Observability for Hybrid Systems

Andrea Balluchi
PARADES
Via S. Pantaleo, 66, 00186 Roma, Italy
balluchi@parades.rm.cnr.it

Luca Benvenuti
DIS, Università di Roma “La Sapienza”
Via Eudossiana 18, 00184 Roma, Italy
luca.benvenuti@uniroma1.it

Maria D. Di Benedetto
DIEL , Università dell’Aquila
Poggio di Roio, 67040 L’Aquila, Italy
dibenede@ing.univaq.it

Alberto L. Sangiovanni–Vincentelli
EECS Dept., Univ. of California at Berkeley
CA 94720, USA
alberto@eecs.berkeley.edu

***Abstract*— The notion of generic final–state asymptotically determinable hybrid system is introduced. Then, sufficient conditions for a linear hybrid system to be generic final–state asymptotically determinable are given. These conditions show that generic final–state asymptotic determinability can be verified even if each of the continuous subsystems of the hybrid system is not observable. More precisely, these conditions are related to the minimum and maximum sojourn time in each location as well as on the dimension and orientation of the unobservable subspaces and on the reset mappings between them.**

I. INTRODUCTION

Hybrid systems are powerful abstractions for modelling complex systems so that they have been used in a number of applications to provide models better reflecting the nature of control problems such as the ones related to embedded system design where discrete controls are routinely applied to continuous processes. Their theoretical properties are still the subject of intense research: in particular, observability is one of the fundamental system properties that form the foundations of control. The notion of observability is non trivial and attention must be paid to the implications of the definitions used. In traditional continuous and discrete–time systems, the concept of observability has been studied extensively. Among the seminal papers on the topic, Sontag in [7] introduces a set of observability–related definitions and examines the implications among the various concepts of observability. Recently several papers appeared that dealt with observability and observer design for hybrid and switching systems (see [3] for a review of these approaches). These papers introduce different definitions of observability so that it is quite difficult to compare them in a rigorous framework. De Santis *et al.* in [1], show that the observability notions based on state indistinguishability do not imply state reconstructability and propose a new definition of observability, and a weaker notion of detectability, based on the possibility of reconstructing the system state for discrete-time switching systems. In [3], we addressed observability for hybrid systems by following the main ideas exposed by Sontag in his seminal paper and generalizing his definitions to the case of hybrid systems. In [7], Sontag established the implications among the six different notions of observability he introduced in the case of polynomial systems. In [3], we showed by counterexamples that none of these implications holds in the case of hybrid systems.

In this paper, we focus on the property called generic final–state asymptotically determinability, since this condition is enough to guarantee the existence of an asymptotic observer, the one we need in most of the applications we are interested in, including automotive engine control problems.

The contribution of this paper are:

- The extension of the definition of generic final–state determinability of Sontag to hybrid systems (Section II). Informally, a hybrid system is generic final–state asymptotically determinable if any generic input/output experiment permits the asymptotic determination of the state. This notion facilitates the constructions of asymptotic state observers.
- Sufficient conditions for generic final–state asymptotically determinability of linear hybrid systems (Section III).
- Examples that show that a hybrid system can be generic final–state asymptotic determinable even if each of the continuous subsystems of the hybrid system is not observable (Section IV). This result is based on the synergy that can be exploited between the continuous and the discrete dynamics.

II. GENERIC FINAL–STATE ASYMPTOTIC DETERMINABILITY FOR HYBRID SYSTEMS

A hybrid system $\mathcal{H}$ is a tuple

$$\mathcal{H} = (Q, \Sigma, \Psi, \varphi, \phi, \eta, X, U, Y, f, h, r)$$

where $Q = \{q_1, \cdots, q_N\}$ is the finite set of discrete states (locations) with $N = |Q|$, Σ is the finite set of possible input and internal events, Ψ is the finite set of discrete outputs, $X \subseteq \mathbb{R}^n$, $U \subseteq \mathbb{R}^m$, and $Y \subseteq \mathbb{R}^p$ are the continuous state, control and output domains, respectively. The functions φ, ϕ and η characterize the dynamics of the discrete states of the system as follows:

$$q(k+1) \in \varphi\left(q(k), \sigma(k+1)\right) \quad (1)$$
$$\sigma(k+1) \in \phi\left(q(k), x(t_{k+1}^-), u(t_{k+1}^-)\right) \quad (2)$$
$$\psi(k+1) \in \eta\left(q(k), \sigma(k+1), q(k+1)\right) \quad (3)$$

where $q(k) \in Q$ and $\psi(k) \in \Psi$ are, respectively, the location and the discrete output after the k-th input event $\sigma(k) \in \Sigma$, and t_k denotes the unknown time at which this event takes place. The finite set Σ is composed by both internal events, auto-generated by the hybrid system on the basis of the values of the continuous state x and input u, and exogenous input events, whose enabling condition may or may not depend on x and u. The set–valued functions $\varphi : Q \times \Sigma \rightarrow 2^Q \setminus \{\}$ and $\eta : Q \times \Sigma \times Q \rightarrow \Psi$ are the transition and output functions respectively. The function $\phi : Q \times X \times U \rightarrow 2^{\Sigma \cup \{\epsilon\}} \setminus \{\}$ is the set–valued function specifying the possible events at each location for given values of the continuous state $x(t) \in X$ and continuous input $u(t) \in U$ of the system.

The event ϵ is the *silent event* and it is introduced to model different possible situations for the discrete dynamics. For example, if $\phi(q, x, u) = \{\epsilon\}$, then there is no discrete transition enabled for the given values of x and u while if $\phi(q, x, u) = \{\overline{\sigma}, \epsilon\}$, then it is possible either to let time pass or to take the discrete transition associated to $\overline{\sigma}$. Moreover, if $\phi(q, x, u) = \{\overline{\sigma}\}$, then the discrete transition associated to $\overline{\sigma}$ is forced to occur. This is useful, for example, to model internal transitions due to the continuous state hitting a guard.

The functions $f : Q \times X \times U \rightarrow \mathbb{R}^n$ and $h : Q \times X \rightarrow Y$ define the dynamics of the continuous variables of the hybrid systems. For linear hybrid systems they are assumed to be linear and time-invariant:

$$\dot{x}(t) = f(q_i, x(t), u(t)) = A_i\ x(t) + B_i\ u(t) \quad (4)$$
$$y(t) = h(q_i, x(t)) = C_i\ x(t) \quad (5)$$

where $y(t) \in Y$ is the continuous output of the system and $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times m}$, $C_i \in \mathbb{R}^{p \times n}$ depend on the current plant location q_i.

Finally, the function $r : Q \times Q \times X \rightarrow X$ describes the continuous state resets associated to the hybrid system transitions. For each transition $q_i \rightarrow q_j$ the reset function is assumed to be affine and described by

$$x(t_k) = x(t_k^+) = r(q_i, q_j, x(t_k^-)) = R_{ij}^1\ x(t_k^-) + R_{ij}^0 \quad (6)$$

where t_k denotes the transition time and $R_{ij}^1 \in \mathbb{R}^{n \times n}$, $R_{ij}^0 \in \mathbb{R}^n$. We assume that the hybrid system $\mathcal{H}$ is subject to an unbounded sequence of events, never reaches a blocking condition and that there are no infinitely fast event sequences. In this paper we investigate generic final–state asymptotically determinability for linear hybrid systems, defined as follows:

Definition 1: A hybrid system $\mathcal{H}$ is *generic final–state asymptotically determinable* if there exists an integer K finite and large enough such that, for any initial configuration (q_0, x_0) and any inputs $\sigma(k)$ and $u(t)$, $q(k)$ can be identified for any $k \geq K$ and $x(t)$ can be identified for $t \rightarrow \infty$ from the outputs $\psi(k)$ and $y(t)$.

Informally, the above definition states that a hybrid system is generic final–state asymptotically determinable if for any initial condition, any generic input/output experiment permits the asymptotic determination of the state.

III. SUFFICIENT CONDITIONS FOR GENERIC FINAL–STATE ASYMPTOTIC DETERMINABILITY

In [2] and [4], a class of generic final–state asymptotically determinable hybrid systems were identified and a scheme of hybrid observers that achieve asymptotic and exponential hybrid state estimation were described. To this end, the notion of current–location observability of hybrid systems was introduced. A hybrid system is *current–location observable* if, after an initial transient of a bounded number K of steps, the location $q(\bar{k})$, can be determined for any $\bar{k} \geq K$, from the knowledge of the output signal $\psi(k)$ from $k = 1$ to $k = \bar{k}$, for any initial configuration (q_0, x_0) and any inputs $\sigma(k)$ and $u(t)$. Hence, current–location observable hybrid systems are generic final–state determinable with respect to the discrete state. In general, as shown in [2], for any $\bar{k} \geq K$, the location $q(\bar{k})$ belongs to an easy–to–compute subset $E_{\mathcal{O}}$ of Q.

The following theorem, obtained by extending the results in [4], gives a class of generic final–state asymptotically determinable hybrid systems:

Theorem 2: A linear hybrid system $\mathcal{H}$ is generic final–state asymptotically determinable if

1) $\mathcal{H}$ is current–location observable;
2) $\mathcal{H}$ remains in each location q_i a bounded time greater than or equal to a time D_m, i.e.

$$0 < D_m \leq t_{k+1} - t_k < \infty \quad (7)$$

3) for each $q_i \in E_{\mathcal{O}}$, there exists a gain matrix G_i such that $A_i - G_iC_i$ has distinct eigenvalues and

$$\alpha(A_i - G_iC_i) + \max\left\{0, \frac{\log\left[\max\limits_{q_j \in \text{Reach}(q_i)} \|T_i R_{ij}^1 T_j^{-1}\|\right]}{D_m}\right\} < 0$$

where $\alpha(\cdot)$ denotes the spectral abscissa, $\text{Reach}(q_i)$ is the set of locations reachable in one step from q_i, and T_i is a transformation matrix diagonalizing $A_i - G_iC_i$.

If one can impose the sojourn time[1] in each location, then the following corollary may be used:

Corollary 3: Given a current–location observable linear hybrid system $\mathcal{H}$, if all the pairs (A_i, C_i) are observable then there exists a minimum sojourn time $D_m > 0$ such that if $\mathcal{H}$ remains in each location q_i a bounded time greater than

[1] Maximum dwell-time in the notation of [6].

or equal to D_m, then $\mathcal{H}$ is generic final–state asymptotically determinable.

The sufficient conditions given in Theorem 2 do not hold, for example, for linear hybrid systems with undetectable (i.e. unstable and unobservable) dynamics associated to at least one location in $E_{\mathcal{O}}$. In fact, in this case. condition 3) cannot be fulfilled.

In the sequel, sufficient conditions for generic final–state asymptotic determinability for linear hybrid systems with undetectable dynamics will be given. To this end, for each $q_i \in E_{\mathcal{O}}$, let us consider the Kalman decomposition of the continuous state dynamics (4–5):

$$\begin{aligned} \dot{x}^{Oi}(t) &= A_i^O \; x^{Oi}(t) + B_i^O \; u(t) && (8)\\ \dot{x}^{Ni}(t) &= A_i^P \; x^{Oi}(t) + A_i^N \; x^{Ni}(t) + B_i^N \; u(t) && (9)\\ y(t) &= C_i^O \; x^{Oi}(t) && (10) \end{aligned}$$

where x^{Oi} and x^{Ni} respectively stand for the observable and unobservable state components, which are related to the original state space by the transformation

$$\left[\begin{array}{c} x^{Oi} \\ x^{Ni} \end{array}\right] = T_i x = \left[\begin{array}{c} T_i^O \\ T_i^N \end{array}\right] x, \tag{11}$$

$$x = T_i^{-1} \left[\begin{array}{c} x^{Oi} \\ x^{Ni} \end{array}\right] = \left[T_i^{\prime O} \;\; T_i^{\prime N}\right] \left[\begin{array}{c} x^{Oi} \\ x^{Ni} \end{array}\right] \tag{12}$$

where $T_i^O \in \mathbb{R}^{n^{Oi} \times n}$, $T_i^N \in \mathbb{R}^{n^{Ni} \times n}$, $T_i^{\prime O} \in \mathbb{R}^{n \times n^{Oi}}$, $T_i^{\prime N} \in \mathbb{R}^{n \times n^{Ni}}$, n^{Ni} is the dimension of the unobservable subspace and $n^{Oi} = n - n^{Ni}$. The matrices $T_i^{\prime O}$ and $T_i^{\prime N}$ are such that Im $\{T_i^{\prime O}\}$ = Im $\{\mathcal{O}(A_i, C_i)^T\}$ and Im $\{T_i^{\prime N}\}$ = Ker $\{\mathcal{O}(A_i, C_i)\}$, where $\mathcal{O}(A_i, C_i)$ is the observability matrix of the pair (A_i, C_i). If for some i the system (4–5) is observable, i.e. rank $\mathcal{O}(C_i, A_i) = n$, then $n^{Oi} = n$ and $n^{Ni} = 0$. Under the given state space transformation, the resets (6) for the k-th transition from q_i to q_j become

$$\begin{aligned} &x^{Oj}(t_k) = r^{Oj}(q_i, q_j, x^{Oi}(t_k^-), x^{Ni}(t_k^-)) = && (13)\\ &T_j^O R_{ij}^1 T_i^{\prime O} \; x^{Oi}(t_k^-) + T_j^O R_{ij}^1 T_i^{\prime N} \; x^{Ni}(t_k^-) + T_j^O R_{ij}^0 \\ &x^{Nj}(t_k) = r^{Nj}(q_i, q_j, x^{Oi}(t_k^-), x^{Ni}(t_k^-)) = && (14)\\ &T_j^N R_{ij}^1 T_i^{\prime O} \; x^{Oi}(t_k^-) + T_j^N R_{ij}^1 T_i^{\prime N} \; x^{Ni}(t_k^-) + T_j^N R_{ij}^0 \end{aligned}$$

The following theorem gives sufficient conditions for generic final–state asymptotic determinability of current–location observable hybrid systems with unobservable (possibly undetectable) subsystems.

Theorem 4: A linear hybrid system $\mathcal{H}$ is generic final–state asymptotically determinable if

1) $\mathcal{H}$ is current–location observable;
2) the graph describing the discrete–evolution of $\mathcal{H}$ contains only one cycle of locations $\mathcal{C}$;
3) $\mathcal{H}$ remains in each location q_i at least a time D_m and no more than a time D_M, i.e.

$$0 \;<\; D_m \;\leq\; t_{k+1} - t_k \;\leq\; D_M \;<\; 1 \tag{15}$$

4) for at least one location $q_{\ell_1} \in \mathcal{C}$ the following condition is verified

$$R_{\ell_S \ell_1}^1 \; T_{\ell_S}^{\prime N} \; T_{\ell_S}^N \; \cdots \; T_{\ell_2}^{\prime N} \; T_{\ell_2}^N \; R_{\ell_1 \ell_2}^1 \; T_{\ell_1}^{\prime N} \; T_{\ell_1}^N = 0 \tag{16}$$

where $\mathcal{C} = \{q_{\ell_1}, q_{\ell_2}, \ldots, q_{\ell_S}\}$;

5) there exist $\epsilon \in (0,1)$ such that

$$D_M \leq \frac{\epsilon}{R_N \left[(\nu_A + 1)^S - 1\right]} \tag{17}$$

and matrices G_i such that

$$e^{(S-1)\alpha_A D_M} \left[e^{\alpha_F D_m} + \nu_P \frac{e^{\alpha_A D_M} - e^{\alpha_F D_M}}{\alpha_A - \alpha_F}\right] < \frac{1-\epsilon}{2^{S-1} R_X} \tag{18}$$

with

$$\begin{aligned} R_N &= \|R_{\ell_S \ell_1}^1 \; T_{\ell_S}^{\prime N}\| \cdot \|T_{\ell_1}^N\| \textstyle\prod_{i=1}^{S-1} \|T_{\ell_{i+1}}^N \; R_{\ell_i \ell_{i+1}}^1 \; T_{\ell_i}^{\prime N}\| \\ \nu_A &= \max_{i=1,\ldots,S} \left\{\|A_{\ell_i}^N\| \cdot \max[1, e^{\alpha(A_{\ell_i}^N)}]\right\} \\ \alpha_A &= \max_{i=1,\ldots,S} |\alpha(A_{\ell_i}^N)| \\ \alpha_F &= \max_{i=1,\ldots,S} \alpha(A_{\ell_i}^O - G_{\ell_i} C_{\ell_i}^O) \\ \nu_P &= \max_{i=1,\ldots,S} \|A_{\ell_i}^P\| \\ R_X &= \max_{X \in \{O,N\}} \|R_{\ell_S \ell_1}^1 \; T_{\ell_S}^{\prime X}\| \cdot \max_{X \in \{O,N\}} \|T_{\ell_1}^X\| \cdot \\ &\quad \prod_{i=1}^{S-1} \max_{X,Z \in \{O,N\}} \|T_{\ell_{i+1}}^X \; R_{\ell_i \ell_{i+1}}^1 \; T_{\ell_i}^{\prime Z}\| \end{aligned} \tag{19}$$

where $\alpha(\cdot)$ denotes the spectral abscissa. Parameters (19) are given for A_i with distinct unobservable eigenvalues, diagonalizable observable closed-loop dynamics, and T_i such that A_i^N and $A_i^O - G_i C_i^O$ are diagonal.

Proof: Extending the methodology for hybrid observer design proposed in [2], [4], the dynamics of the continuous observer is: $\dot{\tilde{x}}(t) = 0$, if $\tilde{q} \notin E_{\mathcal{O}}$, and

$$\begin{aligned} \dot{\tilde{x}}^{Oi}(t) &= F_i \; \tilde{x}^{Oi}(t) + B_i^O \; u(t) + G_i \; y(t) \\ \dot{\tilde{x}}^{Ni}(t) &= A_i^P \; \tilde{x}^{Oi}(t) + A_i^N \; \tilde{x}^{Ni}(t) + B_i^N \; u(t) \end{aligned} \tag{20}$$

if $\tilde{q} = \{q_i\} \in E_{\mathcal{O}}$, where A_i^O, B_i^O, C_i^O, A_i^N, A_i^P and B_i^N are as in (8–10), $F_i = A_i^O - G_i C_i^O$, and the observer gain matrix $G_i \in \mathbb{R}^{n^{Oi} \times p}$ is the design parameter used to set the velocity of convergence of the observable components $\tilde{x}^{Oi}$ to x^{Oi}, in each location $\tilde{q} \in E_{\mathcal{O}}$.

The continuous–time dynamics (20) are integrated over no-empty time intervals $[t_k, t_{k+1})$. At time t_{k+1}, when a transition to location q_j takes place, the plant observer state is reset according to (13–14) as follows

$$\begin{aligned} \tilde{x}^{Oj}(t_{k+1}) &= r^{Oj}(q_i, q_j, \tilde{x}^{Oi}(t_{k+1}^-), \tilde{x}^{Ni}(t_{k+1}^-)) \\ \tilde{x}^{Nj}(t_{k+1}) &= r^{Nj}(q_i, q_j, \tilde{x}^{Oi}(t_{k+1}^-), \tilde{x}^{Ni}(t_{k+1}^-)) \end{aligned} \tag{21}$$

Notice that since the location observer instantaneously identifies the hybrid plant transitions, then the continuous observer dynamics switching and resets occur synchronously with the hybrid transitions and resets at time t_k. To be more precise, from the hypothesis of current-location observability of the hybrid system $\mathcal{H}$ it follows that there exists a positive integer K such that the current location of the hybrid system is properly identified for any $k \geq K$. Moreover, after at most $|E_{\mathcal{O}}|$ additional transitions, say at time $t_{\ell_1} > t_K$, the hybrid system reaches the location q_{ℓ_1} and starts repeating forever the cycle $\mathcal{C}$. Then, asymptotic convergence of the continuous observation error $\zeta(t) = \tilde{x}(t) - x(t)$ is analyzed for $t \geq t_{\ell_1}$. According to the transformation in (11), the observation error ζ is decomposed into the components $(\zeta^{O_i}, \zeta^{N_i})$, where

$$\begin{bmatrix} \zeta^{Oi} \\ \zeta^{Ni} \end{bmatrix} = \begin{bmatrix} \tilde{x}^{Oi} - x^{Oi} \\ \tilde{x}^{Ni} - x^{Ni} \end{bmatrix} = \begin{bmatrix} T_i^O \\ T_i^N \end{bmatrix} \zeta \tag{22}$$

$$\zeta = T_i'^O \zeta^{Oi} + T_i'^N \zeta^{Ni} \tag{23}$$

and is subject to

$$\begin{array}{l} \dot{\zeta}^{Oi}(t) = F_i \zeta^{Oi}(t) \\ \dot{\zeta}^{Ni}(t) = A_i^N \zeta^{Ni}(t) + A_i^P \zeta^{Oi}(t) \end{array} \tag{24}$$

By (24) and (23), for $t \in [t_k, t_{k+1})$, being $q = q_i$,

$$\zeta(t) = \Big[\; T_i'^O e^{F_i(t-t_k)} T_i^O + T_i'^N e^{A_i^N(t-t_k)} T_i^N + T_i'^N \int_0^{t-t_k} e^{A_i^N \tau} A_i^P e^{F_i(t-t_k-\tau)} d\tau \; T_i^O \Big] \zeta(t_k)$$

and, by (6) and (21)

$$\zeta(t_{k+1}) = R_{ij}^1 \; \zeta(t_{k+1}^-) = R_{ij}^1 \; \Big[\; T_i'^O e^{F_i(t_{k+1}-t_k)} T_i^O + T_i'^N e^{A_i^N(t_{k+1}-t_k)} T_i^N + T_i'^N \int_0^{t_{k+1}-t_k} e^{A_i^N \tau} A_i^P e^{F_i(t_{k+1}-t_k-\tau)} d\tau \; T_i^O \Big] \zeta(t_k)$$

Consider now an evolution of the hybrid system $\mathcal{H}$ along the cycle $\mathcal{C} = \{q_{\ell_1}, q_{\ell_2}, \ldots, q_{\ell_S}\}$, starting from location q_{ℓ_1} at time t_k and here ending at time t_{k+S}. We have

$$\zeta(t_{k+S}) = \prod_{i=1}^{S} \Big\{ R_{\ell_i \ell_{i+1}}^1 \Big[\; T_{\ell_i}'^O e^{F_{\ell_i}(t_{k+i}-t_{k+i-1})} T_{\ell_i}^O + T_{\ell_i}'^N e^{A_{\ell_i}^N(t_{k+i}-t_{k+i-1})} T_{\ell_i}^N + T_{\ell_i}'^N \int_0^{t_{k+i}-t_{k+i-1}} e^{A_{\ell_i}^N \tau} A_{\ell_i}^P e^{F_{\ell_i}(t_{k+i}-t_{k+i-1}-\tau)} d\tau \; T_{\ell_i}^O \; \Big] \Big\} \zeta(t_k) \tag{25}$$

The matrix multiplying $\zeta(t_k)$ can be rewritten as the sum of 2^S terms

$$M_{X_{\ell_1},\ldots,X_{\ell_S}} = R_{\ell_S \ell_1}^1 \; Y_{\ell_S}^{X_{\ell_S}} \; \ldots Y_{\ell_2}^{X_{\ell_2}} \; R_{\ell_1 \ell_2}^1 \; Y_{\ell_1}^{X_{\ell_1}}$$

where either $X_{\ell_i} = O$ or $X_{\ell_i} = N$, and

$$Y_{\ell_i}^O = T_{\ell_i}'^O e^{F_{\ell_i}(t_{k+i}-t_{k+i-1})} T_{\ell_i}^O + T_{\ell_i}'^N \cdot \int_0^{t_{k+i}-t_{k+i-1}} e^{A_{\ell_i}^N \tau} A_{\ell_i}^P e^{F_{\ell_i}(t_{k+i}-t_{k+i-1}-\tau)} d\tau \; T_{\ell_i}^O$$

$$Y_{\ell_i}^N = T_{\ell_i}'^N e^{A_{\ell_i}^N(t_{k+i}-t_{k+i-1})} T_{\ell_i}^N$$

The most critical term is

$$M_{N,\ldots,N} = R_{\ell_S \ell_1}^1 \; Y_{\ell_S}^N \; \ldots Y_{\ell_2}^N \; R_{\ell_1 \ell_2}^1 \; Y_{\ell_1}^N \; ,$$

since it is composed of unobservable dynamics only, whose evolutions are not modifiable by observer feedbacks G_i. Terms $Y_{\ell_i}^N$ can be rewritten as

$$Y_{\ell_i}^N = T_{\ell_i}'^N \left[e^{A_{\ell_i}^N(t_{k+i}-t_{k+i-1})} - I \right] T_{\ell_i}^N + T_{\ell_i}'^N T_{\ell_i}^N \; ,$$

where the exponential term, for $D_M < 1$, can be bounded as follows

$$\left\| e^{A_{\ell_i}^N(t_{k+i}-t_{k+i-1})} - I \right\| \leq \nu_A \; D_M \tag{26}$$

with ν_A is as in (19). Matrix $M_{N,\ldots,N}$ can be written as the sum of 2^S terms, where by condition (16) the one without any exponential matrix is null. Then, for $D_M < 1$, we have

$$\begin{aligned} \|M_{N,\ldots,N}\| &< R_N \left[(\nu_A D_M + 1)^S - 1 \right] \\ &< R_N \left[(\nu_A + 1)^S - 1 \right] D_M \end{aligned} \tag{27}$$

where R_N is as in (19). Hence, for ϵ such that condition (17) holds, we have $\|M_{N,\ldots,N}\| < \epsilon$.
Moreover, for the other $2^S - 1$ terms, $M_{X_{\ell_1},\ldots,X_{\ell_S}} \neq M_{N,\ldots,N}$, we have[2]

$$\left\| e^{F_{\ell_i}(t_{k+i}-t_{k+i-1})} \right\| \leq e^{\alpha_F D_m}$$

$$\left\| \int_0^{t_{k+i}-t_{k+i-1}} e^{A_{\ell_i}^N \tau} A_{\ell_i}^P e^{F_{\ell_i}(t_{k+i}-t_{k+i-1}-\tau)} d\tau \right\| \leq \nu_P \frac{e^{\alpha_A D_M} - e^{\alpha_F D_M}}{\alpha_A - \alpha_F}$$

$$\left\| e^{A_{\ell_i}^N(t_{k+i}-t_{k+i-1})} \right\| \leq e^{\alpha_A D_M}$$

where ν_P, α_A and α_F are as in (19), and

$$\|M_{X_{\ell_1},\ldots,X_{\ell_S}}\| \leq R_X e^{k_O \alpha_A D_M} \cdot \left[e^{\alpha_F D_m} + \nu_P \frac{e^{\alpha_A D_M} - e^{\alpha_F D_M}}{\alpha_A - \alpha_F} \right]^{k_N}$$

with k_O and k_N being the number of O and N terms, respectively, in $X_{\ell_1}, \ldots, X_{\ell_S}$, and R_X is as in (19). Then the sum of the $2^S - 1$ terms $M_{X_{\ell_1},\ldots,X_{\ell_S}} \neq M_{N,\ldots,N}$, is upper bounded by

$$2^S R_X e^{(S-1)\alpha_A D_M} \left[e^{\alpha_F D_m} + \nu_P \frac{e^{\alpha_A D_M} - e^{\alpha_F D_M}}{\alpha_A - \alpha_F} \right]$$

Then, by choosing G_i according to (18), we have from (25)

$$\|\zeta(t_{k+S})\| < \left[\sum \|M_{X_{\ell_1},\ldots,X_{\ell_S}}\| \right] \; \|\zeta(t_k)\| < \|\zeta(t_k)\|$$

This shows that the value of the norm of the observation error after each cycle $\mathcal{C} = \{q_{\ell_1}, q_{\ell_2}, \ldots, q_{\ell_S}\}$ decreases and this concludes the proof.

∎

[2]For diagonal matrices D, $\|e^{Dt}\| \leq e^{\alpha(D)t}$, $\forall t \geq 0$ (see [5]).

Remark 5: Constraint $D_M < 1$ in (15) is introduced to obtain a very simple form for condition (17). Similar results can be obtained for $D_M > 1$, replacing (17) with a polynomial constraint for D_M derived from (27); more precisely D_M and ϵ must satisfy the following condition:

$$R_N \left[(\nu_A D_M + 1)^S - 1 \right] \leq \epsilon \tag{28}$$

Remark 6: Slightly more involved expressions for parameters in (19) can be given when either A_i has multiple unobservable eigenvalues or the observable closed-loop dynamics is defective.

Equation (16) is a geometric condition on the dimension and orientation of the unobservable subspaces, as well as on the reset mappings between them, under which continuous state determinability can be achieved for the hybrid system by switching between the subsystems. However, continuous state determinability can be gained only if the switching is fast enough. In fact, constraint (17) provides an upper bound for the sojourn time in each location for the switching to be effective. Finally, condition (18) expresses how fast the observation dynamics should be in order to overcome the possible unstability of the unobservable evolutions.

IV. EXAMPLES

In this section, some examples of final–state asymptotically determinable hybrid systems are given in order to illustrate the results of Theorem 4.

In geometric condition (16) both the orientation of the unobservable subspaces and the reset maps play a role. As an example, consider a hybrid system with a two dimensional continuous space and a two–location cycle with coincident unobservable subspaces, e.g. $T_1'^N = T_2'^N = \left[\begin{array}{cc} 0 & 1 \end{array}\right]^T$. If one of the reset map is the 2×2 permutation matrix, e.g. $R_{12}^1 = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right]$, then (16) is satisfied with $\ell_1 = 1, \ell_2 = 2$ and final–state determinability may be achieved. In fact, the second component of the state space is unobservable for both subsystems, but at each switching from q_1 to q_2 its value is mapped into the first component, which is observable. Hence, after each transition to q_2, the component that was not observable can be recovered.

On the other hand, if the reset maps are the identity matrix but the unobservable subspaces have a trivial intersection, e.g. $T_1'^N = \left[\begin{array}{cc} 1 & 0 \end{array}\right]^T$ and $T_2'^N = \left[\begin{array}{cc} 0 & 1 \end{array}\right]^T$, then (16) is satisfied. In fact, both components from time to time become observable.

As a final example, consider a three–location hybrid system, with locations q_1, q_2, q_3 connected in a cycle $q_1 \to q_2 \to q_3 \to q_1$, continuous dynamics in (4–5) given by

$$A_1 = \begin{bmatrix} -\frac{2258}{3125} & -\frac{1996}{3125} & \frac{1996}{3125} & \frac{998}{3125} \\ \frac{504}{3125} & -\frac{6502}{3125} & -\frac{5998}{3125} & -\frac{2999}{3125} \\ \frac{16049}{12500} & -\frac{2003}{3125} & -\frac{31963}{12500} & -\frac{4011}{3125} \\ \frac{3931}{6250} & -\frac{989}{3125} & -\frac{8047}{6250} & -\frac{3961}{6250} \end{bmatrix} \quad C_1^T = \begin{bmatrix} \frac{1}{10} \\ \frac{6}{5} \\ \frac{4}{5} \\ \frac{2}{5} \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -\frac{3163}{12500} & \frac{777}{6250} & -\frac{436}{3125} & -\frac{401}{6250} \\ \frac{411}{3125} & -\frac{451}{6250} & \frac{311}{6250} & \frac{401}{12500} \\ -\frac{411}{3125} & \frac{777}{12500} & -\frac{747}{12500} & -\frac{401}{12500} \\ -\frac{193}{3125} & \frac{213}{6250} & -\frac{143}{6250} & -\frac{47}{3125} \end{bmatrix} \quad C_2^T = \begin{bmatrix} \frac{4}{5} \\ -\frac{2}{5} \\ \frac{2}{5} \\ \frac{1}{5} \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -\frac{1}{100} & 0 & 0 & \frac{1}{250} \\ 0 & \frac{1}{200} & \frac{1}{250} & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix} \quad C_3^T = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

and no continuous state resets during transitions, i.e. $R_{ij}^1 = I$ and $R_{ij}^0 = 0$ in (6). Computing a Kalman decomposition in observable and unobservable components, we obtain $n^{O1} = n^{N1} = n^{O3} = n^{N3} = 2$, $n^{O2} = 1$, $n^{N2} = 3$ and transformation matrices (11–12) as follows

$$T_1 = \begin{bmatrix} T_1^O \\ T_1^N \end{bmatrix} = \left[\begin{array}{cccc} 9/20 & 2/5 & -2/5 & -1/5 \\ 2/5 & -1/5 & -4/5 & -2/5 \\ \hline -4/5 & 2/5 & -9/10 & 4/5 \\ -4/5 & 2/5 & -2/5 & -1/5 \end{array}\right]$$

$$T_1^{-1} = \left[T_1'^O \quad T_1'^N \right] = \left[\begin{array}{cc|cc} 4/5 & 0 & 0 & -4/5 \\ 8/5 & -1 & 0 & 2/5 \\ 0 & -4/5 & -2/5 & 0 \\ 0 & -2/5 & 4/5 & -1 \end{array}\right]$$

$$T_2 = \begin{bmatrix} T_2^O \\ T_2^N \end{bmatrix} = \left[\begin{array}{cccc} -4/5 & 2/5 & -2/5 & -1/5 \\ \hline 0 & 1/5 & 0 & 2/5 \\ 1/5 & 0 & -2/5 & 0 \\ 0 & 1/5 & 1/5 & 0 \end{array}\right]$$

$$T_2^{-1} = \left[T_2'^O \quad T_2'^N \right] = \left[\begin{array}{c|ccc} -4/5 & -2/5 & 9/5 & 2 \\ 2/5 & 1/5 & 8/5 & 4 \\ -2/5 & -1/5 & -8/5 & 1 \\ -1/5 & 12/5 & -4/5 & -2 \end{array}\right]$$

$$T_3 = \begin{bmatrix} T_3^O \\ T_3^N \end{bmatrix} = \left[\begin{array}{cccc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right]$$

$$T_3^{-1} = \left[T_3'^O \quad T_3'^N \right] = \left[\begin{array}{cc|cc} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right]$$

In the transformed space, dynamics (8–10) are given by

$$A_1^O = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix},\ C_1^O = \begin{bmatrix} 2 & -2 \end{bmatrix}$$

$$A_1^N = \begin{bmatrix} \frac{1}{100} & 0 \\ 0 & -\frac{1}{250} \end{bmatrix},\ A_1^P = \begin{bmatrix} -\frac{1}{250} & 0 \\ 0 & 0 \end{bmatrix}$$

$$A_2^O = -2/5,\ C_2^O = -1$$

$$A_2^N = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{100} & 0 \\ 0 & 0 & -\frac{1}{100} \end{bmatrix},\ A_2^P = \begin{bmatrix} -\frac{1}{500} \\ 0 \\ 0 \end{bmatrix}$$

$$A_3^O = \begin{bmatrix} -5 & 0 \\ 0 & -2 \end{bmatrix},\ C_3^O = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$A_3^N = \begin{bmatrix} \frac{1}{200} & 0 \\ 0 & -\frac{1}{100} \end{bmatrix},A_3^P = \begin{bmatrix} 0 & \frac{1}{250} \\ -\frac{1}{250} & 0 \end{bmatrix}$$

Since

$$\begin{aligned} R_{31}^1\, T_3'^N\, T_3^N\, R_{23}^1\, T_2'^N\, T_2^N\, R_{12}^1\, T_1'^N\, T_1^N &= \\ T_3'^N\, T_3^N\, T_2'^N\, T_2^N\, T_1'^N\, T_1^N &= 0 \end{aligned}$$

condition (16) is verified and final–state determinability can be achieved if the system switches fast enough.

By (19), $R_N = 4.3554$, $\nu_A = 0.01$, and since the cycle has three locations, $S = 3$. The upper bound $D_M < 7.1325$ for the sojourn time is obtained from inequality (28) with $\epsilon = 1$. Choosing $D_M = 7$, inequality (28) is verified with $\epsilon = 0.98$.

To complete the observability test, matrices G_i and the minimum sojourn time D_m for which (18) holds should be determined. By (19), $\alpha_A = 0.01$ and $\nu_P = 0.004$. Setting $D_m = 1.5$, matrices G_i are chosen such that $\alpha_F = -10$. Then, matrices T_i^O and $T_i'^O$ are recomputed to diagonalize the closed loop matrices $A_i^O - G_i C_i^0$. With the resulting transformations, by (19) $R_X = 1825$. Since with this choice of G_i inequality (18) is verified, then, according to Theorem 4, the given hybrid system is final–state asymptotically determinable.

V. CONCLUDING REMARKS

We discussed the notion of generic final–state asymptotic determinability for hybrid systems following the lead of Sontag [7]. This notion facilitates the constructions of asymptotic state observers. Interestingly, generic final–state asymptotic determinability can be verified even if each of the continuous subsystems of the hybrid system is not observable. This result is based on the synergy that can be exploited between the continuous and the discrete dynamics.

VI. ACKNOWLEDGEMENTS

This work has been conducted with partial support by the European Community Projects IST-2001-33520 CC (Control and Computation) and IST-2001-32460 HYBRIDGE.

VII. REFERENCES

[1] E. De Santis, M.D. Di Benedetto, and G. Pola. On observability and detectability of continuous-time linear switching systems. In *These proceedings*, Maui, Hawaii, USA, December 2003.

[2] A. Balluchi, L. Benvenuti, M. D. Di Benedetto, and A. L. Sangiovanni-Vincentelli. Design of observers for hybrid systems. In C.J. Tomlin and J.R. Greenstreet, editors, *Hybrid Systems: Computation and Control*, volume 2289 of *Lecture Notes in Computer Science*, pages 76–89. Springer-Verlag, Stanford, CA, 2002.

[3] A. Balluchi, L. Benvenuti, M. D. Di Benedetto, and A. L. Sangiovanni-Vincentelli. Obsevability and observers synthesis for hybrid systems. Technical report, UCB ERL, 2003.

[4] A. Balluchi, L. Benvenuti, and A. L. Sangiovanni-Vincentelli. Observers for hybrid systems with continuous state resets. In *Proc. 10th Mediterranean Conference on Control and Automation - MED2002*, Lisbon, Porgugal, July 2002.

[5] C. Van Loan. The sensitivity of the matrix exponential. *SIAM Journal of Number Analysis*, 14(6):971–981, December 1977.

[6] A.S. Morse. Supervisory control of a families of linear set-point controllers – part 1: Exact matching. *IEEE Trans. on Automatic Control*, 41(10):1413–1431, October 1996.

[7] E.D. Sontag. On the observability of polynomial systems, i: finite–time problems. *SIAM J. Control and Optimization*, 17(1):139–151, 1979.