CONSTRUCTIVE COMPACT LINEAR MAPPINGS

HAJIME ISHIHARA

ABSTRACT

In this paper, we deal with compact linear mappings of a normed linear space, within the framework of Bishop's constructive mathematics. We prove the constructive substitutes for the classically well-known theorems on compact linear mappings: T is compact if and only if T^* is compact; if S is bounded and if T is compact, then TS is compact; if S and T is compact, then $S+T$ is compact.

1. *Introduction*

In this paper, we deal with compact linear mappings of a normed linear space, within the framework of Bishop's constructive mathematics; we assume familiarity with the foundations of constructive mathematics; see [**1**, **2**, **3**, **4**].

Although constructive investigations of compact linear mappings could be found in [**2**, Chapter 7, Problem 41–43], [**4**, Chapter 2], [**5**], some elementary classical facts on compact linear mappings have not been proved constructively; for example the compactness of the sum of compact linear mappings, which is classically proved by means of the constructively unacceptable property of sequential compactness, even in the special case where the compact linear mappings are normable linear functionals.

Recently, the author proved this theorem. Let A and B be compact operators on a Hilbert space H and let C be an operator on H. Then $A+B$ is compact, A^* is compact, and if C^* exists, then AC is compact; see [**6**]. Since the proof of the theorem depends on the geometric properties of Hilbert spaces, we cannot expect to generalize the theorem to arbitrary normed linear spaces in the same manner.

In the sequel, we shall prove similar results in a more general setting, and to do this we have to deal with quasinormed linear spaces; see [**2**, Chapter 7].

Let E and F be quasinormed linear spaces with quasinorms $\{\|\cdot\|_\alpha : \alpha \in A\}$ and $\{\|\cdot\|_\beta : \beta \in B\}$ respectively, and let $S_E \equiv \{x \in E : \|x\|_\alpha \leqslant 1, \alpha \in A\}$ be the closed unit sphere of E. Then we say that a linear mapping T of E into F is *bounded* if there exists $c > 0$, called a *bound* for T, such that if $x \in E$ and $\|Tx\|_\beta > c$ for some $\beta \in B$, then $\|x\|_\alpha > 1$ for some $\alpha \in A$; *normable* if $\|T\| \equiv \sup\{\|Tx\|_\beta : \beta \in B, x \in S_E\}$ exists and $\|T\|$ is a bound for T; *compact* if for each $\varepsilon > 0$ there exists a finite subset Y of $T(S_E)$, called an *ε-approximation* to $T(S_E)$, such that if $x \in E$ and for each $y \in Y$ there exists $\beta \in B$ with $\|Tx - y\|_\beta > \varepsilon$, then $\|x\|_\alpha > 1$ for some $\alpha \in A$.

It is easy to see that if T is bounded, then $T(S_E)$ is bounded, in the sense that there exists $c > 0$ such that $\|Tx\|_\beta < c$ for all $\beta \in B$ and all $x \in S_E$; if E is a normed linear space and $T(S_E)$ is bounded, then T is bounded; if E is a normed linear space and $\sup\{\|Tx\|_\beta : \beta \in B, x \in S_E\}$ exists, then T is normable; if F is a normed linear space and T is compact, then $T(S_E)$ is totally bounded, in the sense that for each $\varepsilon > 0$ there exists

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a finite subset Y of $T(S_E)$, called an *ε-approximation* to $T(S_E)$, such that for each $x \in S_E$ there exists $y \in Y$ with $\|Tx-y\|_\beta < \varepsilon$ for all $\beta \in B$; if E is a normed linear space and $T(S_E)$ is totally bounded, then T is compact; if each element of E (or F) is normable, then T is compact if and only if T is compact as a mapping from a *normed* (quasinormed) linear space E into a quasinormed (*normed*) linear space F, respectively.

PROPOSITION 1. *If T is a compact linear mapping of a quasinormed linear space E with a quasinorm $\{\|\cdot\|_\alpha : \alpha \in A\}$ into a quasinormed linear space F with a quasinorm $\{\|\cdot\|_\beta : \beta \in B\}$, then T is bounded. If, in addition, for each $x \in E$, Tx is a normable element of F, then T is normable.*

Proof. Let $\{y_1, \ldots, y_n\}$ be a 1-approximation to $T(S_E)$. Then there exist positive numbers $c_1, \ldots, c_n$ such that $\|y_k\|_\beta < c_k$ for all $\beta \in B$ and all $k = 1, \ldots, n$. Let $c \equiv \max\{c_1, \ldots, c_n\} + 1$, and suppose that $\|Tx\|_\beta > c$ for some $\beta \in B$. Then

$$\|Tx - y_k\|_\beta \geqslant \|Tx\|_\beta - \|y_k\|_\beta > c - c_k \geqslant 1 \quad (k = 1, \ldots, n),$$

and hence $\|x\| > 1$ for some $\alpha \in A$. Therefore T is bounded.

Moreover, suppose that for each $x \in E$, Tx is a normable element of F. Let a and b be real numbers with $a < b$, and let $\varepsilon \equiv \frac{1}{4}(b-a)$. Let $\{y_1, \ldots, y_n\}$ be an ε-approximation to $T(S_E)$, and choose k such that $\|y_k\| > \max\{\|y_1\|, \ldots, \|y_n\|\} - \varepsilon$. Either $a < \|y_k\|$ or $\|y_k\| < a + \varepsilon$; in the former case, there exists $\beta \in B$ such that $a < \|y_k\|_\beta$; in the latter case, if $x \in S_E$ and there exists $\beta \in B$ such that $\|Tx\|_\beta > b - \varepsilon$, then

$$\begin{aligned}\|Tx - y_i\|_\beta &\geqslant \|Tx\|_\beta - \|y_i\| \\ &> b - \varepsilon - \max\{\|y_1\|, \ldots, \|y_n\|\} \\ &> b - \varepsilon - \|y_k\| - \varepsilon > b - a - 3\varepsilon = \varepsilon,\end{aligned}$$

and hence $\|x\|_\alpha > 1$ for some $\alpha \in A$, a contradiction; hence $\|Tx\|_\beta \leqslant b - \varepsilon < b$ for all $\beta \in B$ and all $x \in S_E$. Therefore $\|T\| \equiv \sup\{\|Tx\|_\beta : \beta \in B, x \in S_E\}$ exists by [2, Chapter 2, (1.2)]. Suppose that $x \in E$ and $\|Tx\|_\beta > \|T\|$ for some $\beta \in B$. Let $\varepsilon \equiv \frac{1}{2}(\|Tx\|_\beta - \|T\|)$ and let $\{y_1, \ldots, y_n\}$ be an ε-approximation to $T(S_E)$. Then

$$\|Tx\|_\beta > \|T\| + \varepsilon \geqslant \|y_k\|_\beta + \varepsilon \quad (k = 1, \ldots, n),$$

and hence $\|Tx - y_k\|_\beta > \varepsilon$ for all $k = 1, \ldots, n$. Therefore $\|x\|_\alpha > 1$ for some $\alpha \in A$, and hence $\|T\|$ is a bound for T.

PROPOSITION 2. *Let T be a compact linear mapping of a quasinormed linear space E with a quasinorm $\{\|\cdot\|_\alpha : \alpha \in A\}$ into a quasinormed linear space F with a quasinorm $\{\|\cdot\|_\beta : \beta \in B\}$, and let S be a bounded linear mapping of a quasinormed linear space F into a quasinormed linear space G with a quasinorm $\{\|\cdot\|_\gamma : \gamma \in \Gamma\}$. Then ST is compact.*

Proof. Let $c > 0$ be a bound for S, and let Y be an ε/c-approximation to $T(S_E)$. If $x \in E$ and for each $y \in Y$ there exists $\gamma \in \Gamma$ such that $\|STx - Sy\|_\gamma > \varepsilon$, then there exists $\beta \in B$ such that $\|Tx - y\|_\beta > \varepsilon/c$, and hence $\|x\|_\alpha > 1$ for some $\alpha \in A$. Therefore $\{Sy : y \in Y\}$ is an ε-approximation to $ST(S_E)$.

2. *Linear mappings of finite rank*

The linear space $\mathbf{C}^n$ admits a number of natural norms, one of which is given by $\|(\lambda_1, \ldots, \lambda_n)\| = |\lambda_1| + \ldots + |\lambda_n|$. It is easy to see that every bounded linear functional on $\mathbf{C}^n$ is normable, and there exists a linear metric equivalence between the set $(\mathbf{C}^n)^*$ of all bounded linear functionals on $\mathbf{C}^n$ and $\mathbf{C}^n$.

PROPOSITION 3. *If C is a convex subset of $\mathbf{C}^n$, then C is totally bounded if and only if every $f \in (\mathbf{C}^n)^*$ has a supremum on C.*

Proof. Suppose that C is totally bounded. Then trivially, every $f \in (\mathbf{C}^n)^*$ has a supremum on C. Conversely, suppose that every $f \in (\mathbf{C}^n)^*$ has a supremum on C. It suffices to show that C is bounded and located in $\mathbf{C}^n$ by [**2**, Chapter 4, (4.6)]. Let S_n^* denote the closed unit sphere of $(\mathbf{C}^n)^*$. Since S_n^* is compact, there exists a $\frac{1}{2}$-approximation $\{f_1, \ldots, f_m\}$ to S_n^*. Let $M \equiv \max\{\sup_{x \in C} f_i(x) : i = 1, \ldots, m\} + 1$ and let $x \in C$. Then there exists $f \in S_n^*$ with $\|x\| < f(x) + 1$ by [**2**, Chapter 7, (4.5)]; and also there exists i with $\|f - f_i\| < \frac{1}{2}$. Hence we have

$$\begin{aligned} \|x\| < f(x) + 1 &= (f_i(x) + 1) + (f(x) - f_i(x)) \\ &\leqslant \sup_{x \in C} f_i(x) + 1 + |f(x) - f_i(x)| \\ &\leqslant M + \|f - f_i\| \, \|x\| \leqslant M + \tfrac{1}{2}\|x\|, \end{aligned}$$

and hence $\|x\| < 2M$. Therefore C is bounded. To show that C is located, let $x \in \mathbf{C}^n$. Then since C is bounded, the family $\{f \mapsto \operatorname{Re} f(x-z) : z \in C\}$ of mappings of $\mathbf{C}^n$ into $\mathbf{R}$ is equicontinuous, and hence the mapping

$$f \mapsto \inf\nolimits_{z \in C} \operatorname{Re} f(x-z) \,(= \operatorname{Re} f(x) - \sup\nolimits_{z \in C} \operatorname{Re} f(z))$$

is continuous; and also $a \equiv \sup_{f \in S_n^*} \inf_{z f \in C} \operatorname{Re} f(x-z)$ exists and $a \leqslant \|x - z\|$ for all $z \in C$. Let $\varepsilon > 0$. Then there exist $f_1, \ldots, f_{m'} \in S_n^*$ such that for each $f \in S_n^*$ there exists i with $|f(x-z) - f_i(x-z)| < \frac{1}{3}\varepsilon$ for all $z \in C$. Choose $z_1, \ldots, z_{m'} \in C$ so that $f_i(x - z_i) < a + \frac{1}{3}\varepsilon$ for all $i = 1, \ldots, m'$ and let K be the convex hull of $\{z_1, \ldots, z_{m'}\}$. Then K is a totally bounded subset of C, and for each $f \in S_n^*$ there exists i such that $\operatorname{Re} f(x-z) \leqslant \operatorname{Re} f_i(x-z) + \frac{1}{3}\varepsilon$ for all $z \in C$ and hence

$$\inf_{z \in K} \operatorname{Re} f(x-z) \leqslant \inf_{z \in K} \operatorname{Re} f_i(x-z) + \tfrac{1}{3}\varepsilon < a + \tfrac{2}{3}\varepsilon.$$

Hence $\inf_{z \in K} \operatorname{Re} f(x-z) < a + \frac{2}{3}\varepsilon$ for all $f \in S_n^*$. Either $d(x, k) > 0$ or $d(x, K) < \frac{1}{3}\varepsilon$; in the former case, there exists $g \in S_n^*$ such that $\operatorname{Re} g(x) > \operatorname{Re} g(z) + d(x, K) - \frac{1}{3}\varepsilon$ for all $z \in K$ by [**2**, Chapter 7, (4.3)], and hence

$$d(x, K) \leqslant \inf_{z \in K} \operatorname{Re} g(x-z) + \tfrac{1}{3}\varepsilon \leqslant a + \varepsilon;$$

in the latter case, putting $g \equiv 0$, we have

$$d(x, K) < \tfrac{1}{3}\varepsilon = \inf_{z \in K} \operatorname{Re} g(x-z) + \tfrac{1}{3}\varepsilon \leqslant a + \varepsilon.$$

Hence, in both cases, we have $d(x, K) \leqslant a + \varepsilon$, and hence there exists $y \in K \subset C$ with $\|x - y\| < a + 2\varepsilon$. Therefore $d(x, C)$ exists and equals a.

As a consequence of Proposition 3, we have the following Proposition which is a generalization of [**6**, Proposition 1] and plays a crucial rɵle in the proofs of Theorem 1 and Theorem 3.

PROPOSITION 4. *If u is a linear mapping of a quasinormed linear space E with a quasinorm $\{\|\cdot\|_\alpha : \alpha \in A\}$ into $\mathbf{C}^n$, then u is compact if and only if $f \circ u$ is normable for all $f \in (\mathbf{C}^n)^*$.*

Proof. Suppose that u is compact. Since f is bounded, $f \circ u$ is compact by Proposition 2, and hence normable by Proposition 1.

Conversely, suppose that $f \circ u$ is normable for all $f \in (\mathbf{C}^n)^*$. Then for each $f \in (\mathbf{C}^n)^*$, since $(f \circ u)(S_E)$ is a totally bounded subset of $\mathbf{C}$, f has the supremum on $u(S_E)$, and hence $u(S_E)$ is a totally bounded subset of $\mathbf{C}^n$ by Proposition 3. Let $\{u(x_1), \ldots, u(x_n)\}$ be a $\frac{1}{2}\varepsilon$-approximation to $u(S_E)$, and let $x \in E$. Suppose that $\|u(x) - u(x_k)\| > \varepsilon$ for all $k = 1, \ldots, n$. If $d(u(x), u(S_E)) < \frac{1}{2}\varepsilon$, then there exists $y \in S_E$ with $\|u(x) - u(y)\| < \frac{1}{2}\varepsilon$; and also there exists i with $\|u(y) - u(x_i)\| < \frac{1}{2}\varepsilon$; whence we have

$$\|u(x) - u(x_i)\| \leqslant \|u(x) - u(y)\| + \|u(y) - u(x_i)\| < \varepsilon,$$

a contradiction. Therefore $d(u(x), u(S_E)) \geqslant \frac{1}{2}\varepsilon$, and hence there exists $f \in (\mathbf{C}^n)^*$ such that $\operatorname{Re} f(u(x)) > \operatorname{Re} f(u(y)) + \frac{1}{4}\varepsilon$ for all $y \in S_E$ ([**2**, Chapter 7, (4.3)]). Since

$$|(f \circ u)(x)| \geqslant \operatorname{Re} f(u(x)) \geqslant \sup_{y \in S_E} \operatorname{Re} f(u(y)) + \tfrac{1}{4}\varepsilon \geqslant \|f \circ u\| + \tfrac{1}{4}\varepsilon > \|f \circ u\|$$

and $f \circ u$ is normable, $\|x\|_\alpha > 1$ for some $\alpha \in A$.

3. *Compact linear mappings*

Let E be a normed linear space. Then we use the notations E^* and E' to denote the set of all bounded functionals on E, and the set of all normable linear functionals on E, respectively. We consider E^* as a quasinormed linear space with the quasinorm $\{\|\cdot\|_x : x \in S_E\}$ given by $\|f\|_x \equiv |f(x)|$ for all $f \in E^*$. As we shall see in Brouwerian Example 2, E' is not a linear subspace of E^* in general.

Let T be a bounded linear mapping of a normed linear space E into a normed linear space F. Then the mapping $f \mapsto f \circ T$ of F^* into E^* is called the *adjoint* of T, and is written T^*; if c is a bound for T and $|(T^* f)x| > c$ for some $x \in S_E$, then $|f(Tx)| > c$ and hence $|f(Tx/c)| > 1$; whence T^* is a bounded linear mapping with the same bound of T. Also it is easy to see that if F is a separable normed linear space, then T is normable if and only if T^* is normable.

Note that the definition of the adjoint is slightly different from that of [**2**, Chapter 7, p. 371], in the sense that the adjoint in our definition always exists, but the one in [**2**] exists when $T^*(F) \subset E'$.

LEMMA 1. *If E is a separable normed linear space, then for each $x \in E$ the mapping $x : f \mapsto f(x)$ is a normable linear functional on the quasinormed linear space E^*.*

Proof. Let $f \in S_{E^*}$. Then $|x(f)| = |f(x)| \leqslant \|x\|$ and for each $\varepsilon > 0$ there exists $g \in S_{E^*}$ such that $\operatorname{Re} g(x) > \|x\| - \varepsilon$ by [**2**, Chapter 7, (4.5)], and hence $\|x\| = \sup\{|f(x)| : f \in S_{E^*}\}$. If $|f(x)| > \|x\|$, then $|f(x/(\|x\| + \varepsilon))| > 1$ for sufficiently small $\varepsilon > 0$, and hence $\|x\|$ is a bound for x.

Note that if E' is a linear space, then since we can find normable g ([**2**, Chapter 7, (4.5)]), x is also a normable linear functional on a subspace E' of E^*.

We denote the range of a mapping T by $\operatorname{ran}(T)$.

THEOREM 1. *If T is a linear mapping of a normed linear space E into a separable normed linear space F, then T is compact if and only if* $\operatorname{ran}(T^*) \subset E'$ *and T^* is compact.*

Proof. Suppose that T is compact. Then it is easy to see that $\operatorname{ran}(T^*) \subset E'$. Since $T(S_E)$ is totally bounded, for each $\varepsilon > 0$ there exists a $\frac{1}{4}\varepsilon$-approximation $\{Tx_1, \ldots, Tx_n\}$ to $T(S_E)$. Let $u: F^* \to \mathbf{C}^n$ be the linear mapping defined by

$$u(f) \equiv (f(Tx_1), \ldots, f(Tx_n)) \ (f \in F^*).$$

Then $(\lambda \circ u)(f) = f(\lambda_1 Tx_1 + \ldots + \lambda_n Tx_n)$ is normable for all $\lambda \equiv (\lambda_1, \ldots, \lambda_n) \in (\mathbf{C}^n)^*$ by Lemma 1, and hence u is compact by Proposition 4. Therefore there exists a $\frac{1}{3}\varepsilon$-approximation $\{u(f_1), \ldots, u(f_m)\}$ to $u(S_{F^*})$ Let $g \in F^*$ and suppose that for each $k = 1, \ldots, m$ there exists $y_k \in S_E$ such that $|(T^*g)(y_k) - (T^*f_k)(y_k)| > \varepsilon$. Then for each k there exists i_k such that $\|Tx_{i_k} - Ty_k\| < \frac{1}{4}\varepsilon$. Either $|g(Tx_{i_k}) - g(Ty_k)| > \frac{1}{4}\varepsilon$ for some $k = 1, \ldots, n$, or $|g(Tx_{i_k}) - g(Ty_k)| < \frac{1}{3}\varepsilon$ for all $k = 1, \ldots, n$; in the former case, we have

$$|g(Tx_{i_k} - Ty_k)| > \tfrac{1}{4}\varepsilon > \|Tx_{i_k} - Ty_k\|,$$

and hence $|g((Tx_{i_k} - Ty_k)/(\|Tx_{i_k} - Ty_k\| + \delta))| > 1$ for sufficiently small $\delta > 0$; in the latter case, we have

$$\begin{aligned}
\|u(g) - u(f_k)\| &= \sum_{i=1}^{n} |g(Tx_i) - f_k(Tx_i)| \\
&\geqslant |g(T_{i_k}) - f_k(Tx_{i_k})| \\
&\geqslant |g(Ty_k) - f_k(Ty_k)| - |g(Tx_{i_k}) - g(Ty_k)| - |f_k(Tx_{i_k}) - f_k(Ty_k)| \\
&> |(T^*g)\,y_k - (T^*f_k)\,y_k| - |g(Tx_{i_k}) - g(Ty_k)| - \|Tx_{i_k} - Ty_k\| \\
&> \varepsilon - \tfrac{1}{3}\varepsilon - \tfrac{1}{4}\varepsilon \\
&> \tfrac{1}{3}\varepsilon \ (k = 1, \ldots, m),
\end{aligned}$$

and hence there exists $z \in S_F$ with $|g(z)| > 1$.

Conversely, suppose that $\operatorname{ran}(T^*) \subset E'$ and T^* is compact. Since $\operatorname{ran}(T)$ can be regarded as a normed linear space, $T^*(S_{F^*})$ is a totally bounded metric space. Let $\varepsilon > 0$ and let $\{T^*f_1, \ldots, T^*f_n\}$ be an ε-approximation to $T^*(S_{F^*})$. Define the linear mapping $v: E \to \mathbf{C}^n$ by

$$v(x) \equiv ((T^*f_1)\,x, \ldots, (T^*f_n)\,x) \ (x \in E).$$

Then $\lambda \circ v = T^*(\lambda_1 f_1 + \ldots + \lambda_n f_n)$ is normable for all $\lambda \equiv (\lambda_1, \ldots, \lambda_n) \in (\mathbf{C}^n)^*$ and hence $v(S_E)$ is a totally bounded subset of $\mathbf{C}^n$ by Proposition 4. Therefore, when we deal with each element x in S_E as a uniformly continuous function $f \mapsto f(x)$ from $T^*(S_{F^*})$ to $\mathbf{C}$, the equicontinuous subset S_E of $C(T^*(S_{F^*}), \mathbf{C})$ is totally bounded by Ascoli's theorem ([**2**, Chapter 4, (5.2)]); we denote the norm on $C(T^*(S_{F^*}), \mathbf{C})$ by $\|\cdot\|_{T^*(S_{F^*})}$. Let $\{x_1, \ldots, x_m\}$ be an ε-approximation to S_E. Then for each $x \in S_E$ there exists k such that

$$\begin{aligned}
\varepsilon &> \|x - x_k\|_{T^*(S_{F^*})} \\
&= \sup\{|f(x - x_k)| : f \in T^*(S_{F^*})\} \\
&= \sup\{|(T^*f)(x - x_k)| : f \in S_{F^*}\} \\
&= \sup\{|f(Tx - Tx_k)| : f \in S_{F^*}\} = \|Tx - Tx_k\|
\end{aligned}$$

(by [**2**, Chapter 7, (4.5)]), and hence $\{Tx_1, \ldots, Tx_m\}$ is an ε-approximation to $T(S_E)$.

THEOREM 2. *If S is a bounded linear mapping of a normed linear space E into a normed linear space F with $S^*(F') \subset E'$, and if T is a compact linear mapping of F into a separable normed linear space G, then TS is compact.*

Proof. Since S^* is bounded and T^* is compact by Theorem 1, $S^*T^*: G^* \to E^*$ is compact by Proposition 2; and also since $\operatorname{ran}(T^*) \subset F'$ and $S^*(F') \subset E'$, we have $\operatorname{ran}(S^*T^*) \subset E'$. Therefore TS is compacted by Theorem 1.

The following example shows that the above theorem is the best we can hope for in the constructive setting, in the sense that the condition $S^*(F') \subset E'$ cannot be dropped.

BROUWERIAN EXAMPLE 1. *Bounded linear mappings S and T of a Hilbert space into itself such that T is compact, but TS is not compact.*

Let l^2 be the Hilbert space of square-summable sequences of real numbers, for each positive integer n, let e_n be the basis vector with nth component 1 and all other components 0, and for each $x \in l^2$, let x_n be the nth component of x. Let $\{a_n\}$ be a binary sequence such that $a_n = 1$ for at most one value of n and define linear mappings $S, T: l^2 \to l^2$ by

$$Sx \equiv \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m}{\sqrt{2^n}} x_m e_n,$$

$$Tx \equiv \sum_{n=1}^{\infty} \frac{1}{\sqrt{2^n}} x_n e_n.$$

Then it is easy to see that T is compact; also we have

$$\|Sx\|^2 = \sum_{n=1}^{\infty} \frac{1}{2^n} \left| \sum_{m=1}^{\infty} a_m x_m \right|^2 \leqslant \sum_{n=1}^{\infty} \frac{1}{2^n} \|x\|^2 = \|x\|^2$$

and hence S is bounded. However, we have

$$\|TSx\|^2 = \left\| \sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{m=1}^{\infty} a_m x_m e_n \right\|^2 = \sum_{n=1}^{\infty} \frac{1}{4^n} \left| \sum_{m=1}^{\infty} a_m x_m \right|^2 = \frac{1}{3} \left| \sum_{m=1}^{\infty} a_m x_m \right|^2.$$

If TS is compact, then either $\|TS\| > 0$, in which case there exists n such that $a_n = 1$; or else $\|TS\| < \frac{1}{3}$, and $a_n = 0$ for all n. (Note that, for the normable linear functional $f(x) \equiv \sum_{n=1}^{\infty} (1/\sqrt{2^n}) x_n$,

$$f(Sx) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m}{2^n} x_m = \sum_{m=1}^{\infty} a_m x_m,$$

and hence S^*f is not normable.)

THEOREM 3. *If S and T are compact linear mappings of a normed linear space E into a separable normed linear space F, then $S^* + T^*$ is compact. Moreover, if $\operatorname{ran}(S^* + T^*) \subset E'$, then $S + T$ is compact.*

Proof. Since $S(S_E)$ and $T(S_E)$ are totally bounded, for each $\varepsilon > 0$ there exist $\frac{1}{8}\varepsilon$-approximations $\{Sx_1, \ldots, Sx_m\}$ and $\{Ty_1, \ldots, Ty_n\}$ to $S(S_E)$ and $T(S_E)$, respectively. Let $u: F^* \to \mathbf{C}^{m+n}$ be the linear mapping defined by

$$u(f) \equiv (f(Sx_1), \ldots, f(Sx_m), f(Ty_1), \ldots, f(Ty_n)) \quad (f \in F^*).$$

Then u is compact, and hence there exists a $\frac{1}{3}\varepsilon$-approximation $\{u(f_1), \ldots, u(f_N)\}$ to $u(S_{F^*})$. Let $g \in F^*$ and suppose that for each $k = 1, \ldots, N$ there exists $z_k \in S_E$ such that $|((S^*+T^*)g)(z_k) - ((S^*+T^*)f_k)(z_k)| > \varepsilon$. Then for each k there exist i_k and j_k such that $\|Sx_{i_k} - Sz_k\| < \frac{1}{8}\varepsilon$ and $\|Ty_{j_k} - Tz_k\| < \frac{1}{8}\varepsilon$. Either $|g(Sx_{i_k}) - g(Sz_k)| + |g(Ty_{j_k}) - g(Tz_k)| > \frac{1}{4}\varepsilon$ for some $k = 1, \ldots, N$, or $|g(Sx_{i_k}) - g(Sz_k)| + |g(Tx_{i_k}) - g(Tz_k)| < \frac{1}{3}\varepsilon$ for all $k = 1, \ldots, N$; in the former case, we have $|g((Sx_{i_k} - Sz_k)/(\|Sx_{i_k} - Sz_k\| + \delta))| > 1$ or $|g((Ty_{j_k} - Tz_k)/(\|Ty_{j_k} - Tz_k\| + \delta))| > 1$ for sufficiently small $\delta > 0$; in the latter case, we have $\|u(g) - u(f_k)\| > \frac{1}{3}\varepsilon$ for all $k = 1, \ldots, N$ and hence there exists $w \in S_F$ with $|g(w)| > 1$. Moreover, if $\operatorname{ran}(S^* + T^*) \subset E'$, then $S+T$ is compact by Theorem 1.

COROLLARY 1. *Let S and T be compact linear mappings of a normed linear space E into a separable normed linear space F. If E' is a linear space, then $S+T$ is compact.*

Proof. Since $\operatorname{ran}(S^*) \subset E'$, $\operatorname{ran}(T^*) \subset E'$, and E' is linear, we have

$$\operatorname{ran}(S^* + T^*) \subset E'.$$

The following example shows that there exists a Banach space such that E' is not a linear space, and also that, in the constructive setting, we cannot drop the condition $\operatorname{ran}(S^* + T^*) \subset E'$.

BROUWERIAN EXAMPLE 2. *Bounded linear mappings S and T of a Banach space into* $\mathbf{R}$ *such that $S+T$ is not compact.*

Let l^1 be the Banach space of absolutely summable sequences of real numbers, and for each $x \in l^2$, let x_n be the nth component of x. Let $\{a_n\}$ be a binary sequence and define bounded linear mappings $S, T: l^1 \to \mathbf{R}$ by

$$Tx \equiv x_1 + \sum_{n=1}^{\infty} a_n x_{n+1}, \quad Sx \equiv -x_1.$$

Then S and T are normable, with $\|S\| = \|T\| = 1$, and hence compact. If $S+T$ is compact, then either $\|S+T\| > 0$ in which case there exists n such that $a_n = 1$; or else $\|S+T\| < 1$, and $a_n = 0$ for all n.

With the additional assumption that F' is a linear space, we have the following slightly different version of Theorem 1, which corresponds to the definition of the adjoint in [2, Chapter 7, p. 371].

THEOREM 4. *Let T be a linear mapping of a normed linear space E into a separable normed linear space F. If F' is a linear space, then T is compact if and only if* $\operatorname{ran}(T^*|_{F'}) \subset E'$ *and $T^*|_{F'}$ is compact.*

Proof. By replacing F^* with F' and noting the remark after Lemma 1, the proof of Theorem 1 is valid for this theorem.

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Faculty of Integrated Arts and Sciences
Hiroshima University
Higashisenda-machi, Naka-ku
Hiroshima 730
Japan