FUZZY ORDERS IN APPROXIMATE REASONING

by
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Abstract. An approach to fixed point theory based on the notion of fuzzy order is proposed. Such an approach extends both the fixed point theory in ordered sets and the fixed point theory in metric spaces. This since the fuzzy orders are strictly related with the quasi-metrics as defined by A. K. Seda in [10]. The aim is to give new tools for logic programming and for approximate reasoning.

Keywords. Fuzzy orders, fixed points, logic programming, approximate reasoning, relevance.

1. Introduction

Fixed point theory for operators in an ordered set is a basic tool for formal logic. Indeed, usually any logical apparatus enables us to define an "*immediate consequence operator*" T in a complete lattice L in such a way that:
- the elements of L represent "*pieces of information*",
- the order in L is intended with respect to the informative content, i.e. $x \leq y$ means that y is more complete than x,
- $T(x)$ is the information we can obtain from x by one step of the inferential process.

In accordance:
- the fixed points of T are the deductively closed pieces of information,
- given $x \in L$, the least fixed point $D(x)$ of T greater or equal to x is the theory generated by x.

As an example, in a classical logic whose set of formulas is F, L is the class $\mathcal{P}(F)$ of all subsets of F. In a fuzzy logic, L is the class $\mathcal{F}(F)$ of all fuzzy subsets of F (as an example, see [12] for crisp logic and [4] for fuzzy logic). Also, fixed point theory is very useful in logic programming. In such a case, if B is the Herbrand base associate with a program P, then L is the power set $\mathcal{P}(B)$ and T the immediate consequence operator associated with P. The fixed points of T are the Herbrand models for P and therefore the least fixed point of T is the least Herbrand model of P. The same holds for fuzzy logic programming (see [2], [6] and [8]).

Now if T is monotone and therefore the logic under consideration is monotone, it is possible to apply fixed point theory in ordered set as proposed by Knaster, Tarski and other authors. Indeed, one proves that, under suitable hypotheses, the least fixed point $D(x)$ of T greater or equal to x is equal to $Sup_{n\in N}T^n(x)$. Unfortunately, when the immediate consequence operator T is not monotone, for instance if T is associated with a program containing negation, such an approach does not work. Moreover, it is as well inadequate in a monotone fuzzy logic. In fact, in such a case the process leading to the fuzzy set of consequences from a fuzzy set of hypotheses takes place in the continuous context given by the real interval [0,1]. So, such a process is an endless algorithm to approximate the ideal output and therefore the need arises to define someway the notion of *"sufficiently precise result"*. The ordered sets are not the adequate framework to define such a notion.

A tentative to solve these difficulties is to consider an approach metric in nature and to apply fixed point techniques which are derived from Banach-Caccioppoli's theorem (see [8]).

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On the other hand, fixed point theory in ordered sets and fixed point theory in metric spaces can be unified. Indeed, if one introduces the notion of quasi-metric space, lacking in symmetric property, it is possible to demonstrate a theorem extending both the fixed point theorem for ordered structures and for metric spaces (see [8], [9], [10]).

In this paper we observe that the notions of quasi-metric and fuzzy ordering are strictly related (see Section 3). So, it is not surprising that we can find fixed point theorems based on the notion of fuzzy order (Section 4) and that these theorems enables us to unify fixed point theory for ordered structures and for metric spaces. In particular, we are interested to fuzzy inclusion for logic programming. Indeed, since it is possible to define several examples of fuzzy inclusion with a *"logical"* meaning (see Section 5), we hope that the sketched fixed point theory will give intuitive and useful tools for crisp and fuzzy logic programming.

2. Preliminaries

In the following R denotes the real numbers set, $R^+ = \{x\in R : x\geq 0\}$ and $[0,1] = \{x\in R : 0\leq x\leq 1\}$. Moreover, we denote by $\wedge$ and $\vee$ the binary operations in [0,1] defined by setting $x\wedge y = min\{x,y\}$ and $x\vee y = max\{x,y\}$. Trivially, [0,1] is a complete, completely distributive lattice under these operations. Let M be a set, then we call *fuzzy subset* of M or, more simply, *fuzzy set* any function s: $M \rightarrow [0,1]$. The idea is that if $x\in M$, then $s(x)$ is the membership degree of x to s. Given two fuzzy sets s and t, we set $s\subseteq t$ provided that $s(x)\leq t(x)$ for every $x\in M$. Moreover a fuzzy set $s : M \rightarrow [0,1]$ is called *crisp* if $s(x) \in \{0,1\}$ for every $x\in M$. We identify any subset X of M with the crisp subset $c_X : M \rightarrow \{0,1\}$ defined by setting $c_X(x) = 1$ if $x \in X$ and $c_X(x) = 0$ if $x \notin X$, i.e. we identify the subsets of M with the related characteristic functions. Given a fuzzy set s, for every $\lambda \in [0,1]$, the subset

$$C(s, \lambda) = \{x\in M \,/\, s(x) \geq \lambda\}$$

is called the (closed) λ-*cut* of s. Given two sets M_1 and M_2, we call *fuzzy relation* from M_1 to M_2 any fuzzy subset $\mathcal{R} : M_1 \times M_2 \rightarrow [0,1]$ of the Cartesian product $M_1 \times M_2$. We denote by $\mathcal{R}_\lambda$ the cut $C(\mathcal{R}, \lambda) = \{(x,y)\in M_1 \times M_2 \,/\, \mathcal{R}(x,y) \geq \lambda\}$.

Given a first order language L, we define a *fuzzy interpretation* of L as a pair (D, I) where, for any n-ary relation symbol r, $\mathcal{R} = I(r)$ is an n-ary fuzzy relation in D, i.e. a fuzzy subset of D^n. The constants and the function symbols are interpreted as usual. The logical connectives are interpreted by suitable operations in [0,1]. In particular, the conjunction $\wedge$ is interpreted by a triangular norm $\otimes$ in accordance with the following definition.

Definition 2.1. Let M be a non-empty set and $\otimes : [0,1]^2 \rightarrow [0,1]$ a binary operation. Then $\otimes$ is called a *triangular norm* (briefly, a *t-norm*) provided the following conditions hold:
(i) $\otimes$ is associative;
(ii) $\otimes$ is commutative;
(iii) $\otimes$ is order-preserving in both variables;
(iv) $x \otimes 1 = x \quad \forall\, x\in [0, 1]$.
A t-norm is called *continuous* provided that it preserves the least upper bounds.

Examples of continuous triangular norms are the usual product, the minimum $\wedge$ and the Lukasiewicz norm defined by setting $x\otimes y = x+y-1$ if $x+y-1\geq 0$ and $x\otimes y = 0$ otherwise. We interpret the implication by the binary operation $\rightarrow$ defined by setting $x\rightarrow y = Sup\{z \in [0,1] \,/\, x\otimes z \leq y\}$. It is immediate that $x\rightarrow y = 1$ if and only if $x\leq y$. Also, we consider some additional 1-ary connectives called *modifiers* corresponding to words as "*very*", "*almost*", *"completely"* ...

of the natural language. In particular, we are interested to the connective *al* ("*almost*") we interpret by a power function x^c with $0 < c < 1$. (see Zadeh, [14]) and to the connective *compl* (*"completely"*) we interpret by the function C: $[0,1] \to [0,1]$ defined by setting $C(x) = 1$ if $x = 1$ and $C(x) = 0$ otherwise. Finally, as usual, we interpret the universal quantifier by the least upper bound. This enables us, given a fuzzy interpretation (D,I), to associate any formula α, whose variables are among $x_1,...,x_n$ and any $d_1, ...,d_n \in D$, with a value $|\alpha|^{d_1 \cdots d_n}$ in [0,1] (see Hájek, [6]).

Consider a first order language L containing a binary relation name $\leq$. Then we can consider the following formulas, which are basic to define the notion of order and equivalence in classical set theory:

- $\forall x(x\leq x)$; (*reflexivity*)
- $\forall x\forall y\forall z((x\leq y\wedge y\leq z) \to x\leq z)$; (*transitivity*)
- $\forall x\forall y((compl(x\leq y)\wedge compl(y\leq x)) \to x = y)$; (*antisymmetry*)
- $\forall x\forall y(x\leq y \to y\leq x)$ (*simmetry*).

It is evident that a fuzzy interpretation $\mathcal{R} = I(\leq)$ in a domain M satisfies the above axioms if and only if the following are satisfied:

(1) $\mathcal{R}(x,x) = 1$; (*reflexivity*)
(2) $\mathcal{R}(x,y) \otimes \mathcal{R}(y,z) \leq \mathcal{R}(x,z)$; ($\otimes$-*transitivity*)
(3) $\mathcal{R}(x,y) = 1$ and $\mathcal{R}(y,x) = 1 \Rightarrow x = y$; (*antisymmetry*)
(4) $\mathcal{R}(x,y) = \mathcal{R}(y,x)$ (*symmetry*).

for any x, y, z in M.

Definition 2.2. If $\mathcal{R}$ is reflexive and $\otimes$−transitive, $\mathcal{R}$ is called a $\otimes$−*fuzzy preorder*. If $\mathcal{R}$ satisfies also the antisymmetry, $\mathcal{R}$ is called a $\otimes$−*fuzzy order*.

Observe that if $\mathcal{R}$ is a crisp relation, then $\mathcal{R}$ is $\otimes$−fuzzy preorder ($\otimes$−fuzzy order) if and only if $\mathcal{R}$ is the characteristic function of a preorder (order) relation. In the following, we omit the symbol $\otimes$ every time $\otimes$ denotes the minimum. So in such a case we write *transitivity*, *fuzzy preorder*, *fuzzy order* instead of $\otimes$−*transitivity*, $\otimes$−*fuzzy preorder*, $\otimes$−*fuzzy order.*

Proposition 2.3. *Let* $\mathcal{R} : M\times M \to [0,1]$ *be a fuzzy relation. Then* $\mathcal{R}$ *is reflexive (symmetric, transitive) if and only if every cut* $\mathcal{R}_\lambda = C(\mathcal{R},\lambda)$ *of* $\mathcal{R}$ *is reflexive (symmetric, transitive). Consequently* $\mathcal{R}$ *is a fuzzy preorder if and only if every cut* $\mathcal{R}_\lambda$ *is a preorder.*

Proof. Assume that $\mathcal{R}$ is reflexive and let $x\in M$. Then, for any $\lambda \in [0,1]$, $\mathcal{R}(x, x) = 1 \geq \lambda$ and therefore $x\mathcal{R}_\lambda x$. This proves that $\mathcal{R}_\lambda$ is reflexive. Conversely, assume that $x\mathcal{R}_\lambda x$, i.e. that $\mathcal{R}(x, x) \geq \lambda$, for any $\lambda \in [0,1]$. Then $\mathcal{R}(x, x) = 1$ and this proves that $\mathcal{R}$ is reflexive.

Assume that $\mathcal{R}$ is symmetric and let $\lambda \in [0,1]$. Then from $x\mathcal{R}_\lambda y$ it follows that $\mathcal{R}(y,x) = \mathcal{R}(x,y) \geq \lambda$ and therefore $y\mathcal{R}_\lambda x$. This proves that $\mathcal{R}_\lambda$ is symmetric. Conversely assume that each $\mathcal{R}_\lambda$ is symmetric. Then from $\mathcal{R}(x,y) \geq \lambda$ follows that $(x,y) \in \mathcal{R}_\lambda$, and therefore $y\mathcal{R}_\lambda x$. This proves that $\mathcal{R}(y,x) \geq \lambda$. In particular, by setting $\lambda = \mathcal{R}(x,y)$ we have $\mathcal{R}(y,x) \geq \mathcal{R}(x,y)$ and so $\mathcal{R}(y,x) = \mathcal{R}(x,y)$. Thus $\mathcal{R}$ is symmetric.

Finally, assume that $\mathcal{R}$ is transitive and let $\lambda \in [0,1]$. Then from $x\mathcal{R}_\lambda y$ and $y\mathcal{R}_\lambda z$ it follows that $\mathcal{R}(x, y) \geq \lambda$ and $\mathcal{R}(y, z) \geq \lambda$ and therefore $\mathcal{R}(x,z) \geq \mathcal{R}(x,y)\wedge\mathcal{R}(y,z) \geq \lambda$. So, $x\mathcal{R}_\lambda z$ and this proves that $\mathcal{R}_\lambda$ is transitive. Conversely, assume that each $\mathcal{R}_\lambda$ is transitive. Then from $\mathcal{R}(x, y) \geq \lambda$ and $\mathcal{R}(y, z) \geq \lambda$, i.e. $x \mathcal{R}_\lambda y$ and $y\mathcal{R}_\lambda z$ it follows that $x\mathcal{R}_\lambda z$, i.e. $\mathcal{R}(x,z) \geq \lambda$. In particular, by setting $\lambda = \mathcal{R}(x, y)\wedge\mathcal{R}(y, z)$ we obtain $\mathcal{R}(x,z) \geq \mathcal{R}(x,y)\wedge\mathcal{R}(y,z)$. This proves that $\mathcal{R}$ is transitive.

If $\mathcal{R}$ is a $\otimes$−fuzzy preorder, and $\lambda \neq 1$, then the cut $\mathcal{R}_\lambda$ is not a preorder, in general. Instead, $\mathcal{R}_1 = \{(x, y) \in M\times M \,/\, \mathcal{R}(x, y) = 1\}$ is always a preorder we call the *preorder associated with* $\mathcal{R}$. Indeed, trivially $\mathcal{R}_1$ is reflexive. Moreover, from $(x,y) \in \mathcal{R}_1$ and $(y,z)\in \mathcal{R}_1$ it follows that

$$\mathcal{R}(x,z) \geq \mathcal{R}(x,y)\otimes\mathcal{R}(y,z) = 1\otimes 1 = 1,$$

and therefore that $x\mathcal{R}_1 z$. As usual such a preorder defines an equivalence relation $\equiv$, namely we set

$$x\equiv y \quad\Leftrightarrow\quad \mathcal{R}(x, y) = \mathcal{R}(y, x) = 1.$$

If $x\equiv y$ we say that x is *similar* with y. Moreover, we can consider the quotient $M' = \{[x] \,/\, x \in M\}$, where $[x] = \{x' \in M \,/\, x\equiv x'\}$. It is immediate to prove that the fuzzy relation $\mathcal{R}' : M' \times M' \to [0,1]$ such that $\mathcal{R}'([x], [y]) = \mathcal{R}(x, y)$ is well defined and it is a $\otimes$−fuzzy order on M'. By this identification is always possible to change from a $\otimes$−fuzzy preorder relation to a $\otimes$−fuzzy order relation.

Definition 2.4. Given a map f: $M \to M$ and a $\otimes$−fuzzy preorder $\mathcal{R}$, we say that $x \in M$ is a *fixed point* for f (w.r. to $\mathcal{R}$) provided that $f(x) \equiv x$.

Trivially, in the case that $\mathcal{R}$ is a $\otimes$−fuzzy order, x is a fixed point if and only if $f(x) = x$.

3. Quasi-metrics and fuzzy orders.

Let M be a non-empty set and d: $M\times M \to R^+$ a mapping. Also, consider the following axioms for any $x, y, z \in M$:

(d1) $d(x, y) = 0 \Rightarrow x = y$;
(d'1) $d(x, x) = 0$;
(d2) $d(x, y) = d(y, x)$;
(d'2) $d(x, y) = 0$ and $d(y, x) = 0 \Rightarrow x = y$;
(d3) $d(x, y) + d(y, z) \geq d(x, z)$;
(d'3) $d(x, y) \vee d(y, z) \geq d(x, z)$.

The properties (d1), (d'1), (d2) and (d3) characterise the *metric spaces*. When (d3) is substituted with (d'3) we obtain the *ultrametric spaces*. Since (d'3) entails (d3), any ultrametric space is a metric space. We can also consider the following definitions: (M, d) is called a

generalised quasi-metric space if it satisfies (d'1) and (d3);
quasi-metric space if it satisfies (d'1), (d'2) and (d3);
generalised quasi-ultrametric space if it satisfies (d'1) and (d'3);
quasi-ultrametric space if it satisfies (d'1), (d'2) and (d'3).

Then, by referring to the usual definition of metric space, the word "*quasi*" refers to the lack of the symmetric property. The word "*generalised*" refers to the lack of (d'2). Finally, the word "*ultra*" refers to the fact that (d3) is substituted by condition (d'3). The following proposition, whose proof is trivial, extends a connection between similarities and ultrametrics proved in [12].

Theorem 3.1. *Let* $\otimes$ *be the t- norm of the minimum, d:* $M \times M \to [0,1]$ *a map, and set*

$$\mathcal{R}(x, y) = 1 - d(x, y)$$

for any $x, y \in M$. *Then:*

(i) $\mathcal{R}$ *is a fuzzy preorder if and only if d is a generalised quasi-ultrametrics;*

(ii) $\mathcal{R}$ *is a fuzzy order if and only if d is a quasi-ultrametrics.*

Example. Since any crisp preorder is a fuzzy preorder, by Theorem 3.1 we have that any preorder $(M, \leq)$ can be viewed as a generalised quasi-ultrametric space. Namely, the map d defined by setting

$$d(p, q) = \begin{cases} 0 & \text{if } p \leq q \\ 1 & \text{otherwise.} \end{cases}$$

is a generalised quasi-ultrametric (see also [1], [9]). In the same way, any order defines a quasi-ultrametric space.

To prove an analogous result for triangular norms different from the minimum it is useful the following definition. Let $\otimes$ be a t-norm, $x \in [0, 1]$ and $n \in N$. Then, as usual, we define $x^{(n)}$ by setting $x^{(1)} = x$ and $x^{(n+1)} = x \otimes x^{(n)}$ if $n > 1$.

Definition 3.2. A t-norm $\otimes$ is called *Archimedean* if, for any pair $x, y \in [0, 1]$, an integer n exists such that $x^{(n)} < y$.

The usual product is an example of Archimedean t-norm. The minimum is an example of non-Archimedean t-norm. The class of Archimedean *t*-norms admits the following characterization. Denote by $[0, \infty]$ the extension R^+ by the symbol ∞ with the convention $x \leq \infty$ and $x + \infty = \infty + x = \infty + \infty = \infty$, for any $x \in [0, \infty]$. Then we say that $f : [0, 1] \to [0, \infty]$ is an *additive generator* provided that f is a *continuous strictly decreasing function such that* $f(1) = 0$. Also define the *pseudoinverse* $f^{[-1]} : [0,\infty] \to [0, 1]$ of f by setting:

$$f^{[-1]}(x) = \begin{cases} f^{-1}(x) & \text{if } x \in f([0, 1]) \\ 0 & \text{otherwise.} \end{cases}$$

Trivially, we have that $f^{[-1]}(0) = 1$ and $f^{[-1]}(\infty) = 0$. Moreover, $f^{[-1]}$ is order-reversing.

Proposition 3.3. *An operation* $\otimes : [0,1]\times[0,1] \to [0,1]$ *is a a continuous Archimedean t-norm if and only if an additive generator f exists such that*

$$x \otimes y = f^{[-1]}(f(x) + f(y))$$

for all x, y in $[0, 1]$.

Now we are able to establish a connection among ⊗−fuzzy preorders and generalized quasi-metrics.

Theorem 3.4. *Let* ⊗ *be a continuous Archimedean t-norm and* $f : [0,1] \to [0, \infty]$ *an additive generator of* ⊗. *Moreover, let* $d : M\times M \to R^+$ *be a map and define* $\mathcal{R}_d$ *by setting*

$$\mathcal{R}_d(x, y) = f^{[-1]}(d(x,y)).$$

Then:

i) d is a generalised quasi-metrics ⇒ $\mathcal{R}_d$ *is a* ⊗*-fuzzy preorder;*

ii) d is a quasi-metrics ⇒ $\mathcal{R}_d$ *is a* ⊗*-fuzzy order.*

Conversely, let $\mathcal{R} : M\times M \to [0,1]$ *be a map and consider the function* $d_{\mathcal{R}}$ *defined by setting*

$$d_{\mathcal{R}}(x, y) = f(\mathcal{R}(x, y)).$$

Then:

i') $\mathcal{R}$ *is a* ⊗*-fuzzy preorder* ⇒ $d_{\mathcal{R}}$ *is a generalised quasi-metrics;*

ii') $\mathcal{R}$ *is a* ⊗*-fuzzy order* ⇒ $d_{\mathcal{R}}$ *is a quasi-metrics.*

Proof. (*i*) Since $\mathcal{R}_d(x,x) = f^{[-1]}(d(x,x)) = f^{[-1]}(0) = 1$, $\mathcal{R}_d$ is reflexive. To prove the ⊗-transitivity, i.e. that

$$\mathcal{R}_d(x, y) \otimes \mathcal{R}_d(y, z) \leq \mathcal{R}_d(x, z),$$

observe that in the case $d(x,y) \notin f([0,1])$ we have that $\mathcal{R}_d(x, y) = f^{[-1]}(d(x,y)) = 0$ and in the case $d(y,z) \notin f([0,1])$, we have that $\mathcal{R}_d(y, z) = f^{[-1]}(d(y,z)) = 0$ and such an inequality is trivial. Assume that both $d(x,y)$ and $d(y,z)$ are in $f([0,1])$, then,

$$\begin{aligned}\mathcal{R}_d(x, y) \otimes \mathcal{R}_d(y, z) &= f^{[-1]}(d(x,y)) \otimes f^{[-1]}(d(y,z))\\ &= f^{[-1]}(f(f^{[-1]}(d(x,y))) + f(f^{[-1]}(d(y,z))))\\ &= f^{[-1]}(d(x,y) + d(y,z)) \leq f^{[-1]}(d(x, z)) = R_d(x, z).\end{aligned}$$

(*ii*) We have to prove the antisymmetry of $\mathcal{R}_d$. So, let $x, y \in M$ such that $\mathcal{R}_d(x, y) = 1$ and $\mathcal{R}_d(y, x) = 1$. From this conditions it follows that $f^{[-1]}(d(x, y)) = 1 = f^{[-1]}(d(y, x))$, and therefore $f^{1}(d(x,y)) = 1 = f^{1}(d(y,x))$. Then $d(x,y) = 0 = d(y,x)$ and so $x = y$ for the antisymmetry of d.

(*i*') Reflexivity of $d_{\mathcal{R}}$ follows immediately from definitions. Before to prove the transitivity of $d_{\mathcal{R}}$, we observe that

$$f(f^{[-1]}(x)) = \begin{cases} x & \text{if } x \in f([0, 1]) \\ f(0) & \text{otherwise} \end{cases}$$

where $f(0)$ is the maximum of the function. From ⊗-transitivity of $\mathcal{R}_d$ it follows that

$$f(\mathcal{R}(x, y) \otimes \mathcal{R}(y, z)) \geq f(\mathcal{R}(x, z)),$$

because f is strictly decreasing. Then

$$f[f^{[-1]}(f(\mathcal{R}(x, y) \otimes f(\mathcal{R}(y, z)))] \geq f(\mathcal{R}(x, z)).$$

Now, if $f(\mathcal{R}(x, y)) + f(\mathcal{R}(y, z)) \in f([0,1])$, we obtain that

$$f(\mathcal{R}(x, y)) + f(\mathcal{R}(y, z)) \geq f(\mathcal{R}(x, z),$$

and then the thesis. Otherwise, $f(\mathcal{R}(x, y)) + f(\mathcal{R}(y, z)) \geq f(0) \geq f(\mathcal{R}(x, z)$.

(ii') We have to prove the antisymmetry of $d_{\mathcal{R}}$. So, let $x, y \in M$ such that $d_{\mathcal{R}}(x, y) = 0$ and $d_{\mathcal{R}}(y, x) = 0$. Then $f(\mathcal{R}(x,y)) = 0 = f(\mathcal{R}(y,x))$, and hence $\mathcal{R}(x,y) = 1 = \mathcal{R}(y,x)$. From the antisymmetry of $\mathcal{R}$ it follows that $x = y$.

4. Fixed point theorems for fuzzy orders

We extend the notion of order preserving sequence in an ordered set M as follows. Let $\mathcal{R}$ be a binary relation in a set M. Then we say that a sequence $(x_n)_{n\in N}$ in M is *eventually $\mathcal{R}$-preserving* if a natural number $\underline{n}$ exists such that $(x_n,x_m)\in \mathcal{R}$ whenever $m \geq n \geq \underline{n}$. Equivalently, $(x_n)_{n\in N}$ is eventually $\mathcal{R}$-preserving if the implication

$$n\leq m \Rightarrow (x_n,x_m)\in \mathcal{R}$$

holds almost everywhere. Let $\mathcal{R}$ be a binary fuzzy relation in M. Then we say that $(x_n)_{n\in N}$ is eventually *$\mathcal{R}$-preserving* if $(x_n)_{n\in N}$ is eventually $\mathcal{R}_\lambda$-preserving for any $\lambda\in[0,1]$, $\lambda\neq 1$.

Proposition 4.1. *Let $\mathcal{R}$ be a fuzzy preorder. Then $(x_n)_{n\in N}$ is eventually $\mathcal{R}$-preserving if and only if for each ε such that $0<\varepsilon<1$, a natural number $\underline{n}$ exists such that $\mathcal{R}(x_n,x_{n+1})\geq\varepsilon$ for all $n\geq\underline{n}$.*

Proof. It is sufficient to observe that, since each $\mathcal{R}_\varepsilon$ is transitive, from $x_n \mathcal{R}_\varepsilon x_{n+1}$, $x_{n+1} \mathcal{R}_\varepsilon x_{n+2}$, ..., $x_{m-1} \mathcal{R}_\varepsilon x_m$ we have that $x_n \mathcal{R}_\varepsilon x_m$.

Furthermore, we say that an $\mathcal{R}$-preserving sequence $(x_n)_{n\in N}$ *converges to l*, and we write $lim_{n\to\infty}x_n = l$, if

$$\mathcal{R}(l, x) = lim_{n\to\infty}\ \mathcal{R}(x_n, x)$$

for all $x \in M$. As usual, in such a case l is called a *limit* of $(x_n)_{n\in N}$. Observe that if l is a limit of $(x_n)_{n\in N}$, then $lim_{n\to\infty}\mathcal{R}(x_n, l) = 1$. Such a definition extend the classical definition of least upper bound. Indeed, assume that $\mathcal{R}$ is (the characteristic function of) a partial order $\leq$ and that $(x_n)_{n\in N}$ is order preserving with respect to $\leq$. Then the statement $lim_{n\to\infty}x_n = l$ is equivalent with

$$\forall x \in M \quad (l \leq x \Leftrightarrow \exists m\ \forall n \geq m,\quad x_n \leq x),$$

and therefore with $l = Sup\{x_n \mid n \in N\}$.

The structure $(M,\mathcal{R})$ is called *complete* if every eventually $\mathcal{R}$-preserving sequence admits a limit.

Proposition 4.2. *Let $\mathcal{R}$ be a $\otimes$-fuzzy preorder. Then, two limits of a given sequence are similar. Consequently, if $\mathcal{R}$ is a $\otimes$-fuzzy order, then limits are unique.*

Proof. Assume that $lim_{n\to\infty}x_n = l$ and $lim_{n\to\infty}x_n = l'$. Then, by definition,

$$\mathcal{R}(l, x) = lim_{n\to\infty}\mathcal{R}(x_n, x) \text{ and } \mathcal{R}(l', x) = lim_{n\to\infty}\mathcal{R}(x_n, x)$$

for all $x \in M$. In particular, by setting $x = l$, $1 = \mathcal{R}(l, l) = lim_{n\to\infty}\mathcal{R}(x_n, l) = \mathcal{R}(l', l)$ and, by setting $x = l'$, $1 = \mathcal{R}(l', l') = lim_{n\to\infty}\mathcal{R}(x_n, l') = \mathcal{R}(l, l')$. Then $1 = \mathcal{R}(l', l) = \mathcal{R}(l, l')$ and l is similar with l'.

By recalling that a ⊗−fuzzy preorder $\mathcal{R}$ is a suitable interpretation of binary relation ≤, we propose the following definition.

Definition 4.3. We say that *f* is *order preserving* if the following formula

$$\forall x \forall y (x \leq y \rightarrow f(x) \leq f(y))$$

holds, i.e. if $\mathcal{R}(f(x), f(y)) \geq \mathcal{R}(x, y)$ for all $x, y \in M$.

Definition 4.4. Let $\mathcal{R}$ be a ⊗−fuzzy preorder and f: $M \rightarrow M$ an order preserving map; *f* is called *continuous* if from $lim_{n\rightarrow\infty}x_n = l$ it follows $lim_{n\rightarrow\infty}f(x_n) = f(l)$, for every eventually $\mathcal{R}$-preserving sequence $(x_n)_{n\in N}$ in M.

Obviously, when M is a partial ordered set, *f* is continuous if and only if it preserves least upper bounds of chains.

Definition 4.5. Let *al* be the linguistic modifier "*almost*" we interpret by the function x^c, $0 < c < 1$. Then we say that a continuous map *f* is *strongly order preserving* if the formula

$$\forall x \forall y (al(x \leq y) \rightarrow f(x) \leq f(y))$$

holds, i.e. if

$$\mathcal{R}(f(x), f(y)) \geq \mathcal{R}(x, y)^c.$$

We can prove the first fixed point theorem extending analogous theorems in ordered sets.

Theorem 4.6. *Let $\mathcal{R}$ be a ⊗−fuzzy preorder such that $(M, \mathcal{R})$ is complete and let f: $M \rightarrow M$ be a continuous map such that $\mathcal{R}(x, f(x)) = 1$ for a suitable $x \in M$. Then f has a fixed point.*

Proof. Since *f* is order preserving, the sequence x, $f(x)$, $f^2(x)$,..... is eventually $\mathcal{R}$-preserving. Indeed, trivially we have that

$$1 = \mathcal{R}(x, f(x)) \leq \mathcal{R}(f(x), f^2(x)) \leq ... \leq \mathcal{R}(f^n(x), f^{n+1}(x)),$$

and so, for each $m \geq n$,

$$\mathcal{R}(f^n(x), f^m(x)) \geq \mathcal{R}(f^n(x), f^{n+1}(x)) \otimes ... \otimes \mathcal{R}(f^{m-1}(x), f^m(x)) \geq 1 \otimes ... \otimes 1 = 1$$

This proves that $(f^n(x))_{n\in N}$ is eventually $\mathcal{R}$-preserving. Then by the completeness hypothesis, there exists a limit l of the sequence $(f^n(x))_{n\in N}$. Also, since *f* is continuous, we have that $f(l)$ is a limit of $(f^{n+1}(x))_{n\in N}$, and therefore of $(f^n(x))_{n\in N}$. Since $\mathcal{R}$ is a ⊗−fuzzy preorder, limits are similar, i.e. $\mathcal{R}(f(l), l) = 1 = \mathcal{R}(l, f(l))$. Thus l is a fixed point for *f*.

The following theorem corresponds with analogous theorems on contractions in metric spaces.

Theorem 4.7. *Assume that ⊗ is a t-norm greater or equal to the usual product, and let $\mathcal{R}$ be a ⊗−fuzzy order such that $(M, \mathcal{R})$ is complete and $f : M \rightarrow M$ be a strongly order preserving map. Then f has a unique fixed point.*

Proof. The proof consists of an obvious transposition of the proof of the fixed point theorem for contractions in a metric space. Let x_0 be any element of M, and consider the sequence $(x_n)_{n\in N}$ defined as follows: $x_1 = f(x_0)$, $x_2 = f(x_1)$, ... $x_{n+1} = f(x_n)$, ... We have to prove that this sequence is eventually $\mathcal{R}$-preserving. Observe that by hypotheses c exists such that $0 < c < 1$ and:

$$\mathcal{R}(x_1, x_2) = \mathcal{R}(f(x_0), f(x_1)) \geq (\mathcal{R}(x_0, x_1))^{c} = (\mathcal{R}(x_0, f(x_0)))^{c}$$

$$\mathcal{R}(x_2, x_3) = \mathcal{R}(f(x_1), f(x_2)) \geq (\mathcal{R}(x_1, x_2))^{c} \geq (\mathcal{R}(x_0, x_1))^{c^2} = (\mathcal{R}(x_0, f(x_0)))^{c^2}$$

$$\cdots$$

$$\mathcal{R}(x_n, x_{n+1}) = \mathcal{R}(f(x_{n-1}), f(x_n)) \geq (\mathcal{R}(x_{n-1}, x_n))^{c} \geq (\mathcal{R}(x_0, x_1))^{c^n} = (\mathcal{R}(x_0, f(x_0)))^{c^n}$$

Let ε such that $1 > \varepsilon > 0$ and set $d = \mathcal{R}(x_0, f(x_0))$. Then, since $lim_{n\to\infty}\frac{c^n}{1-c} = 0$ and $log_d(\varepsilon) > 0$, a natural number n_0 exists such that $\frac{c^n}{1-c} \leq log_d(\varepsilon)$ for any $n \geq n_0$. Since

$$c^n + c^{n+1} + \ldots + c^{n+r-1} = \frac{c^n(1-c^r)}{1-c} \leq \frac{c^n}{1-c},$$

this entails that $c^n + c^{n+1} + \ldots + c^{n+r-1} \leq log_d(\varepsilon)$ for any $n \geq n_0$. Consequently,

$$\mathcal{R}(x_n, x_{n+r}) \geq \mathcal{R}(x_n, x_{n+1}) \otimes \mathcal{R}(x_{n+1}, x_{n+2}) \otimes \ldots\ldots \otimes \mathcal{R}(x_{n+r-1}, x_{n+r}) \geq$$

$$\geq d^{c^n} d^{c^{n+1}} \ldots\ldots d^{c^{n+r-1}} = d^{c^n + c^{n+1} + \ldots + c^{n+r-1}} \geq d^{\log_d(\varepsilon)} = \varepsilon$$

for any $n \geq n_0$. and $r \in N$, and then $(x_n)_{n\in N}$ is eventually $\mathcal{R}$-preserving. From completeness it follows that there is a limit l of the sequence. Because of f is continuous, $f(l) = lim_{n\to\infty}f(x_n) = lim_{n\to\infty}x_{n+1} = l$, and then l is a fixed point for f. Suppose l_1 is another fixed point. Then, $\mathcal{R}(l, l_1) = \mathcal{R}(f(l), f(l_1)) \geq (\mathcal{R}(l, l_1))^{c}$, and hence $\mathcal{R}(l, l_1) = 1$. In the same way we can prove that $\mathcal{R}(l_1, l) = 1$, so from antisymmetry it follows $l = l_1$.

Observe that, in particular, Theorem 4.7 is true for the t-norm of the minimum. Indeed, since $x \cdot y \leq x \cdot 1 = x$ and $x\cdot y \leq 1\cdot y = y$, we have that $x\cdot y \leq x \wedge y$.

5. Examples for logic programming

In order to apply the proposed notions to logic programming, it is useful to consider fuzzy orders defined in a power set $\mathcal{P}(M)$. We will call *generalised ⊗−fuzzy inclusion* any ⊗−fuzzy preorder extending the set theoretical inclusion i.e. any symmetric ⊗−transitive fuzzy relation $Incl : \mathcal{P}(M)\times\mathcal{P}(M)\to[0,1]$ such that $X \subseteq Y \Rightarrow Incl(X, Y) = 1$. We say that $Incl$ is a *⊗−fuzzy inclusion* if $Incl$ is a ⊗−fuzzy order. Trivially, if $Incl$ is a generalised ⊗−fuzzy inclusion, we have that:

a) $X_1 \subseteq X_2 \Rightarrow Incl(X_1, Y) \geq Incl(X_2, Y)$;

b) $Y_1 \subseteq Y_2 \Rightarrow Incl(X, Y_1) \leq Incl(X, Y_2)$.

Indeed, from transitivity follows that

$$Incl(X_1, Y) \geq Incl(X_1, X_2)\otimes Incl(X_2, Y) \geq 1\otimes Incl(X_2, Y) = Incl(X_2, Y),$$

and then a). Likewise one proves b). We say that $Incl$ is a *⊗−fuzzy inclusion* if $Incl$ is a fuzzy order (a ⊗−fuzzy order).

In order to give some examples of fuzzy inclusion, we fix a fuzzy subset rel: $M \to [0,1]$ of M, we interpret as the fuzzy subset of elements which are "*relevant*". Such a fuzzy subset is associated with a map μ: $\mathcal{P}(M) \to [0,1]$ such that $\mu(\emptyset) = 0$ and, if $X \neq \emptyset$,

$$\mu(X) = Sup\{rel(x) \,/\, x \in X\}. \tag{5.1}$$

We call such a map the *possibility measure defined by rel*. By recalling that the existential quantifier is interpreted in [0,1] by the least upper bound, we can interpret $\mu(X)$ as the truth-degree of the claim "*there is a relevant element in X*".

Proposition 5.1. *Let* μ *be the possibility measure defined by* (5.1) *and define the fuzzy relation Incl* : $\mathcal{P}(M)\times\mathcal{P}(M) \to [0,1]$ *by setting*

$$Incl(X, Y) = 1 - \mu(X - Y). \quad (5.2)$$

Then, Incl is a generalised fuzzy inclusion. If $rel(x) \neq 0$ *for any* $x \in M$, *then Incl is a fuzzy inclusion.*

Proof. Assume that $X \subseteq Y$, then $X-Y = \emptyset$, and therefore $Incl(X, Y) = 1$. This proves that *Incl* is an extension of $\subseteq$. By Theorem 3.1, to prove that *Incl* is a fuzzy preorder it is sufficient to prove that $d(X,Y) = \mu(X-Y)$, is a generalised quasi-ultrametric. Indeed, (d'1) follows from the definition; to prove (d'3), i.e. that $\mu(X-Z) \leq \mu(X-Y) - \mu(Y-Z)$, observe that

$$X-Z \subseteq ((X-Y)\cup(Y-Z)). \quad (5.3)$$

Indeed, assume that $x \in X-Z$. Then, in the case $x \in Y$ we have $x \in Y-Z$, in the case $x \notin Y$ we have $x \in X-Y$. From (5.3) we have

$$Sup\{rel(x) \,/\, x \in X-Z\} \leq Sup\{rel(x) \,/\, x \in (X-Y)\cup(Y-Z)\}$$
$$= Sup\{rel(x) \,/\, x \in X-Y\} \vee Sup\{rel(x) \,/\, x \in Y-Z\},$$

so (d'3) is satisfied.

Assume that $rel(x) \neq 0$ for any $x \in M$. Then $\mu(X) = 0$ entails that $X = \emptyset$. So from $\mu(X-Y) = 0$ we have that $X-Y = \emptyset$, and therefore $X \subseteq Y$. Likewise, from $\mu(Y-X) = 0$ we have that $Y-X = \emptyset$. So $Y \subseteq X$ and (d'2) is satisfied.

We can interpret $Incl(X,Y)$ as the truth degree of the claim "*there is no relevant element belonging in X and not in Y*", i.e., *"all the relevant elements of X are in Y"*. A different way to define such a fuzzy inclusion is the following. Given $\lambda \in [0,1]$, we call *λ-relevant* any element $x \in M$ such that $rel(x) \geq \lambda$ and we denote by $M_\lambda = C(rel,\lambda)$ the set of all λ-relevant elements. In accordance, a condition like $X \cap M_\lambda \subseteq Y$ means that every λ-relevant element of X belongs to Y.

Proposition 5.2. *Define d* : $\mathcal{P}(M)\times\mathcal{P}(M) \to [0,1]$ *by setting*

$$d(X,Y) = Inf\{\lambda \in [0,1] \,/\, X \cap M_\lambda \subseteq Y\} \quad (5.4)$$

and Incl by setting

$$Incl(X, Y) = 1 - d(X, Y).$$

Then Incl coincides with the generalized fuzzy inclusion defined by (5.1) *and* (5.2).

Proof. Obviously,

$$X \cap M_\lambda \subseteq Y \text{ and } \mu \geq \lambda \Rightarrow X \cap M_\mu \subseteq Y$$

and this means that $\{\lambda \in [0,1] \,/\, X \cap M_\lambda \subseteq Y\}$ is an interval. Consequently,

$$Inf\{\lambda \in [0,1] \,/\, X \cap M_\lambda \subseteq Y\} = Sup\{\lambda \in [0,1] \,/\, X \cap M_\lambda \text{ is not contained in } Y\}$$
$$= Sup\{\lambda \in [0,1] \,/\, x \in X \text{ exists such that } x \in M_\lambda \text{ and } x \notin Y\}$$
$$= Sup\{\lambda \in [0,1] \,/\, x \in X\text{-}Y \text{ exists such that } rel(x) \geq \lambda\}$$

$= Sup\{rel(x) / x \in X\text{-}Y\} = \mu(X\text{-}Y)$.

The just defined class of fuzzy inclusions extends the class given by A. K. Seda in the framework of programming logic (see [10]), in a sense. Namely, let n: $M \to N$ be any map and set, for every subset X of M,

$$I(X, \lambda) = \{x \in X / n(x) \leq \lambda\}.$$

Then, A. K. Seda defines the map d': $\mathcal{P}(M) \times \mathcal{P}(M) \to [0,1]$ by setting

$$d'(X, Y) = Inf\{2^{-\lambda} / I(X, \lambda) \subseteq I(Y, \lambda)\} \quad (5.5)$$

and it proves that d' is a quasi-ultrametric.

Proposition 5.3. *Consider the fuzzy set rel* : $M \to [0, 1]$ *such that* $rel(x) = 2^{-n(x)}$, *and let d and d' be the maps defined by* (5.4) *and* (5.5), *respectively. Then* $d = d'$.

Proof. Observe that

$$d(X, Y) = Inf\{\lambda \in [0, 1] / \{x / 2^{-n(x)} \geq \lambda\} \cap X \subseteq Y\}$$
$$= Inf\{\lambda \in [0, 1] / \{x / log_2 2^{-n(x)} \geq log_2(\lambda)\} \cap X \subseteq Y\}$$
$$= Inf\{\lambda \in [0, 1] / \{x / n(x) \leq -log_2(\lambda)\} \cap X \subseteq Y\}$$
$$= Inf\{2^{-\lambda} / \{x / n(x) \leq \lambda\} \cap X \subseteq Y\} = d'(X, Y).$$

Another class of $\otimes$−fuzzy inclusions is obtained by assuming that rel: $M \to [0,1]$ satisfies $\sum_{x \in M} rel(x) = 1$. Then, we can define the *finitely additive probability with density* "*rel*" i.e. the map η : $\mathcal{P}(M) \to [0,1]$ such that $\eta(\emptyset) = 0$ and, if $X \neq \emptyset$,

$$\eta(X) = \sum_{x \in X} rel(x). \quad (5.6)$$

Differently from $\mu(X)$, $\eta(X)$ takes in account the number of relevant elements in X and therefore we can interpret $\eta(X)$ as a measure of the truth degree of the claim *"there are several relevant elements in X"*.

Proposition 5.4. *Let f be an additive generator and set*

$$Incl(X, Y) = f^{[-1]}(\eta(X - Y)). \quad (5.7)$$

Then Incl is a generalised $\otimes$−*fuzzy inclusion with respect to t-norm generated by f. If rel* $(x) \neq 0$ *for any* $x \in M$, *then Incl is a* $\otimes$−*fuzzy inclusion.*

Proof. Firstly we prove that $d(X, Y) = \eta(X-Y)$ is a generalised quasi-metrics, and then from Theorem 3.4 it follows that i is a $\otimes$−fuzzy order. (d'1) follows from the definition. To prove (d'3), i.e. that $\eta(X-Z) \leq \eta(X-Y) + \eta(Y-Z)$, recall the relation (5.3). Since η is a measure and $(X-Y)\cap(Y-Z) = \emptyset$,

$$\eta(X-Z) \leq \eta((X-Y)\cup(Y-Z)) = \eta(X-Y) + \eta(Y-Z).$$

Trivially, i satisfies conditions a), b) and c), and so is a $\otimes$−generalised fuzzy inclusion.
Finally, observe that if $rel(x) \neq 0$ for any $x \in M$, then $\eta(X) = 0$ entails that $X = \emptyset$ for any $X \subseteq M$;. So from $\eta(X-Y) = 0$ we have that $X-Y = \emptyset$, and therefore $X \subseteq Y$. Likewise, from $\eta(Y-X) = 0$ we have that $Y-X = \emptyset$. So $Y \subseteq X$ and (d'2) is satisfied.

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