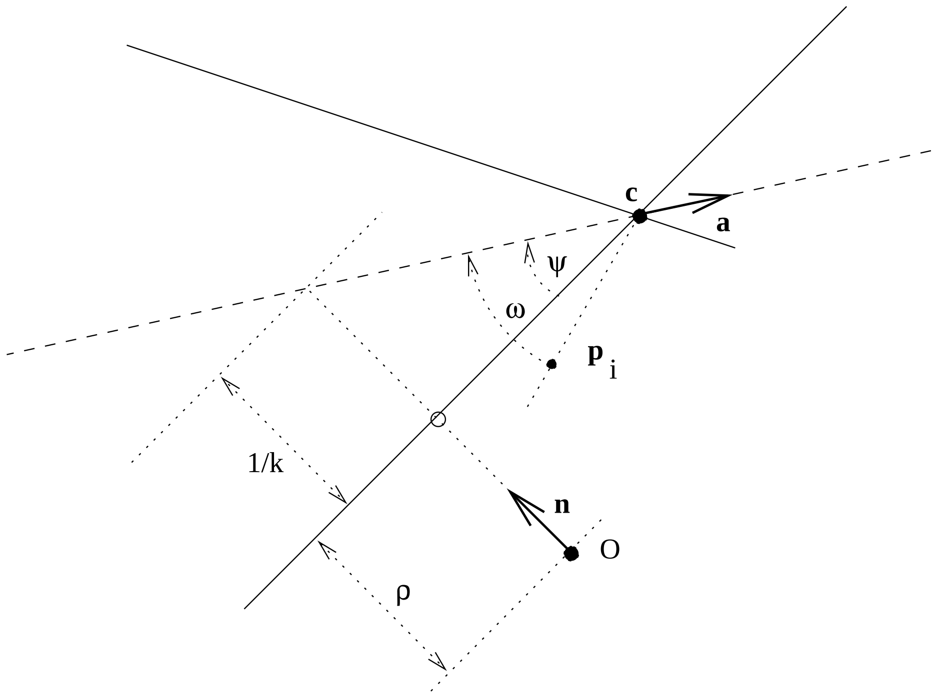

Figure 4: Parameterisation of the cone

5.4 Right circular cone fitting

Again, the parameterisation used for the cone will be quite similar to that for the cylinder. Let $\varrho\mathbf{n}$ with $|\mathbf{n}| = 1$ be that point on the cone surface for which a line in the direction of the surface normal passes through the origin. (Hence $\mathbf{n}$ is a normal to the cone.) Let the non-zero principal curvature of the cone at the point $\varrho\mathbf{n}$ be k (*i.e.* the radius of the osculating sphere is $1/k$). Let us denote the unit direction of the axis of the cone by $\mathbf{a}$. $\mathbf{n}$ will be parameterized by φ and ϑ as above in Eqn. 5.8. Since $\mathbf{n}$ and $\mathbf{a}$ are not in this case perpendicular, $\mathbf{a}$ can be parameterised freely by two polar coordinate angles, σ and τ:

$$\mathbf{a} = (\cos\sigma\sin\tau, \sin\sigma\sin\tau, \cos\tau)\,, \tag{5.27}$$

where τ is the angle between $\mathbf{a}$ and the z axis and σ is the angle of the projection of $\mathbf{a}$ onto the plane $z = 0$ with the x axis. The six parameters $(\varrho, \varphi, \vartheta, k, \sigma, \tau)$ entirely characterise the right circular cone surface.

In order to understand how this works, let the half angle of the cone be ψ (see Fig. 4), and denote the position of the apex of the cone by $\mathbf{c}$. (We shall express ψ and $\mathbf{c}$ using the above parameters later.) Let us compute the distance of $\mathbf{p}$, an arbitrary point, from the cone surface. Let us denote the angle between axis of the cone and $\mathbf{p} - \mathbf{c}$ by ω. Then the distance from the mantle is given by[3]:

$$d(\mathbf{s}, \mathbf{p}) = |\mathbf{p} - \mathbf{c}|\sin(\omega - \psi) = |\mathbf{p} - \mathbf{c}|\sin\omega\cos\psi - |\mathbf{p} - \mathbf{c}|\cos\omega\sin\psi.$$

Obviously without loss of generality one can suppose that both ψ and ω are acute angles. Since the direction of the axis, $\mathbf{a}$ is a unit vector we have:

$$d(\mathbf{s}, \mathbf{p}) = |(\mathbf{p} - \mathbf{c}) \times \mathbf{a}|\cos\psi - |\langle\mathbf{p} - \mathbf{c}, \mathbf{a}\rangle|\sin\psi \tag{5.28}$$

[3]Note that $\mathbf{p}$ may not lie in the plane of $\mathbf{n}$ and $\mathbf{a}$.

Moreover since the angle between the normal $\mathbf{n}$ and the axis $\mathbf{a}$ is the complementary angle to ψ we have:

$$\cos\psi = |\mathbf{n}\times\mathbf{a}| \qquad \sin\psi = |\langle\mathbf{n},\mathbf{a}\rangle|. \tag{5.29}$$

Thus from Eqn. 5.28 one obtains:

$$\begin{aligned} d(\mathbf{s},\mathbf{p}) &= |(\mathbf{p}-\mathbf{c})\times\mathbf{a}|\,|\mathbf{n}\times\mathbf{a}| - |\langle\mathbf{p}-\mathbf{c},\mathbf{a}\rangle\langle\mathbf{n},\mathbf{a}\rangle| \\ &= |\mathbf{n}\times\mathbf{a}|\sqrt{|\mathbf{p}-\mathbf{c}|^2 - \langle\mathbf{p}-\mathbf{c},\mathbf{a}\rangle^2} - |\langle\mathbf{n},\mathbf{a}\rangle\langle\mathbf{p}-\mathbf{c},\mathbf{a}\rangle|\,. \end{aligned} \tag{5.30}$$

As in the cylinder case this function is again of the form Eqn. 4.1 so one can apply approximation Eqn. 4.2 and obtain

$$\tilde{d}(\mathbf{s},\mathbf{p}) = \frac{|\mathbf{p}-\mathbf{c}|^2\cos^2\psi - \langle\mathbf{p}-\mathbf{c},\mathbf{a}\rangle^2}{2\langle\mathbf{p}-\mathbf{c},\mathbf{a}\rangle\sin\psi} = \frac{|\mathbf{p}-\mathbf{c}|^2\,|\mathbf{n}\times\mathbf{a}|^2 - \langle\mathbf{p}-\mathbf{c},\mathbf{a}\rangle^2}{2\langle\mathbf{p}-\mathbf{c},\mathbf{a}\rangle\langle\mathbf{n},\mathbf{a}\rangle}. \tag{5.31}$$

Statement 5.3. *A non-linear distance function for right circular cones which is faithful up to the first derivative is:*

$$\tilde{d}(\mathbf{s},\mathbf{p}) = \frac{\frac{k}{2}(|\mathbf{n}\times\mathbf{a}|^2|\widehat{\mathbf{p}}|^2 - \langle\widehat{\mathbf{p}},\mathbf{a}\rangle^2) - \langle\widehat{\mathbf{p}},\mathbf{n}\rangle|\mathbf{n}\times\mathbf{a}|^2}{k\langle\widehat{\mathbf{p}},\mathbf{a}\rangle\langle\mathbf{n},\mathbf{a}\rangle + |\mathbf{n}\times\mathbf{a}|^2}, \tag{5.32}$$

(the proof follows) where as in Eqn. 5.6 $\widehat{\mathbf{p}} = \mathbf{p} - \varrho\mathbf{n}$*. Thus this function* $\tilde{d}$ *depends on six unconstrained parameters:* ϱ, φ, ϑ, σ, τ *and* k*. If we introduce the notation*

$$\lambda(\varrho,\varphi,\vartheta,\sigma,\tau) = (|\mathbf{n}\times\mathbf{a}|^2|\widehat{\mathbf{p}}|^2 - \langle\widehat{\mathbf{p}},\mathbf{a}\rangle^2)/2 \tag{5.33}$$
$$\xi(\varrho,\varphi,\vartheta,\sigma,\tau) = -\langle\widehat{\mathbf{p}},\mathbf{n}\rangle|\mathbf{n}\times\mathbf{a}|^2 \tag{5.34}$$
$$\mu(\varrho,\varphi,\vartheta,\sigma,\tau) = \langle\widehat{\mathbf{p}},\mathbf{a}\rangle\langle\mathbf{n},\mathbf{a}\rangle \tag{5.35}$$
$$\eta(\varphi,\vartheta,\sigma,\tau) = |\mathbf{n}\times\mathbf{a}|^2, \tag{5.36}$$

the partial derivatives of the function $\tilde{d}$ *are the following:*

$$\frac{\partial\tilde{d}}{\partial\varrho} = \frac{1}{(\mu k+\eta)^2}\left((\lambda^\varrho\mu - \lambda\mu^\varrho)\,k^2 + (\lambda^\varrho\eta + \xi^\varrho\mu - \xi\mu^\varrho)\,k + \xi^\varrho\eta\right) \tag{5.37}$$
$$\frac{\partial\tilde{d}}{\partial\varphi} = \frac{1}{(\mu k+\eta)^2}\left((\lambda^\varphi\mu - \lambda\mu^\varphi)\,k^2 + (\lambda^\varphi\eta + \xi^\varphi\mu - \lambda\eta^\varphi - \xi\mu^\varphi)\,k + \xi^\varphi\eta - \xi\eta^\varphi\right) \tag{5.38}$$
$$\frac{\partial\tilde{d}}{\partial\vartheta} = \frac{1}{(\mu k+\eta)^2}\left((\lambda^\vartheta\mu - \lambda\mu^\vartheta)\,k^2 + \left(\lambda^\vartheta\eta + \xi^\vartheta\mu - \lambda\eta^\vartheta - \xi\mu^\vartheta\right)k + \xi^\vartheta\eta - \xi\eta^\vartheta\right) \tag{5.39}$$
$$\frac{\partial\tilde{d}}{\partial k} = \frac{\lambda\eta - \mu\xi}{(\mu k+\eta)^2}, \tag{5.40}$$
$$\frac{\partial\tilde{d}}{\partial\sigma} = \frac{1}{(\mu k+\eta)^2}\left((\lambda^\sigma\mu - \lambda\mu^\sigma)\,k^2 + (\lambda^\sigma\eta + \xi^\sigma\mu - \lambda\eta^\sigma - \xi\mu^\sigma)\,k + \xi^\sigma\eta - \xi\eta^\sigma\right) \tag{5.41}$$
$$\frac{\partial\tilde{d}}{\partial\tau} = \frac{1}{(\mu k+\eta)^2}\left((\lambda^\tau\mu - \lambda\mu^\tau)\,k^2 + (\lambda^\tau\eta + \xi^\tau\mu - \lambda\eta^\tau - \xi\mu^\tau)\,k + \xi^\tau\eta - \xi\eta^\tau\right). \tag{5.42}$$

The derivatives of the auxiliary functions λ *and* ξ *are:*

$$\begin{aligned}
\lambda^\varrho &= \varrho(|\mathbf{n}\times\mathbf{a}|^2 - 2\langle\mathbf{n},\mathbf{a}\rangle^2) + 2\langle\mathbf{p},\mathbf{a}\rangle\langle\mathbf{n},\mathbf{a}\rangle - \langle\mathbf{p},\mathbf{n}\rangle|\mathbf{n}\times\mathbf{a}|^2 & \xi^\varrho &= 2|\mathbf{n}\times\mathbf{a}|^2 \\
\lambda^\varphi &= -\langle\mathbf{n}^\varphi,\mathbf{a}\rangle\langle\mathbf{n},\mathbf{a}\rangle|\widehat{\mathbf{p}}|^2 - \varrho|\mathbf{n}\times\mathbf{a}|^2\langle\mathbf{n}^\varphi,\mathbf{p}\rangle + 2\varrho\langle\widehat{\mathbf{p}},\mathbf{a}\rangle\langle\mathbf{n}^\varphi,\mathbf{a}\rangle & \xi^\varphi &= 2\langle\mathbf{n}^\varphi,\mathbf{a}\rangle\langle\mathbf{n},\mathbf{a}\rangle - \langle\widehat{\mathbf{p}},\mathbf{n}^\varphi\rangle|\mathbf{n}\times\mathbf{a}|^2 \\
\lambda^\vartheta &= -\langle\mathbf{n}^\vartheta,\mathbf{a}\rangle\langle\mathbf{n},\mathbf{a}\rangle|\widehat{\mathbf{p}}|^2 - \varrho|\mathbf{n}\times\mathbf{a}|^2\langle\mathbf{n}^\vartheta,\mathbf{p}\rangle + 2\varrho\langle\widehat{\mathbf{p}},\mathbf{a}\rangle\langle\mathbf{n}^\vartheta,\mathbf{a}\rangle & \xi^\vartheta &= 2\langle\mathbf{n}^\vartheta,\mathbf{a}\rangle\langle\mathbf{n},\mathbf{a}\rangle - \langle\widehat{\mathbf{p}},\mathbf{n}^\vartheta\rangle|\mathbf{n}\times\mathbf{a}|^2 \\
\lambda^\sigma &= -\left(\langle\mathbf{n},\mathbf{a}^\sigma\rangle\langle\mathbf{n},\mathbf{a}\rangle|\widehat{\mathbf{p}}|^2 + \langle\widehat{\mathbf{p}},\mathbf{a}\rangle\langle\widehat{\mathbf{p}},\mathbf{a}^\sigma\rangle\right) & \xi^\sigma &= \langle\widehat{\mathbf{p}},\mathbf{n}\rangle\langle\mathbf{n},\mathbf{a}^\sigma\rangle\langle\mathbf{n},\mathbf{a}\rangle \\
\lambda^\tau &= -\left(\langle\mathbf{n},\mathbf{a}^\tau\rangle\langle\mathbf{n},\mathbf{a}\rangle|\widehat{\mathbf{p}}|^2 + \langle\widehat{\mathbf{p}},\mathbf{a}\rangle\langle\widehat{\mathbf{p}},\mathbf{a}^\tau\rangle\right) & \xi^\tau &= \langle\widehat{\mathbf{p}},\mathbf{n}\rangle\langle\mathbf{n},\mathbf{a}^\tau\rangle\langle\mathbf{n},\mathbf{a}\rangle.
\end{aligned}$$

The derivatives of the auxiliary functions μ and η are:

$$\begin{aligned}
\mu^{\varrho} &= -\langle \mathbf{n}, \mathbf{a}\rangle^2 & & \\
\mu^{\varphi} &= \langle \mathbf{n}^{\varphi}, \mathbf{a}\rangle\langle \mathbf{p} - 2\varrho\mathbf{n}, \mathbf{a}\rangle & \eta^{\varphi} &= -2\langle \mathbf{n}^{\varphi}, \mathbf{a}\rangle\langle \mathbf{n}, \mathbf{a}\rangle \\
\mu^{\vartheta} &= \langle \mathbf{n}^{\vartheta}, \mathbf{a}\rangle\langle \mathbf{p} - 2\varrho\mathbf{n}, \mathbf{a}\rangle & \eta^{\vartheta} &= -2\langle \mathbf{n}^{\vartheta}, \mathbf{a}\rangle\langle \mathbf{n}, \mathbf{a}\rangle \\
\mu^{\sigma} &= \langle \widehat{\mathbf{p}}, \mathbf{a}^{\sigma}\rangle\langle \mathbf{n}, \mathbf{a}\rangle + \langle \widehat{\mathbf{p}}, \mathbf{a}\rangle\langle \mathbf{n}, \mathbf{a}^{\sigma}\rangle & \eta^{\sigma} &= -2\langle \mathbf{n}, \mathbf{a}^{\sigma}\rangle\langle \mathbf{n}, \mathbf{a}\rangle \\
\mu^{\tau} &= \langle \widehat{\mathbf{p}}, \mathbf{a}^{\tau}\rangle\langle \mathbf{n}, \mathbf{a}\rangle + \langle \widehat{\mathbf{p}}, \mathbf{a}\rangle\langle \mathbf{n}, \mathbf{a}^{\tau}\rangle & \eta^{\tau} &= -2\langle \mathbf{n}, \mathbf{a}^{\tau}\rangle\langle \mathbf{n}, \mathbf{a}\rangle.
\end{aligned}$$

PROOF: Let us express the position of the apex $\mathbf{c}$ through the normal vector $\mathbf{n}(\varphi, \vartheta)$, the distance ϱ, the curvature k and through the axis of the cone $\mathbf{a}(\sigma, \tau)$:

$$\begin{aligned}
\mathbf{c} &= (\varrho + \frac{1}{k})\mathbf{n} + \gamma\mathbf{a} \\
\langle \mathbf{c}, \mathbf{n}\rangle &= \varrho.
\end{aligned}$$

Hence $\gamma = -1/(k\langle \mathbf{n}, \mathbf{a}\rangle)$ that is:

$$\mathbf{c} = (\varrho + \frac{1}{k})\mathbf{n} - \frac{\mathbf{a}}{k\langle \mathbf{n}, \mathbf{a}\rangle}. \tag{5.43}$$

If one substitutes this into Eqn. 5.31 one obtains:

$$\tilde{d} = \frac{|\mathbf{n} \times \mathbf{a}|^2 \left(|\widehat{\mathbf{p}} - \mathbf{n}/k|^2 - \langle \widehat{\mathbf{p}} - \mathbf{n}/k, \mathbf{a}\rangle^2\right) - \left(\langle \widehat{\mathbf{p}} - \mathbf{n}/k, \mathbf{a}\rangle\langle \mathbf{n}, \mathbf{a}\rangle + 1/k\right)^2}{2\left(\langle \widehat{\mathbf{p}} - \mathbf{n}/k, \mathbf{a}\rangle\langle \mathbf{n}, \mathbf{a}\rangle + 1/k\right)}$$

Using Pythagoras' theorem it is easy to see that the coefficient of $1/k^2$ in the numerator is zero. Multiplying both the numerator and the denominator by k we obtain Eqn. 5.32. Applying Lemma 5.1 Eqn. 5.37, Eqn. 5.38, Eqn. 5.39, Eqn. 5.41, Eqn. 5.42 and Eqn. 5.40 are consequences of the derivatives of a rational linear function. ■

5.5 Torus fitting

Basically our approach for tori is similar to those for previous surface types. The torus can be parameterized through seven unconstrained parameters. Note that a torus can be obtained as a surface swept by a circular disc rotated around an axis lying in the plane of the circle. The radius of the disc is called the minor radius and the distance of the centre of the disc from the axis called the major radius of the torus. Tori where the major radius is *smaller* then the minor one can also be considered. In this case the resulting surface is self-intersecting, and it is necessary to distinguish the different parts in the following. The smaller arcs sweep a 'lemon-torus' (*i.e.* the inner part of the torus surface), while the larger arcs sweep an 'apple-torus' (*i.e.* the outer part of the torus surface). (We will also refer to a non-self-intersecting torus as 'apple-shaped'.) In special cases the torus may degenerate into a sphere, when the major radius vanishes, or or into a cone, when the minor radius tends to infinity. This will be reflected in our equations in such a way that the equations gradually reduce to those of for sphere or cone fitting[4]. The parameterization used for the torus is the following. As in previous cases, the point on the torus where a line through the surface normal passes through the origin is $\varrho\mathbf{n}$, where $|\mathbf{n}| = 1$. The principal curvature value of the torus corresponding to the minor radius at the point $\varrho\mathbf{n}$ is k (*i.e.* the radius of the circular disk is $1/k$). The other principal curvature is s, and the corresponding centre of curvature lies on the axis of the torus (see Fig. 5). Let the unit direction vector of the torus axis be $\mathbf{a}$. We parameterize $\mathbf{n}$ by φ and ϑ as above in Eqn. 5.8. The unit vector $\mathbf{a}$ will be parameterised in a similar way to Eqn. 5.27:

$$\mathbf{a} = (\cos\sigma \sin\tau, \sin\sigma \sin\tau, \cos\tau).$$

[4]If the major radius tends to infinity then the torus becomes a cylinder. This case will be singular, but mathematically will be close to the cylinder fit.

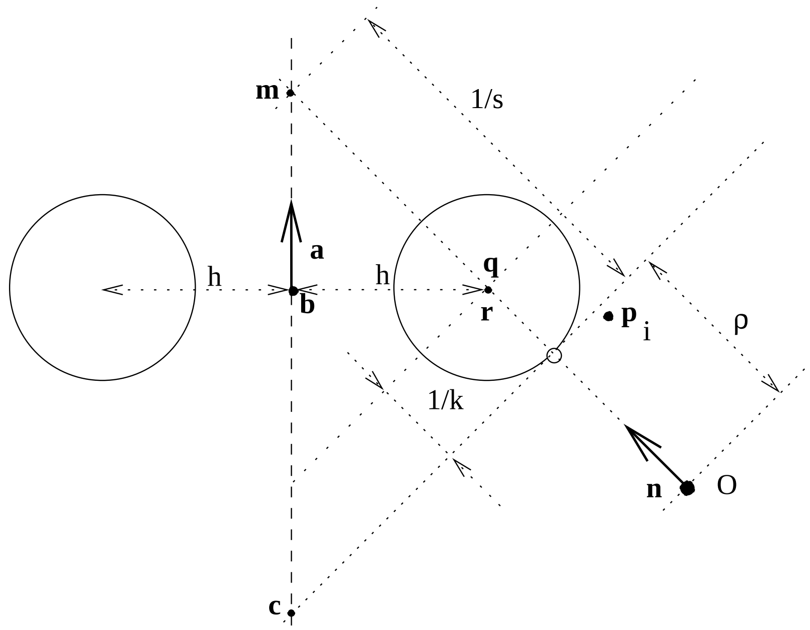

Figure 5: Parameterisation of the torus

Hence the unconstrained parameters $(\varrho, \varphi, \vartheta, k, s, \sigma, \tau)$ entirely characterise the torus surface. Now the following statement holds:

Statement 5.4. *A non-linear distance function for tori which is faithful up to the first derivative is:*

$$\tilde{d}(\mathbf{s}, \mathbf{p}) = \tilde{d}_0(\varrho, \varphi, \vartheta, k, \mathbf{p}) - \delta_\epsilon(\varrho, \varphi, \vartheta, k, s, \sigma, \tau, \mathbf{p}) \tag{5.44}$$

where $\tilde{d}_0$ *is the approximate distance function for the sphere Eqn. 5.5:*

$$\tilde{d}_0 = \frac{k}{2}\left(|\mathbf{p}|^2 - 2\varrho\langle\mathbf{p}, \mathbf{n}\rangle + \varrho^2\right) + \varrho - \langle\mathbf{p}, \mathbf{n}\rangle = \frac{k}{2}|\widehat{\mathbf{p}}|^2 - \langle\widehat{\mathbf{p}}, \mathbf{n}\rangle,$$

while

$$\delta_\epsilon = (\frac{k}{s} - 1)\left[\epsilon \cdot \mathrm{sign}(\frac{k^2}{s} - k)|(\widehat{\mathbf{p}} - \mathbf{n}/s) \times \mathbf{a}||\mathbf{n} \times \mathbf{a}| + \langle(\widehat{\mathbf{p}} - \mathbf{n}/s) \times \mathbf{a}, \mathbf{n} \times \mathbf{a}\rangle\right] \tag{5.45}$$

where $\epsilon = +1$ *for an apple torus surface and* $\epsilon = -1$ *for a lemon torus surface.*

Proof: First compute the distance of a point $\mathbf{p}$ from the torus surface. Assume that k and s are not zero. Let us call the symmetry centre of the torus $\mathbf{b}$ and let h be the major radius of the torus. (Note that $h \geq 0$.)

Let $\mathbf{e}$ be defined by

$$\mathbf{e} = \frac{\mathbf{p} - \mathbf{b} - \langle\mathbf{p} - \mathbf{b}, \mathbf{a}\rangle\mathbf{a}}{|\mathbf{p} - \mathbf{b} - \langle\mathbf{p} - \mathbf{b}, \mathbf{a}\rangle\mathbf{a}|} = \frac{\mathbf{p} - \mathbf{b} - \langle\mathbf{p} - \mathbf{b}, \mathbf{a}\rangle\mathbf{a}}{|(\mathbf{p} - \mathbf{b}) \times \mathbf{a}|},$$

which is a unit vector formed by the projection of the vector $\mathbf{p} - \mathbf{b}$ onto the central symmetry plane of the torus. Cut the torus with the plane which includes the axis and contains $\mathbf{p}$. The intersection consists of two circles, and unless $\mathbf{p}$ is on the axis, the two centres can be computed by the following formula:

$$\mathbf{q}_\epsilon = \mathbf{b} \pm h\mathbf{e} = \mathbf{b} + \epsilon h\mathbf{e} \tag{5.46}$$

and the distance from the torus surface is clearly:

$$d_\epsilon(\mathbf{s},\mathbf{p}) = |\mathbf{p} - \mathbf{q}_\epsilon| - 1/k. \tag{5.47}$$

Here it is appropriate to use $\epsilon = +1$ if we are interested in an apple torus surface, that is if $|hk| > 1$ (a non-self-intersecting torus), or the apple torus sheet in the self-intersecting case. Using $\epsilon = -1$ corresponds to the lemon torus sheet. Note that usually a priori we do not know if the input data best fits a lemon torus or an apple torus, so in general we have to fit both sheets to the input points points and choose the one having the lower overall error.

Let us decompose $\mathbf{p} - \mathbf{q}_\epsilon$ into components in the directions of $\mathbf{a}$ and $\mathbf{e}$. Substituting into Eqn. 5.47 we obtain:

$$d_\epsilon(\mathbf{s},\mathbf{p}) = \sqrt{\langle \mathbf{p} - \mathbf{b}, \mathbf{a}\rangle^2 + (|(\mathbf{p} - \mathbf{b}) \times \mathbf{a}| - \epsilon h)^2} - \frac{1}{k}. \tag{5.48}$$

It is clear that the major radius (see Fig. 5) is given by:

$$h = \left|\frac{1}{s} - \frac{1}{k}\right| |\mathbf{n} \times \mathbf{a}|.$$

Let us introduce the following quantities (see Fig. 5):

$$\mathbf{m} = \left(\varrho + \frac{1}{s}\right)\mathbf{n}$$
$$\mathbf{r} = \left(\varrho + \frac{1}{k}\right)\mathbf{n}.$$

Thus[5]

$$\langle \mathbf{p} - \mathbf{b}, \mathbf{a}\rangle = \langle \mathbf{p} - \mathbf{r}, \mathbf{a}\rangle$$
$$(\mathbf{p} - \mathbf{b}) \times \mathbf{a} = (\mathbf{p} - \mathbf{m}) \times \mathbf{a},$$

so we can rewrite Eqn. 5.48 as further:

$$d_\epsilon(\mathbf{s},\mathbf{p}) = \sqrt{\langle \mathbf{p} - (\varrho + \frac{1}{k})\mathbf{n}, \mathbf{a}\rangle^2 + \left(|(\mathbf{p} - (\varrho + \frac{1}{s})\mathbf{n}) \times \mathbf{a}| - \epsilon\left|\frac{1}{s} - \frac{1}{k}\right| |\mathbf{n} \times \mathbf{a}|\right)^2} - \frac{1}{k}. \tag{5.49}$$

This expression of distance satisfies the conditions of Statement 4.1, so it can approximated as

$$\tilde{d}_\epsilon(\mathbf{s},\mathbf{p}) = \frac{k}{2}\left[\langle \mathbf{p} - (\varrho + \frac{1}{k})\mathbf{n}, \mathbf{a}\rangle^2 + \left(|(\mathbf{p} - (\varrho + \frac{1}{s})\mathbf{n}) \times \mathbf{a}| - \epsilon\left|\frac{1}{s} - \frac{1}{k}\right| |\mathbf{n} \times \mathbf{a}|\right)^2 - \frac{1}{k^2}\right]. \tag{5.50}$$

Using the notation $\widehat{\mathbf{p}} = \mathbf{p} - \varrho\mathbf{n}$ as usual, we have

$$\begin{aligned}\tilde{d}_\epsilon &= \frac{k}{2}\left[\langle \widehat{\mathbf{p}} - \mathbf{n}/k, \mathbf{a}\rangle^2 + (|(\widehat{\mathbf{p}} - \mathbf{n}/s) \times \mathbf{a}| - \epsilon\left|\frac{1}{s} - \frac{1}{k}\right| |\mathbf{n} \times \mathbf{a}|)^2 - \frac{1}{k^2}\right] \\ &= \frac{k}{2}\Bigg[\langle \widehat{\mathbf{p}}, \mathbf{a}\rangle^2 - \frac{2}{k}\langle \widehat{\mathbf{p}}, \mathbf{a}\rangle\langle \mathbf{n}, \mathbf{a}\rangle + \frac{1}{k^2}\langle \mathbf{n}, \mathbf{a}\rangle^2 + |\widehat{\mathbf{p}} \times \mathbf{a}|^2 - \frac{2}{s}\langle \widehat{\mathbf{p}} \times \mathbf{a}, \mathbf{n} \times \mathbf{a}\rangle + \frac{1}{s^2}|\mathbf{n} \times \mathbf{a}|^2 \\ &- 2\epsilon\left|\frac{1}{s} - \frac{1}{k}\right| |(\widehat{\mathbf{p}} - \mathbf{n}/s) \times \mathbf{a}||\mathbf{n} \times \mathbf{a}| + \left(\frac{1}{s^2} - \frac{2}{sk} + \frac{1}{k^2}\right)|\mathbf{n} \times \mathbf{a}|^2 - \frac{1}{k^2}\Bigg].\end{aligned}$$

Furthermore using the formula

$$\langle \widehat{\mathbf{p}} \times \mathbf{a}, \mathbf{n} \times \mathbf{a}\rangle = \langle \widehat{\mathbf{p}}, \mathbf{n}\rangle - \langle \widehat{\mathbf{p}}, \mathbf{a}\rangle\langle \mathbf{n}, \mathbf{a}\rangle$$

[5] It should be noted that the plane containing $\mathbf{p}$, $\mathbf{q}_\epsilon$, $\mathbf{e}$ is not necessarily the same as that defined by the origin, $\mathbf{n}$, $\mathbf{a}$ and $\mathbf{r}$. However both are symmetry planes of the torus, thus containing the axis, *i.e.* $\mathbf{b}$, $\mathbf{a}$ and $\mathbf{m}$.

which follows from Lemma 5.1 due to $|\mathbf{a}| = 1$, and using the identity

$$|\widehat{\mathbf{p}}|^2 = \langle \widehat{\mathbf{p}}, \mathbf{a}\rangle^2 + |\widehat{\mathbf{p}} \times \mathbf{a}|^2$$

we find that the term involving $1/k^2$ vanishes. We have:

$$\begin{aligned}
\tilde{d}_\epsilon &= \frac{k}{2}\Bigg[|\widehat{\mathbf{p}}|^2 - \frac{2}{k}\langle \widehat{\mathbf{p}}, \mathbf{a}\rangle\langle \mathbf{n}, \mathbf{a}\rangle - \frac{2}{s}\langle \widehat{\mathbf{p}}, \mathbf{n}\rangle + \frac{2}{s}\langle \widehat{\mathbf{p}}, \mathbf{a}\rangle\langle \mathbf{n}, \mathbf{a}\rangle + \left(\frac{2}{s^2} - \frac{2}{sk}\right)|\mathbf{n} \times \mathbf{a}|^2 \\
&- 2\epsilon\left|\frac{1}{s} - \frac{1}{k}\right| |(\widehat{\mathbf{p}} - \mathbf{n}/s) \times \mathbf{a}||\mathbf{n} \times \mathbf{a}|\Bigg] \\
&= \frac{k}{2}|\widehat{\mathbf{p}}|^2 - \frac{k}{s}\langle \widehat{\mathbf{p}}, \mathbf{n}\rangle \\
&+ \left(\frac{k}{s} - 1\right)\left[\langle \widehat{\mathbf{p}}, \mathbf{a}\rangle\langle \mathbf{n}, \mathbf{a}\rangle + \frac{1}{s}|\mathbf{n} \times \mathbf{a}|^2 - \epsilon \cdot \operatorname{sign}\left(\frac{k}{s} - 1\right) \cdot \operatorname{sign}(k) \cdot |(\widehat{\mathbf{p}} - \mathbf{n}/s) \times \mathbf{a}||\mathbf{n} \times \mathbf{a}|\right] \\
&= \frac{k}{2}|\widehat{\mathbf{p}}|^2 - \langle \widehat{\mathbf{p}}, \mathbf{n}\rangle \\
&- \left(\frac{k}{s} - 1\right)\left[\langle \widehat{\mathbf{p}}, \mathbf{n}\rangle - \langle \widehat{\mathbf{p}}, \mathbf{a}\rangle\langle \mathbf{n}, \mathbf{a}\rangle - \frac{1}{s}|\mathbf{n} \times \mathbf{a}|^2 + \epsilon \cdot \operatorname{sign}\left(\frac{k^2}{s} - k\right)|(\widehat{\mathbf{p}} - \mathbf{n}/s) \times \mathbf{a}||\mathbf{n} \times \mathbf{a}|\right],
\end{aligned} \tag{5.51}$$

from which Eqn. 5.44 follows. ■

Let us see what happens if we approach the limits mentioned at the beginning of this section. If $k = s$ then from Eqn. 5.45 we simply get back the distance expression for spheres, Eqn. 5.7.

If $k \to 0$ and s is bounded from below then $\left(\frac{k}{s} - 1\right) \to -1$ and from Eqn. 5.51 one obtains:

$$\begin{aligned}
\lim_{k\to 0} \tilde{d}_\epsilon &= \langle \widehat{\mathbf{p}}, \mathbf{n}\rangle + \langle \widehat{\mathbf{p}}, \mathbf{n}\rangle - \langle \widehat{\mathbf{p}}, \mathbf{a}\rangle\langle \mathbf{n}, \mathbf{a}\rangle - \frac{1}{s}|\mathbf{n} \times \mathbf{a}|^2 \\
&- \epsilon \cdot \operatorname{sign}(k)|(\widehat{\mathbf{p}} - \mathbf{n}/s) \times \mathbf{a}||\mathbf{n} \times \mathbf{a}| \\
&= -\epsilon \cdot \operatorname{sign}(k)|(\widehat{\mathbf{p}} - \mathbf{n}/s) \times \mathbf{a}||\mathbf{n} \times \mathbf{a}| - \langle \widehat{\mathbf{p}}, \mathbf{a}\rangle\langle \mathbf{n}, \mathbf{a}\rangle - \frac{1}{s} + \frac{\langle \mathbf{n}, \mathbf{a}\rangle^2}{s}
\end{aligned}$$

Actually if one introduces:

$$\mathbf{c} = \mathbf{m} - \frac{\mathbf{a}}{k\langle \mathbf{n}, \mathbf{a}\rangle} = (\varrho + \frac{1}{s})\mathbf{n} - \frac{\mathbf{a}}{s\langle \mathbf{n}, \mathbf{a}\rangle}$$

it is be the intersection of the tangent plane in $\varrho\mathbf{n}$ with the axis of the torus (see Fig. 5 and cf. Eqn. 5.43). This means that

$$\widehat{\mathbf{p}} - \mathbf{n}/s = \mathbf{p} - \mathbf{c}$$

and since one can add freely a multiple of $\mathbf{a}$ in the cross product we arrive at:

$$\lim_{k\to 0} \tilde{d}_\epsilon = -\epsilon \cdot \operatorname{sign}(k) \cdot |(\mathbf{p} - \mathbf{c}) \times \mathbf{a}||\mathbf{n} \times \mathbf{a}| - \langle \mathbf{p} - \mathbf{c}, \mathbf{a}\rangle\langle \mathbf{n}, \mathbf{a}\rangle.$$

Thus, either the lemon case ($\epsilon = -1$) or the apple case ($\epsilon = +1$) gives back the original distance equation for cones Eqn. 5.30.

If we look at the formula Eqn. 5.45 we can see δ_ϵ has a singularity at $s = 0$, *i.e.* as the major radius tends to infinity. This is certainly a drawback since it is quite possible that our fitting algorithm will be called upon to work under circumstances where this singularity can occur. We shall show that mathematically that our formula behaves nicely as $s \to 0$, more exactly:

Statement 5.5. *The distance function of an apple-torus Eqn. 5.44 has the following limiting behaviour:*

$$\lim_{s\to 0} \delta_{+1} = \frac{k}{2}\langle \mathbf{p}, \mathbf{n} \times \mathbf{a}/|\mathbf{n} \times \mathbf{a}|\rangle^2. \tag{5.52}$$

Thus, it degenerates to the distance function of a cylinder (see Eqn. 5.16) with axis $\mathbf{n} \times \mathbf{a}/|\mathbf{n} \times \mathbf{a}|$.

PROOF: Without loss of generality one can assume that

$$\frac{k}{s} - 1 > 0.$$

Then for δ_{+1} all the conditions of Eqn. 4.4 in Statement 4.1 are satisfied. That is

$$\begin{aligned}
\lim_{s\to 0} \delta_{+1} &= \lim_{s\to 0} \left(\frac{k-s}{s}\right) \cdot \frac{|(\widehat{\mathbf{p}} - \mathbf{n}/s) \times \mathbf{a}|^2 |\mathbf{n}\times\mathbf{a}|^2 - \langle(\widehat{\mathbf{p}} - \mathbf{n}/s)\times\mathbf{a}, \mathbf{n}\times\mathbf{a}\rangle^2}{2\langle(\widehat{\mathbf{p}} - \mathbf{n}/s)\times\mathbf{a}, \mathbf{n}\times\mathbf{a}\rangle} \\
&= \lim_{s\to 0} \frac{(k-s)|\,((\widehat{\mathbf{p}} - \mathbf{n}/s)\times\mathbf{a})\times(\mathbf{n}\times\mathbf{a})|^2}{2s\langle(\widehat{\mathbf{p}} - \mathbf{n}/s)\times\mathbf{a}, \mathbf{n}\times\mathbf{a}\rangle} \\
&= \frac{k}{2}\langle\widehat{\mathbf{p}}, \mathbf{n}\times\mathbf{a}/|\mathbf{n}\times\mathbf{a}|\rangle^2 = \frac{k}{2}\langle\mathbf{p} - \varrho\mathbf{n}, \mathbf{n}\times\mathbf{a}/|\mathbf{n}\times\mathbf{a}|\rangle^2 \\
&= \frac{k}{2}\langle\mathbf{p}, \mathbf{n}\times\mathbf{a}/|\mathbf{n}\times\mathbf{a}|\rangle^2.
\end{aligned}$$

■

We now give the derivatives of the approximate distance function of the torus. The derivatives of $\tilde{d}_0$ in Eqn. 5.44 are Eqn. 5.11, Eqn. 5.12, Eqn. 5.13 and Eqn. 5.14 in Statement 5.1.

Statement 5.6. *Assume that the normal vector* $\mathbf{n}$ *is not parallel to the axis* $\mathbf{a}$ *and that the point* $\mathbf{p}$ *does not lie on the axis (these assumptions will be discussed later). We can then introduce the following notation*

$$\mathbf{v} = \frac{(\widehat{\mathbf{p}} - \mathbf{n}/s)\times\mathbf{a}}{|(\widehat{\mathbf{p}} - \mathbf{n}/s)\times\mathbf{a}|} \qquad \mathbf{u} = \frac{\mathbf{n}\times\mathbf{a}}{|\mathbf{n}\times\mathbf{a}|}. \tag{5.53}$$

The derivatives of the auxiliary functions δ_ϵ *are the following:*

$$\delta_\epsilon^\varrho = -|\mathbf{n}\times\mathbf{a}|^2 \left(\frac{k}{s} - 1\right)\left[\epsilon\cdot\mathrm{sign}(\frac{k^2}{s} - k)\langle\mathbf{v},\mathbf{u}\rangle + 1\right] \tag{5.54}$$

$$\delta_\epsilon^\varphi = -\left(\varrho + \frac{1}{s}\right)\left(\frac{k}{s} - 1\right)\left(\epsilon\cdot\mathrm{sign}(\frac{k^2}{s} - k)|(\widehat{\mathbf{p}} - \mathbf{n}/s)\times\mathbf{a}| + |\mathbf{n}\times\mathbf{a}|\right) \cdot \left(\epsilon\cdot\mathrm{sign}(\frac{k^2}{s} - k)\langle\mathbf{n}^\varphi\times\mathbf{a},\mathbf{v}\rangle + \langle\mathbf{n}^\varphi\times\mathbf{a},\mathbf{u}\rangle\right) \tag{5.55}$$

$$\delta_\epsilon^\vartheta = -\left(\varrho + \frac{1}{s}\right)\left(\frac{k}{s} - 1\right)\left(\epsilon\cdot\mathrm{sign}(\frac{k^2}{s} - k)|(\widehat{\mathbf{p}} - \mathbf{n}/s)\times\mathbf{a}| + |\mathbf{n}\times\mathbf{a}|\right) \cdot \left(\epsilon\cdot\mathrm{sign}(\frac{k^2}{s} - k)\langle\mathbf{n}^\vartheta\times\mathbf{a},\mathbf{v}\rangle + \langle\mathbf{n}^\vartheta\times\mathbf{a},\mathbf{u}\rangle\right) \tag{5.56}$$

$$\delta_\epsilon^k = \frac{|(\widehat{\mathbf{p}} - \mathbf{n}/s)\times\mathbf{a}||\mathbf{n}\times\mathbf{a}|}{s}\left[\epsilon\cdot\mathrm{sign}(\frac{k^2}{s} - k) + \langle\mathbf{v},\mathbf{u}\rangle\right] \tag{5.57}$$

$$\delta_\epsilon^s = \frac{|\mathbf{n}\times\mathbf{a}|}{s^2}\left(\epsilon\cdot\mathrm{sign}(\frac{k^2}{s} - k) + \langle\mathbf{v},\mathbf{u}\rangle\right)\left[\epsilon\cdot\mathrm{sign}(k)\left|\frac{k}{s} - 1\right||\mathbf{n}\times\mathbf{a}| - k|(\widehat{\mathbf{p}} - \mathbf{n}/s)\times\mathbf{a}|\right] \tag{5.58}$$

$$\delta_\epsilon^\sigma = \left(\frac{k}{s} - 1\right)\Big[\epsilon\cdot\mathrm{sign}(\frac{k^2}{s} - k)\left(\langle(\widehat{\mathbf{p}} - \mathbf{n}/s)\times\mathbf{a}^\sigma,\mathbf{v}\rangle|\mathbf{n}\times\mathbf{a}| + \langle\mathbf{n}\times\mathbf{a}^\sigma,\mathbf{u}\rangle|(\widehat{\mathbf{p}} - \mathbf{n}/s)\times\mathbf{a}|\right) + \langle(\widehat{\mathbf{p}} - \mathbf{n}/s)\times\mathbf{a}^\sigma,\mathbf{u}\rangle|\mathbf{n}\times\mathbf{a}| + \langle\mathbf{n}\times\mathbf{a}^\sigma,\mathbf{v}\rangle|(\widehat{\mathbf{p}} - \mathbf{n}/s)\times\mathbf{a}|\Big] \tag{5.59}$$

$$\delta_\epsilon^\tau = \left(\frac{k}{s} - 1\right)\Big[\epsilon\cdot\mathrm{sign}(\frac{k^2}{s} - k)\left(\langle(\widehat{\mathbf{p}} - \mathbf{n}/s)\times\mathbf{a}^\tau,\mathbf{v}\rangle|\mathbf{n}\times\mathbf{a}| + \langle\mathbf{n}\times\mathbf{a}^\tau,\mathbf{u}\rangle|(\widehat{\mathbf{p}} - \mathbf{n}/s)\times\mathbf{a}|\right) + \langle(\widehat{\mathbf{p}} - \mathbf{n}/s)\times\mathbf{a}^\tau,\mathbf{u}\rangle|\mathbf{n}\times\mathbf{a}| + \langle\mathbf{n}\times\mathbf{a}^\tau,\mathbf{v}\rangle|(\widehat{\mathbf{p}} - \mathbf{n}/s)\times\mathbf{a}|\Big] \tag{5.60}$$

PROOF: Straightforward differentiation. ■

We can always satisfy the first assumption above by suitable choice of the position of the origin. The second assumption may not always be satisfied (if the torus is self-intersecting, or if a point has an error in an unlucky direction), but such points are always exceptional cases, and can be discarded.

6 Initial estimates

To find the solution of any of the above nonlinear least-squares problems, we use some iterative technique (*e.g.* Gauss-Newton or Levenberg-Marquardt — see [Bjö96]). All of these algorithms require some good initial estimate of the minimum which is then refined. Here we give one method of finding such initial estimates, but other geometric ideas could be used instead.

The first step this process is to find an estimate of the rotational axis (except in the case of sphere fitting). A method is given to do this in the following section; it is based only on estimates of the surface normal vector. We then pick a point at which we have a normal vector estimate and put the origin at this point. We shall call this point the 'base point' below. Thus we have an estimate[6] for $\mathbf{n}$ and hence φ, ϑ; the initial estimate for ϱ is 0.

For sphere fitting, these values alone are sufficient, and an estimate for k can now be found by solving a linear least-squares problem as mentioned earlier.

In the case of cylinder fitting, we adjust the normal to be perpendicular to the axis. We then compute α as the angle between $\mathbf{a}$ and $\mathbf{n}^{\vartheta}$ and compute the distance of the base point from the axis, which is $1/k$ signed with the direction of $\mathbf{n}$.

In the case of cone fitting, after estimating the rotational axis $\mathbf{a}$ we compute the distance of the base point from the axis *along* the estimated normal line in order to obtain $1/k$.

As far as the torus is concerned, for better conditioning one has to pick a base point at which the normal subtends as large an angle as possible with the estimated axis. We get $1/s$ as the distance along the normal between the base point and the estimated axis. For simplicity, we can put $k = 0$ (*i.e.* start from a cone) and try fitting both $\epsilon = +1$ (apple-torus) and $\epsilon = -1$ (lemon-torus). More robustly, one can opt to estimate principal curvatures of the surface at the base point. (Any straightforward method will do, for example by fitting a second order paraboloid to a few surface points near the base point.) Taking into account that s is one of the principal curvatures we can compute the other one, k, even if we estimate just the Gaussian curvature at the base point. We can then choose a suitable value for ϵ for example by determining on which sheet of the torus the base point lies. Thus $\epsilon = +1$ (apple) if $|k| > |s|$ or $k \cdot s < 0$ and $\epsilon = -1$ (lemon) otherwise (the distinction should be quite clear provided that we have chosen a well placed base point as described above). We assume here that the point set being fitted does not contain points belonging to both the apple and lemon sheets of the same torus simultaneously. This is very unlikely to happen in practice, but if it is considered to be a possibility, before fitting we should separate the points into two sets, one for each sheet, based on curvature estimates and the above mentioned criterion.

6.1 Estimating the axis of rotation

Suppose we have a number of points and corresponding surface normal vectors on a surface of revolution. We wish to compute the axis of rotation of the surface. This task can be expressed in the following geometric form.

Given m straight lines in three dimensions, compute the straight line intersecting all of them (if such a line exists).

Let us denote by $\mathbf{p}_i$ a fixed point on the ith straight line and by $\mathbf{n}_i$ the direction of that straight line $i = 0, 1, \ldots, m-1$. Now let us draw a straight line through two generic points of the 0th and 1st straight lines respectively. Assume that these points have the parameters t_0 and t_1. A necessary condition that a straight line passing through the point $\mathbf{p}_i$ in the direction $\mathbf{n}_i$ ($i = 2, 3, \ldots, m-1$)

[6] Note that the solution surface need not pass through the base point, since ϱ can change.

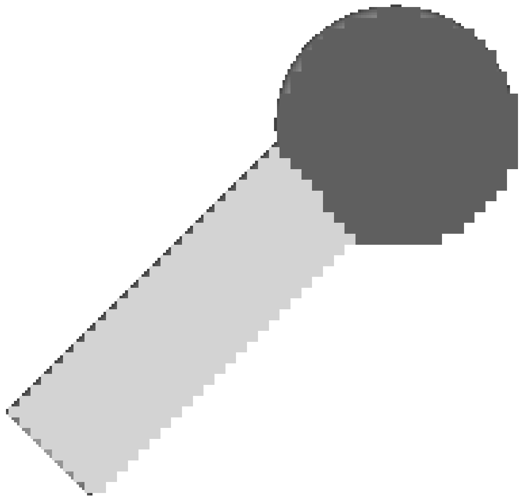

Figure 6: Final segmentation

intersects the previous line is that the vectors $\mathbf{p}_0 + t_0\mathbf{n}_0 - \mathbf{p}_i$, $\mathbf{p}_1 + t_1\mathbf{n}_1 - \mathbf{p}_0 - t_0\mathbf{n}_0$ and $\mathbf{n}_i$ are coplanar. This means that the following triple product must vanish:

$$((\mathbf{p}_0 + t_0\mathbf{n}_0 - \mathbf{p}_i)\,(\mathbf{p}_1 + t_1\mathbf{n}_1 - \mathbf{p}_0 - t_0\mathbf{n}_0)\,\mathbf{n}_i) = 0, \tag{6.1}$$

where $(\mathbf{a}\,\mathbf{b}\,\mathbf{c}) = \langle \mathbf{a} \times \mathbf{b}, \mathbf{c} \rangle$. On expansion this simplifies to:

$$(\mathbf{n}_0\,\mathbf{n}_1\,\mathbf{n}_i) \cdot t_0 t_1 + ((\mathbf{p}_i - \mathbf{p}_1)\,\mathbf{n}_0\,\mathbf{n}_i) \cdot t_0 + ((\mathbf{p}_0 - \mathbf{p}_i)\,\mathbf{n}_1\,\mathbf{n}_i) \cdot t_1 + ((\mathbf{p}_0 - \mathbf{p}_i)\,(\mathbf{p}_1 - \mathbf{p}_0)\,\mathbf{n}_i) = 0.$$

Now taking $\mathbf{p}_i = \mathbf{p}_2, \mathbf{p}_3$ and $\mathbf{n}_i = \mathbf{n}_2, \mathbf{n}_3$ respectively in turn, we get an equation system for t_0 and t_1 of the form

$$a_{01} t_0 t_1 + a_0 t_0 + a_1 t_1 + a = 0 \tag{6.2}$$

$$b_{01} t_0 t_1 + b_0 t_0 + b_1 t_1 + b = 0. \tag{6.3}$$

¿From here we can eliminate $t_0 t_1$ and express t_1 or t_0 as a linear function of the other. After substituting this expression for t_1 or t_0 in one of the above equations we obtain a second degree equation which we can solve.

This leads to two solutions for the rotational axis. Various methods using further $\mathbf{p}_i$ can be used to choose between them, or to find an overall best axis for the whole set of $\mathbf{p}_i$. In this application perhaps it is simpler just to try to fit the rotational surface using both axes and selecting the result with the smaller least-squares residual.

7 Results and Conclusion

The above fitting routines have been tested using the segmentation approach described earlier. Simulated 3D point data was used that was accurate to five significant digits. In all cases the average error of fit of the models recovered was accurate to at least four significant digits, and segmentations consistent with the underlying geometric primitives were obtained. An example of the final segmentation obtained for the peg composed of a cylinder and sphere shown in Fig 1 is given in Fig 6.

In summary we have given new methods for the least-squares fitting of spheres, cylinders, cones and tori to point data. We have outlined how they can be used in a segmentation strategy that is capable of extracting spheres, cylinders, cones and tori from three-dimensional data. Our fitting methods have the particular advantage of being robust in the sense that as the principal curvatures of the surfaces being fitted decrease (or become more equal), the results which are returned naturally become closer and closer to the surfaces of "simpler type", *i.e.* planes, cylinders, or cones (or spheres) which best describe the data.

Whilst our motivation has the reverse engineering of boundary representation solid models from three-dimensional depth maps of scanned objects, we believe that the fitting methods described in this paper will be of interest to the computer vision and CAD communities in general. The initial results obtained so far appear very promising and we intend to extend these tests to a greater range of cases, and to real as well as simulated data.

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