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Fair cost allocations under conflicts — a game-theoretic point of view —**

Abstract. Optimization theory resolves problems to minimize the total costs when the agents are involved in some conflicts. In this paper, we consider how to allocate the minimized total cost among the agents. To do that, the allocation is required to be fair in a certain sense. We use a game-theoretic point of view, and provide algorithms to compute fair allocations in polynomial time for a certain conflict situation. More specifically, we study a minimum coloring game, introduced by Deng, Ibaraki & Nagamochi [X. Deng, T. Ibaraki and H. Nagamochi: Algorithmic aspects of the core of combinatorial optimization games. *Math. Oper. Res.* **24** (1999) 751–766], and investigate the core, the nucleolus, the τ -value, and the Shapley value. In particular, we provide the following four results. (1) The characterization of the core for a perfect graph in terms of its extreme points. This leads to polynomial-time algorithms to compute a vector in the core, and to determine whether a given vector belongs to the core. (2) A polynomial-time algorithm to compute the τ -value for a perfect graph. (3) A characterization of the nucleolus for some classes of the graphs, including the complete multipartite graphs and the chordal graphs. This leads to a polynomial-time algorithm to compute the nucleolus for these classes of graphs. (4) A polynomial-time algorithm to compute the Shapley value for a forest. The investigation of this paper gives several insights to the relationship of algorithm theory with cooperative games.

1. Introduction

The main concern of optimization theory is to optimize a certain objective under some constraints. This turns out to be very useful, and the optimization paradigm is applied to most of the disciplines. Among them, some real-world situations require to solve the following type of problems: minimizing the total cost for units (or agents like people, nations, companies, jobs, processes, requests and so on) under their conflict. Then, optimization theory can solve this type of problems, and the algorithmic studies for optimization problems provide how to compute the optimal solution. However, we have more to solve. The next requirement is to solve the following type of problems: computing an allocation of the minimized total cost to all units so that each unit can be satisfied. Since a usual optimization theory itself does not answer this type of questions, we need another theory.

However, this is exactly the scope of game theory, especially cooperative game theory. Cooperative game theory has been developing the theorems on fair cost allocations, and we are interested in algorithms to compute such fair cost allocations.

This paper studies fair cost allocations over conflict situations from the viewpoint of cooperative game theory. As a simple model of a conflict, we use a conflict graph. In a conflict graph, each vertex represents an agent involved in the situation. Two vertices are adjacent through an edge if the corresponding agents are in conflict under the situation. We model the total cost of the conflict simply as it is proportional to the chromatic number of the conflict graph. So we want to divide the chromatic number of the graph (possibly fractionally) and allocate it to each agent. This kind of cooperative games was first introduced by Deng, Ibaraki & Nagamochi [4] by the name of a minimum coloring game. Their paper [4] and a subsequent paper by Deng, Ibaraki Nagamochi & Zang [5] included some investigations of the core (a sort of fair allocations) of a minimum coloring game.

In this paper, we study minimum coloring games more thoroughly. Among many kinds of concepts on fair allocations in cooperative game theory, we consider the core (introduced by Gillies [13]), the τ -value

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(by Tijs [35]), the nucleolus (by Schmeidler [30]) and the Shapley value (by Shapley [31]). For general graphs, it turns out hard to compute any of these fair cost allocations (Proposition 1). Therefore, we concentrate on a special class of graphs, namely perfect graphs. This makes sense since perfect graphs give rise to totally balanced minimum coloring games [5]. For perfect graphs, we show the following theorems.

- The core of the minimum coloring game on a perfect graph is characterized as the convex hull of the characteristic vectors of the maximum cliques of the graph (Theorem 2). This is a generalization of a result by Deng, Ibaraki & Nagamochi [4] for a bipartite graph, and furthermore it implies that we can compute a core allocation in polynomial time and that we can decide whether a given vector belongs to the core or not in polynomial time for perfect graphs (Corollary 5).
- The τ -value for a perfect graph can be computed in polynomial time. This is a consequence of Theorem 6 and a polynomial-time algorithm for computing the chromatic number of a perfect graph [15]. So far, we only knew that this could be done for complete multipartite graphs (since complete multipartite graphs give rise to submodular minimum coloring games [28] and the τ -value of a submodular cost game can be computed in polynomial time by the application of a polynomial-time algorithm for the submodular function minimization [15]).
- The nucleolus is the barycenter of the characteristic vectors of the maximum cliques for some classes of perfect graphs (Theorem 8). This class includes the complete multipartite graphs and the chordal graphs, which appear in some application contexts like scheduling [14] (Proposition 13). As a consequence, for chordal graphs, we obtain a polynomial-time algorithm for computing the nucleoli. We can see that this is much simplified for forests. In addition, for a complete multipartite graph, we can see that the Shapley value and the nucleolus coincide (Corollary 16).
- The Shapley value can be computed efficiently for the minimum coloring game on a forest (Section 3.5). However, the same technique cannot be applied to bipartite graphs in general (Proposition 18).

Perhaps, it was Megiddo [24] who first noticed the computational issue on cooperative game theory. Since then, there have been many studies on the computational complexity and algorithms for the core and the nucleolus and a little about the Shapley value (here we omit the vast references). Especially, cooperative games arising from combinatorial optimization problems fit very much into the framework of algorithm theory. These kinds of cooperative games are called “combinatorial optimization games” and some of these cooperative games and their properties are discussed in Curiel’s book [3]. However, none of the past works dealt with minimum coloring games in spite of the importance, except for a small amount of papers [4, 5, 28]. Since these works [4, 5, 28] only studied the core, this paper provides the first treatment of solution concepts for minimum coloring games other than cores, and sheds more light on relationship of algorithm theory with cooperative games.

The paper is organized as follows. In the next section, we introduce some graph-theoretic and game-theoretic concepts which we need in the rest of the paper. Section 3 is the main part of the paper. Section 4 concludes the paper with some open problems.

2. Preliminaries

First, we remark a general notation. Throughout the paper, for a vector $\mathbf{x} \in \mathbb{R}^N$ and $S \subseteq N$, we write $\mathbf{x}(S) := \sum\{x_i : i \in S\}$. When $S = \emptyset$, set $\mathbf{x}(S) := 0$. For a subset $S \subseteq N$ of a finite set N , the *characteristic vector* of S is a vector $\mathbb{1}_S \in \{0, 1\}^N$ defined as $(\mathbb{1}_S)_i = 1$ if $i \in S$ and $(\mathbb{1}_S)_i = 0$ if not. Note that, for $S, T \subseteq N$, we have $\mathbb{1}_S(T) = |S \cap T|$.

2.1. Graphs

We assume the reader a moderate familiarity with graphs. In this paper, a graph is always simple and finite. The *(open) neighborhood* of v in G is the set of vertices adjacent to v , and denoted by $N_G(v)$. The *closed neighborhood* of v in G is $N_G(v) \cup \{v\}$, and denoted by $N_G[v]$. For a subset $U \subseteq V(G)$, the subgraph of G induced by U is denoted by $G[U]$.

A k -coloring of a graph $G = (V, E)$ is a function $c : V \rightarrow \{1, \dots, k\}$. A k -coloring c of G is called *proper* if $c(u) \neq c(v)$ for any $\{u, v\} \in E(G)$. Note that there may not exist a proper k -coloring of a graph for some k . From the definition of a proper k -coloring, we can immediately see that if a graph G has a proper k -coloring then G also has a proper k' -coloring for $k' \geq k$. A graph G is k -colorable if there exists a proper k -coloring of G . The *chromatic number* of a graph G is the minimum k such that a proper k -coloring of G exists. The chromatic number of G is denoted by $\chi(G)$.

A *clique* of a graph G is a vertex subset $K \subseteq V(G)$ such that $G[K]$ is a complete graph. An *independent set* of a graph G is a vertex subset $I \subseteq V(G)$ such that $G[I]$ has no edge. A clique (or an independent set, respectively) of G is *maximal* if there is no other clique (or independent set, respectively) of G which contains it. A clique (or an independent set, respectively) is called of G *maximum* if it has the largest size among all cliques (or independent sets, respectively) of G . The size of the maximum clique of a graph G is denoted by $\omega(G)$. Note that it holds that $\omega(G) \leq \chi(G)$ for any graph G .

A graph G is called *perfect* if $\omega(H) = \chi(H)$ for all induced subgraphs H of G . The class of perfect graphs includes a lot of interesting subclasses of graphs like forests (i.e., graphs without cycles), bipartite graphs (i.e., 2-colorable graphs), complete multipartite graphs, and chordal graphs. A graph G is a *complete r -partite graph* if $\chi(G) = r$ and a proper r -coloring of G is unique. This is equivalent to saying that there exists a partition of the vertex set V of G into V_1, \dots, V_r such that the edge set of G is $E = \{\{u, v\} : u \in V_i, v \in V_j \text{ for some distinct } i, j \in \{1, \dots, r\}\}$. A graph is *chordal* if every induced cycle is a triangle. Facts on perfect graphs can be found in books by Golumbic [14] and by Grötschel, Lovász & Schrijver [15].

2.2. Cooperative games

A *cooperative game* (a *transferable utility (TU) game* or simply a *game*) is a pair (N, γ) of a finite set N and a function $\gamma : 2^N \rightarrow \mathbb{R}$ satisfying $\gamma(\emptyset) = 0$. Often, an element of N is called a *player* of the game, and γ is called the *characteristic function* of the game. Furthermore, each subset $S \subseteq N$ is called a *coalition*. Literally, for $S \subseteq N$ the value $\gamma(S)$ is interpreted as the total profit (or the total cost) for the players in S when they work in cooperation. In particular, $\gamma(N)$ represents the total profit (or cost) for the whole players when they all agree with working together. When γ represents a profit, we call the game a *profit game*. On the other hand, when γ represents a cost, we call the game a *cost game*.¹ In this paper, we mainly consider a certain class of cost games.

One of the aims of cooperative game theory is to provide a concept of “fairness,” namely, how to allocate the total cost (or profit) $\gamma(N)$ to each player in a “fair” manner when we take all $\gamma(S)$ ’s into account. Here, we describe some allocation rules which are considered fair in cooperative game theory. Formally, a cost allocation is defined as a preimputation in the terminology of cooperative game theory. A *preimputation* of a game (N, γ) is a vector $\mathbf{x} \in \mathbb{R}^N$ satisfying $\mathbf{x}(N) = \gamma(N)$. Each component x_i expresses how much the player $i \in N$ should owe according to the cost allocation \mathbf{x} .

Now we define some fair allocations, namely a core allocation, the τ -value, the nucleolus and the Shapley value. See Driessen’s book [7] for details (why they are seen fair) and other kinds of fair allocation concepts from cooperative game theory. Let (N, γ) be a cost game. A vector $\mathbf{x} \in \mathbb{R}^N$ is called a *core allocation* if \mathbf{x} satisfies the following conditions: $\mathbf{x}(N) = \gamma(N)$ and $\mathbf{x}(S) \leq \gamma(S)$ for all $S \subseteq N$. The set of all core allocations of a cost game (N, γ) is called the *core* of (N, γ) . The core was introduced by Gillies [13], and many aspects of the core are surveyed by Kannai [18] and Peleg [26]. Note that a core might be empty. Therefore, a cost game with a nonempty core is especially interesting, and such a cost game is called *balanced*.² Moreover, we call a cost game *totally balanced* if each of the subgames has a nonempty core. (Here, a *subgame* of a cost game (N, γ) is a cost game $(T, \gamma^{(T)})$ for some nonempty $T \subseteq N$ defined as $\gamma^{(T)}(S) = \gamma(S)$ for each $S \subseteq T$.) Naturally, a totally balanced game is also balanced. A special subclass of the totally balanced games consists of submodular games (Shapley [33]), where a cost game (N, γ) is

¹ Thus, the terms “profit game” and “cost game” are not mathematically determined. They are just determined by the interpretation of a game.

² Originally, Shapley [32] introduced a balanced game in a different form, and showed that it is equivalent to that the game has a nonempty core. Note that Bondareva [1] also showed the same statement. In this paper, we adapt the core nonemptiness as the definition of balancedness.

called *submodular* (or *concave*) if it satisfies the following condition: $\gamma(S) + \gamma(T) \geq \gamma(S \cup T) + \gamma(S \cap T)$ for any $S, T \subseteq N$. Therefore, we have the following chain of implications:

submodularity \Rightarrow total balancedness \Rightarrow balancedness.

These implications are fundamental in cooperative game theory.

Next, we introduce the τ -value due to Tijs [35]. Let (N, γ) be a cost game. Define two vectors $\underline{\mathbf{m}}, \overline{\mathbf{m}} \in \mathbb{R}^N$ as

$$\begin{aligned} \underline{m}_i &:= \gamma(N) - \gamma(N \setminus \{i\}), \\ \overline{m}_i &:= \min\{\gamma(S) - \underline{\mathbf{m}}(S \setminus \{i\}) : i \in S \subseteq N\} \end{aligned}$$

for each $i \in N$. It is known [35] that, if a cost game (N, γ) is balanced, it holds that $\underline{m}_i \leq \overline{m}_i$ for each $i \in N$ and $\underline{\mathbf{m}}(N) \leq \gamma(N) \leq \overline{\mathbf{m}}(N)$.³ Then, the τ -value of a balanced cost game (N, γ) is defined as a vector $\mathbf{x} \in \mathbb{R}^N$ uniquely represented by $\mathbf{x} := \lambda \underline{\mathbf{m}} + (1 - \lambda) \overline{\mathbf{m}}$ where we choose $\lambda \in [0, 1]$ so that $\mathbf{x}(N) = \gamma(N)$ is satisfied. From the definition we can see that the τ -value can be computed in polynomial time for submodular games by a polynomial-time algorithm for the submodular function minimization [15].

Next, we define the nucleolus, which was introduced by Schmeidler [30]. Maschler [22] gave a detailed survey of the nucleolus. It is known that the nucleolus always belongs to the core if it is not empty. For our purpose, it is better to use an algorithmic definition of the nucleolus due to Peleg (referred by Kopelowitz [19]) than the original one by Schmeidler [30]. In this scheme, we successively solve a series of linear programming problems (P_1) , (P_2) , and so on. For a cost game (N, γ) , the i -th problem (P_i) is described as follows: find $(\mathbf{x}, \epsilon) \in \mathbb{R}^N \times \mathbb{R}$ to

$$\begin{aligned} (P_i): \quad & \text{maximize} \quad \epsilon \\ & \text{subject to} \quad \mathbf{x}(N) = \gamma(N) \\ & \quad \mathbf{x}(S) = \gamma(S) - \epsilon_\ell \quad \text{for all } S \in \mathcal{C}_\ell \\ & \quad \quad \quad \text{for all } \ell \in \{1, \dots, i-1\} \\ & \quad \mathbf{x}(S) \leq \gamma(S) - \epsilon \quad \text{for all } S \in \mathcal{C}_0 \setminus \bigcup_{\ell=1}^{i-1} \mathcal{C}_\ell, \end{aligned}$$

where $\mathcal{C}_0 = 2^N \setminus \{N, \emptyset\}$, ϵ_j is the optimal value of (P_j) , and $\mathcal{C}_j = \{S \in \mathcal{C}_0 \setminus \bigcup_{\ell=1}^{j-1} \mathcal{C}_\ell : \mathbf{x}(S) = \gamma(S) - \epsilon_j \text{ for all optimal solutions } (\mathbf{x}, \epsilon_j) \text{ of } (P_j)\}$ for every $j = 1, \dots, i-1$. It is known that finally, say, at the t -th step, the problem has a unique optimal solution $(\mathbf{x}^*, \epsilon_t)$. Then the vector \mathbf{x}^* is called the *nucleolus* of the game. Notice that the procedure above is not a polynomial-time algorithm. Indeed it is NP-hard to compute a nucleolus for a totally balanced game [9]. On the other hand, the nucleolus can be computed in polynomial time for a submodular game [10, 20].

The Shapley value is another cost allocation we study. The *Shapley value* of a game (N, γ) is a vector $\mathbf{x} \in \mathbb{R}^N$ which is defined as

$$x_i := \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N| - |S| - 1)!}{|N|!} (\gamma(S \cup \{i\}) - \gamma(S))$$

for each $i \in N$. The Shapley value was first introduced by Shapley [31] (why not!), and is surveyed by Winter [37]. It is not difficult to show that the Shapley value is indeed a cost allocation (i.e., a preimputation). By the definition, the Shapley value always uniquely exists even if the game is not balanced. However, the Shapley value may not be a core allocation. On an algorithmic side, there exists an exponential lower bound for the computation of the Shapley value in general [8].

³ More precisely, a cost game satisfying these two conditions is called *quasi-balanced*, and it is known that a balanced game is quasi-balanced.

3. Minimum coloring games

As we saw in Section 1, the minimum coloring problem is a simple model of conflicts. The corresponding cost game is called a minimum coloring game. In Sections 3.1 we define a minimum coloring game, and subsequently we investigate the core, the τ -value, the nucleolus and the Shapley value.

3.1. Definition and a hardness statement

Let $G = (V, E)$ be a graph. The *minimum coloring game* on G is a cost game (V, χ_G) where $\chi_G : 2^V \rightarrow \mathbb{R}$ is defined as $\chi_G(S) := \chi(G[S])$ for all $S \subseteq V$. Remember that $G[S]$ represents the subgraph of G induced by S , and note that we set $\chi_G(\emptyset) = 0$.

Here, before proceeding to the algorithmic aspects of minimum coloring games, we notice the issue on inputs. Usually, when we think of a computational task for a cooperative game (N, γ) , we have the characteristic function γ as an oracle which outputs the value $\gamma(S)$ for a queried set $S \subseteq N$, and consider that one oracle call can be done in a constant time. This is because, in order to store the function values $\gamma(S)$ for all $S \subseteq N$, this requires $2^{|N|}$ space, and it is already exponential of $|N|$. The remarks given so far in this paper about the computational complexity on cooperative games rely on this model. However, in a minimum coloring game, we take a graph G as an input, and the running time of algorithms are measured by the encoding length of G , not by the oracle complexity model. This makes a real difference since the computation of $\chi_G(V) = \chi(G)$ is generally NP-hard but in the oracle model it could be done just by a single query (i.e., constant time).

Deng, Nagamochi & Ibaraki [4] proved that it is NP-complete to decide whether the minimum coloring game on a given graph is balanced. Later, Deng, Ibaraki, Nagamochi & Zang [5] showed that the minimum coloring game on a graph G is totally balanced if and only if G is perfect. So the decision problem on the total balancedness of a minimum coloring game is as hard as recognizing perfect graphs. Furthermore, Okamoto [28] showed that the minimum coloring game on a graph G is submodular if and only if G is complete multipartite. So we can decide whether a given graph yields a submodular minimum coloring game in polynomial time.

In this paper, we are interested in the computation of a core allocation, the τ -value, the nucleolus and the Shapley value of a minimum coloring game. The following observation is important although it is not difficult to show.

Proposition 1. *It is NP-hard to compute a preimputation of the minimum coloring game on a graph given from a class for which the computation of the chromatic number is NP-hard.*

Proof. Suppose that a preimputation \mathbf{x} can be computed in polynomial time. Then we have $\mathbf{x}(V) = \chi_G(V) = \chi(G)$. (The first equality comes from the definition of a preimputation, and the second equality comes from the definition of a minimum coloring game.) In this way, we could know $\chi(G)$ in polynomial time. However, this would be possible only if $P = NP$ since we are considering a class of graphs in which the minimum coloring problem is NP-hard. \square

Proposition 1 suggests that, in order to obtain a polynomial-time algorithm to compute a certain preimputation of a minimum coloring game, we would better concentrate on a class of graphs for which the chromatic number can be computed in polynomial time. Perfect graphs form such a class [15]. From now on, we concentrate on perfect graphs, and in Sections 3.2–3.5, we investigate the core, the τ -value, the nucleolus and the Shapley value respectively.

3.2. The core of a minimum coloring game

As described above, the minimum coloring game on a perfect graph is totally balanced [5], which implies that the core is nonempty. Now we characterize the core for a perfect graph in terms of its extreme points.

Theorem 2. *The core of the minimum coloring game on a perfect graph is the convex hull of the characteristic vectors of the maximum cliques of the graph.*

Notice that Theorem 2 generalizes a theorem by Deng, Ibaraki & Nagamochi [4] on bipartite graphs to all perfect graphs. They showed that the core of the minimum coloring game on a bipartite graph is the convex hull of the characteristic vectors of the edges of the graph.

To prove Theorem 2, we use two facts. The first one is due to Deng, Ibaraki & Nagamochi [4], which is related to the linear programming relaxation of an integer programming formulation of the minimum coloring problem. A minimum coloring game was originally introduced as a subclass of the minimum covering games by Deng, Nagamochi & Ibaraki [4]. Namely, they used a formulation of the minimum coloring problem as the minimum covering problem. Let \mathcal{M}_G be the family of all maximal independent sets of G and consider the following minimum covering formulation of the minimum coloring problem: for a graph $G = (V, E)$ find $\mathbf{y} \in \mathbb{R}^{\mathcal{M}_G}$ to

$$\begin{aligned} (\text{IP}(G)): \quad & \text{minimize} && \sum_{I \in \mathcal{M}_G} y_I \\ & \text{subject to} && \sum_{\substack{I \in \mathcal{M}_G \\ \text{s.t. } v \in I}} y_I \geq 1 \quad \text{for all } v \in V, \\ & && y_I \in \{0, 1\} \quad \text{for all } I \in \mathcal{M}_G. \end{aligned}$$

Let us denote the linear programming relaxation of (IP(G)) by (LP(G)), which looks as follows:

$$\begin{aligned} (\text{LP}(G)): \quad & \text{minimize} && \sum_{I \in \mathcal{M}_G} y_I \\ & \text{subject to} && \sum_{\substack{I \in \mathcal{M}_G \\ \text{s.t. } v \in I}} y_I \geq 1 \quad \text{for all } v \in V, \\ & && y_I \geq 0 \quad \text{for all } I \in \mathcal{M}_G. \end{aligned}$$

Further, denote the dual of (LP(G)) by (DLP(G)), namely, the problem finding $\mathbf{x} \in \mathbb{R}^V$ to

$$\begin{aligned} (\text{DLP}(G)): \quad & \text{maximize} && \sum_{v \in V} x_v \\ & \text{subject to} && \sum_{v \in I} x_v \leq 1 \quad \text{for all } I \in \mathcal{M}_G, \\ & && x_v \geq 0 \quad \text{for all } v \in V. \end{aligned}$$

Deng, Ibaraki & Nagamochi [4] proved the following.

Lemma 3 ([4]). *Let $G = (V, E)$ be a graph and (V, χ_G) be the minimum coloring game on G . Then, the core of (V, χ_G) is nonempty if and only if there exists an integral optimal solution to (LP(G)). In case it is nonempty, the core is exactly the set of all optimal solutions of (DLP(G)).*

The second fact which we use is due to Lovász [21], Fulkerson [11] and Chvátal [2].

Lemma 4 ([2, 11, 21]). *Let $Q(G)$ be the convex hull of the characteristic vectors of the cliques of G . Then G is perfect if and only if $Q(G) = \{\mathbf{x} \in \mathbb{R}^V : \mathbf{x}(I) \leq 1 \text{ for all independent sets } I \text{ of } G, \mathbf{x} \geq \mathbf{0}\}$.*

Lemma 4 is well known in the field of combinatorial optimization and polyhedral combinatorics. You can find a proof of Lemma 4 in several books (like Grötschel, Lovász & Schrijver [15]).

Now it is time to prove Theorem 2.

Proof (of Theorem 2). Since it is known that the minimum coloring game on a perfect graph G is totally balanced [5], the core is the set of the optimal solutions of (DLP(G)) by Lemma 3.

Let us denote by $\text{Feasible}(\text{DLP}(G))$ the set of feasible solutions of $(\text{DLP}(G))$. We claim that $\text{Feasible}(\text{DLP}(G)) = Q(G)$ if G is a perfect graph. First, from Lemma 4, we can see that the set of constraints for $Q(G)$ includes those for $\text{Feasible}(\text{DLP}(G))$. Therefore, it holds that $\text{Feasible}(\text{DLP}(G)) \supseteq Q(G)$. To show the reverse inclusion, choose an arbitrary vector $\tilde{\mathbf{x}} \in \text{Feasible}(\text{DLP}(G))$. Namely, $\tilde{\mathbf{x}}$ satisfies that $\tilde{\mathbf{x}}(I) \leq 1$ for all maximal independent sets I of G and $\tilde{\mathbf{x}} \geq \mathbf{0}$. Now we have to show that $\tilde{\mathbf{x}}(I) \leq 1$ for all (not necessarily maximal) independent sets I of G . If I is maximal, we are done. If it is not, there exists a maximal independent set I' of G such that $I \subseteq I'$. Since $\tilde{\mathbf{x}} \geq \mathbf{0}$, it holds that $\tilde{\mathbf{x}}(I) \leq \tilde{\mathbf{x}}(I') \leq 1$. Hence we have that $\text{Feasible}(\text{DLP}(G)) \subseteq Q(G)$. This concludes the proof of the claim.

Let $G = (V, E)$ be a perfect graph. From the claim above, the problem $(\text{DLP}(G))$ is seen to be equivalent to the following one:

$$\begin{aligned} (\text{DLP}'(G)): \quad & \text{maximize} \quad \sum_{v \in V} x_v \\ & \text{subject to} \quad \mathbf{x} \in Q(G). \end{aligned}$$

By Lemma 3, the core is the set of optimal solutions of $(\text{DLP}'(G))$.

Consider the following $\{0, 1\}$ -version of $(\text{DLP}'(G))$:

$$\begin{aligned} (\text{DIP}'(G)): \quad & \text{maximize} \quad \sum_{v \in V} x_v \\ & \text{subject to} \quad \mathbf{x} \in Q(G), \\ & \quad \mathbf{x} \in \{0, 1\}^V. \end{aligned}$$

Then, we can find that the feasible solutions to $(\text{DIP}'(G))$ exactly correspond to the cliques of G by the definition of $Q(G)$. Namely, the feasible solutions to $(\text{DIP}'(G))$ are the characteristic vectors of the cliques of G . Since the objective function of $(\text{DIP}'(G))$ is to maximize the sum of the components of the characteristic vector of a clique in G , namely the size of a clique, the optimal solutions of $(\text{DIP}'(G))$ are the characteristic vectors of the maximum cliques of G .

Now let us go back to $(\text{DLP}'(G))$. By the definitions of $(\text{DLP}'(G))$ and $(\text{DIP}'(G))$ and the claim above, we can see that the set of optimal solutions to $(\text{DLP}'(G))$ is the convex hull of the set of optimal solutions to $(\text{DIP}'(G))$. From the observation above, this is equivalent to saying that the set of optimal solutions of $(\text{DLP}'(G))$ is the convex hull of the characteristic vectors of the maximum cliques of G . Hence, \mathbf{x} is an optimal solution of $(\text{DLP}'(G))$ if and only if \mathbf{x} can be written as a convex combination of the characteristic vectors of the maximum cliques of G . \square

Making use of a result by Grötschel, Lovász & Schrijver [15] with Theorem 2, we conclude the following algorithmic consequences.

Corollary 5. *The following problems can be solved in polynomial time. (1) The problem to compute a core allocation of the minimum coloring game on a perfect graph. (2) The problem to decide whether a given vector belongs to the core of the minimum coloring game on a perfect graph.*

Proof. The consequence (1) comes from Theorem 2 and the fact that a maximum clique can be computed in polynomial time for a perfect graph. The consequence (2) comes from Theorem 2 and the fact that we can solve the membership problem for the polytope $Q(G)$, when G is perfect, in polynomial time. Both facts are explained in Grötschel, Lovász & Schrijver [15]. \square

3.3. The τ -value of a minimum coloring game

As we observed before, the τ -value can be computed in polynomial time for a submodular game by a polynomial-time algorithm for the submodular function minimization. However, for minimum coloring games, this only works for complete multipartite graphs because of the characterization due to Okamoto [28]. In this subsection, we describe how to compute the τ -values in polynomial time for perfect graphs. Remark that the τ -value for a perfect graph is well-defined since the minimum coloring game on a perfect graph is totally balanced [5]. The next theorem is crucial.

Theorem 6. Let $G = (V, E)$ be a perfect graph. Consider the minimum coloring game (V, χ_G) on G . If we denote all the maximum cliques of G by K_1, \dots, K_k and $K = K_1 \cap \dots \cap K_k$, then we have that, for each $v \in V$,

$$\underline{m}_v = \begin{cases} 1 & (v \in K) \\ 0 & (\text{otherwise}); \end{cases}$$

$$\overline{m}_v = \begin{cases} 1 & (v \in K \text{ or } K \subseteq N_G(v)) \\ 0 & (\text{otherwise}). \end{cases}$$

Proof. First we prove the formula for \underline{m} . Assume that $v \in K$. Then $\omega(G[V \setminus \{v\}]) = \omega(G) - 1$. Therefore, we have $\underline{m}_v = \chi_G(V) - \chi_G(V \setminus \{v\}) = \omega(G) - \omega(G[V \setminus \{v\}]) = 1$ by the perfectness of G . On the other hand, assume that there exists an index $i \in \{1, \dots, k\}$ such that $v \notin K_i$. Then $G[V \setminus \{v\}]$ still has K_i as its maximum clique. Therefore, we have $\underline{m}_v = \chi_G(V) - \chi_G(V \setminus \{v\}) = \omega(G) - \omega(G[V \setminus \{v\}]) = 0$ by the perfectness of G .

Next we prove the formula for \overline{m} . As a first observation, we have that

$$\begin{aligned} \overline{m}_v &= \min\{\chi_G(S) - \underline{m}(S \setminus \{v\}) : v \in S \subseteq V\} \\ &\leq \chi_G(\{v\}) - \underline{m}(\{v\} \setminus \{v\}) \\ &= \chi(G[\{v\}]) = 1. \end{aligned}$$

Here, from the first line to the second line, we have chosen $\{v\}$ as S . Now assume that $v \in K$. From the proof of the first half, we have $\underline{m}_v = 1$. So it holds that $1 = \underline{m}_v \leq \overline{m}_v \leq 1$ since the minimum coloring game on a perfect graph is balanced (implying that $\underline{m}_v \leq \overline{m}_v$). It concludes that $\overline{m}_v = 1$ if $v \in K$. Next assume that there exists an index $i \in \{1, \dots, k\}$ such that $v \notin K_i$. From the proof of the first half, we have $\underline{m}_v = 0$. If there exists a vertex $u \in K$ which is not a neighbor of v , then we have

$$\begin{aligned} \overline{m}_v &= \min\{\chi_G(S) - \underline{m}(S \setminus \{v\}) : v \in S \subseteq V\} \\ &\leq \chi_G(\{u, v\}) - \underline{m}(\{u, v\} \setminus \{v\}) \\ &= \chi(G[\{u, v\}]) - \underline{m}_u \\ &= 1 - 1 = 0. \end{aligned}$$

Here, from the first line to the second line, we have chosen $\{u, v\}$ as S , and from the third line to the fourth line, we used $\underline{m}_u = 1$, which follows since $u \in K$ and the first half of this theorem. Consider the case in which all the vertices in K are neighbors of v (i.e., $K \subseteq N_G(v)$). Suppose that $\overline{m}_v = 0$. This means that there exists $S \subseteq V$ such that $v \in S$ and $\chi_G(S) - \underline{m}(S \setminus \{v\}) = 0$ by the definition of \overline{m} . From the perfectness of G , we have

$$\begin{aligned} 0 &= \chi_G(S) - \underline{m}(S \setminus \{v\}) \\ &= \omega(G[S]) - (\underline{m}((S \setminus \{v\}) \cap K) + \underline{m}((S \setminus \{v\}) \setminus K)) \\ &= \omega(G[S]) - (|(S \setminus \{v\}) \cap K| + 0) \\ &= \omega(G[S]) - |S \cap K|. \end{aligned}$$

(Here, for the third equality we have used the formula for \underline{m} which we have already proved. For the fourth equality we have used the assumption that $v \notin K$.) Thus we can see that $\omega(G[S]) = |S \cap K|$. Since $S \cap K$ is a clique in G , so is in $G[S]$ as well. This means that $S \cap K$ is a maximum clique of $G[S]$. However, $S \cap (K \cup \{v\})$ is also a clique of $G[S]$ because $K \subseteq N_G(v)$. Since $v \notin S \cap K$ we can also see that $\omega(G[S]) = |S \cap K| < |S \cap (K \cup \{v\})|$. This is a contradiction to the maximality of $S \cap K$. Hence we have $\overline{m}_v = 1$ since $\overline{m}_v \leq 1$ and \overline{m} is integral. \square

Based on Theorem 6, we are able to establish an algorithm to compute the τ -value of the minimum coloring game on a perfect graph. First we compute \underline{m} . To do that, we just compute $\chi_G(V)$ and $\chi_G(V \setminus \{v\})$ and conclude that $\underline{m}_v = \chi_G(V) - \chi_G(V \setminus \{v\})$ for all $v \in V$. Since the chromatic number of a perfect graph can be computed in polynomial time [15], \underline{m} can also be computed in polynomial time. From \underline{m} ,

we can see what K is, by using Theorem 6, in linear time. Namely, $K = \{v \in V : m_v = 1\}$. After we know K , we can immediately compute \overline{m} again with help of Theorem 6. Finally, we can also compute the appropriate λ in a straightforward manner because we have already known $\chi_G(V)$. Namely,

$$\lambda = \frac{\chi_G(V) - \overline{m}(V)}{\underline{m}(V) - \overline{m}(V)}.$$

Thus we have obtained the τ -value and this algorithm runs in polynomial time. The dominant step for the running time is the computations of $\chi_G(V)$ and $\chi_G(V \setminus \{v\})$ for all $v \in V$. This gives the running time $O(nT_\chi(n))$, where $T_\chi(n)$ is the time complexity to compute the chromatic number of a perfect graph with n vertices.

Note that Theorem 6 enables us to compute the τ -value for a complete multipartite graph without using an algorithm for the submodular function minimization.

3.4. The nucleolus of a minimum coloring game

Now we investigate the nucleolus. Unfortunately, we are not aware of an efficient algorithm for general perfect graphs. So we restrict to a subclass of the perfect graphs. To do that, we introduce some terms.

Let $G = (V, E)$ be a perfect graph, \mathcal{I}_G be the family of nonempty independent sets of G , and K_1, K_2, \dots, K_k be the maximum cliques of G . For every nonempty independent set $I \in \mathcal{I}_G$, define

$$g(I) := |\{i \in \{1, \dots, k\} : |I \cap K_i| = 1\}|.$$

Then consider the set $\Gamma := \{g(I) : I \in \mathcal{I}_G\}$. Note that $|\Gamma| < |\mathcal{I}_G|$ in case some independent sets have identical $g(I)$'s. Let us enumerate the elements of Γ in decreasing order as: $g_1 > g_2 > \dots > g_{|\Gamma|}$. The following lemma tells us $g_1 = k$.

Lemma 7. *For a graph G with $\chi(G) = \omega(G)$, there exists an independent set of G which has a nonempty intersection with each of the maximum cliques of G .*

Proof. Let $G = (V, E)$ be a graph with $\chi(G) = \omega(G)$, and consider a minimum coloring of G . Then, take the vertices colored by an identical color. Denote by I the set of these vertices. Clearly, I is an independent set. Moreover, in each maximum clique K of G all colors used to color G can be found since $\chi(G) = \omega(G) = |K|$. Namely, K intersects I . \square

On the other hand, we have less control for g_2 in general. This motivates the definition of O -goodness. A perfect graph G is called *O -good* if either $|\Gamma| = 1$ or G satisfies the following condition: for each $j \in \{1, \dots, k\}$ there exists an independent set $I^{(j)} \in \mathcal{I}_G$ such that $|I^{(j)} \cap K_j| = 0$ and $|\{i \in \{1, \dots, k\} \setminus \{j\} : |I^{(j)} \cap K_i| = 1\}| = g_2$.

Our theorem on the nucleolus is as follows.

Theorem 8. *The nucleolus of the minimum coloring game on an O -good perfect graph is the barycenter of the characteristic vectors of the maximum cliques of the graph.*

To prove Theorem 8, first we use a fact that we can restrict to the so-called essential coalitions. Let (N, γ) be a cost game. Then a nonempty set $S \subseteq N$ is called an *essential coalition* if, for every proper partition \mathcal{P} of S , we have $\gamma(S) < \sum_{P \in \mathcal{P}} \gamma(P)$. Huberman [17] showed that when we compute the nucleolus through a sequence of the linear programming problems (P_1) , (P_2) and so on, we only have to take the essential coalitions into account rather than handling all nonempty proper subsets of N . Namely, instead of considering \mathcal{C}_0 in (P_1) , we consider the family of essential coalitions, and keep the same track as the original case through the computation. The next lemma characterizes the essential coalitions of a minimum coloring game.

Lemma 9. *Let $G = (V, E)$ be a graph (not necessarily perfect). Then $S \subseteq V$ is an essential coalition of the minimum coloring game (V, χ_G) on G if and only if S is a nonempty independent set of G .*

Proof. We show the if-part first. Assume that S is a nonempty independent set of G . Then every nonempty proper subset P of S satisfies that $\chi_G(P) = 1$ since P is also an independent set. Therefore, for every proper partition \mathcal{P} of S we have that $\chi_G(S) = 1 < 2 \leq |\mathcal{P}| = \sum_{P \in \mathcal{P}} \chi_G(P)$. Hence, S is an essential coalition.

Next we show the converse. Assume that S is not an independent set of G . If we put $\ell = \chi_G(S)$, then we have $\ell \geq 2$ since S is not independent. Fix a minimum coloring of $G[S]$. Then the color classes of this minimum coloring yield a proper partition of S since $k \geq 2$. Let us denote this proper partition by \mathcal{P} . Since $\chi_G(P) = 1$ for each $P \in \mathcal{P}$, it follows that $\chi_G(S) = \ell = |\mathcal{P}| = \sum_{P \in \mathcal{P}} \chi_G(P)$. Thus we can see that S is not an essential coalition. \square

With help of Lemma 9, our problem (P_1) can be written as follows.

$$(P_1): \quad \begin{aligned} &\text{maximize} && \epsilon \\ &\text{subject to} && \mathbf{x}(V) = \chi_G(V) \\ &&& \mathbf{x}(I) \leq 1 - \epsilon \quad \text{for all } I \in \mathcal{I}_0, \end{aligned}$$

where $\mathcal{I}_0 = \mathcal{I}_G$ is the family of all nonempty independent sets of G . Now we consider the optimal solutions of (P_1) .

Lemma 10. *Let G be a perfect graph and consider the minimum coloring game (V, χ_G) on G . Then a feasible solution (\mathbf{x}, ϵ) to the problem (P_1) is optimal if and only if \mathbf{x} belongs to the core of (V, χ_G) and $\epsilon = 0$.*

Proof. First, notice that for any feasible solution (\mathbf{x}, ϵ) to (P_1) , \mathbf{x} belongs to the core if $\epsilon \geq 0$. This is a direct consequence from Section 3.2. Therefore, if \mathbf{x} belongs to the core and $\epsilon = 0$, then (\mathbf{x}, ϵ) is a feasible solution to (P_1) . Now suppose there exists a feasible solution (\mathbf{x}, ϵ) to (P_1) such that $\epsilon > 0$. From the observation above, \mathbf{x} should belong to the core. By Theorem 2, \mathbf{x} can be written as a convex combination of the characteristic vectors of the maximum cliques K_1, \dots, K_k of G since G is perfect. Let

$$\mathbf{x} = \sum_{i=1}^k \lambda_i \mathbb{1}_{K_i},$$

where $\lambda_1 + \dots + \lambda_k = 1$ and $\lambda_i \geq 0$ for all $i \in \{1, \dots, k\}$. Now, we use Lemma 7, which states that there exists an independent set I which has a nonempty intersection with every maximum clique. Namely, we have $|K_i \cap I| = 1$ for all $i = 1, \dots, k$. Then it follows that

$$\mathbf{x}(I) = \sum_{i=1}^k \lambda_i \mathbb{1}_{K_i}(I) = \sum_{i=1}^k \lambda_i |K_i \cap I| = \sum_{i=1}^k \lambda_i = 1 > 1 - \epsilon.$$

This means that (\mathbf{x}, ϵ) violates the constraint $\mathbf{x}(I) \leq 1 - \epsilon$, which contradicts the feasibility of (\mathbf{x}, ϵ) . Thus we conclude that the optimal value of (P_1) is zero. \square

Remark that so far we did not use O-goodness, but just the perfectness of G .

If a perfect graph G has only one maximum clique, then we can see that (P_1) has a unique optimal solution from Lemma 10 and Theorem 2. In such a case, the characteristic vector of this unique maximum clique is the nucleolus. Thus Theorem 8 holds for this case. Now consider the case in which a perfect graph G has at least two maximum cliques. Then we should consider the second problem (P_2) . To solve (P_2) , we must know what is the set $\mathcal{I}_1 := \{I \in \mathcal{I}_0 : \mathbf{x}(I) = 1 \text{ for all optimal solutions } (\mathbf{x}, 0) \text{ of } (P_1)\}$, which is the analogue of \mathcal{C}_1 in the definition of the nucleolus. The next lemma characterizes \mathcal{I}_1 .

Lemma 11. *For the minimum coloring game on a perfect graph G , we have*

$$\mathcal{I}_1 = \{I \in \mathcal{I}_0 : |K_i \cap I| = 1 \text{ for all } i \in \{1, \dots, k\}\},$$

where we denote all the maximum cliques of G by K_1, \dots, K_k .

Proof. From the definition of \mathcal{I}_1 and Lemma 10, we have $\mathcal{I}_1 = \{I \in \mathcal{I}_0 : \mathbf{x}(I) = 1 \text{ for all core allocations } \mathbf{x}\}$.

Let $I \in \mathcal{I}_0$. First assume that I satisfies that $\mathbf{x}(I) = 1$ for all core allocations \mathbf{x} . Let K_i be an arbitrary maximum clique of G . Since $\mathbb{1}_{K_i}$ is a core allocation by Theorem 2, we have that $|K_i \cap I| = \mathbb{1}_{K_i}(I) = 1$. Thus, I satisfies that $|K_i \cap I| = 1$ for all $i \in \{1, \dots, k\}$.

Conversely, assume that I satisfies that $|K_i \cap I| = 1$ for all $i \in \{1, \dots, k\}$. Take an arbitrary core allocation \mathbf{x} . By Theorem 2, \mathbf{x} can be written as a convex combination of the characteristic vectors of the maximum cliques of G . Let

$$\mathbf{x} = \sum_{i=1}^k \lambda_i \mathbb{1}_{K_i},$$

where $\lambda_1 + \dots + \lambda_k = 1$ and $\lambda_i \geq 0$ for all $i = 1, \dots, k$. Then it holds that

$$\mathbf{x}(I) = \sum_{i=1}^k \lambda_i \mathbb{1}_{K_i}(I) = \sum_{i=1}^k \lambda_i |K_i \cap I| = \sum_{i=1}^k \lambda_i = 1.$$

(Here, the third identity comes from our assumption that $|K_i \cap I| = 1$ for all $i \in \{1, \dots, k\}$.) Thus, we can see that I satisfies that $\mathbf{x}(I) = 1$ for all core allocations \mathbf{x} . To conclude, we have shown that $\mathcal{I}_1 = \{I \in \mathcal{I}_0 : |K_i \cap I| = 1 \text{ for all } i \in \{1, \dots, k\}\}$. \square

Remark that Lemma 7 ensures the nonemptiness of \mathcal{I}_1 .

Now we are ready to solve (P₂). To make it clearer to us, we write down (P₂) here.

$$\begin{aligned} \text{(P}_2\text{):} \quad & \text{maximize} \quad \epsilon \\ & \text{subject to} \quad \mathbf{x}(V) = \chi_G(V) \\ & \quad \mathbf{x}(I) = 1 \quad \text{for all } I \in \mathcal{I}_1 \\ & \quad \mathbf{x}(I) \leq 1 - \epsilon \quad \text{for all } I \in \mathcal{I}_0 \setminus \mathcal{I}_1. \end{aligned}$$

The following lemma tells us, surprisingly, (P₂) already gives the nucleolus for an O-good perfect graph. (That is why we call the graph “good!”)

Lemma 12. *For the minimum coloring game on an O-good perfect graph G with at least two maximum cliques, the problem (P₂) has a unique optimal solution $(\boldsymbol{\nu}, (k - g_2)/k)$, where $\boldsymbol{\nu}$ is the barycenter of the characteristic vectors of the maximum cliques of G .*

Note that Lemma 12 completes the proof of Theorem 8.

Proof. First, remark that we can write $\boldsymbol{\nu}$ as $\boldsymbol{\nu} = \sum_{i=1}^k \mathbb{1}_{K_i}/k$. Now we show the following claim.

Claim. $(\boldsymbol{\nu}, (k - g_2)/k)$ is feasible to (P₂).

Proof (of Claim). The constraint “ $\mathbf{x}(V) = \chi_G(V)$ ” can be verified as

$$\boldsymbol{\nu}(V) = \sum_{i=1}^k \frac{1}{k} \mathbb{1}_{K_i}(V) = \sum_{i=1}^k \frac{1}{k} |K_i| = \sum_{i=1}^k \frac{1}{k} \chi_G(V) = \chi_G(V),$$

since the definition of a perfect graph gives us $\chi_G(V) = |K_i|$.

The constraint “ $\mathbf{x}(I) = 1$ for all $I \in \mathcal{I}_1$ ” can be verified as follows.

$$\boldsymbol{\nu}(I) = \sum_{i=1}^k \frac{1}{k} \mathbb{1}_{K_i}(I) = \sum_{i=1}^k \frac{1}{k} |K_i \cap I| = \sum_{i=1}^k \frac{1}{k} = 1.$$

Here, we used Lemma 11. For the constraint “ $\mathbf{x}(I) \leq 1 - \epsilon$ for all $I \in \mathcal{I}_0 \setminus \mathcal{I}_1$,” we again use Lemma 11. By Lemma 11 and the definition of g_2 , for $I \in \mathcal{I}_0 \setminus \mathcal{I}_1$, we have $|\{i : |K_i \cap I| = 1\}| \leq g_2$. Therefore, it holds that

$$\boldsymbol{\nu}(I) = \sum_{i=1}^k \frac{1}{k} \mathbb{1}_{K_i}(I) = \sum_{i=1}^k \frac{1}{k} |K_i \cap I| \leq \frac{g_2}{k} = 1 - \frac{k - g_2}{k}.$$

This concludes the proof. \square

Let us get back to the proof of Lemma 12. The next step is to perturb $(\boldsymbol{\nu}, (k - g_2)/k)$ a little bit to get another vector and we show that the perturbed vector should have a smaller objective value than $(k - g_2)/k$ as long as it is feasible. To do that, first let us fix an index $j \in \{1, \dots, k\}$ arbitrarily and consider a vector $\mathbf{x}_\delta^{(j)}$ defined as

$$\mathbf{x}_\delta^{(j)} := \sum_{i \neq j} \left(\frac{1}{k} + \delta \right) \mathbb{1}_{K_i} + \left(\frac{1}{k} - (k-1)\delta \right) \mathbb{1}_{K_j},$$

where δ is a very small positive real number. Observe that $\mathbf{x}_\delta^{(j)}$ is a convex combination of the characteristic vectors of the maximum cliques of G since δ is very small. Therefore, $\mathbf{x}_\delta^{(j)}$ is a core allocation (by Theorem 2). Now suppose that $(\mathbf{x}_\delta^{(j)}, \epsilon)$ is a feasible solution to (P_2) for some $\epsilon \geq (k - g_2)/k$.

It is time to use the O-goodness of G . By the O-goodness of G , there exists an independent set $I^{(j)}$ of G such that $|I^{(j)} \cap K_j| = 0$ and $|\{i \in \{1, \dots, k\} \setminus \{j\} : |I^{(j)} \cap K_i| = 1\}| = g_2$. From Lemma 11 and the definition of g_2 , we can see that $I^{(j)} \notin \mathcal{I}_1$. So if $(\mathbf{x}_\delta^{(j)}, \epsilon)$ is feasible to (P_2) , we should have that

$$\mathbf{x}_\delta^{(j)}(I^{(j)}) \leq 1 - \epsilon \leq 1 - \frac{k - g_2}{k} = \frac{g_2}{k}.$$

On the other hand, we have that

$$\begin{aligned} \mathbf{x}_\delta^{(j)}(I^{(j)}) &= \sum_{i \neq j} \left(\frac{1}{k} + \delta \right) \mathbb{1}_{K_i}(I^{(j)}) + \left(\frac{1}{k} - (k-1)\delta \right) \mathbb{1}_{K_j}(I^{(j)}) \\ &= \sum_{i \neq j} \left(\frac{1}{k} + \delta \right) |K_i \cap I^{(j)}| + \left(\frac{1}{k} - (k-1)\delta \right) |K_j \cap I^{(j)}| \\ &= g_2 \left(\frac{1}{k} + \delta \right) > \frac{g_2}{k}. \end{aligned}$$

So they contradict each other. Hence, $(\mathbf{x}_\delta^{(j)}, \epsilon)$ cannot be feasible for any j if $\epsilon \geq (k - g_2)/k$. Then, from the convexity of the linear programming problem (P_2) , we can conclude that $(\boldsymbol{\nu}, 1/k)$ is a unique optimal solution of (P_2) . \square

Now we observe what kind of perfect graphs are O-good.

Proposition 13. *1. A perfect graph with only one maximum clique is O-good. 2. A forest is O-good. 3. A complete multipartite graph is O-good. 4. A chordal graph is O-good.*

Proof. 1. Immediately from the definition of O-goodness.

2. Since a forest is a chordal graph, the statement follows from 4. However, this case is much simpler than the case of chordal graphs. So we give such a simpler proof here. We prove that for a tree. The proof for a forest can be similarly done.

Let G be a tree. In a tree, the maximum cliques are the edges. Let us choose an edge of G , say e . Then, delete e from G which results in two components. They are trees. Look at one of them. Say, this component is T . We can see that T contains a vertex r which was an endpoint of e in G . Now we consider the distance between each vertex of T and r , and collect all vertices in T to which the distance from r is an odd number. These vertices form an independent set of T , and have a nonempty intersection with all edges of T . We do the same thing for the other component. Then gather these two sets of vertices. This gives an independent set of G which has a nonempty intersection with all edges except for e . (Thus, g_2 is always the number of edges minus one for the tree.)

3. Let G be a complete r -partite graph for some $r \geq 1$, and V_1, V_2, \dots, V_r be a partition of the vertex set of G such that each V_i is independent. (Such a partition is unique because of the complete r -partiteness of G .) Also let $n_i = |V_i|$ for every $i \in \{1, \dots, r\}$. The first observation is that every independent set is a subset of V_i for some $i \in \{1, \dots, r\}$ due to the complete r -partiteness. (Here, the choice of i depends on each independent set.) Assume that an independent set I is a subset of V_i . Then,

since every vertex of I is an element of $n_1 n_2 \cdots n_{i-1} n_{i+1} \cdots n_r$ maximum cliques of G , we have that $g(I) = |I| n_1 n_2 \cdots n_{i-1} n_{i+1} \cdots n_r$. Thus we can see that $g_2 = \max_i n_1 n_2 \cdots n_{i-1} (n_i - 1) n_{i+1} \cdots n_r$. Let i^* be a maximizer of this expression.

Now consider a maximum clique K of G . Let v be a unique vertex lying on the intersection $V_{i^*} \cap K$. Now we choose an independent set $V_{i^*} \setminus \{v\}$. This is what we have wanted to obtain.

4. We prove the following stronger statement.

Proposition 14. *Let G be a chordal graph and K_1, K_2, \dots, K_k be the maximum cliques of G ($k \geq 1$). If $k \geq 2$, then for each $i \in \{1, \dots, k\}$ there exists an independent set $I^{(i)}$ of G such that $|I^{(i)} \cap K_i| = 0$ and $|\{j \in \{1, \dots, k\} \setminus \{i\} : |I^{(i)} \cap K_j| = 1\}| = k - 1$.*

We prove Proposition 14 by the induction on the number n of vertices of a chordal graph G . If $n = 1$, then $k = 1$. Therefore the statement is true.

Consider the case in which $n > 1$, and let us assume that the statement is true for all $n' < n$. Now we invoke a property of chordal graphs due to Dirac [6]. Remember that a vertex of a graph is called *simplicial* if the neighborhood of the vertex forms a clique.

Lemma 15 ([6]). *A chordal graph has a simplicial vertex.*

Let $G = (V, E)$ be a chordal graph with n vertices such that $k \geq 2$. By Lemma 15 above, G has a simplicial vertex v . Then $T := N_G(v) \cup \{v\}$ is a maximal clique of G (by the definitions of a simplicial vertex and a neighbor). Now we consider the following two cases.

Case 1: T is not a maximum clique of G . Then the maximum cliques of $G[V \setminus \{v\}]$ are exactly those of G . So, by applying the induction hypothesis to $G[V \setminus \{v\}]$, we obtain for each $i \in \{1, \dots, k\}$ there exists an independent set $I^{(i)}$ of $G[V \setminus \{v\}]$ such that $|I^{(i)} \cap K_i| = 0$ and $|\{j \in \{1, \dots, k\} \setminus \{i\} : |I^{(i)} \cap K_j| = 1\}| = k - 1$. Since $I^{(i)}$ is also an independent set of G , this is exactly what we have wanted.

Case 2: T is a maximum clique of G . Without loss of generality, let $T = K_k$. Then the maximum cliques of $G[V \setminus \{v\}]$ are K_1, K_2, \dots, K_{k-1} . Now we have two subcases.

Case 2-1: $k = 2$. Namely, G has two maximum cliques K_1 and $K_2 = T$. Then there exist vertices $v_1 \in K_1 \setminus K_2$ and $v_2 \in K_2 \setminus K_1$. We can see that $\{v_2\}$ plays a role of $I^{(1)}$ and $\{v_1\}$ plays a role of $I^{(2)}$.

Case 2-2: $k > 2$. Then, by induction hypothesis, for each $i \in \{1, \dots, k-1\}$ there exists an independent set $J^{(i)}$ of $G[V \setminus \{v\}]$ such that $|J^{(i)} \cap K_i| = 0$ and $|\{j \in \{1, \dots, k-1\} \setminus \{i\} : |J^{(i)} \cap K_j| = 1\}| = k-2$. Now we turn $J^{(i)}$ into $I^{(i)}$ which we have wanted to obtain. In case $J^{(i)} \cap N_G(v) = \emptyset$, set $I^{(i)} := J^{(i)} \cup \{v\}$, which is an independent sets of G . Then, we have that $|I^{(i)} \cap K_i| = |(J^{(i)} \cup \{v\}) \cap K_i| = 0$ and

$$\begin{aligned} & |\{j \in \{1, \dots, k\} \setminus \{i\} : |I^{(i)} \cap K_j| = 1\}| \\ &= |\{j \in \{1, \dots, k-1\} \setminus \{i\} : |I^{(j)} \cap K_j| = 1\}| + 1 \\ &= |\{j \in \{1, \dots, k-1\} \setminus \{i\} : |(J^{(j)} \cup \{v\}) \cap K_j| = 1\}| + 1 \\ &= |\{j \in \{1, \dots, k-1\} \setminus \{i\} : |J^{(j)} \cap K_j| = 1\}| + 1 \\ &= (k-2) + 1 = k-1 \end{aligned}$$

because $I^{(i)} \cap K_k = (J^{(i)} \cup \{v\}) \cap (N_G(v) \cup \{v\}) = \{v\}$. In case $J^{(i)} \cap N_G(v) \neq \emptyset$, set $I^{(i)} := J^{(i)}$. Then we have that $|I^{(i)} \cap K_i| = |J^{(i)} \cap K_i| = 0$ and

$$\begin{aligned} & |\{j \in \{1, \dots, k\} \setminus \{i\} : |I^{(i)} \cap K_j| = 1\}| \\ &= |\{j \in \{1, \dots, k-1\} \setminus \{i\} : |I^{(j)} \cap K_j| = 1\}| + 1 \\ &= |\{j \in \{1, \dots, k-1\} \setminus \{i\} : |J^{(j)} \cap K_j| = 1\}| + 1 \\ &= (k-2) + 1 = k-1 \end{aligned}$$

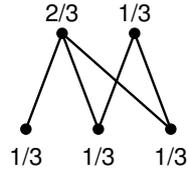


Fig. 1. A bipartite graph which is not O-good.

because it follows that $I^{(i)} \cap K_k = J^{(i)} \cap (N_G(v) \cup \{v\}) = J^{(i)} \cap N_G(v) \neq \emptyset$.

Finally, we should find an independent set $I^{(k)}$ of G such that $|I^{(k)} \cap T| = 0$ and $|\{j \in \{1, \dots, k-1\} : |I^{(i)} \cap K_j| = 1\}| = k-1$. Remembering the proof of Lemma 7, we can find an independent set I of G such that $v \in I$ and I has a nonempty intersection with every maximum clique K_ℓ ($\ell = 1, \dots, k$). Now set $I^{(k)} := I \setminus \{v\}$. Then we can check that $I^{(k)}$ satisfies the conditions above. This concludes the proof of Proposition 14. \square

Note that there exists a bipartite graph which is not O-good and for which the nucleolus is not given as in Theorem 8. Consider the bipartite graph in Fig. 1. The associated numbers express the components of the nucleolus of the minimum coloring on this graph. We can see that this is not the barycenter of the characteristic vectors of the maximum cliques. (Note that, since it is bipartite, the maximum cliques are the edges.) We can also see that $g_2 = 4$. However, if we look at the rightmost edge e , then every independent set avoiding e can intersect at most 3 other edges. Therefore, this graph is not O-good.

Now we describe how to compute the nucleolus for classes of O-good perfect graphs in Proposition 13. In a forest $G = (V, E)$, the maximum cliques are the edges of G . Let $d_G(v)$ be the degree of $v \in V$ in G . Since a forest is O-good (Proposition 13.2), Theorem 8 concludes that the v -th component of the nucleolus for the forest G is $d_G(v)/|E|$. So the nucleolus for a forest can be easily computed.

Let $G = (V, E)$ be a complete r -partite graph in which V is partitioned into V_1, \dots, V_r and $E = \{\{u, v\} : u \in V_i, v \in V_j, i, j \in \{1, \dots, r\}, i \neq j\}$. Then a maximum clique K is a vertex subset which satisfies that $|V_i \cap K| = 1$ for all $i \in \{1, \dots, r\}$. Since a complete multipartite graph is O-good (Proposition 13.3), with help of Theorem 8, the v -th component of the nucleolus for the complete r -partite graph G can be computed as $1/|V_i|$ if $v \in V_i$. In a past work, the author investigated the Shapley value for a complete multipartite graph [28]. Since this is exactly the same as the expression above, we immediately obtain the following corollary.

Corollary 16. *The Shapley value and the nucleolus coincide for the minimum coloring game on a complete multipartite graph.*

Note that this is not true for a general submodular game.

For a chordal graph, we are not aware of a closed formula for the nucleolus. However, still we can compute the nucleolus in polynomial time. The strategy is to enumerate all the maximum cliques. A polynomial-time algorithm to enumerate all the maximal cliques of a chordal graph is known (and described in Golumbic's book [14]), which is based on a proposition by Fulkerson & Gross [12]. (Note that the number of the maximal cliques of a chordal graph is bounded by the number of the vertices from above [12].) Therefore, from this enumeration, we are able to obtain the list of all maximum cliques of the chordal graph. Then, since a chordal graph is O-good (Proposition 13.4), we can compute the nucleolus from this list by utilizing Theorem 8.

3.5. The Shapley value of a minimum coloring game

In the former work [28], the author provided the formula of the Shapley value for a complete multipartite graph which can be easily evaluated (see also Corollary 16 above). Here we give a polynomial-time algorithm to compute the Shapley value of a minimum coloring game on a forest. Note that a forest is bipartite. On the Shapley value for a bipartite graph, the following lemma holds.

Lemma 17. For a bipartite graph $G = (V, E)$, the v -th component of the Shapley value ϕ of the minimum coloring game on G can be written as

$$\phi_v = \frac{1}{n} + \sum_{k=1}^{n-1} a_v^k(G) \frac{k!(n-k-1)!}{n!},$$

where $n = |V|$ and $a_v^k(G)$ is the number of independent sets I of G such that $v \notin I$, $|I| = k$ and $I \cup \{v\}$ is not independent.

Proof. Recall the definition of the Shapley value ϕ of the minimum coloring game (V, χ_G) :

$$\phi_v = \sum_{S \subseteq V \setminus \{v\}} \frac{|S|!(|V| - |S| - 1)!}{|V|!} (\chi_G(S \cup \{v\}) - \chi_G(S)).$$

Since G is a bipartite graph, we have that

$$\chi_G(S) = \begin{cases} 0 & (S = \emptyset) \\ 1 & (S \text{ is nonempty and independent}) \\ 2 & (\text{otherwise}) \end{cases}$$

for each $S \subseteq V$. Therefore, it follows that

$$\chi_G(S \cup \{v\}) - \chi_G(S) = \begin{cases} 1 & \left(\begin{array}{l} S = \emptyset \text{ or} \\ S \text{ is independent} \\ \text{but } S \cup \{v\} \text{ is not} \end{array} \right) \\ 0 & (\text{otherwise}) \end{cases}$$

for each $S \subseteq V \setminus \{v\}$.

So now, we manipulate the v -th component of the Shapley value as follows:

$$\begin{aligned} \phi_v &= \sum_{S \subseteq V \setminus \{v\}} \frac{|S|!(|V| - |S| - 1)!}{|V|!} (\chi_G(S \cup \{v\}) - \chi_G(S)) \\ &= \frac{|\emptyset|!(|V| - |\emptyset| - 1)!}{|V|!} + \sum_{\substack{S \subseteq V \setminus \{v\}: S \text{ independent,} \\ S \cup \{v\} \text{ not independent}}} \frac{|S|!(|V| - |S| - 1)!}{|V|!} \\ &= \frac{1}{n} + \sum_{k=1}^{n-1} a_v^k(G) \frac{k!(n-k-1)!}{n!}. \end{aligned}$$

This concludes the proof. \square

Therefore, if we have a polynomial-time algorithm to compute $a_v^k(G)$ for every $k \in \{1, \dots, n-1\}$, we are able to obtain the Shapley value in polynomial time when G is bipartite. However we are not aware of such a procedure for a general bipartite graph. Actually, we have an easy hardness result again. (This is just a simple application of the fact that counting the number of independent sets in a bipartite graph is #P-complete [27], even if it is in addition planar and with maximum degree 4 [36].)

Proposition 18. For planar bipartite graphs G with n vertices and with maximum degree 4 and for $k \in \{1, \dots, n-1\}$, it is #P-complete to compute the value $a_v^k(G)$.

Proof. Clearly, the problem belongs to #P. So we only have to show the #P-hardness. For this purpose, we use a #P-complete problem to count the number of independent sets in a planar bipartite graph with maximum degree 4 [36]. Let $H = (V_H, E_H)$ be an arbitrarily given planar bipartite graph with $n-2$ vertices and with maximum degree 4.

Now we construct a bipartite graph $G = (V_G, E_G)$ as follows: $V_G = V_H \cup \{x, y\}$ and $E_G = E_H \cup \{\{x, y\}\}$. (Here, we assume that $\{x, y\} \cap V_H = \emptyset$.) Apparently, G is still planar and bipartite, and has maximum degree 4. We can see that, for each $k \in \{1, \dots, n-1\}$, $a_x^k(G)$ is the number of independent sets in G of size k which contains y and does not contain x , and this is exactly the same as the number of independent sets in H of size $k-1$ by the construction. Therefore, the number of independent sets in H is equal to $\sum_{k=1}^{n-1} a_x^k(G)$. This concludes the reduction. \square

Therefore, we consider a special case in which G is a forest. As a result, we give a polynomial-time algorithm to compute $a_v^k(G)$ for every $k \in \{1, \dots, n-1\}$ when G is a forest with n vertices.

To do that, let us first fix $k \in \{1, \dots, n-1\}$ and $v \in V(G)$, and concentrate on computing $a_v^k(G)$. Let $N_G(v) = \{v_1, v_2, \dots, v_d\}$. Every independent set I of G such that $v \notin I$ and $I \cup \{v\}$ is not independent must contain at least one of v_1, \dots, v_d . If I contains v_1 , then it cannot contain any vertices in $N_G(v_1)$, since it is independent. Namely, it must contain other $k-1$ vertices among $V(G) \setminus N_G[v_1]$. (Remember that $N_G[v_1]$ denotes the closed neighborhood of v_1 in G .) If I does not contain v_1 , then I is an independent set of size k in $G[V(G) \setminus \{v_1\}]$ which contains at least one of v_2, \dots, v_d . Therefore, this consideration gives the following formula:

$$a_v^k(G) = I^{k-1}(G[V(G) \setminus N_G[v_1]]) + a_v^k(G[V(G) \setminus \{v_1\}]),$$

where $I^k(G)$ represents the number of independent sets of size k in G . Now, applying the same trick to the second term of the right-hand side, we obtain the following lemma.

Lemma 19. *In the setting above, let G_i be the graph obtained from G by removing the vertices in $\{v_1, \dots, v_{i-1}\} \cup N_G[v_i]$. Then, it follows that*

$$a_v^k(G) = \sum_{i=1}^d I^{k-1}(G_i).$$

Note that, when G is a forest, G_i is also a forest for each $i \in \{1, \dots, d\}$. Therefore, to compute $a_v^k(G)$ in polynomial time, it is enough to calculate the number of independent sets of a given size for a forest in polynomial time.

To compute $I^k(G)$, we use the generating function method. Define the following polynomial

$$I(G; x) := \sum_{k=0}^n I^k(G)x^k,$$

where n is the number of vertices in G , and x is an indeterminant. We call $I(G; x)$ the *independent set polynomial* of G . (This concept seems first defined by Hoede & Li [16].) If we can compute $I(G; x)$ in polynomial time, then, from $I(G; x)$, we can obtain $I^k(G)$ for every $k \in \{0, \dots, n\}$. Therefore, we now concentrate on a polynomial-time computation of $I(G; x)$ when G is a forest.

For independent set polynomials, the following two facts are known.

Lemma 20 ([16]). *Let $G = (V, E)$ be a graph, which is not necessarily a forest.*

1. *If G is disconnected and G_1, \dots, G_ℓ are the connected components of G , then it holds that*

$$I(G; x) = \prod_{j=1}^{\ell} I(G_j; x).$$

2. *For each vertex $v \in V$, it holds that*

$$I(G; x) = I(G[V \setminus \{v\}]; x) + xI(G[V \setminus N_G[v]]; x).$$

Now, we concentrate on the computation of $I(G; x)$. From the first part of Lemma 20, it is enough to consider $I(G; x)$ when G is connected, i.e., G is a tree, as far as the polynomial-time computation is concerned. Therefore, we assume that G is a tree. Let us fix one vertex $r \in V(G)$ and regard it as a root of G . Then, we can naturally associate the parent-child relations among pairs of adjacent vertices. Denote by v_i the i -th child of r in G , and $v_{i,j}$ the j -th child of the i -th child of r in G , if exists. (So, $v_{i,j}$ is a grandchild of r .) Let $G^{[v]}$ be the subgraph induced by the set of descendants of v and v itself. Then, by the second part of Lemma 20, we have that

$$I(G; x) = \sum_i I(G^{[v_i]}; x) + x \sum_{i,j} I(G^{[v_{i,j}]}; x).$$

Due to this recurrence, we can establish the bottom-up dynamic programming algorithm. Namely, first we compute $I(G^{[v]}; x)$ for each vertex v farthest from r . Next, we compute it for the second farthest vertices, which only uses the outcome from the farthest vertices. In this way, we keep computing until getting to the root r . Thus, we obtain $I(G; x)$ in polynomial time, and it concludes the polynomial-time algorithm to compute the Shapley value of the minimum coloring game on a forest.

4. Conclusion

We have investigated some fair cost allocations (the core, the τ -value, the nucleolus, the Shapley value) under conflict situations by modeling them as minimum coloring games. Our investigation on the core and the nucleolus suggests that results from polyhedral combinatorics are really useful in the study of cooperative games arising from combinatorial optimization games. This was indeed true in the work by Deng, Ibaraki & Nagamochi [4]. Furthermore, our approach to the nucleolus gives a viewpoint different from the literature. In the literature, polynomial-time algorithms for nucleoli of special classes of games were based on properties of the classes such as the number of essential coalitions is small (e.g. [34]) or the nucleolus is a unique vector in the intersection of the core and the kernel (e.g. [9]). Furthermore, usually the core is treated as a system of linear inequalities. On the contrary, in our approach we first look at the core as the convex hull of its extreme points (Theorem 2) and represent the nucleolus as a convex combination of them (Theorem 8). This should also be useful for other cooperative games. In addition, our investigation on the Shapley value suggests the use of a recurrence and the dynamic programming technique. Indeed, it is known that for the power indices of a weighted majority game the dynamic programming is quite useful [23]. Since there seemed to exist no application of the dynamic programming to cooperative games other than weighted majority games so far, it should be interesting to investigate more possibility of the dynamic programming technique for the Shapley value.

Finally, we mention some open questions which this paper raises. First of all, we do not know how to compute the nucleolus for a general perfect graph or even for a bipartite graph in polynomial time. It could be NP-hard or #P-hard. The situation is the same for the Shapley values. Another question is the computation of fair cost allocations for the graphs which are not perfect but for which the chromatic number can be computed in polynomial time, like outer-planar graphs.

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References

1. O.N. Bondareva: Some applications of linear programming methods to the theory of cooperative games. (In Russian.) *Problemy Kibernet.* **10** (1963) 119–139.
2. V. Chvátal: On certain polytopes associated with graphs. *J. Combin. Theory Ser. B* **18** (1975) 138–154.
3. I.J. Curiel: *Cooperative Game Theory and Applications: Cooperative Games Arising from Combinatorial Optimization Problems.* Kluwer Academic Publishers, Dordrecht, 1997.
4. X. Deng, T. Ibaraki and H. Nagamochi: Algorithmic aspects of the core of combinatorial optimization games. *Math. Oper. Res.* **24** (1999) 751–766.

5. X. Deng, T. Ibaraki, H. Nagamochi and W. Zang: Totally balanced combinatorial optimization games. *Math. Program.* **87** (2000) 441–452.
6. G.A. Dirac: On rigid circuit graphs. *Abh. Math. Sem. Univ. Hamburg* **25** (1961) 71–76.
7. T. Driessen: *Cooperative Games, Solutions and Applications*. Kluwer Academic Publishers, Dordrecht, 1988.
8. U. Faigle and W. Kern: The Shapley value for cooperative games under precedence constraints. *Internat. J. Game Theory* **21** (1992) 249–266.
9. U. Faigle, W. Kern and J. Kuipers: Computing the nucleolus of min-cost spanning tree games is *NP*-hard. *Internat. J. Game Theory* **27** (1998) 443–450.
10. U. Faigle, W. Kern and J. Kuipers: On the computation of the nucleolus of a cooperative game. *Internat. J. Game Theory* **30** (2001) 79–98.
11. D.R. Fulkerson: Anti-blocking polyhedra. *J. Combin. Theory Ser. B* **12** (1972) 50–71.
12. D.R. Fulkerson and O.A. Gross: Incidence matrices and interval graphs. *Pacific J. Math.* **15** (1965) 835–855.
13. D.B. Gillies: Some theorems on n -person games. Ph.D. Thesis, Princeton University, 1953.
14. M.C. Golumbic: *Algorithmic Graph Theory and Perfect Graphs*. Academic Press, New York, 1980.
15. M. Grötschel, L. Lovász and A. Schrijver: *Geometric algorithms and combinatorial optimization*. Second edition. Springer-Verlag, Berlin, 1993.
16. C. Hoede and X. Li: Clique polynomials and independent set polynomials of graphs. *Discrete Math.* **125** (1994) 219–228.
17. G. Huberman: The nucleolus and the essential coalitions. *Analysis and Optimization of Systems, Lecture Notes in Control and Information Sci.* **28** (1980) 416–422.
18. Y. Kannai: The core and balancedness. In: *Handbook of Game Theory with Economic Applications Volume 1*, R.J. Aumann and S. Hart eds., North Holland, Amsterdam, 1992, 355–395.
19. A. Kopelowitz: Computation of the kernels of simple games and the nucleolus of n -person games. RM 31, Research Program in Game Theory and Mathematical Economics, The Hebrew University of Jerusalem, 1967.
20. J. Kuipers: A polynomial time algorithm for computing the nucleolus of convex games. Report M 96-12, Maastricht University, 1996.
21. L. Lovász: Normal hypergraphs and the perfect graph conjecture. *Discrete Math.* **2** (1972) 253–267.
22. M. Maschler: The bargaining set, kernel, and nucleolus. *Handbook of Game Theory with Economic Applications Volume 1*, R.J. Aumann and S. Hart eds., North Holland, Amsterdam, 1992, 591–667.
23. T. Matsui and Y. Matsui: A survey of algorithms for calculating power indices of weighted majority games. *J. Oper. Res. Soc. Japan* **43** (2000) 71–86.
24. N. Megiddo: Computational complexity and the game theory approach to cost allocation for a tree. *Math. Oper. Res.* **3** (1978) 189–196.
25. H. Mohri and Y. Okamoto: Discrete optimization and cooperative games (2). (In Japanese.) *Comm. Oper. Res. Soc. Japan* **48** (2003) 114–120.
26. B. Peleg: Axiomatizations of the core. In: *Handbook of Game Theory with Economic Applications Volume 1*, R.J. Aumann and S. Hart eds., North Holland, Amsterdam, 1992, 397–412.
27. J.S. Provan and M.O. Ball: The complexity of counting cuts and of computing the probability that a graph is connected. *SIAM J. Comput.* **12** (1983) 777–788.
28. Y. Okamoto: Submodularity of some classes of the combinatorial optimization games. *Math. Methods Oper. Res.* **58** (2003) 131–139.
29. Y. Okamoto: Fair cost allocations under conflicts — a game-theoretic point of view —, *Proceedings of 14th Annual International Symposium on Algorithms and Computation (ISAAC 2003)*, *Lect. Notes in Comput. Sci.* **2906** (2003) 686–695.
30. D. Schmeidler: The nucleolus of a characteristic function game. *SIAM J. Appl. Math.* **17** (1969) 1163–1170.
31. L.S. Shapley: A value for n -person games. In: *Contributions to the Theory of Games II*, H. Kuhn and A.W. Tucker eds., Princeton University Press, Princeton, New Jersey, 1953, 307–317.
32. L.S. Shapley: On balanced sets and cores. *Naval Research Logistics Quarterly* **14** (1967) 453–460.
33. L.S. Shapley: Cores of convex games. *Internat. J. Game Theory* **1** (1971) 11–26. Errata is in the same volume, 1972, pp. 199.
34. T. Solymosi and T.E.S. Raghavan: An algorithm for finding the nucleolus of assignment games. *Internat. J. Game Theory* **23** (1994) 119–143.
35. S.H. Tijs: Bounds for the core and the τ -value. In: *Game Theory and Mathematical Economics*, O. Moeshlin and P. Pallaschke, eds., North Holland, Amsterdam, 1981, 123–132.
36. S.P. Vadhan: The complexity of counting in sparse, regular, and planar graphs. *SIAM J. Comput.* **31** (2001) 398–427.
37. E. Winter: The Shapley value. In: *Handbook of Game Theory with Economic Applications Volume 3*, R.J. Aumann and S. Hart eds., North Holland, Amsterdam, 2002, 2025–2054.