

Monotone paths in line arrangements with a small number of directions

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Abstract

We give subquadratic bounds on the maximum length of an x -monotone path in an arrangement of n lines with at most $C \log \log n$ directions, where C is a suitable constant. For instance, the maximum length of an x -monotone path in an arrangement of n lines having at most ten slopes is $O(n^{67/34})$. In particular, we get tight estimates for the case of lines having at most five directions, by showing that previous constructions – $\Omega(n^{3/2})$ for arrangements with four slopes and $\Omega(n^{5/3})$ for arrangements with five slopes – due to Sharir and Matoušek respectively, are (asymptotically) best possible.

1 Introduction

Consider a set \mathcal{L} of n lines in the plane. The lines of \mathcal{L} induce a cell complex, $\mathcal{A}(\mathcal{L})$, called the *arrangement* of \mathcal{L} , whose *vertices* are the intersection points of the lines, whose *edges* are the maximal portions of lines containing no vertices and whose two-dimensional *cells* are the connected components of $\mathbb{R}^2 \setminus \cup_{\ell \in \mathcal{L}} \ell$. It will be further assumed that none of the lines is parallel to the y -axis. Other than that, parallel or concurrent lines are permitted. One of the properties of a line arrangement with n lines is the maximum possible length, denoted by λ_n , of an x -monotone polygonal line (path) composed of edges of the arrangement. The length is defined as the number of turns of the polygonal line plus one (i.e., the number of segments of the polygonal path). See [6] for other properties of line arrangements.

The problem to estimate λ_n was posed in [4]. An interesting application of this problem can be found in [11]. Sharir established an $\Omega(n^{3/2})$ lower bound (see [3, 4]), which was later improved to $\Omega(n^{5/3})$ by Matoušek [5]. Recently Radoičić and Tóth [9] have raised it to $\Omega(n^{7/4})$, and very recently, Balogh, Regev, Smyth, Steiger and Szegedy [1] have brought it up to $\Omega(n^2/C^{\sqrt{\log n}})$ for some suitable constant $C > 1$.

Clearly the maximum number of vertices in an arrangement of n lines is $\binom{n}{2}$, so $\lambda_n \leq \binom{n}{2} + 1 = O(n^2)$. A very slight improvement of this bound, which only concerns the constant factor was mentioned in [9]: $\lambda_n < 5n^2/12$. This was later improved by Pór to $\lambda_n < 3n^2/10$ [8].

As a — perhaps intimately — related problem, Matoušek [5] has also considered arrangements of pseudolines, and longest monotone paths in such arrangements.¹ For this case, a lower bound of $\Omega(n^2/\log n)$ was given (and obviously the $O(n^2)$ upper bound still holds).

Another related problem is the k -level (or its dual, the k -set) problem in the plane. The k -level of an arrangement of n lines is the closure of the set of points on the lines with the property that there are exactly

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¹A pseudoline is a continuous x -monotone curve, and an arrangements of pseudolines is a finite collection of such curves, such that any two meet at at most one point where they cross.

k lines below them ($k = 0, \dots, n - 1$). The k -level of a line arrangement is also a x -monotone polygonal path, which turns at each vertex of the arrangement that lies on the path. The k -level problem asks for the maximum complexity of the k -level in an arrangement of lines. The best upper and lower bounds are both quite recent [2, 10].

In this note, we give subquadratic bounds on the maximum length of x -monotone paths in arrangements of n lines with at most $C \log \log n$ directions, where C is a suitable constant. In particular, we obtain tight estimates for the case of lines having at most five directions. Theorem 1 below gives a partial answer to the open problem of determining the maximum length of a monotone path in an n -line arrangement whose lines have at most k distinct slopes, posed in [1].

Our bounds depend on the well known Fibonacci numbers, defined by the following recurrence:

$$F_0 = 1, F_1 = 1, F_i = F_{i-1} + F_{i-2}, \text{ for } i \geq 2.$$

Theorem 1 *Let $L_k(n)$ be the maximum length of a monotone path in an n -line arrangement whose lines have at most k distinct slopes. Then*

- (i) $L_1(n) = 1$.
- (ii) $L_2(n) = n$.
- (iii) $L_3(n) = 2n - 3$, for $n \geq 4$.
- (iv) $L_4(n) = \Theta(n^{3/2})$.
- (v) $L_5(n) = \Theta(n^{5/3})$.
- (vi) $L_6(n) = O(n^{9/5})$.
- (vii) $L_7(n) = O(n^{15/8})$.
- (viii) For any $k \geq 4$, $L_k(n) \leq 25 \cdot k \cdot n^{2 - \frac{1}{F_{k-2}}}$, where F_k is the k -th Fibonacci number.

Remark. The upper bounds in (iv),(v), (vi) and (vii) are special cases of (viii), as it is also the example mentioned in the abstract. Note that (viii) also yields (ii) and (iii) up to multiplicative constant factors.

2 Proof of the Theorem

The lower bounds in (i),(ii) are immediate. For (iii), the following construction due to Pálvölgyi [7] gives an exact bound. If n is even, use a bundle of m horizontal lines, a bundle of m near vertical parallel lines of positive slope, so that they form a regular grid - like structure and two (diagonal) lines of slope 1; an example with $m = 5$ is illustrated in Fig. 1. If n is odd, add one more horizontal line, see Fig. 1. The path outlined in the figure has length $2n - 3$.

For (iv), we note that the $\Omega(n^{3/2})$ construction of Sharir mentioned in the Introduction (see [3, 4]) uses only four slopes. For illustration, we reproduce it in the Appendix (Fig. 4). For (v), we note that the $\Omega(n^{5/3})$ construction of Matoušek [5] uses only five slopes. Both constructions can in fact be realized by arrangements of lines in which no two lines are parallel, but they are most easily described in terms of four or five directions.

Now we show the upper bounds. First we introduce some general notation. Consider a set \mathcal{L} of n lines in the plane, with k distinct slopes $s_1 < \dots < s_k$, and write n_i for the number of lines having slope s_i ,

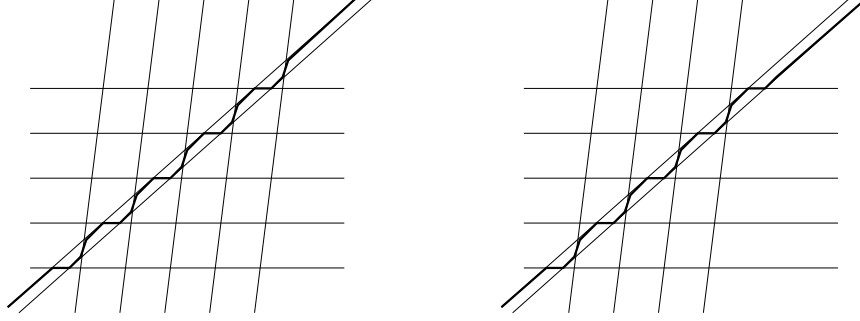


Figure 1: Arrangement of lines with three slopes which admits a monotone path of length $2n - 3$.

$i = 1, \dots, k$. Denote by \mathcal{A} the corresponding arrangement. For convenience we identify the slopes with their indices. For any subset of slopes $I \subseteq [n]$, consider the subset of lines \mathcal{L}_I which consists of all the lines in \mathcal{L} whose slopes are in I . Denote by \mathcal{A}_I the corresponding arrangement (a sub-arrangement of \mathcal{A}).

The following argument will be used repeatedly. Consider the arrangement $\mathcal{A} = \mathcal{A}_{1, \dots, k}$, and fix a monotone path p in \mathcal{A} . Shortcut p to get p' a *monotone shortcut path* in $\mathcal{A}_{1, k}$ in the following way. Replace the portion of p in the interior of each cell of $\mathcal{A}_{1, k}$ by one edge (segment, or half-line for unbounded cells), namely the segment determined by the endpoints of the portion of p in each cell. Clearly, p' is also an x -monotone polygonal path (however it is not a monotone path in $\mathcal{A}_{1, k}$). Put $l(p')$ for its length. It is easy to see that $l(p') = O(n)$, see Lemma 1 below. A cell of $\mathcal{A}_{1, k}$ is said to be *visited* by p if p intersects its interior. Put $l(p)$ for the length of p , and let $Q(\mathcal{A}_{1, k}, p)$ be the set of cells of $\mathcal{A}_{1, k}$ which are visited by p and in which p turns. Denote by l_c the length of the portion of p inside a cell $c \in Q(\mathcal{A}_{1, k})$. Then

$$l(p) \leq l(p') + \sum_{c \in Q(\mathcal{A}_{1, k}, p)} (l_c - 1). \quad (1)$$

A similar inequality holds if one considers instead a monotone shortcut path in \mathcal{A}_1 (or \mathcal{A}_k). In the former case, denote by $Q(\mathcal{A}_1, p)$ the set of cells of \mathcal{A}_1 which are visited by p and in which p turns. Then

$$l(p) \leq l(p') + \sum_{c \in Q(\mathcal{A}_1, p)} (l_c - 1). \quad (2)$$

Lemma 1 *Let p' and p'' be shortcut monotone paths in the arrangements \mathcal{A}_1 and $\mathcal{A}_{1, k}$, respectively. Put $q' = |Q(\mathcal{A}_1, p')|$ and $q'' = |Q(\mathcal{A}_{1, k}, p'')|$. Then*

- (i) $l(p') \leq 2n_1 + 1$ and $q' \leq n_1 + 1$.
- (ii) $l(p'') \leq 2n_1 + 2n_k + 1$ and $q'' \leq n_1 + n_k + 1$.

Proof. (i) p' cannot revisit any of the lines with slopes 1, since the slope of each shortcut edge lies in the open interval (s_1, ∞) . By the slope condition, p' can have at most two turns on each of the n_1 lines of minimum slope, which implies the bound on $l(p')$. The number of cells of \mathcal{A}_1 is $n_1 + 1$, thus $q' \leq n_1 + 1$. (ii) p'' cannot revisit any of the lines with slopes 1 or k , since the slope of each shortcut edge lies in the open interval (s_1, s_k) . By the slope condition, p'' can have at most two turns on each of the $n_1 + n_k$ lines of minimum and maximum slope, which implies the bound on $l(p'')$. The number of cells of $\mathcal{A}_{1, k}$ visited by p'' is at most $n_1 + n_k + 1$, since none of the $n_1 + n_k$ lines of minimum and maximum slope can be revisited by p'' . \square

Lemma 2 Consider an arrangement $\mathcal{A} = \mathcal{A}_{1,\dots,k}$ of n lines having k distinct slopes ($k \geq 4$), and let p be a monotone path in \mathcal{A} . Let c be a convex cell $c \in Q(\mathcal{A}_{1,k}, p)$, and let p_c be the portion of p which lies in the interior of c . Assume that ℓ_i and ℓ_j are two lines of minimum and maximum slope respectively, which intersect p_c . Then ℓ_i and ℓ_j intersect in the interior of c .

Proof. Clearly p_c is the contiguous portion of a monotone path in $\mathcal{A}_{2,\dots,k-1}$ which lies in c . Since p_c turns in the interior of c , $i \neq j$, thus $2 \leq i < j \leq k - 1$. Suppose that the two lines intersect on the boundary of c or outside c . We have three cases.

Case 1: ℓ_i and ℓ_j intersect before entering c — see Fig. 2(left). Assume without loss of generality (flipping the figure gives the other case) that p_c intersects ℓ_j at some point A , before intersecting ℓ_i at some point B . Draw a line ℓ parallel to ℓ_i passing through A . Since ℓ_i is a line of minimum slope which intersects p_c , the portion of p_c right of A must lie on or above ℓ and thus cannot intersect ℓ_i inside c , which is a contradiction.

Case 2: ℓ_i and ℓ_j intersect after leaving c — see Fig. 2(right). Assume without loss of generality that p_c intersects ℓ_i at some point A , before intersecting ℓ_j at some point B . Since ℓ_i is a line of minimum slope which intersects p_c , the portion of p_c right of A must lie on or above ℓ_i and thus cannot intersect ℓ_j inside c , which is a contradiction.

Case 3: The point of intersection of ℓ_i and ℓ_j lies on the portion of ℓ_i left of its intersection with c , and on the portion of ℓ_j right of its intersection with c . The argument is similar with that in the previous two cases. (The case when the intersection point lies on the portion of ℓ_i right of its intersection with c , and on the portion of ℓ_j left of its intersection with c is symmetric).

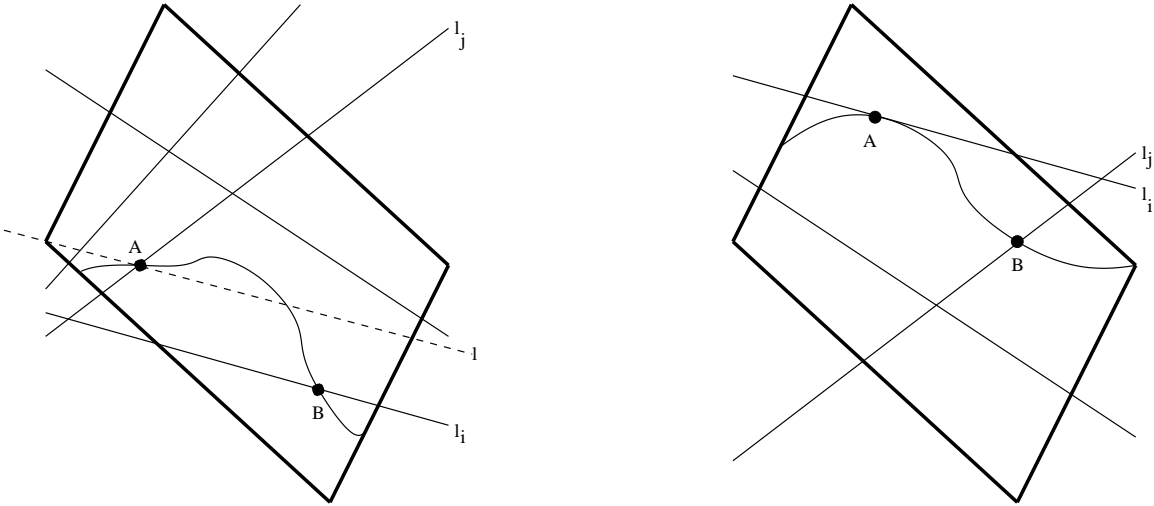


Figure 2: Proof of Lemma 2. A cell $c \in Q(\mathcal{A}_{1,\dots,k}, p)$, and the portion p_c of p which lies in the interior of c .

The argument remains the same for unbounded cells. See also Fig. 3 for an illustration of the case $k = 4$. □

We return to the proof of Theorem 1.

(i) Trivially, since all lines are parallel, each monotone path follows exactly one line and the number of turns on any such path is 0.

(ii) The bound follows from the next lemma.

Lemma 3 *Let $m \geq 1$ be the number of vertices in a line arrangement \mathcal{A} of n lines having two distinct slopes, and let p be a monotone path in \mathcal{A} . Then $l(p) \leq \min(n, 2\sqrt{m} + 1)$. Further on, consider an arrangement of n lines having four distinct slopes, a convex cell $c \in Q(\mathcal{A}_{1,4}, p)$, and the portion p_c of p which lies in the interior of c . Then $l(p_c) \leq 2\sqrt{m_c} + 1 \leq 3\sqrt{m_c}$, where m_c is the number of vertices of $\mathcal{A}_{2,3}$ in the interior of c .*

Proof. Write $n = n_1 + n_2$, where n_1 and n_2 are the two slope multiplicities. Assume that $n_1 \leq n_2$. Then $n \geq 2n_1$ and $m = n_1n_2 \geq n_1^2$. Clearly there are at most two turns per line, as no line can be revisited by the path once the path has left that line. Since each turn must involve one of the n_1 lines, the number of turns is at most $2n_1$. This gives $l(p) \leq 2n_1 + 1 \leq n + 1$. In fact, by charging each turn of the path to the line on which the path goes before the turn, we get that the number of turns is at most $n - 1$ (no turn is charged to the line containing the last edge on the path), thus $l(p) \leq n$. We also have $n_1 \leq \sqrt{m}$, which implies $l(p) \leq 2n_1 + 1 \leq 2\sqrt{m} + 1$.

For the second part, let L_2 (resp. L_3) be the set lines of slope 2 (resp. slope 3) which intersect p_c . Note that p_c is the contiguous portion of a monotone path in $\mathcal{A}_{2,3}$ which lies in c . Since $c \in Q(\mathcal{A}_{1,4}, p)$, $|L_2|, |L_3| \geq 1$. By Lemma 2 for $k = 4$, all the $|L_2| \cdot |L_3|$ vertices of the arrangement of the lines in $L_2 \cup L_3$ are in the interior of c . See Fig. 3. Hence $m_c \geq |L_2| \cdot |L_3|$. Assuming $|L_2| \leq |L_3|$, we have $m_c \geq |L_2|^2$.

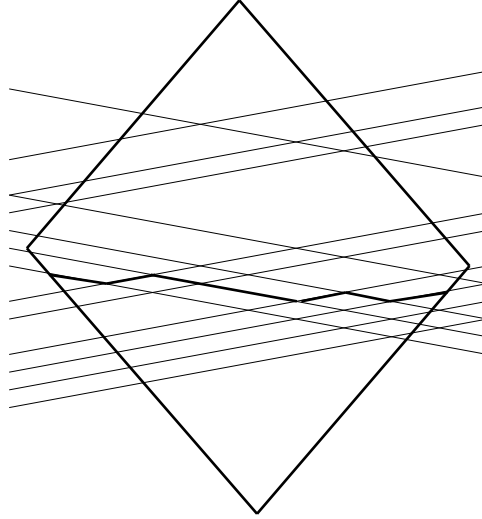


Figure 3: A cell $c \in Q(\mathcal{A}_{1,4}, p)$, and the portion p_c of p which lies in the interior of c . All the $3 \times 4 = 12$ vertices of the arrangement of the lines in $L_2 \cup L_3$ are in the interior of c .

Thus $l(p_c) \leq 2|L_2| + 1 \leq 2\sqrt{m_c} + 1 \leq 3\sqrt{m_c}$. □

(iii) The bound follows from the next lemma.

Lemma 4 *Let $m \geq 1$ be the number of vertices in a line arrangement \mathcal{A} of n lines having three distinct slopes, and let p be a monotone path in \mathcal{A} . Then $l(p) \leq \min(2n_1 + 2n_3 + 1, 6m^{2/3})$. Further on, consider an arrangement of n lines having five distinct slopes, a convex cell $c \in Q(\mathcal{A}_{1,5}, p)$, and the portion p_c of p which lies in the interior of c . Then $l(p_c) \leq 6m_c^{2/3}$, where m_c is the number of vertices of $\mathcal{A}_{2,3,4}$ in the interior of c .*

Proof. Let $\mathcal{A} = \mathcal{A}_{1,2,3}$. The lines with minimum and maximum slope (1 and 3) can contribute at most one edge to the path, and thus can contribute at most two turns. On the other hand, each turn on the path must involve either a line of slope 1 or one of slope 3. Consequently,

$$l(p) \leq 2n_1 + 2n_3 + 1 \leq 2n + 1. \quad (3)$$

A more careful calculation gives $l(p) \leq 2n - 3$, for $n \geq 4$ (i.e., the bound in Theorem 1):

If $n_2 = 0$, $l(p) \leq n \leq 2n - 3$, for $n \geq 4$. If $n_2 = 1$, when the path returns to the diagonal line, it has to go from a ‘‘vertical’’ line to a horizontal line (or the other way around), so instead of $l(p) \leq 2n_1 + 2n_3 + 1$, we get $l(p) \leq 3(n_1 + n_3)/2 + 1 = (3n - 1)/2$. It follows that $l(p) \leq 2n - 3$, for $n \geq 4$. Finally, if $n_2 \geq 2$, $l(p) \leq 2n_1 + 2n_3 + 1 \leq 2(n - 2) + 1 = 2n - 3$.

Continuing with the proof of the lemma, observe that (3) is a special case of (1). Without loss of generality, we can assume that $n_1 \leq n_3$. Consider the arrangement \mathcal{A}_1 and use (2). By Lemma 1(i), $l(p') \leq 2n_1 + 1 \leq 2n + 1$, and $q = |Q(\mathcal{A}_1, p)| \leq n_1 + 1 \leq n + 1$.

$$l(p) \leq 2n_1 + 1 + \sum_{c \in Q(\mathcal{A}_1, p)} (l_c - 1) \leq 2n_1 + 1 + 2 \sum_{c \in Q(\mathcal{A}_1, p)} \sqrt{m_c}.$$

Consequently, by Jensen’s inequality

$$l(p) \leq 2n_1 + 1 + 2q \sqrt{\frac{m}{q}} \leq 2n_1 + 1 + 2\sqrt{n_1 + 1} \sqrt{m}. \quad (4)$$

We distinguish two cases.

Case 1: $n_1 \leq m^{1/3}$. (4) implies that

$$l(p) \leq 2m^{1/3} + 1 + 2(m^{1/3} + 1)^{1/2} m^{1/2} \leq 6m^{2/3}.$$

Case 2: $n_1 \geq m^{1/3}$. Clearly $m \geq n_1 n_3$. This implies $n_3 \leq m/m^{1/3} = m^{2/3}$. Now (3) gives

$$l(p) \leq 4m^{2/3} + 1 \leq 6m^{2/3}.$$

This proves the first inequality.

The second part follows from a similar argument to that in the proof of Lemma 3. Let $\mathcal{A} = \mathcal{A}_{1,2,3,4,5}$. Note that p_c is the contiguous portion of a monotone path in $\mathcal{A}_{2,3,4}$ which lies in c . Assume that L_i and L_j are the set of lines of minimum and maximum slope respectively, which intersect p_c . Since p_c turns in the interior of c , $i \neq j$ and $|L_i|, |L_j| \geq 1$. By Lemma 2 for $k = 5$, all the vertices of the arrangement of the lines in $L_i \cup L_j$ are in the interior of c . Thus $m_c \geq |L_i| \cdot |L_j|$, and assume without loss of generality that $|L_i| \leq |L_j|$. Two inequalities, similar to (3) and (4) bound the length of p_c .

$$l(p_c) \leq 2|L_i| + 1 + 2\sqrt{|L_i| + 1} \sqrt{m_c}.$$

$$l(p_c) \leq 2|L_i| + 2|L_j| + 1.$$

Since $m_c \geq |L_i| \cdot |L_j|$, a two-case analysis as in the first part of the proof ($|L_i| \leq m_c^{1/3}$, and $|L_i| \geq m_c^{1/3}$) yields the claimed bound. \square

(viii)

Lemma 5 *Let $k \geq 2$. Let $m \geq 1$ be the number of vertices in a line arrangement \mathcal{A} of n lines having k distinct slopes, and let p be a monotone path in \mathcal{A} . Then*

$$l(p) \leq c_k \cdot m^{1 - \frac{1}{F_k}},$$

where

$$c_k = 5 \cdot k \cdot 3^{\sum_{i=2}^{k-1} \frac{1}{F_i}},$$

and F_k is the k -th Fibonacci number. Further on, consider an arrangement of n lines having $k+2$ distinct slopes, a convex cell $c \in Q(\mathcal{A}_{1,k+2}, p)$, and the portion p_c of p which lies in the interior of c . Then

$$l(p_c) \leq c_k \cdot m_c^{1 - \frac{1}{F_k}},$$

where m_c is the number of vertices of $\mathcal{A}_{2,\dots,k+1}$ in the interior of c .

Proof. The argument is similar to the one used in proving Lemma 3 and Lemma 4. We proceed by induction on k . For $k = 2$ and $k = 3$, the statements follow from Lemma 3 and Lemma 4: $c_2 = 5 \cdot 2 = 10 \geq 3$ and $c_3 = 5 \cdot 2 \cdot 3^{1/2} \geq 6$. Assume now that $k \geq 4$, and the lemma holds for all smaller values of k' , $k' = 2, \dots, k-1$. Let $\mathcal{A} = \mathcal{A}_{1,\dots,k}$. Without loss of generality, $n_1 \leq n_k$.

Consider the arrangement \mathcal{A}_1 and use (2). By Lemma 1(i), $l(p') \leq 2n_1 + 1$, and $|Q(\mathcal{A}_1, p)| \leq n_1 + 1$. The induction hypothesis for $k' = k-1$ implies that

$$l(p) \leq 2n_1 + 1 + \sum_{c \in Q(\mathcal{A}_1, p)} (l_c - 1) \leq 2n_1 + 1 + c_{k-1} \sum_{c \in Q(\mathcal{A}_1, p)} m_c^{1 - \frac{1}{F_{k-1}}}.$$

By Jensen's inequality,

$$l(p) \leq (2n_1 + 1) + c_{k-1} \cdot (n_1 + 1)^{\frac{1}{F_{k-1}}} \cdot m^{1 - \frac{1}{F_{k-1}}}. \quad (5)$$

Consider also the arrangement $\mathcal{A}_{1,k}$ and use (1). By Lemma 1(ii), $l(p'') \leq 2n_1 + 2n_k + 1$, and $|Q(\mathcal{A}_{1,k}, p)| \leq n_1 + n_k + 1$. The induction hypothesis for $k' = k-2$ implies that

$$l(p) \leq (2n_1 + 2n_k + 1) + c_{k-2} \sum_{c \in Q(\mathcal{A}_{1,k}, p)} m_c^{1 - \frac{1}{F_{k-2}}}.$$

By Jensen's inequality,

$$l(p) \leq (2n_1 + 2n_k + 1) + c_{k-2} \cdot (n_1 + n_k + 1)^{\frac{1}{F_{k-2}}} \cdot m^{1 - \frac{1}{F_{k-2}}}. \quad (6)$$

We distinguish two cases:

Case 1: $n_1 \leq m^{\frac{F_{k-2}}{F_k}}$. This implies using (5) that

$$l(p) \leq (2m^{\frac{F_{k-2}}{F_k}} + 1) + c_{k-1} \cdot (m^{\frac{F_{k-2}}{F_k}} + 1)^{\frac{1}{F_{k-1}}} \cdot m^{1 - \frac{1}{F_{k-1}}}. \quad (7)$$

Note that

$$(m^{\frac{F_{k-2}}{F_k}} + 1)^{\frac{1}{F_{k-1}}} \leq 3^{\frac{1}{F_{k-1}}} \cdot m^{\frac{F_{k-2}}{F_k \cdot F_{k-1}}}, \quad (8)$$

since (8) is equivalent to

$$m^{\frac{F_{k-2}}{F_k}} + 1 \leq 3 \cdot m^{\frac{F_{k-2}}{F_k}}, \quad (9)$$

which is clearly satisfied. So (7) implies that

$$\begin{aligned} l(p) &\leq (2m^{\frac{F_{k-2}}{F_k}} + 1) + c_{k-1} \cdot 3^{\frac{1}{F_{k-1}}} \cdot m^{\frac{F_{k-2}}{F_k \cdot F_{k-1}}} \cdot m^{1 - \frac{1}{F_{k-1}}} \\ &\leq 3 \cdot m^{\frac{F_{k-2}}{F_k}} + 5 \cdot (k-1) \cdot 3^{\sum_{i=2}^{k-2} \frac{1}{F_i}} \cdot 3^{\frac{1}{F_{k-1}}} \cdot m^{\frac{1}{F_{k-1}} - \frac{1}{F_k}} \cdot m^{1 - \frac{1}{F_{k-1}}} \\ &= 3 \cdot m^{\frac{F_{k-2}}{F_k}} + 5 \cdot (k-1) \cdot 3^{\sum_{i=2}^{k-1} \frac{1}{F_i}} \cdot m^{1 - \frac{1}{F_k}} \\ &\leq 5 \cdot k \cdot 3^{\sum_{i=2}^{k-1} \frac{1}{F_i}} \cdot m^{1 - \frac{1}{F_k}} \\ &= c_k \cdot m^{1 - \frac{1}{F_k}}. \end{aligned}$$

Case 2: $n_1 \geq m^{F_{k-2}/F_k}$. Clearly $m \geq n_1 n_k$. This implies $n_k \leq m/m^{F_{k-2}/F_k} = m^{F_{k-1}/F_k}$. Using (6) we have

$$l(p) \leq (4m^{\frac{F_{k-1}}{F_k}} + 1) + c_{k-2} \cdot (2m^{\frac{F_{k-1}}{F_k}} + 1)^{\frac{1}{F_{k-2}}} \cdot m^{1 - \frac{1}{F_{k-2}}}. \quad (10)$$

Note that

$$(2m^{\frac{F_{k-1}}{F_k}} + 1)^{\frac{1}{F_{k-2}}} \leq 3^{\frac{1}{F_{k-2}}} \cdot m^{\frac{F_{k-1}}{F_k \cdot F_{k-2}}}, \quad (11)$$

since (11) is equivalent to

$$2m^{\frac{F_{k-1}}{F_k}} + 1 \leq 3 \cdot m^{\frac{F_{k-1}}{F_k}}, \quad (12)$$

which clearly holds. So (10) implies that

$$\begin{aligned} l(p) &\leq (4m^{\frac{F_{k-1}}{F_k}} + 1) + c_{k-2} \cdot 3^{\frac{1}{F_{k-2}}} \cdot m^{\frac{F_{k-1}}{F_k \cdot F_{k-2}}} \cdot m^{1 - \frac{1}{F_{k-2}}} \\ &\leq 5 \cdot m^{\frac{F_{k-1}}{F_k}} + 5 \cdot (k-2) \cdot 3^{\sum_{i=2}^{k-3} \frac{1}{F_i}} \cdot 3^{\frac{1}{F_{k-2}}} \cdot m^{\frac{1}{F_{k-2}} - \frac{1}{F_k}} \cdot m^{1 - \frac{1}{F_{k-2}}} \\ &= 5 \cdot m^{\frac{F_{k-1}}{F_k}} + 5 \cdot (k-2) \cdot 3^{\sum_{i=2}^{k-2} \frac{1}{F_i}} \cdot m^{1 - \frac{1}{F_k}}. \end{aligned}$$

Since for $k \geq 4$, $F_{k-1} \leq F_k - 1$, we have

$$\begin{aligned} l(p) &\leq (5 + 5(k-2)) \cdot 3^{\sum_{i=2}^{k-2} \frac{1}{F_i}} \cdot m^{1 - \frac{1}{F_k}} \\ &\leq 5 \cdot (k-1) \cdot 3^{\sum_{i=2}^{k-1} \frac{1}{F_i}} \cdot m^{1 - \frac{1}{F_k}} \\ &\leq 5 \cdot k \cdot 3^{\sum_{i=2}^{k-1} \frac{1}{F_i}} \cdot m^{1 - \frac{1}{F_k}} \\ &= c_k \cdot m^{1 - \frac{1}{F_k}}. \end{aligned}$$

This proves the first inequality.

The second part follows from an argument similar to that in the proof of Lemma 3. Let $\mathcal{A} = \mathcal{A}_{1, \dots, k+2}$. Note that p_c is the contiguous portion of a monotone path in $\mathcal{A}_{2, \dots, k+1}$ which lies in c . Assume that L_i

and L_j are the set lines of minimum and maximum slope respectively, which intersect p_c . Since p_c turns in the interior of c , $i \neq j$ (so $2 \leq i < j \leq k+1$) and $|L_i|, |L_j| \geq 1$. By Lemma 2, all the vertices of the arrangement of the lines in $L_i \cup L_j$ are in the interior of c . Thus $m_c \geq |L_i| \cdot |L_j|$. Assuming without loss of generality that $|L_i| \leq |L_j|$, one can write two inequalities similar to (5) and (6), since by definition, none of the lines in L_i or L_j can be revisited by p_c .

$$l(p_c) \leq (2|L_i| + 1) + c_{k-1} \cdot (|L_i| + 1)^{\frac{1}{F_{k-1}}} \cdot m_c^{1 - \frac{1}{F_{k-1}}}. \quad (13)$$

$$l(p_c) \leq (2|L_i| + 2|L_j| + 1) + c_{k-2} \cdot (|L_i| + |L_j| + 1)^{\frac{1}{F_{k-2}}} \cdot m_c^{1 - \frac{1}{F_{k-2}}}. \quad (14)$$

Since $m_c \geq |L_i| \cdot |L_j|$, a two-case analysis as above ($|L_i| \leq m_c^{F_{k-2}/F_k}$, and $|L_i| \geq m_c^{F_{k-2}/F_k}$) yields the claimed bound. \square

Corollary 1 For any $k \geq 4$, $L_k(n) \leq 25 \cdot k \cdot n^{2 - \frac{1}{F_{k-2}}}$, where F_k is the k -th Fibonacci number.

Proof. Let $m \leq \binom{n}{2}$ be the number of vertices of the arrangement \mathcal{A} , and p be a monotone path in \mathcal{A} . By (6),

$$l(p) \leq (2n + 1) + 5 \cdot 3^{\sum_{i=2}^{\infty} \frac{1}{F_i}} \cdot (k-2) \cdot (n+1)^{\frac{1}{F_{k-2}}} \cdot \left(\frac{n^2}{2}\right)^{1 - \frac{1}{F_{k-2}}}. \quad (15)$$

Using well known bounds on the Fibonacci numbers (or by induction), we have

$$F_k \geq \left(\frac{3}{2}\right)^k, \quad \text{for } k \geq 5.$$

Thus

$$\sum_{i=2}^{\infty} \frac{1}{F_i} \leq \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \left(\frac{2}{3}\right)^5 \cdot \sum_{i=0}^{\infty} \left(\frac{2}{3}\right)^i \leq 1.43.$$

Plugging this bound into (15) yields (using the inequality $F_{k-2} \geq 2$, for $k \geq 4$)

$$l(p) \leq 3n + 25 \cdot (k-2) \cdot \frac{2^{\frac{1}{F_{k-2}}}}{2^{1 - \frac{1}{F_{k-2}}}} \cdot n^{\frac{1}{F_{k-2}}} \cdot n^{2 - \frac{2}{F_{k-2}}} \leq 25 \cdot k \cdot n^{2 - \frac{1}{F_{k-2}}}. \quad \square$$

Corollary 2 There exists an absolute constant $C > 0$, so that if $k \leq C \log \log n$, then $L_k(n) = o(n^2)$.

Proof. From well known bounds on the Fibonacci numbers (or by induction), we have

$$F_k \leq \left(\frac{5}{3}\right)^k, \quad \text{for } k \geq 1,$$

and the corollary follows. \square

3 Concluding remarks

Possibly, the factor k in Theorem 1(viii) could be removed by a more careful calculation. However this would not extend the range of k — in terms of n — for which Corollary 2 gives a subquadratic bound.

Most likely, the Fibonacci numbers present in the exponents of our bounds have little to do with the right order of magnitude of $L_k(n)$ for small values of k , and their presence is probably only an artifact of our proofs. For example, one can check that the $\Omega(n^{7/4})$ construction of Radoičić and Tóth can be realized using only six slopes — thus $L_6(n) = \Omega(n^{7/4})$ — while our upper bound is $O(n^{9/5})$. This is a possible start point, if one wants to establish better bounds for small values of k . Similarly, for fixed k , the recent construction due to Balogh, Regev, Smyth, Steiger and Szegedy gives $L_k(n) = \Omega(n^{2-2/k})$.

Of course, the most interesting question is to give a subquadratic upper bound for an arbitrary number of directions (as it is generally believed to hold). Which of the ideas from our note (if any) may be useful for this purpose, remains to be seen.

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Appendix

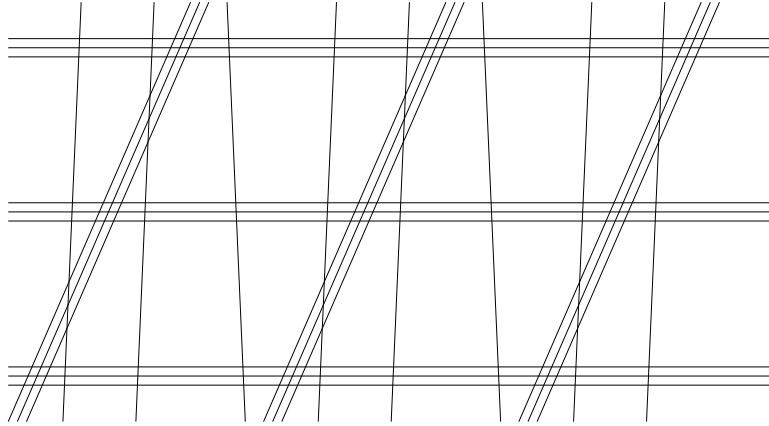


Figure 4: Arrangement of lines with four slopes which admits a monotone path of length $\Omega(n^{3/2})$. It consists of: m bundles of m horizontal lines each; m bundles of m lines each at (say) 60° ; $m(m-1)$ near vertical parallel lines of positive slope, and $m-1$ near vertical parallel lines of negative slope; $m=3$ in this example.