A uniform tableau method for intuitionistic modal logics I

Giambattista Amati*
Fondazione Ugo Bordoni
e-mail: fubdpt5 @ itcaspur.caspur.it
via Baldassarre Castiglione 59,
00142 Roma, Italy

Fiora Pirri
Dipartimento di Informatica e Sistemistica
e-mail: Pirri@assi.ing.uniroma1.it
Università di Roma "La Sapienza"
via Salaria 113, 00198 Roma, Italy

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Abstract

We present tableau systems and sequent calculi for the intuitionistic analogues *IK, ID, IT, IKB, IKDB, IB, IK4, IKD4, IS4, IKB4, IK5, IKD5, IK45, IKD45* and *IS5* of the normal classical modal logics. We provide soundness and completeness theorems with respect to the models of intuitionistic logic enriched by a modal accessibility relation, as proposed by G. Fischer Servi. We then show the disjunction property for *IK, ID, IT, IKB, IKDB, IB, IK4, IKD4, IS4, IKB4, IK5, IK45* and *IS5*. We also investigate the relationship of these logics with some other intuitionistic modal logics proposed in the literature.

1 Introduction

In this paper we provide tableau systems and sequent calculi for normal propositional intuitionistic modal logics. The semantics is defined through Kripke-type models, where the forcing relation on modal connectives and the frame properties are those proposed by Fischer Servi in [13]. The

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frames, endowed with two accessibility relations, are those of intuitionistic logic enriched by a Kripkean accessibility relation. Given any combination of the properties of seriality, reflexivity, transitivity, symmetry and euclideanness for the modal accessibility relation, an analogue of the classical modal system is defined.

We use the notations and the method due to Fitting [14]. Strong completeness theorems are provided with the assumption of the cut rule. The cut is not eliminable for the systems with symmetry and for both $IK5$ and $IKD5$.

Although the literature on intuitionistic modal logics does not seem to be very extensive, there are several proposals which define intuitionistic analogues of classical modal logics. Here we briefly survey the literature that is strictly related to the present work.

The problem of presenting an intuitionistic concept of modality was faced by Fitch and Prior in early papers [15, 24]. Prior proposes a modal extension of the intuitionistic propositional calculus which turns out to be S5 once the excluded middle is added. In [2, 3] Bull studies Prior's propositional modal calculus and shows that it is related to the intuitionistic monadic calculus just as S5 is related to the classic monadic calculus. The relationship between intuitionistic modal logics and intermediate predicate logics has been further investigated for instance in [22, 25, 26].

In [21] a lattice of intuitionistic modal logics is given, with the necessity operator $\Box$ as primitive, both in Hilbert and in Gentzen-style formulations. These logics are intermediate between the intuitionistic S4 and S5 in the sense of [3]. It is worth noticing that in [21] a semantics with a Kripke-type frame is also adopted. The modal relation is introduced as an extension of the intuitionistic accessibility relation.

General criteria to find the intuitionistic counterpart of many classical modal systems are both proposed in [1, 6] and in [8, 9, 10, 11, 12].

In [1] two systems HK$\Box$ and HK$\Diamond$ are introduced, as intuitionistic counterparts of the classical system K. The first one deals only with $\Box$ and the latter only with $\Diamond$. HK$\Diamond$ turns out to be sound and complete with respect to the Kripke-type frames satisfying one of the two connecting properties we are concerned with (see the second condition of Definition 1 and [13]). On the other hand, HK$\Box$ is sound and complete with respect to three classes of frames ordered by inclusion and where the forcing relation for $\Box$ is given as in classical logic. When bringing together both the property of HK$\Diamond$ and the strongest connecting property of HK$\Box$-frames (i.e. the strictly condensed frames or also $\Box$-normal logics in [28]), $\Diamond$ turns out to be definable with $\Box$ as primitive. Here, it is further shown that this logic, called HK$\Box\Diamond$, is stronger than the logic obtained from HK$\Box$ by adding the axiom schema $\Diamond \leftrightarrow \neg\Box\neg$ (see also [16]). We show that HK$\Box\Diamond$ can also be obtained by a proper subclass of IK-frames of [13] (See Remark 2). Further studies on these logics can be found in [6, 16, 23].

In Fisher Servi's works the two modal operators of possibility $\Diamond$ and necessity $\Box$ are required to be non-interdefinable. Two connecting axioms are proposed to strengthen the weak connection between them. As one should expect from the intuitionistic behaviour of quantifiers, both $\neg\Box\neg A \supset \Diamond A$ and $\neg\Diamond\neg A \supset \Box A$, as well as $\neg\Box A \supset \Diamond\neg A$, cannot be derived on an intuitionistic basis. Following such a proposal, in [13] some intuitionistic modal logics and Kripke-type semantics are given. It is further shown that IS5 is equivalent to Bull's MIPC (see [2, 3, 21]). The forcing relations for both modal operators are analogous to those of the intuitionistic quantifiers on Kripke frames ([5], see Definition 3.6). Two connecting properties for the Kripke-type frames are given to characterize the intuitionistic modal logics (see also [7]). We show that IK, ID, IT, IKB, IKDB, IB, IK4, IKD4, IS4, IKB4, IK5, IK45 and IS5 have the Disjunction Property.

A similar path in tense logic has been followed by Ewald in [7]. In his paper, besides a set of states of knowledge (worlds) and the intuitionistic partial order relation, a set of times and a temporal ordering within each world are introduced. Two dual derived relations on the set of worlds can be defined: one of them turns out to satisfy both the connecting properties of Kripke-type frames of [13] and the forcing relation on the modal operators.

In [29] Wijesekera introduces an intuitionistic predicate modal tableau system with a Kripke-type semantics, whose frames are free from connecting properties. We show that Wijesekera's system is contained in Fischer Servi's minimal system IK. For other works on intuitionistic modal logics and their applications see [18, 19].

2 Preliminaries

Let us first recall that I^* denotes any intuitionistic modal calculus introduced in [13]. We consider I^* where $*$ is K, D, T, KB, KDB, B, K4, KD4, S4, KB4, K5, KD5, K45, KD45 and S5. Let $\mathcal{L}$ denote the modal language $\{\nu; \vee, \wedge, \supset, \neg, \Box, \Diamond\}$, where ν is the set of propositional variables.

In [13] the intuitionistic modal models are defined using two relations, namely a partial order $\leq$ over the set of worlds W and an accessibility relation R, meant to extend the usual modal concepts on an intuitionistic basis. The properties of seriality, reflexivity, transitivity, symmetry and euclideanness of R yield fifteen intuitionistic analogues of the classical modal systems, K, D, T, KB, KDB, B, K4, KD4, S4, KB4, K5, KD5, K45, KD45 and S5 (see Table 1, 4.6). According to [4]'s terminology D is KD, T is KDT , B is KTB, S4 is KT4, S5 is KDB4.

DEFINITION 1 *Let F be $< W, \leq, R >$ where W is a set of worlds. We say that F is an I^*-frame, or shortly a frame, where $*$ is K, D, T, KB, KDB, B, K4, KD4, S4, KB4, K5, KD5, K45, KD45 or S5, when the following conditions are satisfied:*

i) $\leq$ is a partial order on W, that is $\leq$ is reflexive and transitive. The relation R of the I^*-frame has the same accessibility properties of the analogous classical frame.

ii) If $w \leq v$ and wRu then there exists a world z s.t. vRz and $u \leq z$.

iii) If wRu and $u \leq v$ then there exists a world z s.t. $w \leq z$ and zRv.

Figure 1

Notational conventions. w^* denotes a world v s.t. $w \leq v$; $w^\bullet$ a world v s.t. wRv, $w^{*^\bullet}$ a world $(w^*)^\bullet$ and $w^{\bullet^*}$ a world $(w^\bullet)^*$.

Condition ii) can be shortly stated as $\leq^{-1} \circ R \subseteq R \circ \leq^{-1}$, while iii) as $R \circ \leq \subseteq \leq \circ R$.

THEOREM 1 *If R is symmetric then iii) follows from ii).*

Proof. Straightforward (see also [10]).

DEFINITION 2 *Let F be a frame, $M =< F, \models >$ is an I^*-model if, for all w, $w \in W$:*

1) for all propositional variables A, $w \models A \Rightarrow$ for all w^*, $w^* \models A$.

2) $w \models A \wedge B \Leftrightarrow w \models A$ and $w \models B$.

3) $w \models A \vee B \Leftrightarrow w \models A$ or $w \models B$.

4) $w \models \neg A \Leftrightarrow$ for all w^*, $w^* \not\models A$.

5) $w \models A \supset B \Leftrightarrow$ for all w^*, $w^* \not\models A$ or $w^* \models B$.

6) $w \models \Box A \Leftrightarrow$ for all w^* and v s.t. w^*Rv, $v \models A$.

7) $w \models \Diamond A \Leftrightarrow$ there exists v s.t. wRv, $v \models A$.

Moreover:

(a) A formula A *is I^*-satisfiable* if there exist an I^*-model $M =< W, \leq, R, \models >$ and a world w, $w \in W$, s.t. $w \models A$.

(b) A formula A *is valid in a model* M, or M*-valid*, i.e. $M \models A$, if for all w, $w \in W$, $w \models A$.

(c) A *set* Γ *of formulas is* M*-valid* in a model M, i.e. $M \models \Gamma$, if every formula in Γ is M-valid.

(d) A formula A is I^**-valid* , i.e. $\models A$, if, for every I^*-model M, A is M-valid.

(e) Let Δ and Γ be sets of formulas, Γ *forces* Δ *to be true* , i.e. $\Gamma \models \Delta$, if for every model M s.t. $M \models \Gamma$, there exists a formula $B \in \Delta$, s.t. $M \models B$.

Notational convention. Note that $\models$ depends on the properties of the relation R, so that we should use the notation $\models_{I^*}$ rather than $\models$. We use, however, the simplest notation unless confusion arises. For any A, $A \wedge \neg A$ is denoted by the symbol $\perp$.

THEOREM 2 *If $M =< W, \leq, R, \models >$ is any I^*-model then, for every formula A and $w \in W$:*

a) (Heredity Property) $w \models A \Rightarrow$ for all w^*, $w^* \models A$.

b) $w \models \Diamond A \Leftrightarrow$ for all w^* there exists $w^{*\bullet}$ s.t. $w^{*\bullet} \models A$.

Proof.

a) See [13], Theorem 3.

b) Note that from a) and reflexivity of $\leq$ we have $w^* \models A$ iff $w \models A$.

REMARK 1 In [29] the predicative intuitionistic modal logic is based on the forcing properties 1) - 6) of Definition 2 and the interpretation of $\Diamond$ is given by b) of Theorem 2. Moreover, [29] uses frames which are free from connecting properties (the intuitionistic heredity property of the formulas is guaranteed by the forcing relation). The class of [29]'s models thus contains the class of IK-models. The inclusion is proper: in fact, $\Diamond(A \vee B) \supset \Diamond A \vee \Diamond B$ and $(\Diamond A \supset \Box B) \supset \Box(A \supset B)$ are not derivable in [29], while they are IK-valid.

REMARK 2 In [1] the forcing relation on modal operators is given as in classical modal logic. This implies that, even considering the most general frames for $\Box$ and $\Diamond$ of [1], where both the frame conditions for $\Diamond$, i.e.

ii)$\leq^{-1} \circ R \subseteq R \circ \leq^{-1}$,
and that for $\Box$, i.e.
iv) $\leq \circ R \subseteq R \circ \leq$,
are brought together, all the intuitionistic analogues of classical modal systems must have the schema $\Box(A \vee B) \supset (\Diamond A \vee \Box B)$ valid. This axiom schema is not derivable in the Bull's analogue of S5 (consider Bull's translation in intuitionistic monadic first order logic [3]). In particular non-serial systems cannot satisfy the Disjunction Property. In fact from $\models \Box((A \supset A) \vee B) \supset (\Diamond(A \supset A) \vee \Box B)$ and $\models \Box((A \supset A) \vee B)$, we have $\models \Diamond(A \supset A) \vee \Box B$, for all B; Disjunction Property and non-seriality would imply validity of $\Box\neg(A \supset A)$, i.e. $\models \Box\bot$.

REMARK 3 Now, consider the intuitionistic modal logic given by the models of Definitions 1 and 2, with R any relation and where condition ii) is replaced by condition
ii') If $w \leq v$ and $w \leq w'Ru$ then there exists a world z s.t. vRz and $u \leq z$
we prove that it is exactly the HK$\Box\Diamond$ system given in [1].

Let $< W, \leq, R, \models >$ be an IK- model satisfying condition ii'). Let R_1 be the relation $\leq \circ R$ $=\{< v, w > \mid$ there exists u with $v \leq uRw\}$. We show that $< W, \leq, R_1, \models >$ is a HK$\Box\Diamond$ model. First note that $R_1 = \leq \circ R_1 = R_1 \circ \leq$ (use iii of Definition 1); so $< W, \leq, R_1 >$ is a strictly condensed frame (see [1]). Also, $w \models \Box A$ iff $w' \models A$ for all w' s.t. wR_1w' (use definition of R_1). Moreover, $w \models \Diamond A$ iff there is v s.t. wR_1v and $v \models A$ (use Theorem 2 b) and ii')).

Conversely, if $< W, \leq, R, \models >$ is a HK$\Box\Diamond$ model, then, by definition, $< W, \leq, R >$ is strictly condensed and both ii) and iii) of Definition 1 hold; hence ii') follows. Since $< W, \leq, R >$ is strictly condensed, we have that $w \models \Box A$ iff for all w' and u s.t. $w \leq w'Ru$, $u \models A$.

3 The Fischer Servi axiomatization

In the following we list a set of intuitionistic modal systems. We recall the axiom schemata and the rules of intuitionistic modal logic as presented in Fischer Servi [13].

(1) the propositional intuitionistic valid formulas
(2) $\Diamond(A \vee B) \supset \Diamond A \vee \Diamond B$
(3) $\Box A \wedge \Box B \supset \Box(A \wedge B)$
(4) $\neg\Diamond(A \wedge \neg A)$
(5) $\Diamond(A \supset B) \supset (\Box A \supset \Diamond B)$
(6) $(\Diamond A \supset \Box B) \supset \Box(A \supset B)$
(7) from A and $A \supset B$, infer B
(8) from $A \supset B$, infer $\Diamond A \supset \Diamond B$
(9) from $A \supset B$, infer $\Box A \supset \Box B$

(10) $\Box A \supset \Diamond A$

(11) $\Box A \supset A$

(12) $A \supset \Diamond A$

(13) $\Box A \supset \Box\Box A$
(14) $\Diamond\Diamond A \supset \Diamond A$

(15) $A \supset \Box\Diamond A$
(16) $\Diamond\Box A \supset A$

(17) $\Diamond A \supset \Box\Diamond A$
(18) $\Diamond\Box A \supset \Box A$

The system IK is defined by means of axioms and rules (1) - (9).
The system ID: IK + (10)
The system IT: IK + (11) + (12)
The system IK4: IK + (13) + (14)
The system IKB: IK + (15) + (16)
The system IK5: IK + (17) + (18)
The system IKDB: ID + (15) + (16)
The system IKD5: ID + (17) + (18)
The system IB: IT + (15) + (16)
The system IS4: IT + (13) + (14)
The system IKD4: IK4 + (10)
The system IKB4: IK4 + (15) + (16)
The system IK45: IK4 + (17) + (18)
The system IS5: IS4 + (15) + (16)
The system IKD45: IKD4 + (17) + (18)

THEOREM 3 *A is a theorem of I^* iff $\models_{I^*} A$, where $*$ is K, D, T, KB, KDB, B, K4, KD4, S4, KB4, K5, KD5, K45, KD45 and S5.*

Proof. See [13] for $* =$ K, D, T, B, S4, S5. The proof of [13] can be easily extended to the other systems.

THEOREM 4 *The following sentences are theorems and derived rules of IK:*
a) $\Box(A \supset B) \supset (\Box A \supset \Box B)$
b) $\Diamond\bot \supset \bot$
c) $\Box A \wedge \Diamond(A \supset B) \supset \Diamond B$
d) $\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$
e) From A, infer $\Box A$

Proof. a)-d) are straightforward. Part e) is in [13].

We now prove that IK, ID, IT, IKB, IKDB, IB, IK4, IKD4, IS4, IKB4, IK5, IK45 and IS5 have the Disjunction Property. In order to prove it, let us show the following Lemma.

LEMMA 1 *For all I^* if $\vdash_{I^*} \Box A$ then $\vdash_{I^*} A$, where $*$ is K, D, T, KDB, B, K4, KD4, S4 and S5.*

Proof. For the systems with axiom (11), namely T, B, S4 and S5, the claim is trivial.
As for IKDB consider the following proof:

$\vdash \Box A$	hypothesis
$\vdash \Box\Box A$	Theorem 4 e)
$\vdash \Box\Box A \supset \Diamond\Box A$	(10)
$\vdash \Diamond\Box A \supset A$	(16)
$\vdash \Box\Box A \supset A$	tautology
$\vdash A$	MP

We extend the *safe extension* theorem given in [4] (see exercises 3.62 and 3.63). Let $< W, \leq, R, \models >$ be an I^*-model with $* = K, D, K4, KD4$. Let $W' = W \cup \{x\}$ with $x \notin W$ and extend R, $\leq, \models$ as follows: $R' = R \cup \{< x, w > | w \in W\}$, $\leq' = \leq \cup \{< x, x >\}$, for all propositional variables p, $w \models' p$ iff $w \models p$ and $x \not\models' p$. It is easy to see that $< W', \leq', R', \models' >$ is an I^*-model also. Moreover if R is serial, transitive, serial and transitive then R' is respectively serial, transitive, serial and transitive. It is easy to verify by induction on the complexity of α that for all $w \in W$ is $w \models' \alpha$ iff $w \models \alpha$. Suppose now that for some model $< W, \leq, R, \models >$ and $w \in W$, $w \not\models A$. Consider the safe extension $< W', \leq', R', \models' >$ of $< W, \leq, R, \models >$. This implies $x \not\models \Box A$.

THEOREM 5 (Disjunction Property) *For all I^* $\vdash_{I^*} A \vee B$ iff either $\vdash_{I^*} A$ or $\vdash_{I^*} B$, where $*$ is K, D, T, KB, KDB, B, K4, KD4, S4, KB4, K5, K45 and S5.*

Proof. (Part A), $*$ is D, T, KDB, B, KD4, S4 and S5. We use the variant of Kleene's slash introduced in [1] (also see [17]):

$\vert p$	$\Leftrightarrow$	$\vdash_{I^*} p$
$\vert A \wedge B$	$\Leftrightarrow$	$\vert A$ *and* $\vert B$
$\vert A \vee B$	$\Leftrightarrow$	$\Vert\text{-}A$ *or* $\Vert\text{-}B$
$\vert \neg A$	$\Leftrightarrow$	*not* $\Vert\text{-}A$
$\vert A \supset B$	$\Leftrightarrow$	$\Vert\text{-}A$ *implies* $\vert B$
$\vert \Box A$	$\Leftrightarrow$	$\Vert\text{-}A$
$\vert \Diamond A$	$\Leftrightarrow$	$\Vert\text{-}A$

and where $\Vert\text{-}A$ means both $\vdash_{I^*} A$ and $|A$.

We prove by induction on the length of derivation that $\vdash_{I^*} A$ implies $|A$. The thesis then follows from the fact that if $\vdash_{I^*} A \vee B$ then $|A \vee B$ and thus either $\vdash_{I^*} A$ or $\vdash_{I^*} B$, by the definition of $|$ on $\vee$ (see also [1]).

For (1), (3), (4), (7), (8), (9) the proof is as in [1]. For (2), (11), (12), (13), (14) the proof is as in [6]. Notice first that the relation R in I^* where $*$ is D, T, KDB, B, KD4, S4 and S5 is serial, so if $\vdash_{I^*} A$ then $\vdash_{I^*} \Diamond A$. From this property and the definition of $|$ on $\Diamond$, we have that $\Vert\text{-}A$ implies $\Vert\text{-}\Diamond A$. This last property is used below in the proofs of (6) and (15).

Proof of (5). Suppose $\|\text{-}\Diamond(A \supset B)$ and $\|\text{-}\Box A$: we have to show $|\Diamond B$, namely $\|\text{-}B$. $\|\text{-}\Diamond(A \supset B)$ implies $\|\text{-}A \supset B$ and $\|\text{-}\Box A$ implies $\|\text{-}A$. $\|\text{-}A \supset B$ implies that from $\|\text{-}A$ we get $|B$. From $\|\text{-}A$ and $\|\text{-}A \supset B$ we have $\vdash_{I^*} B$.

Proof of (6). Suppose $\|\text{-}\Diamond A \supset \Box B$: we have to show $|\Box(A \supset B)$, namely $|A \supset B$ and $\vdash_{I^*} A \supset B$.

Let us first show $|A \supset B$, that is $\|\text{-}A$ implies $|B$. Suppose $\|\text{-}A$: we have $\|\text{-}\Diamond A$, by the above remark. The hypothesis $|\Diamond A \supset \Box B$ and $\|\text{-}\Diamond A$ imply $|\Box B$, so we have $|B$. On the other hand, the hypothesis $\vdash_{I^*} \Diamond A \supset \Box B$ and axiom (6) imply $\vdash_{I^*} \Box(A \supset B)$, hence $\vdash_{I^*} (A \supset B)$, by Lemma 1.

Proof of (10). Suppose $\|\text{-}\Box A$ then we have $|\Box A$ and $\|\text{-}A$, i.e. $|\Diamond A$.

Proof of (15). Suppose $\|\text{-}A$: we have to show $|\Box\Diamond A$, i.e. $\|\text{-}\Diamond A$, but this follows from the above remark.

Proof of (16). Suppose $\|\text{-}\Diamond\Box A$: we have to show $|A$. $\|\text{-}\Diamond\Box A$ implies $\|\text{-}\Box A$ and thus$|\Box A$, i.e.$\|\text{-}A$.

(Part B). $*$ is K, K4, K45, K5, KB, KB4. We show the Disjunction Property for $*$ via a semantic proof, based on modal intuitionistic Kripke models[1] (see Theorem 4.1 [5]). Consider two models $M_1 =< W_1, \leq_1, R_1, \models_1>$ and $M_2 =< W_2, \leq_2, R_2, \models_2>$ s.t. for some $w_1 \in W_1$ and $w_2 \in W_2$, $w_1 \not\models_1 A$ and $w_2 \not\models_2 B$. Construct the new Kripke model M by taking the disjoint union of M_1 and M_2 and add w, s.t. $w \notin W_1 \bigcup W_2$, $w \leq w_1$ and $w \leq w_2$. We stipulate that no propositional variable is forced at w. M is a modal intuitionistic Kripke model since it satisfies conditions ii) and iii) of Definition 1. By hypothesis $w \models A \vee B$ so that $w \models A$ or $w \models B$. We have a contradiction by the Heredity Property.

REMARK 4 In Part A of Theorem 5, both the rule $\vdash \Box A \Rightarrow \vdash A$ and the dual one $\vdash A \Rightarrow \vdash \Diamond A$ are used. Note that seriality, i.e. axiom (10), is equivalent to the rule $\vdash A \Rightarrow \vdash \Diamond A$. In fact, from $\vdash A \supset A$ we have $\vdash \Diamond(A \supset A)$ and, by (5), $\vdash \Box A \supset \Diamond A$. We have not proved the Disjunction Property for the serial systems IKD5 and IKD45, since Lemma 1 does not hold. In fact, $\Box(A \supset \Diamond A)$ is IKD5 and IKD45-valid, while $A \supset \Diamond A$ is not (see Example 4 in Section 4).

On the other hand, the proof of part B is only given for non-serial systems.

Note, finally, that Lemma 1 does not hold also for IK5, IKB4, IK45 and IKB. As for IK5, IKB4 and IK45 use the previous counterexample. For IKB consider the following counterexample: $\Box(A \supset \Diamond\Diamond A)$ is IKB valid while $A \supset \Diamond\Diamond A$ is not (see example 5 in section 4).

A consequence of Part A of Theorem 5 is that in I^*, with $* =$ D, T, KDB, B, KD4, S4 and S5, if $\vdash \Diamond A$ then $|\Diamond A$ and thus $\vdash A$, namely $\vdash A \Leftrightarrow \vdash \Diamond A$.

4 Semantic tableaux for intuitionistic modal logics

In this section we introduce semantic tableaux for intuitionistic modal logic and prove soundness and completeness theorems with respect to the semantics given in Section 2. We usethe notational

[1]After a suggestion of D.Gabbay

conventions of [14]. We thus use signed formulas, namely formulas prefixed by T and F. TA and FA are called signed formulas. For any formula A, we say that FA and TA are conjugate formulas. In the following upper-case Greek letters denote sets of signed or unsigned formulas.

DEFINITION 3 *Let A be a formula and $M =< W, \leq, R, \models>$ a model. For all w, $w \in W$ we use :*

i)	$w \models TA$	*for*	$w \models A$	
ii)	$w \models FA$	*for*	$w \not\models A$	
iii)	$w \models \delta_1 A, \delta_2 B$	*for*	$w \models \delta_1 A$ *and* $w \models \delta_2 B$,	*where* δ_i*(i=1,2) is F or T*
iv)	$w \models \delta_1 A \vert \delta_2 B$	*for*	$w \models \delta_1 A$ *or* $w \models \delta_2 B$,	*where* δ_i *(i=1,2) is F or T*

DEFINITION 4 *A signed formula A is I^*-satisfiable iff there exist an I^*-model $M =< W, \leq, R, \models>$ and w, $w \in W$, s.t. $w \models A$.*

A set Ω of signed formulas is I^*-satisfiable if there are an I^*-model M, $M =< W, \leq, R, \models>$ and w, $w \in W$, s.t. for all A, $A \in \Omega$, $w \models A$.

A set Ω of signed formulas is I^*-unsatisfiable if Ω is not I^*-satisfiable.

DEFINITION 5 *Let Γ be a set of signed formulas then:*

a) $\Gamma^{\#} = \{TA : TA \in \Gamma\}$

b)Γ^N *is defined as follows:*

for K, D, T as	$\{TA : T\Box A \in \Gamma\} \cup \{FA : F\Diamond A \in \Gamma\}$;
for K4, KD4 as	$\{TA, T\Box A : T\Box A \in \Gamma\} \cup \{FA, F\Diamond A : F\Diamond A \in \Gamma\}$;
for KB, KDB, B as	$\{TA : T\Box A \in \Gamma\} \cup \{FA : F\Diamond A \in \Gamma\} \cup$ $\cup \{T\Diamond A : TA \in \Gamma\} \cup \{F\Box A : FA \in \Gamma\}$;
for S4 as	$\{T\Box A : T\Box A \in \Gamma\} \cup \{F\Diamond A : F\Diamond A \in \Gamma\}$;
for KB4 as	$\{TA, T\Box A : T\Box A \in \Gamma\} \cup \{FA, F\Diamond A : F\Diamond A \in \Gamma\} \cup$ $\cup \{T\Diamond A : TA \in \Gamma\} \cup \{F\Box A : FA \in \Gamma\} \cup$ $\cup \{T\Diamond A : T\Diamond A \in \Gamma\} \cup \{F\Box A : F\Box A \in \Gamma\}$;
for K5, KD5 as	$\{TA : T\Box A \in \Gamma\} \cup \{FA : F\Diamond A \in \Gamma\} \cup$ $\cup \{T\Diamond A : T\Diamond A \in \Gamma\} \cup \{F\Box A : F\Box A \in \Gamma\}$;
for K45, KD45 as	$\{TA, T\Box A : T\Box A \in \Gamma\} \cup \{FA, F\Diamond A : F\Diamond A \in \Gamma\} \cup$ $\cup \{T\Diamond A : T\Diamond A \in \Gamma\} \cup \{F\Box A : F\Box A \in \Gamma\}$;
for S5 as	$\{T\Box A : T\Box A \in \Gamma\} \cup \{F\Diamond A : F\Diamond A \in \Gamma\} \cup$ $\cup \{T\Diamond A : T\Diamond A \in \Gamma\} \cup \{F\Box A : F\Box A \in \Gamma\}$.

4.1 Structural rules

Sets of signed formulas are treated as sets. Moreover we use the following rule:

$$w \perp rule) \quad \frac{\Gamma}{\Delta} \quad \text{with } \Gamma \supseteq \Delta$$

$$Cut)\quad \frac{\Gamma}{\Gamma, TA|\Gamma, FA}$$

4.2 Regular rules

Let Γ be a set of signed formulas then we have:

$$R\wedge)\quad \frac{\Gamma, TA\wedge B}{\Gamma, TA, TB}\quad \frac{\Gamma, FA\wedge B}{\Gamma, FA|\Gamma, FB}$$

$$R\vee)\quad \frac{\Gamma, TA\vee B}{\Gamma, TA|\Gamma, TB}\quad \frac{\Gamma, FA\vee B}{\Gamma, FA, FB}$$

$$R\neg)\quad \frac{\Gamma, T\neg A}{\Gamma, FA}$$

$$R\supset)\quad \frac{\Gamma, TA\supset B}{\Gamma, FA|\Gamma, TB}$$

4.3 Special rules for the intuitionistic connectives

$$S\neg)\quad \frac{\Gamma, F\neg A}{\Gamma^{\#}, TA}$$

$$S\supset)\quad \frac{\Gamma, FA\supset B}{\Gamma^{\#}, TA, FB}$$

4.4 Modal rules

Notational conventions

$$\frac{\Pi}{\overline{\Omega_1|\ldots|\Omega_n}}$$

means that $\Omega_1, \ldots, \Omega_n$ are obtained from Π by applying only structural rules, regular or special rules for the intuitionistic connectives.

Let Γ be a set of signed formulas. We denote by $T\Diamond\Gamma$ the set $\{T\Diamond A : TA \in \Gamma\}$ and by $F\Box\Gamma$ the set $\{F\Box A : FA \in \Gamma\}$.

Let Ψ be a set of unsigned formulas. We define $\Psi^{\top} = T\Psi = \{TA|A \in \Psi\}$, $\Psi^{-} = F\Psi = \{FA|A \in \Psi\}$, $T\Diamond\Psi = \{T\Diamond A|A \in \Psi\}$, $F\Diamond\Psi = \{F\Diamond A|A \in \Psi\}$, $T\Box\Psi = \{T\Box A|A \in \Psi\}$ and $F\Box\Psi = \{F\Box A|A \in \Psi\}$.

π-rules

$$S\Diamond)\quad \frac{\Gamma, T\Diamond A}{\Gamma^N, TA}$$

$$S\Box)\quad \frac{\Gamma, F\Box A}{\Gamma^{\#}, T\Diamond\Delta_1, F\Box\Omega_1|\ldots|\Gamma^{\#}, T\Diamond\Delta_p, F\Box\Omega_p|\Gamma^{\#N}, T\Delta_{p+1}, F\Omega_{p+1}|\ldots|\Gamma^{\#N}, T\Delta_n, F\Omega_n}$$

with $n \geq p \geq 0$ and where

$$\frac{\Gamma^{\#N}, FA}{\overline{\Gamma^{\#N}, T\Delta_1, F\Omega_1|\ldots|\Gamma^{\#N}, T\Delta_n, F\Omega_n}}$$

ν-rules

$$R\Box)\quad \frac{\Gamma, T\Box A}{\Gamma, TA}$$

$$R\Diamond)\quad \frac{\Gamma, F\Diamond A}{\Gamma, FA}$$

$$S\nu)\quad \frac{\Gamma}{\Gamma^N}$$

DEFINITION 6 *Let Γ be a set of signed formulas.*

(a) A thread of Γ is a finite sequence $\Gamma_0, \ldots, \Gamma_n$ of signed formulas such that

i) $\Gamma_0 = \Gamma$,

ii) for all i, $0 \leq i < n$, there exists a rule where Γ_i is above the line and Γ_{i+1} occurs below the line.

(b) An element of a thread is called a node.

(c) A node is called closed when it contains both TA and FA for some A.

(d) An I^*-T-tableau for Γ, or simply a tableau, is a set $\mathcal{T}$ of threads of Γ s.t. for any thread $t \in \mathcal{T}$ and for any node Γ_i, $\Gamma_i \in t$ and $i > 0$, the sets of formulas below the line of the rule applied on Γ_{i-1}, $\Gamma_{i-1} \in t$, are nodes of some thread $t' \in \mathcal{T}$.

(e) A tableau is called closed if each thread has a node that is closed.

(f) A closed tableau of Γ is called a proof for Γ.

(g) An I^*-T-tableau is called an I^*-T proof for the unsigned formula X if it is a proof for $\{FX\}$.

EXAMPLE 1 *$\neg\Diamond A \supset \Box\neg A$ has an IK-T proof:*

$$\frac{F\neg\Diamond A \supset \Box\neg A}{T\neg\Diamond A, F\Box\neg A}\quad S\supset)$$

$$S\Box)$$

$$\boxed{\frac{F\neg A}{TA}\quad \mathrm{S}\neg)}$$

$$\frac{T\neg\Diamond A, T\Diamond A}{F\Diamond A, T\Diamond A}\quad R\neg)$$

EXAMPLE 2 *$(\Diamond A \supset \Box B) \supset \Box((A \supset B) \vee C)$ has an IK-T proof:*

$$\frac{F(\Diamond A \supset \Box B) \supset \Box((A \supset B) \vee C))}{T(\Diamond A \supset \Box B), F\Box((A \supset B) \vee C)}\, S\supset$$

$$S\Box)$$

$$\boxed{\frac{\dfrac{F(A \supset B)\ \vee C}{FA \supset B, FC}\quad R\vee)}{TA, FB}\quad S\supset}$$

$$\frac{T(\Diamond A \supset \Box B), T\Diamond A, F\Box B}{F\Diamond A, T\Diamond A, F\Box B | T\Box B, T\Diamond A, F\Box B}\quad R\supset)$$

EXAMPLE 3 *$\Box C \wedge (\Diamond A \supset \Box B) \supset \Box((A \supset B) \wedge C))$ has an IK-T proof:*

$$\frac{\dfrac{F\Box C \wedge (\Diamond A \supset \Box B) \supset \Box((A \supset B) \wedge C))}{T\Box C \wedge (\Diamond A \supset \Box B), F\Box((A \supset B) \wedge C)}\quad S\supset)}{T\Box C, T(\Diamond A \supset \Box B), F\Box((A \supset B) \wedge C)}\, R\wedge)$$

$$S\Box)$$

$$\boxed{\frac{TC, F(A \supset B) \wedge C}{\dfrac{TC, FA \supset B}{TC, TA, FB}\ S\supset) | \quad TC, FC}\quad R\wedge)}$$

$$\frac{T\Box C, T(\Diamond A \supset \Box B), T\Diamond C, T\Diamond A, F\Box B \quad R\supset |}{T\Box C, T\Diamond C, F\Diamond A, T\Diamond A, F\Box B | T\Diamond C, T\Box C, T\Box B, T\Diamond A, F\Box B}\quad TC, FC$$

EXAMPLE 4 *$\Box(A \supset \Diamond A)$ has an IK5-T proof:*

$$F\Box(A \supset \Diamond A)\quad S\Box)$$

$$\boxed{\frac{FA \supset \Diamond A}{TA, F\Diamond A}\, S\supset}$$

$$\frac{T\Diamond A, F\Box\Diamond A}{T\Diamond A, F\Diamond A}\quad S\Box)$$

EXAMPLE 5 $\Box(A \supset \Diamond\Diamond A)$ *has an IKB-T proof:*

$$\begin{array}{c} F\Box(A \supset \Diamond\Diamond A) \qquad S\Box) \\ \boxed{\dfrac{FA \supset \Diamond\Diamond A}{TA, F\Diamond\Diamond A} S \supset} \\ \dfrac{T\Diamond A, F\Box\Diamond\Diamond A}{T\Diamond\Diamond A, F\Diamond\Diamond A} \qquad S\Box) \end{array}$$

Note that $FA \supset \Diamond\Diamond A$ cannot close.

4.5 Remarks and notational conventions

a) $S\Box)$ can be denoted as follows:

$$\begin{array}{c} \Gamma, F\Box A \\ \boxed{\Gamma^{\#N}, FA} \\ \boxed{\Gamma^{\#N}, T\Delta_1, F\Omega_1 | \ldots | \Gamma^{\#N}, T\Delta_p, F\Omega_p | \Gamma^{\#N}, T\Delta_{p+1}, F\Omega_{p+1} | \ldots | \Gamma^{\#N}, T\Delta_n, F\Omega_n} \\ \Gamma^{\#}, T\Diamond\Delta_1, F\Box\Omega_1 | \ldots | \Gamma^{\#}, T\Diamond\Delta_p, F\Box\Omega_p | \Gamma^{\#N}, T\Delta_{p+1}, F\Omega_{p+1} | \ldots | \Gamma^{\#N}, T\Delta_n, F\Omega_n \end{array}$$

with $n \geq p \geq 0$

b) If $p = 0$ and no structural rules, regular or special rules for the intuitionistic connectives are applied on $\Gamma^{\#N}, FA$ then we obtain the following derived rule:

$$S\Box^*) \quad \frac{\Gamma, F\Box A}{\Gamma^{\#N}, FA}$$

c) If no rules are applied on $\Gamma^{\#N}, FA$ and $p = n = 1$ we obtain the following derived rule:

$$\frac{\Gamma, F\Box A}{\Gamma^{\#}, T\Diamond\Gamma^{\#N}, F\Box A}$$

d) The following rule is derived:

$$S\Box \supset) \quad \frac{\Gamma, F\Box(A \supset B)}{\Gamma^{\#}, T\Diamond A, F\Box B}$$

In fact

$$\begin{array}{c} \Gamma, F\Box(A \supset B) \qquad S\Box) \\ \boxed{\dfrac{\Gamma^{\#N}, FA \supset B}{\Gamma^{\#N}, TA, FB} S \supset} \\ \Gamma^{\#}, T\Diamond A, F\Box B \end{array}$$

e) in [29] the modal rules are:

π-rules

$$S\Diamond)' \qquad \frac{\Gamma, T\Diamond A, F\Diamond B}{\Gamma\#^N, TA, FB} \qquad \frac{\Gamma, T\Diamond A}{\Gamma\#^N, TA}$$

$$S\Box)' \qquad \frac{\Gamma, F\Box A}{\Gamma\#^N, FA}$$

where $\Gamma^N = \{TA : T\Box A \in \Gamma\} \cup \{FA : F\Diamond A \in \Gamma\}$. Wijesekera's sequent system allows at most one formula to be on the right of a sequent (or, equivalently, at most one F signed formula to be below the tableau rules). Wijesekera gives a syntactic proof of the cut elimination theorem and thus the Disjunction Property holds in his intuitionistic K-system.

f) For systems with the axioms (15) and (16) the cut rule is not eliminable, the proof is as in [14]. Moreover in these systems $S\Box)$ is eliminable by cut and $S\Box^*)$. In fact, let $\Gamma = T\Psi, F\Phi$ where Ψ and Φ are set of formulas. Consider a tableau that closes. Consider the finite subset Ψ^* of Ψ used in the proof and let C be the conjunction of the formulas in Ψ^*. Without loss of generality we can suppose that the first rule applied in the proof is $S\Box)$ and $\Gamma = T\Psi^*$ namely

$$\begin{array}{c}
T\Psi^*, F\Box A \\
\hline
T\Psi^{*N}, FA \\
\hline\hline
T\Psi^{*N}, T\Delta_1, F\Omega_1 | \ldots | T\Psi^{*N}, T\Delta_p, F\Omega_p | T\Psi^{*N}, T\Delta_{p+1}, F\Omega_{p+1} | \ldots | T\Psi^{*N}, T\Delta_n, F\Omega_n \\
\hline
T\Psi^*, T\Diamond\Delta_1, F\Box\Omega_1 | \ldots | T\Psi^*, T\Diamond\Delta_p, F\Box\Omega_p | T\Psi^{*N}, T\Delta_{p+1}, F\Omega_{p+1} | \ldots | T\Psi^{*N}, T\Delta_n, F\Omega_n
\end{array}$$

Consider the following proof without $S\Box)$

$$\begin{array}{c}
T\Psi^*, F\Box A \quad \text{cut on } C) \\
\hline
\dfrac{T\Psi^*, F\Box A, TC \quad S\Box^*)}{T\Psi^{*N}, FA, T\Diamond C} \qquad \dfrac{T\Psi^*, F\Box A, FC}{\text{any node closes for some D in } \Psi^*} \quad R\wedge) \\
\hline\hline
\dfrac{\dfrac{T\Psi^{*N}, T\Delta_1, F\Omega_1, T\Diamond C \quad S\Diamond)}{T\Diamond\Delta_1, F\Box\Omega_1, TC \quad R\wedge)}}{T\Diamond\Delta_1, F\Box\Omega_1, T\Psi^*} \; | \ldots | \; \dfrac{\dfrac{T\Psi^{*N}, T\Delta_p, F\Omega_p, T\Diamond C \quad S\Diamond)}{T\Diamond\Delta_p, F\Box\Omega_p, TC \quad R\wedge)}}{T\Diamond\Delta_p, F\Box\Omega_p, T\Psi^*} \; | \ldots | \; \dfrac{T\Psi^{*N}, T\Delta_n, F\Omega_n, T\Diamond C \quad w)}{T\Psi^{*N}, T\Delta_n, F\Omega_n}
\end{array}$$

g) For the euclidean systems IK5 and IKD5 the cut is not eliminable. Consider the set $\Psi = \{T\Box A, T\Box B, F\Box\Diamond(A \wedge B)\}$. It is easy to see that all tableaux for Ψ without using cut can be reduced to the following ones:

$$
\begin{array}{ll}
T\Box A, T\Box B, F\Box\Diamond(A\wedge B) & S\Box \\
\boxed{TA, TB, F\Diamond(A\wedge B)} & \\
T\Box A, T\Box B, T\Diamond A, T\Diamond B, F\Box\Diamond(A\wedge B) & S\Box) \\
\hline
TA,\ TB,\ T\Diamond A, T\Diamond B, F\Diamond(A\wedge B) & S\Diamond)\text{ on } T\Diamond A \text{ [or } T\Diamond B] \\
\hline
TA, [TB], T\Diamond A, T\Diamond B, F(A\wedge B) & R\wedge) \\
\hline
TA, [TB], T\Diamond A, T\Diamond B, FA \mid TA, [TB], T\Diamond A, T\Diamond B, FB &
\end{array}
$$

while the following tableau closes:

$$
\begin{array}{ll}
T\Box A, T\Box B, F\Box\Diamond(A\wedge B) & \text{cut on } \Box(A\wedge B) \\
\hline
T\Box A, T\Box B, T\Box(A\wedge B), F\Box\Diamond(A\wedge B) \quad S\Box)| & \dfrac{T\Box A, T\Box B, F\Box(A\wedge}{TA, TB, FA\wedge B, F\Box\Diamond(} \\
\boxed{TA, TB, TA\wedge B, F\Diamond(A\wedge B), F\Box\Diamond(A\wedge B)} & \\
T\Diamond A, T\Diamond B, T\Diamond(A\wedge B), F\Box\Diamond(A\wedge B) & S\Box) \\
\boxed{T\Diamond A, T\Diamond B, T\Diamond(A\wedge B), F\Diamond(A\wedge B)} &
\end{array}
$$

Note that transitivity would make the first two tableaux close ($T\Box A$ and $T\Box B$ would be in the fourth node of the tableaux and so both TA and TB would belong to the fifth node).

4.6 Tableau systems

We denote by I^*-T the tableau system of I^*.

IK-T , IK4-T, IK5 -T, IK45 -T:
Intuitionistic rules + Cut rule + $S\Diamond)$ rule + $S\Box)$ rule.
ID-T , IKD4-T, IKD5-T, IKD45-T:
Intuitionistic rules + Cut rule + $S\Diamond)$ rule + $S\Box)$ rule + $S\nu)$rule.
IT-T , IS4-T:
Intuitionistic rules + Cut rule + $S\Diamond)$ rule + $S\Box)$ rule + $R\Box)$ and $R\Diamond)$ rules.
IKB-T, IKB4-T:
Intuitionistic rules + Cut rule + $S\Diamond)$ rule + $S\Box^*)$ rule.
IKDB-T:
Intuitionistic rules + Cut rule + $S\Diamond)$ rule + $S\Box^*)$ rule + $S\nu)$ rule.
IB-T , IS5-T:
Intuitionistic rules + Cut rule + $S\Diamond)$rule + $S\Box^*)$ rule+ $R\Box)$ and $R\Diamond)$ rules.

DEFINITION 7 *We recall some notations used in [14]. A β-formula is of the form: $FA\wedge B$ or $TA\vee B$ or $TA\supset B$. A regular α-formula is of the form $TA\wedge B$ or $FA\vee B$ or $T\neg A$. A special α-formula is of the form $FA\supset B$ or $F\neg A$. When breaking down a β-formula (α-formula) by means of regular rules we generate two new signed formulas below the line. Regular rules can be thus condensed according to the following notation (α_1, α_2 coincide in case of negation):*

$$R\beta) \quad \frac{\Gamma, \beta}{\Gamma, \beta_1 | \Gamma, \beta_2}$$

Analogously, in the case of special rules we have:

$$S\alpha) \quad \frac{\Gamma, \alpha}{\Gamma^{\#}, \alpha_1, \alpha_2}$$

We call $F\Diamond X$ and $T\Box Y$ ν-formulas and FX and TY, generated by them by means of the modal ν-rules, ν_0-formulas. For the ν-rules we have two different rules:

$$\nu \perp rule) \quad \frac{\Gamma}{\Gamma^N} \quad \frac{\Gamma, \nu}{\Gamma, \nu_0}$$

We call $T\Diamond X$ and $F\Box Y$ π-formulas and TX and FY, generated by them by means of the π-rules, π_0-formulas. Note that we cannot use a uniform notation for the π-rules.

The table below summarises the intuitionistic modal calculi along with their tableau systems as introduced so far.

Table 1

	Γ^N	ν-rule	π-rules	R	IK axioms +
IK-T	$\{\nu_0 : \nu \in \Gamma\}$	none	$S\Box$ $S\Diamond$	any	none
ID-T	$\{\nu_0 : \nu \in \Gamma\}$	$\frac{\Gamma}{\Gamma^N}$	$S\Box$ $S\Diamond$	serial	$\Box A \supset \Diamond A$
IK4-T	$\{\nu_0 : \nu \in \Gamma\} \cup \{\nu : \nu \in \Gamma\}$	none	$S\Box$ $S\Diamond$	transitive	$\Box A \supset \Box\Box A$ $\Diamond\Diamond A \supset \Diamond A$
IKD4-T	$\{\nu_0 : \nu \in \Gamma\} \cup \{\nu : \nu \in \Gamma\}$	$\frac{\Gamma}{\Gamma^N}$	$S\Box$ $S\Diamond$	serial transitive	$\Box A \supset \Diamond A$ $\Box A \supset \Box\Box A, \Diamond\Diamond A \supset \Diamond A$
IT-T	$\{\nu_0 : \nu \in \Gamma\}$	$\frac{\Gamma, \nu}{\Gamma, \nu_0}$	$S\Box$ $S\Diamond$	reflexive	$\Box A \supset A$ $A \supset \Diamond A$
IS4-T	$\{\nu : \nu \in \Gamma\}$	$\frac{\Gamma, \nu}{\Gamma, \nu_0}$	$S\Box$ $S\Diamond$	reflexive transitive	$\Box A \supset A, A \supset \Diamond A$ $\Box A \supset \Box\Box A, \Diamond\Diamond A \supset \Diamond A$
IKB-T	$\{\nu_0 : \nu \in \Gamma\} \cup \{\pi : \pi_0 \in \Gamma\}$	none	$S\Box^*$ $S\Diamond$	symmetric	$A \supset \Box\Diamond A$ $\Diamond\Box A \supset A$
IKDB-T	$\{\nu_0 : \nu \in \Gamma\} \cup \{\pi : \pi_0 \in \Gamma\}$	$\frac{\Gamma}{\Gamma^N}$	$S\Box^*$ $S\Diamond$	serial symmetric	$\Box A \supset \Diamond A$ $A \supset \Box\Diamond A, \Diamond\Box A \supset A$
IB-T	$\{\nu_0 : \nu \in \Gamma\} \cup \{\pi : \pi_0 \in \Gamma\}$	$\frac{\Gamma, \nu}{\Gamma, \nu_0}$	$S\Box^*$ $S\Diamond$	reflexive symmetric	$\Box A \supset A, A \supset \Diamond A$ $A \supset \Box\Diamond A, \Diamond\Box A \supset A$
IKB4-T	$\{\nu : \nu \in \Gamma\} \cup \{\pi : \pi \in \Gamma\} \cup$ $\{\nu_0 : \nu \in \Gamma\} \cup \{\pi : \pi_0 \in \Gamma\}$	none	$S\Box^*$ $S\Diamond$	symmetric transitive	$A \supset \Box\Diamond A, \Diamond\Box A \supset A$ $\Box A \supset \Box\Box A, \Diamond\Diamond A \supset \Diamond A$
IK5-T	$\{\nu_0 : \nu \in \Gamma\} \cup \{\pi : \pi \in \Gamma\}$	none	$S\Box$ $S\Diamond$	euclidean	$\Diamond A \supset \Box\Diamond A, \Diamond\Box A \supset \Box A$
IKD5-T	$\{\nu_0 : \nu \in \Gamma\} \cup \{\pi : \pi \in \Gamma\}$	$\frac{\Gamma}{\Gamma^N}$	$S\Box$ $S\Diamond$	serial euclidean	$\Box A \supset \Diamond A$ $\Diamond A \supset \Box\Diamond A, \Diamond\Box A \supset \Box A$
IK45-T	$\{\nu : \nu \in \Gamma\} \cup \{\pi : \pi \in \Gamma\} \cup$ $\{\nu_0 : \nu \in \Gamma\}$	none	$S\Box$ $S\Diamond$	transitive euclidean	$\Box A \supset \Box\Box A, \Diamond\Diamond A \supset \Diamond A$ $\Diamond A \supset \Box\Diamond A, \Diamond\Box A \supset \Box A$
IKD45-T	$\{\nu : \nu \in \Gamma\} \cup \{\pi : \pi \in \Gamma\} \cup$ $\{\nu_0 : \nu \in \Gamma\}$	$\frac{\Gamma}{\Gamma^N}$	$S\Box$ $S\Diamond$	serial transitive euclidean	$\Box A \supset \Diamond A$ $\Box A \supset \Box\Box A, \Diamond\Diamond A \supset \Diamond A$ $\Diamond A \supset \Box\Diamond A, \Diamond\Box A \supset \Box A$
IS5-T	$\{\nu : \nu \in \Gamma\} \cup \{\pi : \pi \in \Gamma\}$	$\frac{\Gamma, \nu}{\Gamma, \nu_0}$	$S\Box^*$ $S\Diamond$	reflexive symmetric transitive	$\Box A \supset A, A \supset \Diamond A$ $A \supset \Box\Diamond A, \Diamond\Box A \supset A$ $\Box A \supset \Box\Box A, \Diamond\Diamond A \supset \Diamond A$

LEMMA 2 *For all I^* models, where $*$ stands for K, D, T, KB, KDB, B, K4, KD4, S4, KB4, K5, KD5, K45, KD45 and S5, and for all w and $w^\bullet$, if $w \models \Gamma$ then $w^\bullet \models \Gamma^N$.*

Proof. We show the proof only for the euclidean models. For the ν-formulas the proof is as in [14]. Suppose that for some $T\Diamond A$, $w \models T\Diamond A$. There exists a world w' with wRw' s.t. $w' \models TA$. By euclideanness, for all $w^\bullet$ is $w^\bullet R w'$, and thus for all $w^\bullet$, $w^\bullet \models T\Diamond A$.

THEOREM 6 (Soundness) *Let Γ be a set of signed formulas. If Γ has an I^*-T-tableau proof, then Γ is I^*-unsatisfiable, where $*$ stands for K, D, T, KB, KDB, B, K4, KD4, S4, KB4, K5, KD5, K45, KD45 and S5.*

Proof. First we say that a tableau is satisfiable if for some thread all nodes are satisfiable. We have to prove that if T is a satisfiable tableau then the tableau T', obtained from T by applying a single tableau rule, is also satisfiable. We show this only for modal rules (see [14] for intuitionistic rules: the proof for the ν-rules is the same as in [14]). The theorem then follows easily: if Γ is satisfiable, then all tableaux for Γ are satisfiable, hence no tableau for Γ closes.

$S\Diamond$) Assume that $\Gamma, T\Diamond A$ is satisfiable. Let w be a world such that $w \models \Gamma, T\Diamond A$. Then there is a $w^\bullet$ such that $w^\bullet \models TA$. By Lemma 2, $w^\bullet \models \Gamma^N$ and hence Γ^N, TA is satisfiable.

$S\Box$) Choose $p \leq n$. Assume that for no $i \leq p$, $\Gamma^\#$, $T\Diamond\Delta_i$, $F\Box\Omega_i$ and for no p, $i < p \leq n$, $\Gamma^{\#N}$, $T\Delta_i$, $F\Omega_i$ are satisfiable. Assume, however, that there is a world w, s.t. $w \models \Gamma, F\Box A$. Then for some $w^{*\bullet*}$ and some i, $i \leq n$, $w^{*\bullet*} \models \Gamma^{\#N}$, $T\Delta_i$, $F\Omega_i$. Then by our assumption, $p > i$. Since $w^* R w^{*\bullet} \leq w^{*\bullet*}$, there is a world u such that $w^* \leq uRw^{*\bullet*}$. Then $w \leq u$ and so $u \models \Gamma^\#$. But also $u \models T\Diamond\Delta_i$, $F\Box\Omega_i$, which is a contradiction.

4.7 Completeness

In [14], a strong completeness theorem for the analytic systems without cut rule is given. This theorem thus provides a pure semantic proof of the cut elimination theorem. The method works for all logics where both one-half direction of the forcing relation on connectives over maximal consistent sets of signed formulas, and the frame properties of the canonical model can be shown without using cut.

In our context, we have not applied Fitting's method for the analogues of the classical analytic modal logics because the proofs of connecting properties of the canonical frame and the forcing relation for $\Box$ do not seem to carry through without cut. On the other hand, counterexamples showing that in such systems cut is independent are still missing.

In order to prove the completeness theorem for I^*-T we give some definitions and properties.

DEFINITION 8 *Let **C** be a collection of sets of signed formulas. **C** is an intuitionistic modal consistency property (imcp) if for all $S \in$ **C** the following holds:*

a) S contains no signed atomic formula and its conjugate;
b) if $\beta \in S$, then $S \cup \{\beta_1\} \in \mathbf{C}$ or $S \cup \{\beta_2\} \in \mathbf{C}$;
c) if $\alpha \in S$ is regular, then $S \cup \{\alpha_1, \alpha_2\} \in \mathbf{C}$;

d) if $\alpha \in S$ is special, then $S^{\#} \cup \{\alpha_1, \alpha_2\} \in \mathbf{C}$;

e) if $S \in \mathbf{C}$ then $S \cup \{TB\} \in \mathbf{C}$ or $S \cup \{FB\} \in \mathbf{C}$;

We write $S_1 \Rightarrow S_2$ to denote that $S_1, S_2 \in \mathbf{C}$ and S_2 is obtained from S_1 by one of the conditions b), c), d) and e).

f) if $F\Box A \in S$ then $S^{\#N} \cup \{FA\} \in \mathbf{C}$. Moreover if $S^{\#N} \cup \{FA\} \Rightarrow S_1 \Rightarrow \ldots \Rightarrow S_n = T\Delta \cup F\Omega$ then $S^{\#} \cup T\Diamond\Delta \cup F\Box\Omega \in \mathbf{C}$;

g) if $T\Diamond A \in S$ then $S^N \cup \{TA\} \in \mathbf{C}$.

LEMMA 3 *Let $\mathbf{C}$ be an intuitionistic modal consistency property (imcp), the following properties hold:*

(i) Let $\mathbf{C}'$ be the set of all the subsets of sets of $\mathbf{C}$. $\mathbf{C}'$ is an imcp.

(ii) Let $\mathbf{C}'$ be defined as in (i) and $\mathbf{C}''$ be the set of S s.t. $S' \in \mathbf{C}'$, for any finite subset S' of S (in symbol $S' \subseteq_f S$). Then $\mathbf{C}''$ is an imcp.

Let $\mathbf{C}$ be a set of sets of signed formulas closed under subsets and having the following finite subset property:

$S \in \mathbf{C} \Leftrightarrow F \in \mathbf{C}$, for all F, s.t. $F \subseteq_f S$.

Then the following holds:

iii) any element of $\mathbf{C}$ is contained in a maximal element of $\mathbf{C}$.

Proof.

(i) Let us now close $\mathbf{C}$ with respect to the inclusion relation.

We have to show that $\mathbf{C}'$ satisfies the conditions of Definition 8. It is easily seen that if $R \supseteq S$ and R' and S' (S' and S'' in case of β-formulas or cut condition e)) are obtained from R and S by applying one of the conditions b)-g) of Definition 8, then $R' \supseteq S'$. Consider such S and S'. If $S \in \mathbf{C}'$ then $S' \in \mathbf{C}'$ (either $S' \in \mathbf{C}'$ or $S'' \in \mathbf{C}'$ in case of β-formulas or cut condition e)).

(ii) We now expand $\mathbf{C}'$ to $\mathbf{C}''$, whose elements S are all the sets of signed formulas s.t. $F \in \mathbf{C}'$ for any finite subset $F \subseteq_f S$. Note that $\mathbf{C}'$ and $\mathbf{C}''$ agree on finite sets. We have to show that $\mathbf{C}''$ is an imcp. We prove the condition f) of Definition 8: the other cases are analogous.

f) Suppose $F\Box A \in S \in \mathbf{C}''$. We have to show that $S^{\#N} \cup \{FA\} \in \mathbf{C}''$. Let F be an arbitrary finite subset of $S^{\#N} \cup \{FA\}$. Let F^* be $\{T\Box A | TA \in F\}$. Since $F^* \cup \{F\Box A\}$ is a finite subset of S, it belongs to $\mathbf{C}'$ by definition of $\mathbf{C}''$, hence $F \in \mathbf{C}'$ by f) of Definition 8, applied to $\mathbf{C}'$. This proves that $S^{\#N} \cup \{FA\}$ belongs to $\mathbf{C}''$. Suppose that $S^{\#N} \cup \{FA\} \Rightarrow S_1 \Rightarrow \ldots \Rightarrow S_n = T\Delta \cup F\Omega$ with $S_i \in \mathbf{C}'$. It is easy to check that $S_n = T\Delta \cup F\Omega \in \mathbf{C}''$ also. Let F be an arbitrary finite subset of $S^{\#} \cup T\Diamond\Delta \cup F\Box\Omega$. Let F^* be $S^{\#N} \cup \{TA | T\Diamond A \in F\} \cup \{FA | F\Box A \in F\}$. Since $S_n \in \mathbf{C}''$, then $F^* \in \mathbf{C}'$ by definition of $\mathbf{C}''$, and thus $F \in \mathbf{C}'$ by f) of Definition 8 for $\mathbf{C}'$. This proves $S^{\#} \cup T\Diamond\Delta \cup F\Box\Omega \in \mathbf{C}''$.

(iii) Let $\mathbf{C}$ be a set having the finite subset property and $S \in \mathbf{C}$. Every chain of elements of $\mathbf{C}$, $\ldots S_\alpha \supseteq \ldots \supseteq S_2 \supseteq S_1 \supseteq S_0 = S$ such that $\alpha < \beta$ for some ordinal β, has an upper bound with respect to the inclusion relation. In fact, let $T = \bigcup_{\alpha<\beta} S_\alpha$ and F be any finite subset of T. For some ordinal α, $S_\alpha \supseteq F$, that is $F \in \mathbf{C}$. This implies that $T \in \mathbf{C}$, because $\mathbf{C}$ has the finite subset property. By Zorn's lemma, there exists a maximal element M in $\mathbf{C}$ with respect to the inclusion with $M \supseteq S$.

DEFINITION 9 *A consistent set S is a set of signed formulas such that no tableau for S closes. Let* **C** *be the collection of all consistent sets. A maximal consistent set is a consistent set which is maximal in* **C** *w.r.t. inclusion.*

LEMMA 4 *Let* **C** *be the collection of all consistent sets.* **C** *has the intuitionistic modal consistency property.*

Proof. See [14] (see pp. 61-62).

LEMMA 5 *For every X either $FX \in M$ or $TX \in M$, with M maximal in* **C**.

Proof. It is a consequence of e) of Definition 8.

Now we prove the completeness theorem for I*-T .

THEOREM 7 (Strong Completeness Theorem) *Let Γ be a set of signed formulas. If Γ is I^*-unsatisfiable then Γ has an I^*-T-tableau proof, where $*$ is K, D, T, KB, KDB, B, K4, KD4, S4, KB4, K5, KD5, K45, KD45 and S5.*

Proof. We prove the theorem only for IK.

We prove the contrapositive. Suppose Γ has no IK-T-tableau proof. We have to show that Γ is satisfiable. Let **C** be the collection of all consistent sets such that no IK-T-tableau closes. **C** is imcp by Lemma 4. It is $\Gamma \in$**C**.

We define an IK canonical model via **C**. Let W be the collection of all maximal sets in **C**. W is not empty by (iii) of Lemma 3.

$\leq$ is defined as follows:

$\Delta \leq \Gamma$ if and only if $\Gamma \supseteq \Delta^{\#}$

R is defined as follows:

$\Delta R \Gamma$ if and only if $\Gamma \supseteq \Delta^N$

The following properties are easily verified:

i) $\Gamma \supseteq \Delta \Rightarrow \Gamma^{\#} \supseteq \Delta^{\#}$

ii) $\Gamma \supseteq \Gamma^{\#}$

iii) $\Gamma^{\#} = \Gamma^{\#\#}$

$\leq$ is reflexive by ii).

$\leq$ is transitive. If $\Delta \leq \Gamma$ and $\Gamma \leq \Omega$ then $\Gamma \supseteq \Delta^{\#}$ and $\Omega \supseteq \Gamma^{\#}$. By i) and iii) we have $\Omega \supseteq \Delta^{\#}$, i.e. $\Delta \leq \Omega$.

LEMMA 6 *R and $\leq$ satisfy the connecting properties ii), iii) of Definition 1.*

Proof.

Property ii) of Definition 1. Suppose $\Delta \leq \Omega$ and $\Delta R \Gamma$. We have to show that there exists a maximal consistent set Z, s.t. $\Omega R Z$ and $\Gamma \leq Z$. Consider the set of formulas $\Gamma^{\#} \cup \Omega^N$. If $\Gamma^{\#} \cup \Omega^N$ is in **C**, then there exists a maximal consistent set $Z \supseteq \Gamma^{\#} \cup \Omega^N$, by (iii) of Lemma 3. Z is obviously the requested set. We only have to show that $\Gamma^{\#} \cup \Omega^N \in$ **C**.

Note that $TA, TB \in \Omega^N \Leftrightarrow TA \wedge B \in \Omega^N$ and $FA, FB \in \Omega^N \Leftrightarrow FA \vee FB \in \Omega^N$, by the cut condition e) of Definition 8, $R\wedge$) and $R\vee$). Then it is sufficient to show, by the finite subset property of $\mathbf{C}$, that $\Gamma^{\#} \cup \{FY\} \cup \{TX\}$ is a consistent set, for all X and Y, s.t. $T\Box X \in \Omega$ and $F\Diamond X \in \Omega$. Suppose not. Let X and Y be s.t. $TX \in \Omega^N$, $FY \in \Omega^N$ and $\Gamma^{\#} \cup \{FY\} \cup \{TX\} \notin \mathbf{C}$. We have $\Gamma^{\#} \cup \{FX \supset Y\}$ is not in $\mathbf{C}$ by condition d) of Definition 8, hence $TX \supset Y \in \Gamma$ by Lemma 5. It follows that $T\Diamond(X \supset Y) \in \Delta$ by Lemma 5 and the hypothesis $\Delta R \Gamma$. Since $\Delta \leq \Omega$, we get $T\Diamond(X \supset Y) \in \Omega$. By g) and c) of Definition 8, either $\Omega^N \cup \{FX\}$ or $\Omega^N \cup \{TY\}$ is a consistent set, which is a contradiction.

Property iii) of Definition 1. Suppose $\Gamma R \Delta \leq \Omega$. We have to show that there exists a maximal consistent set Z, s.t. $\Gamma \leq Z R \Omega$. Consider the set of formulas $\Omega^{\Pi} = \{T\Diamond A : TA \in \Omega\}$ $\cup \{F\Box A : FA \in \Omega\}$ and the set of formulas $\Gamma^{\#} \cup \Omega^{\Pi}$. If $\Gamma^{\#} \cup \Omega^{\Pi}$ is a consistent set, then there exists a maximal consistent set $Z \supseteq \Gamma^{\#} \cup \Omega^{\Pi}$, by (iii) of Lemma 3. Z is the requested set. In fact $Z \supseteq \Gamma^{\#}$, by construction. We have to show $Z^N \subseteq \Omega$. Suppose that $T\Box X \in Z$ and $TX \notin \Omega$, for some X. By Lemma 5 $FX \in \Omega$ and thus $F\Box X \in \Omega^{\Pi}$, hence $F\Box X \in Z$, while $T\Box X \in Z$. In analogous way, one can show that $FX \in Z^N$ implies $FX \in \Omega$.

It remains to show that $\Gamma^{\#} \cup \Omega^{\Pi} \in \mathbf{C}$. Suppose not. Then there exists a finite subset $\Omega^{\Pi}_f =$ $\{T\Diamond A_i$: $TA_i \in \Omega$ with $i \leq n\} \cup \{F\Box B_i : FB_i \in \Omega$ with $i \leq m\}$ of Ω^{Π}, s.t. that $\Gamma^{\#} \cup \Omega^{\Pi}_f \notin \mathbf{C}$.

Let us now show that $T\bigwedge_{i\leq n} \Diamond A_i \supset \bigvee_{i \leq m} \Box B_i \in \Gamma$. If $\Gamma^{\#} \cup \Omega^{\Pi}_f \notin \mathbf{C}$, then $\Gamma^{\#} \cup \{T\bigwedge_{i\leq n} \Diamond A_i\}$ $\cup \{F\bigvee_{i\leq m} \Box B_i\} \notin \mathbf{C}$ by b) and c) of Definition 8, and thus $\Gamma^{\#} \cup \{F\bigwedge_{i\leq n} \Diamond A_i \supset \bigvee_{i\leq m} \Box Bi\} \notin \mathbf{C}$ by d) of Definition 8. By Lemma 5, $T\bigwedge_{i\leq n} \Diamond A_i \supset \bigvee_{i\leq m} \Box B_i \in \Gamma$.

Let us now show that $F\Box(\bigwedge_{i\leq n} A_i \supset \bigvee_{i\leq m} B_i) \in \Gamma$. Suppose not. Thus, by Lemma 5, $T\Box(\bigwedge_{i\leq n} A_i \supset \bigvee_{i\leq m} B_i) \in \Gamma$. Then some FA_i or some TB_j belongs to $\Gamma^{\#N}$, by b) and c) of Definition 8. But $\Omega \supseteq \Gamma^{\#N}$ would imply a contradiction. We can suppose both $T\bigwedge_{i\leq n} \Diamond A_i$ $\supset \bigvee_{i\leq m} \Box B_i \in \Gamma$ and $F\Box \bigwedge_{i\leq n} A_i \supset \bigvee_{i\leq m} B_i \in \Gamma$. Consider the following closed tableau for Γ.

$$T\bigwedge_{i<n} \Diamond A_i \supset \bigvee_{i<m} \Box B_i, F\Box \bigwedge_{i<n} A_i \supset \bigvee_{i<m} B_i \qquad S\Box)$$

$$\boxed{\begin{array}{ll} \dfrac{\dfrac{F\bigwedge_{i\leq n} A_i \supset \bigvee_{i\leq m} B_i}{T\bigwedge_{i\leq n} A_i, F\bigvee_{i\leq m} B_i}}{TA_1, \ldots, TA_n, FB_1, \ldots, FB_m} & \begin{array}{l} S\supset) \\ R\wedge) \text{ and } R\vee) \end{array} \end{array}}$$

$$\dfrac{T\bigwedge_{i<n} \Diamond A_i \supset \bigvee_{i<m} \Box B_i, T\Diamond A_1, \ldots, T\Diamond A_n, F\Box B_1, \ldots, F\Box B_m \quad R\supset)}{F\bigwedge_{i\leq n} \Diamond A_i, T\Diamond A_1, \ldots, T\Diamond A_n, F\Box B_1, \ldots, F\Box B_m | T\bigvee_{i\leq m} \Box B_i, T\Diamond A_1, \ldots, T\Diamond A_n, F\Box B_1, \ldots, F\Box B_m}$$

Both nodes close by $R\wedge$) and $R\vee$), while $\Gamma \in \mathbf{C}$. This proves that $\Gamma^{\#} \cup \Omega^{\Pi}$ is consistent.

We have proved that $< W, \leq, R >$ is a frame.

We define an interpretation as follows:

$$\Gamma \models A \Leftrightarrow TA \in \Gamma;$$

This defines a model. The proof is by induction on the degree of A. The only troublesome step of the proof is for the connective $\Box$. In order to show that $F\Box A \in \Gamma$ implies that there are Δ and Ω s.t. $\Gamma \leq \Delta R \Omega$ with $FA \in \Omega$ we prove the following lemma:

LEMMA 7 *If $F\Box A \in \Gamma$ then there exists Z, $Z \in \mathbf{C}$, $Z \supseteq \Gamma^{\#} \cup \{F\Box A\} \cup \{T\Diamond B : T\Box B \in \Gamma\}$ and s.t. $Z^N \cup \{FA\} \in \mathbf{C}$.*

Proof. Suppose $F\Box A \in \Gamma$. Note first that $\Gamma^{\#} \cup \{F\Box A\} \cup \{T\Diamond B | T\Box B \in \Gamma\}$ is consistent, by f) of Definition 8 and Remark c) of paragraph 4.5. By iii) of Lemma 3, there exists a maximal consistent set Δ s.t. $\Delta \supseteq \Gamma^{\#} \cup \{F\Box A\} \cup \{T\Diamond B | T\Box B \in \Gamma\}$. Let Z be any such a Δ which is also maximal w.r.t. $\leq$. It is easy to show that Z exists by Zorn's lemma. Now we show that Z is the requested set. Let us first prove a useful property of maximal consistent sets.

For all maximal consistent sets Δ with $F\Box A \in \Delta$ there exists a maximal consistent set Z, $\Delta \leq Z$ s.t. $F\Box A \in Z$ and for all formulas B, $T\Diamond B \in Z$ if $\Delta^{\#N} \cup \{FA\} \cup \{FB\} \notin \mathbf{C}$.

Construct Z as follows. Consider an ordering of all the formulas of the language:

$$B_1, B_2, \ldots, B_n, \ldots$$ [2]

and construct a sequence of consistent sets of formulas:

$\Delta_0 = \Delta^{\#} \cup \{F\Box A\} \cup \{T\Diamond B : T\Box B \in \Delta\} \subseteq \Delta_1 \subseteq \Delta_2 \subseteq \ldots \subseteq \Delta_n \ldots$

where $\Delta_n = \Delta^{\#}_{n-1} \cup \{F\Box A\} \cup \{T\Diamond B_n\}$ if $\Delta^{\#N} \cup \{FA\} \cup \{FB_n\} \notin \mathbf{C}$,

$\Delta_n = \Delta_{n-1}$ otherwise.

Let us show by induction that Δ_n, $n < \omega$, is a consistent set. We have already shown that Δ_0 is consistent. Suppose by induction that Δ_{n-1} is consistent. We need to show that Δ_n is consistent. Note that $\Delta^{\#N} = \Delta_n^{\#N}$. Suppose that $\Delta_n \neq \Delta_{n-1}$, i.e. $\Delta^{\#N}_{n-1} \cup \{FA\} \cup \{FB_n\} \notin \mathbf{C}$. By construction, $F\Box A \in \Delta^{\#}_{n-1}$, so $\Delta^{\#N}_{n-1} \cup \{FA\} \in \mathbf{C}$, by f) of Definition 8. Then $\Delta^{\#N}_{n-1} \cup \{FA\} \cup \{TB_n\} \in \mathbf{C}$, from $\Delta^{\#N}_{n-1} \cup \{FA\} \in \mathbf{C}$ and e) of Definition 8.

Again, by f) of Definition 8, $\Delta^{\#}_{n-1} \cup \{F\Box A\} \cup \{T\Diamond B_n\} = \Delta_n \in \mathbf{C}$. By iii) of Lemma 3 there is a Z which is a maximal consistent set containing $\bigcup_{n<\omega} \Delta_n$.

It is easy to check that $T\Diamond B \in Z$ whenever $\Delta^{\#N} \cup \{FA\} \cup \{FB\} \notin \mathbf{C}$. Now, if Δ is maximal with respect to $\leq$, then $Z^{\#} = \Delta^{\#}$ and so $T\Diamond B \in Z$ if $Z^{\#N} \cup \{FA\} \cup \{FB\} \notin \mathbf{C}$.

In the hypothesis of the maximality of Z w.r.t. $\leq$, then $Z^N \cup \{FA\}$ is a consistent set. Suppose not. Without loss of generality we can suppose that $Z^{\#N} \cup \{FA\} \cup \{FB\} \notin \mathbf{C}$, for some B with $F\Diamond B \in Z$, by $R\vee$) and c) of Definition 8. This implies $T\Diamond B \in Z$, by construction of Z, which is a contradiction.

This concludes the proof of the lemma.

Now, $F\Box A \in \Gamma$, so Lemma 3 and Lemma 7 imply that for some maximal consistent set Z and Ω, $\Gamma \models F\Box A$ and $\Omega \models FA$, with $\Gamma \leq ZR\Omega$.

This proves completeness.

COROLLARY 1 *Let X be an unsigned formula. X is I^* valid if and only if X has an I^*-T proof, where $*$ is K, D, T, KB, KDB, B, K4, KD4, S4, KB4, K5, KD5, K45, KD45 and S5.*

Proof. Note that X is I^*-valid if and only if FX is I^*-unsatisfiable. The claim follows from Theorems 6 and 7.

[2]Without loss of generality we can suppose the language denumerable

5 Sequent calculi for Intuitionistic Modal Logics

In this section we present a Gentzen type formulation [20] for I*.

We now recall some notations and definitions from [27]. In the following Greek capital letters Γ, Δ , Λ, ..., Π, denote finite (possible empty) unsigned sequences of formulas separated by commas. We introduce an auxiliary symbol $\rightarrow$.

DEFINITION 10 *For arbitrary Γ and Δ, $\Gamma \rightarrow \Delta$ is called a sequent. Γ and Δ are called the antecedents and succedents, respectively, of the sequent and each formula in Γ and Δ is called a sequent formula.*

DEFINITION 11 *A rule is of the form:*

$$\frac{S_1}{S} \quad or \; \frac{S_1 \quad S_2}{S}$$

where S_1 S_2 and S are sequents, the sequents above (below) the line of the rules are called upper *(*lower*) sequents.*

We now give sequent calculi for I^*-systems.

5.1 Axioms

$$\Gamma, A \rightarrow A, \Delta$$

5.2 Structural rules

Weakening

$$w : left \quad \frac{\Gamma \rightarrow \Delta}{D, \Gamma \rightarrow \Delta} \quad w : right \quad \frac{\Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, A}$$

Contraction

$$c : left \quad \frac{D, D, \Gamma \rightarrow \Delta}{D, \Gamma \rightarrow \Delta} \quad c : right \quad \frac{\Gamma \rightarrow \Delta, A, A}{\Gamma \rightarrow \Delta, A}$$

Exchange

$$e : left \quad \frac{\Gamma, C, D, \Gamma \rightarrow \Delta}{\Gamma, D, C, \Gamma \rightarrow \Delta} \quad e : right \quad \frac{\Gamma \rightarrow \Delta, C, D, \Lambda}{\Gamma \rightarrow \Delta, D, C, \Lambda}$$

Cut Rule

$$\frac{\Gamma \rightarrow \Delta, A \quad A, \Pi \rightarrow \Lambda}{\Gamma, \Pi \rightarrow \Delta, \Lambda}$$

5.3 Logical rules

$$\wedge : left \quad \frac{C, D, \Gamma \rightarrow \Delta}{C \wedge D, \Gamma \rightarrow \Delta} \quad \wedge : right \quad \frac{\Gamma \rightarrow \Delta, A \quad \Gamma \rightarrow \Delta, B}{\Gamma \rightarrow \Delta, A \wedge B}$$

$$\vee : left \quad \frac{C, \Gamma \rightarrow \Delta \quad D, \Gamma \rightarrow \Delta}{C \vee D, \Gamma \rightarrow \Delta} \quad \vee : right \quad \frac{\Gamma \rightarrow \Delta, A, B}{\Gamma \rightarrow \Delta, A \vee B}$$

$$\neg : left \quad \frac{\Gamma \rightarrow \Delta, A}{\Gamma, \neg A \rightarrow \Delta} \quad \neg : right \quad \frac{A, \Gamma \rightarrow}{\Gamma \rightarrow \neg A}$$

$$\supset : left \quad \frac{\Gamma \rightarrow \Delta, C \quad D, \Pi \rightarrow \Lambda}{C \supset D, \Gamma, \Pi \rightarrow \Delta, \Lambda} \quad \supset : right \quad \frac{\Gamma, A \rightarrow B}{\Gamma \rightarrow A \supset B}$$

5.4 Modal rules

$$\Box : right \quad \frac{\Pi, \Gamma^\nu, \Diamond\Lambda_1 \rightarrow \Box\Omega_1 \ldots \Pi, \Gamma^\nu, \Diamond\Lambda_p \rightarrow \Box\Omega_p \qquad \Gamma, \Lambda_{p+1} \rightarrow \Omega_{p+1} \ldots \Gamma, \Lambda_n \rightarrow \Omega_n}{\Pi, \Gamma^\nu \rightarrow \Box A}$$

with $n \geq p \geq 0$ and where

$$\frac{\Gamma, \Lambda_1 \rightarrow \Omega_1 \ldots \Gamma, \Lambda_n \rightarrow \Omega_n}{\overline{\begin{array}{c}\cdots\\ \text{apply only structural and logical rules}\\ \cdots\end{array}}\atop \Gamma \rightarrow A}$$

where $\Gamma^\nu = \{\Box A : A \in \Gamma\}$ for K, D, T;
$\Gamma^\nu = \{\Box A : A \in \Gamma\} \cup \{\Box A : \Box A \in \Gamma\}$ for KD4, K4;
$\Gamma^\nu = \{\Box A : A \in \Gamma\} \cup \{A : \Diamond A \in \Gamma\}$ for KB, KDB, B;
$\Gamma^\nu = \{\Box A : \Box A \in \Gamma\}$ for S4;
$\Gamma^\nu = \{\Box A : A \in \Gamma\} \cup \{\Box A : \Box A \in \Gamma\} \cup \{A : \Diamond A \in \Gamma\} \cup \{\Diamond A : \Diamond A \in \Gamma\}$ for KB4;
$\Gamma^\nu = \{\Box A : A \in \Gamma\} \cup \{\Diamond A : \Diamond A \in \Gamma\}$ for K5, D5;
$\Gamma^\nu = \{\Box A : A \in \Gamma\} \cup \{\Box A : \Box A \in \Gamma\} \cup \{\Diamond A : \Diamond A \in \Gamma\}$ for K45, KD45;
$\Gamma^\nu = \{\Box A : \Box A \in \Gamma\} \cup \{\Diamond A : \Diamond A \in \Gamma\}$for S5;

$$\Diamond : left \quad \frac{\Gamma, A \rightarrow \Delta}{\Gamma^\nu, \Diamond A \rightarrow \Delta^\pi}$$

where $\Delta^\pi = \{\Diamond A : A \in \Delta\}$ for K, D, T;
$\Delta^\pi = \{\Diamond A : A \in \Delta\} \cup \{\Diamond A : \Diamond A \in \Delta\}$ for KD4, K4;

$\Delta^\pi = \{\Diamond A : A \in \Delta\} \cup \{A : \Box A \in \Delta\}$ for KB, KDB, B;
$\Delta^\pi = \{\Diamond A : \Diamond A \in \Delta\}$ for S4;
$\Delta^\pi = \{\Diamond A : A \in \Delta\} \cup \{\Diamond A : \Diamond A \in \Delta\} \cup \{A : \Box A \in \Delta\} \cup \{\Box A : \Box A \in \Delta\}$for KB4;
$\Delta^\pi = \{\Diamond A : A \in \Delta\} \cup \{\Box A : \Box A \in \Delta\}$ for K5, D5;
$\Delta^\pi = \{\Diamond A : A \in \Delta\} \cup \{\Diamond A : \Diamond A \in \Delta\} \cup \{\Box A : \Box A \in \Delta\}$ for K45, KD45;
$\Delta^\pi = \{\Diamond A : \Diamond A \in \Delta\} \cup \{\Box A : \Box A \in \Delta\}$ for S5.

For ID, IKDB, IKD4, IKD5, IKD45:

$$\nu : rule \quad \frac{\Gamma \to \Delta}{\Gamma^\nu \to \Delta^\pi}$$

For IT, IB, IS4 and IS5:

$$\Box : left \quad \frac{\Gamma, A \to \Delta}{\Gamma, \Box A \to \Delta}$$

$$\Diamond : right \quad \frac{\Gamma \to \Delta, A}{\Gamma \to \Delta, \Diamond A}$$

DEFINITION 12 *An I^*-sequent proof of a sequent $\Gamma \to \Delta$ is a tree of sequents, s.t.:*

a) Every sequent in the proof is an upper sequent except the lowest one (endsequent) that is $\Gamma \to \Delta$.

b) Every sequent in the proof is a lower sequent except the topmost ones (initial sequents) which are axioms.

Any I^*-sequent proof of a formula X is an I^*-sequent proof of $\to X$

DEFINITION 13 *We say that the sequent $\Gamma \to \Delta$ is I^*-satisfiable if there are an I^*- model M and some world w of M, s.t. $w \models \Gamma$ implies $w \models \delta$, for some formula δ, $\delta \in \Delta$ (in symbols $w \models \Gamma \to \Delta$).*

$\Gamma \to \Delta$ is I^*-valid if $w \models \Gamma \to \Delta$ for every I^*-model M and every world w of M.

$\Gamma \to \Delta$ is I^*-unsatisfiable if $\Gamma \to \Delta$ is not I^*-satisfiable.

$\Gamma \to \Delta$ is I^*-valid iff $\Gamma^\top, \Delta^-$ I^*-unsatisfiable.

THEOREM 8 (Completeness) *A sequent $\Gamma \to \Delta$ is I^*-valid iff $\Gamma \to \Delta$ has an I^*-sequent proof.*

Proof. We associate with an I^*-T-tableau proof any I^*-sequent proof, and viceversa. The completeness thus comes out from the completeness theorem of the semantic tableaux. Let T be a sequent tree.

1) Transform any sequent according to the following definition:

$$(\Gamma \to \Delta)^* = \Gamma^\top, \Delta^-$$

.

2) Put the tree upside-down.
3) By reading the sequent rules upside-down, we get the tableaux rules.
Note that the endsequents are transformed into closed nodes.
In analogous way, from an I^*-T-tableau proof T we get an I^*-sequent proof.
1) Transform any node into a sequent according to the following definition:

$$(\Gamma^\top, \Delta^-)^* = \Gamma \to \Delta$$

2) Put the tableau upside-down.
3) By reading the tableau rules upside-down, we get easily the sequent rules.
Note that the closed nodes in a tableau proof are transformed in axioms (endsequents).

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