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Robust local problem error estimation for a singularly perturbed reaction-diffusion problem on anisotropic finite element meshes

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Abstract

Singularly perturbed reaction-diffusion problems exhibit in general solutions with anisotropic features, e.g. strong boundary and/or interior layers. This anisotropy is reflected in the discretization by using meshes with anisotropic elements. The quality of the numerical solution rests on the robustness of the a posteriori error estimator with respect to both the perturbation parameters of the problem and the anisotropy of the mesh. An estimator that has shown to be one of the most reliable for reaction-diffusion problem is the *equilibrated residual method* and its modification done by Ainsworth and Babuška for singularly perturbed problem. However, even the modified method is not robust in the case of anisotropic meshes.

The present work modifies the equilibrated residual method for anisotropic meshes. The resulting error estimator is equivalent to the equilibrated residual method in the case of isotropic meshes and is proved to be robust on anisotropic meshes as well. A numerical example confirms the theory.

Key words: a posteriori error estimation, singular perturbations, reaction-diffusion problem, robustness, anisotropic solution, stretched elements

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1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be an open domain with polyhedral boundary $\partial\Omega$. Consider the reaction-diffusion problem with homogeneous Dirichlet boundary conditions

$$-\Delta u + \kappa^2 u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{1.1}$$

where κ is a nonnegative constant.

If $\kappa \gg 1$, then we have a singularly perturbed problem. Many physical phenomena lead to singularly perturbed problems. For instance boundary value problems formulated on thin domains [16], where κ is proportional to the inverse of the domain thickness. They also arise in mathematical models of physical problems, where diffusion is small compared with reaction and convection.

Such problems yield solutions with local anisotropic behavior, e.g. boundary and/or interior layers. In these cases special mesh adaptivity is desirable. Triangles should not only adapt in size but also in shape, to better fit the function to be approximated. While standard finite element meshes consist of isotropic elements, in the current work so-called anisotropic elements are investigated. They are characterized by a large aspect ratio (the ratio of the diameters of the circumscribed and inscribed spheres). The singularly perturbed reaction diffusion problem typically requires triangles stretched along the boundary or in the direction of the interior layer [4, 5, 8].

An error estimator that has shown to be one of the most reliable for the singularly perturbed reaction-diffusion problem is the modified equilibrated residual method [1]. The main purpose of the current work is to consider this estimator on anisotropic meshes and to construct upper and lower error bounds. It turns out that the equilibrated residual method fails on anisotropic meshes due to a (potentially unbounded) factor appearing in the lower bound. This factor is equivalent to 1 on isotropic meshes, but it can be arbitrarily large on anisotropic meshes.

A new modification for anisotropic elements leads to a robust error estimator. The upper error bound of the modification contains the factor $m_1(e, \mathcal{T})$ which is in accordance with the results by Kunert in [14].

The paper is organized as follows. After describing the model problem and its discretization in §2, the standard equilibrated residual method and its modification for the singularly perturbed case are briefly overviewed in §3. Moreover the upper error bound is obtained. In §4 some properties of the equilibrated residual method on anisotropic meshes are proved. In §5 one finds the lower error bound for the standard estimator. Furthermore, in §6 the modification of the equilibrated residual method for the anisotropic case is introduced and the resulting estimator is proved to be robust. A numerical example completes the discussion.

2 The model problem, its discretization and some notation

Let ω be an open subset of Ω. Introduce the usual L_2 scalar product $(u,v)_{L_2(\omega)} = \int_\omega uv\,dx$ and energy scalar product $B_\omega(u,v) = \int_\omega \left(\nabla^\top u \nabla v + \kappa^2 uv\right)\,dx$, which lead to the appropriate norms $\|v\|^2_{L_2(\omega)} = (u,u)_{L_2(\omega)}$ and $|||v|||^2_\omega = B_\omega(u,u)$. When $\omega = \Omega$ the subscript will be omitted. For an edge γ and an element K, let $|\gamma| = \mathrm{meas}_1(\gamma)$ and $|K| = \mathrm{meas}_2(K)$ denote its length and area, respectively. In what follows $\widetilde{K}$ will denote the patch of elements around K that satisfy such relation: $K' \subset \widetilde{K}$ iff $\overline{K} \cap \overline{K'}$ is nonempty. Analogously we define the patch $\widetilde{\gamma}$ of an edge γ: $K \subset \widetilde{\gamma}$ iff $\gamma \subset \partial K$.

Consider the problem (1.1) and assume $f \in L_2(\Omega)$. The Sobolev space of functions from $H^1(\Omega)$ that vanish on $\partial\Omega$ is denoted by $H^1_0(\Omega)$ as usual. The corresponding variational formulation for (1.1) becomes:

$$\text{Find } u \in H^1_0(\Omega): \quad B(u,v) = (f,v) \quad \forall v \in H^1_0(\Omega). \tag{2.1}$$

We utilize a family $\mathcal{F} = \{\mathcal{T}\}$ of triangulations $\mathcal{T}$ of Ω. Let $X \subset H^1_0(\Omega)$ be the space of continuous, piecewise linear functions over $\mathcal{T}$ that vanish on $\partial\Omega$. Then the finite element solution $u_X \in X$ is uniquely defined by

$$B(u_X, v_X) = (f, v_X) \quad \forall v_X \in X. \tag{2.2}$$

Due to the Lax-Milgram Lemma both problems (2.1) and (2.2) admit unique solutions.

Let $\mathcal{N}$ be the set of all the nodes in triangulation $\mathcal{T}$, then we denote by $\mathcal{N}(K)$ and $\mathcal{N}(\gamma)$ the set of all vertices of a triangle K and an edge γ respectively. Let $x_n \in \mathcal{N}$ be any node and let θ_n be the Lagrange basis function associated with that node. Let $\widetilde{x}_n = \mathrm{supp}\theta_n$ be the patch of elements around vertex x_n. Similarly, the set of edges $\mathcal{E}_n$ consists of those edges having a vertex at x_n.

We will require an extension operator $F_{ext} : \mathbb{P}^0(\gamma) \mapsto \mathbb{P}^0(K)$ defined by

$$F_{ext}(\varphi)(x) := \varphi|_\gamma \equiv const.$$

Now we introduce so-called bubble functions which are defined as usual, cf. [15]. They play an important role in deriving lower error bounds. Denote by $\lambda_{K,1}, \lambda_{K,2}, \lambda_{K,3}$ the barycentric coordinates of an arbitrary triangle K. The *element bubble function* b_K is defined by

$$b_K := 27\lambda_{K,1} \cdot \lambda_{K,2} \cdot \lambda_{K,3} \text{ on } K$$

Let $\gamma = \mathrm{int}(\overline{K}_1 \cap \overline{K}_2)$ be an inner face (edge) of $\mathcal{T}$. Enumerate the vertices of K_1 and K_2 such that the vertices of γ are numbered first. Define the *face bubble function* b_γ by

$$b_\gamma := 4\lambda_{K_i,1} \cdot \lambda_{K_i,2} \text{ on } K_i,\ i = 1,2.$$

For boundary face the bubble function is defined analogously with the obvious modification. For simplicity assume that b_K and b_γ are extended by zero outside their original domain of definition. There holds $0 \le b_K(x), b_\gamma(x) \le 1$ and $\|b_K\|_{L_\infty(K)} = \|b_\gamma\|_{L_\infty(K)} = 1$.

We use the abbreviation $a \preceq b$ for $a \le Cb$, with a positive constant C independent of a and b. We also write $a \sim b$ when $a \preceq b$ and $a \succeq b$. We would like to mention that all constants in this work are independent of h, κ and the aspect ratio.

3 The equilibrated residual method

In this section a brief overview over the equilibrated residual method is done since we require parts of this method for our subsequent analysis. The equilibrated residual method may be found in [3] and its modification for the singularly perturbed case is in [1].

3.1 The equilibrated residual method

Consider the model problem of §2. Then the error $e := u - u_X$ belongs to the space $H_0^1(\Omega)$ and satisfies the variational formulation

$$B(e, v) = B(u, v) - B(u_X, v) = (f, v) - B(u_X, v) \quad \forall v \in H_0^1(\Omega). \tag{3.1}$$

For an element K with boundary ∂K, let n_K be the outer normal vector. Next we introduce a set of boundary fluxes $\{g_K : K \in \mathcal{T}\}$ that approximate the actual fluxes of the true solution on the element boundaries $g_K \approx n_K \cdot \nabla u|_K$. Taking into account that the trace of the true solution is continuous on the interelement boundaries, we construct the approximate fluxes for which this condition holds true:

$$g_K + g_{K'} = 0 \text{ on } \partial K \cap \partial K'. \tag{3.2}$$

It is easy to show that the residual on the right hand side of (3.1) may be decomposed into contributions from the individual elements

$$(f, v) - B(u_X, v) = \sum_{K \in \mathcal{T}} \left\{ (f, v)_K - B_K(u_X, v) + \int_{\partial K} g_K v \, ds \right\}. \tag{3.3}$$

The term in parentheses may be represented in terms of the solution $\phi_K \in V_K$ of the local residual problem

$$B_K(\phi_K, v) = (f, v)_K - B_K(u_X, v) + \int_{\partial K} g_K v \, ds \quad \forall v \in V_K \tag{3.4}$$

where V_K is the space of the locally admissible functions,

$$V_K = \{v \in H^1(K) : v = 0 \text{ on } \partial\Omega \cap \partial K\}.$$

The solution ϕ_K is treated as an approximation of the true error on the element K. It yields the a posteriori error estimation $|||e|||^2 \sim \sum\limits_{K\in\mathcal{T}} |||\phi_K|||_K^2$.

Note that the local problem (3.4) is infinite dimensional. Here it is assumed to have a solution ϕ_K which always exists and is unique if the coefficient κ is not zero. If κ vanishes then the problem will have a solution iff the collection of fluxes $\{g_K : K \in \mathcal{T}\}$ satisfies the so-called *equilibration condition*

$$0 = (f,1)_K - B_K(u_X, 1) + \int_{\partial K} g_K \, ds. \tag{3.5}$$

This condition means that the boundary flux g_K is in equilibrium with the interior load.

By substituting (3.4) into (3.3), it follows that for all $v \in V$,

$$B(e,v) = (f,v) - B(u_X,v) = \sum_{K\in\mathcal{T}} B_K(\phi_K, v).$$

One immediate consequence of this result is the upper bound on the true error. It follows from the Cauchy-Schwarz inequality,

$$|B(e,v)| \le \sum_{K\in\mathcal{T}} |||\phi_K|||_K |||v|||_K \le \left\{ \sum_{K\in\mathcal{T}} |||\phi_K|||_K^2 \right\}^{1/2} |||v|||.$$

Finally, it leads to the conclusion

$$|||e||| = \sup_{v\in H_0^1(\Omega):|||v|||=1} B(e,v) \le \left\{ \sum_{K\in\mathcal{T}} |||\phi_K|||_K^2 \right\}^{1/2}.$$

These developments lead to the following theorem:

Theorem 3.1. Upper error bound. *Let $\{g_K : K \in \mathcal{T}\}$ be any set of boundary fluxes satisfying condition (3.2). Additionally, if κ vanishes, then (3.5) is assumed to hold on all elements that do not abut the boundary $\partial\Omega$. Then, the global error in the finite element approximation may be bounded by*

$$|||e|||^2 \le \sum_{K\in\mathcal{T}} |||\phi_K|||_K^2.$$

Proof. For the proof see above. □

3.2 Construction of the equilibrated fluxes

For the convenience of the reader we repeat now part of the theory developed in [3].

It will be assumed that the finite element subspace X is constructed using linear elements on a partition $\mathcal{T}$ of the domain Ω into triangular elements. The key issue of the lower bound of the error is the construction of the appropriate approximate fluxes.

3.2.1 First-order equilibration conditions

The procedure that will be developed produces sets of fluxes $\{g_K\}$ that satisfy the *first-order equilibration conditions*:

$$\left.\begin{array}{rcl} (f,\theta_n)_K - B_K(u_X,\theta_n) + \int_{\partial K} g_K\theta_n\,ds &=& 0 \quad \forall n \in \mathcal{N}(K) \\ g_K + g_{K'} &=& 0 \text{ on } \partial K \cap \partial K'. \end{array}\right\} \tag{3.6}$$

It is convenient to look for $g_K|_\gamma$ belonging to $span\{\theta_n : n \in \mathcal{N}(\gamma)\}$ on all edges. The work of Ainsworth and Oden [3] provides the idea of choosing the degrees of freedom for the fluxes to be the moments with respect to the FEM basis functions $\mu^\gamma_{K,n} = \int_\gamma g_K\theta_n\,ds$, where $x_n \in \mathcal{N}(\gamma)$. This choice gives the possibility of avoiding a global problem by reducing the construction of fluxes to computations of the moments over local patches of elements.

Let $\mathcal{N}(\gamma) = \{x_l, x_r\}$, then it can be shown that the actual flux may be reconstructed from its moments :

$$g_K|_\gamma = \frac{2}{|\gamma|}\left\{(2\mu^\gamma_{K,l} - \mu^\gamma_{K,r})\theta_l + (-\mu^\gamma_{K,l} + 2\mu^\gamma_{K,r})\theta_r\right\}. \tag{3.7}$$

Note that (3.7) could be rewritten in the form

$$g_K|_\gamma = \mu^\gamma_{K,l}\psi_l + \mu^\gamma_{K,r}\psi_r$$

where ψ_l and ψ_r are the functions of the dual basis to θ_l and θ_r, i.e. $(\psi_l,\theta_r)_{L_2(\gamma)} = \delta_{lr}$:

$$\psi_l = \frac{2}{|\gamma|}(2\theta_l - \theta_r), \quad \psi_r = \frac{2}{|\gamma|}(-\theta_l + 2\theta_r).$$

So, in order to determine the boundary fluxes, it is sufficient to determine the moments of the flux with respect to the basis functions. The first-order equilibration conditions (3.6) for the flux g_K may be rewritten in terms of the flux moments in the form:

$$\left.\begin{array}{rcll} \sum_{\gamma\subset\partial K}\mu^\gamma_{K,n} &=& \Delta_K(\theta_n) & \forall n \in \mathcal{N}(K) \\ \mu^\gamma_{K,n} + \mu^\gamma_{K',n} &=& 0 & \forall n \in \mathcal{N}(\gamma), \quad \gamma = \partial K \cap \partial K' \end{array}\right\} \tag{3.8}$$

where

$$\Delta_K(\theta_n) = B_K(u_X,\theta_n) - (f,\theta_n)_K. \tag{3.9}$$

In (3.8) we used the convention that $\mu^\gamma_{K,n} = 0$ if $n \notin \mathcal{N}(\gamma)$.

The conditions (3.8) take one of two distinct structures depending on the location of the node x_n. Here we omit the case of boundary vertex. See [3] for details. Assume x_n to be an interior vertex. The elements and edges are labeled as shown in Figure 1. The moment equilibration conditions (3.8) for the elements $K \in \widetilde{x}_n$ associated with the node x_n may be rewritten in the form

$$\left.\begin{array}{rcl} \mu^{\gamma_1}_{1,n} + \mu^{\gamma_2}_{1,n} &=& \Delta_1(\theta_n) \\ \mu^{\gamma_2}_{2,n} + \mu^{\gamma_3}_{2,n} &=& \Delta_2(\theta_n) \\ &\vdots& \\ \mu^{\gamma_N}_{N,n} + \mu^{\gamma_1}_{N,n} &=& \Delta_N(\theta_n) \end{array}\right\}$$

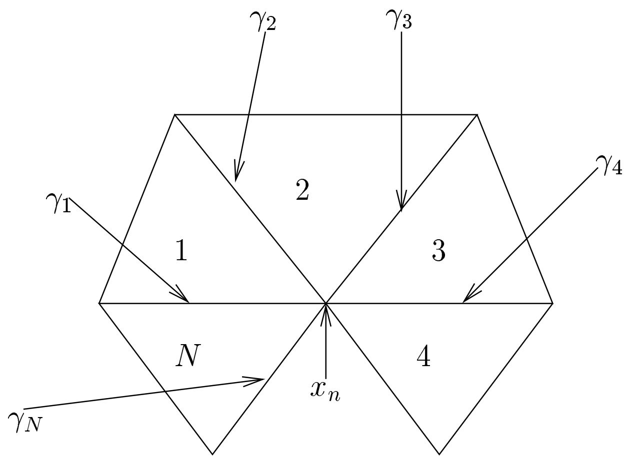

Figure 1: The patch of elements influenced by the basis function θ_n .

with constraints on the edges

$$\left.\begin{array}{rcl} \mu_{1,n}^{\gamma_1} + \mu_{N,n}^{\gamma_1} & = & 0 \\ \mu_{2,n}^{\gamma_2} + \mu_{1,n}^{\gamma_2} & = & 0 \\ & \vdots & \\ \mu_{N,n}^{\gamma_N} + \mu_{N-1,n}^{\gamma_N} & = & 0. \end{array}\right\}$$

These conditions may be fulfilled as shown in [3]. However, they do not uniquely define the moments. The next paragraph provides the selection of the solution from the one-parametric family described by the conditions above.

3.2.2 Resolution of patch problems

The following procedure is described in [3] in details. Here we give only an overview.

The ideal situation would be to choose the flux moments $\{g_K\}$ satisfying $\mu_{K,n}^{\gamma} \approx \int_{\gamma} \theta_n n_K \cdot \nabla u \, ds$. Since the true fluxes are unknown, the flux moments are selected so that $\mu_{K,n}^{\gamma} \approx \widetilde{\mu}_{K,n}^{\gamma} := \int_{\gamma} \theta_n n_K \cdot \nabla u_X|_K \, ds$. We seek flux moments that minimize the objective

$$\frac{1}{2} \sum_{K \in \widetilde{x}_n} \sum_{\gamma \subset \partial K} \left(\mu_{K,n}^{\gamma} - \widetilde{\mu}_{K,n}^{\gamma} \right)^2 . \tag{3.10}$$

Introducing Lagrange multipliers we come to the optimality condition. The Lagrangian is given by

$$\mathcal{L}\left(\{\widetilde{\mu}_{K,n}^{\gamma}\}, \{\lambda_{\gamma}\}, \{\sigma_K\}\right) = \tfrac{1}{2} \sum_{K \in \widetilde{x}_n} \sum_{\gamma \subset \partial K} \left(\mu_{K,n}^{\gamma} - \widetilde{\mu}_{K,n}^{\gamma} \right)^2$$

$$+ \sum_{K \in \widetilde{x}_n} \sigma_{K,n} \left(\Delta_K(\theta_n) - \sum_{\gamma \subset \partial K} \mu_{K,n}^{\gamma} \right) + \sum_{\gamma = \partial K \cap \partial K'} \lambda_{\gamma,n} \left(\mu_{K,n}^{\gamma} + \mu_{K',n}^{\gamma} \right) .$$

Here we used the convention that $\lambda_{\gamma,n} = 0$ on $\gamma \subset \partial\Omega$. We conclude that the conditions for a stationary point consist of two parts; the first part is (3.8), the second part is:

$$\mu^{\gamma}_{K,n} - \widetilde{\mu}^{\gamma}_{K,n} - \sigma_{K,n} + \lambda_{\gamma,n} = 0. \tag{3.11}$$

Using the second part of (3.8) we obtain:

$$\lambda_{\gamma,n} = \begin{cases} \frac{1}{2}\left(\sigma_{K,n} + \sigma_{K',n} + \widetilde{\mu}^{\gamma}_{K,n} + \widetilde{\mu}^{\gamma}_{K',n}\right) & \gamma = \partial K \cap \partial K' \\ 0 & \gamma = \partial K \cap \partial\Omega \end{cases}$$

Using the last formula together with (3.11) the flux moments are expressed as:

$$\mu^{\gamma}_{K,n} = \begin{cases} \frac{1}{2}\left(\sigma_{K,n} - \sigma_{K',n} + \widetilde{\mu}^{\gamma}_{K,n} - \widetilde{\mu}^{\gamma}_{K',n}\right) & \gamma = \partial K \cap \partial K', \\ \sigma_{K,n} + \widetilde{\mu}^{\gamma}_{K,n} & \gamma = \partial K \cap \partial\Omega. \end{cases} \tag{3.12}$$

Substituting this into the first equation of (3.8) leads to the following conditions for $\{\sigma_{K,n} : K \in \widetilde{x}_n\}$:

$$\frac{1}{2} \sum_{\gamma = \partial K \cap \partial K'} (\sigma_{K,n} - \sigma_{K',n}) + \sum_{\gamma \subset \partial K \cap \partial\Omega} \sigma_{K,n} = \widetilde{\Delta}_K(\theta_n) \quad \forall K \in \widetilde{x}_n, \tag{3.13}$$

where

$$\widetilde{\Delta}_K(\theta_n) := B_K(u_X, \theta_n) - (f, \theta_n)_K - \int_{\partial K} \left\langle \frac{\partial u_X}{\partial n_K} \right\rangle \theta_n \, ds, \tag{3.14}$$

$$\left\langle \frac{\partial u_X}{\partial n_K} \right\rangle := \begin{cases} \frac{1}{2} n_K \cdot \{(\nabla u_X)_K + (\nabla u_X)_{K'}\} & \text{on } \partial K \cap \partial K' \\ n_K \cdot (\nabla u_X)_K & \text{on } \partial K \cap \partial\Omega. \end{cases} \tag{3.15}$$

The conditions (3.13) form a linear system of equations over the element patches $\widetilde{x}_n$ with unknowns $\{\sigma_{K,n} : K \in \widetilde{x}_n\}$ corresponding to the elements in the patch. The specific form for an interior vertex is

$$\frac{1}{2} \begin{bmatrix} 2 & -1 & & \dots & & -1 \\ -1 & 2 & -1 & \dots & & 0 \\ \vdots & & & & & \vdots \\ 0 & & \dots & -1 & 2 & -1 \\ -1 & & \dots & & -1 & 2 \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_{N-1} \\ \sigma_N \end{bmatrix} = \begin{bmatrix} \widetilde{\Delta}_1(\theta_n) \\ \widetilde{\Delta}_2(\theta_n) \\ \vdots \\ \widetilde{\Delta}_{N-1}(\theta_n) \\ \widetilde{\Delta}_N(\theta_n) \end{bmatrix}.$$

The kernel of this matrix is the vector $\mathbf{1} = [1, 1, \dots, 1]^\top$ implying that a solution exists if and only if the sum of the components of the right-hand data vanishes. This may be easily verified thanks to the Galerkin property (see [3]).

Since the system (3.13) is singular the least square solution is selected. As a consequence, there exists a constant C, depending only on the number of elements in the patch surrounding a vertex $\widetilde{x}$, such that (for proof see for ex. [10])

$$\sum_{K \in \widetilde{x}_n} \sigma^2_{K,n} \preceq \sum_{K \in \widetilde{x}_n} \widetilde{\Delta}_K(\theta_n)^2. \tag{3.16}$$

3.3 Minimum energy extensions

Minimum energy extensions were first introduced in the work of Ainsworth and Babuška [1]. These extensions play a key role in construction of estimator stable with respect to the perturbation parameter κ. The equilibrated residual method has the following history. The original equilibrated residual method is described in the work of Ainsworth and Oden [2]. However, as it is shown in [1], it is not stable with respect to κ. The work [1] proposes the following modification of the previous method for the singularly perturbed case. *The functions* θ_n *in* (3.13) *are replaced by an approximate minimum energy extension* θ_n^* *to* $\theta_n|_{\partial K}$. *The system* (3.13) *then is solved in a least-square sense*, since it has no solution in general. As in (3.16), one gets the solution that depends continuously on the data:

$$\sum_{K\in\widetilde{x}_n} \sigma_{K,n}^2 \preceq \sum_{K\in\widetilde{x}_n} \widetilde{\Delta}_K(\theta_n^*)^2. \tag{3.17}$$

The error estimator we propose is derived from the estimator of work [1] but differs in two details. First, more attention is paid to the minimization of the appropriate function energy norm and even the minimum is obtained. We will develop this here. The second modification is described in §6.

Let K be any element and let $v \in H^{1/2}(\partial K)$. The *minimum energy extension* $\mathcal{E}v$ of v to the interior of the element is characterized by the conditions

$$\mathcal{E}v \in H^1(K) : \mathcal{E}v = v \,\text{on}\, \partial K, \quad B_K(\mathcal{E}v, \omega) = 0 \;\forall \omega \in H_0^1(K).$$

The definition of the minimum energy extension has an advantageous property. *Let* $v \in H^{1/2}(\partial K)$. *The minimum energy extension* $\mathcal{E}v$ *of* v *to the interior of the element has the minimal energy norm among all functions coinciding with* v *on the boundary* ∂K. Indeed, consider the energy norm of the function $\mathcal{E}v + \omega$:

$$|||\mathcal{E}v + \omega|||^2 = |||\mathcal{E}v|||^2 + |||\omega|||^2 + 2B_K(\mathcal{E}v, \omega) = |||\mathcal{E}v|||^2 + |||\omega|||^2 \geq |||\mathcal{E}v|||^2. \tag{3.18}$$

The proof easily follows from (3.18) observing that $\mathcal{E}v + \omega$ coincides with $\mathcal{E}v$ on the boundary ∂K.

For a one-dimensional case it is possible to find a minimum energy extension explicitly (see [1]).

Consider now the two-dimensional case. We look for an approximation for the minimum energy extension of the first-order basis function. Let element $K = \Delta ABC$ be a triangle. Consider the basis function θ corresponding to the vertex A. We seek an approximation to the minimum energy extension $\mathcal{E}\theta$ in the following class Λ of functions. Set

$$\begin{aligned} \Lambda \;:=\; &\{v \in C^0(K) : \; v = \theta \text{ on } \partial K, \; v = 0 \text{ in } \Delta CDB, \\ & v \text{ is linear in each triangle } \Delta CAD \text{ and } \Delta BAD, \; D \in \Delta ABC\}. \end{aligned}$$

We obtain now an approximation for the minimum energy extension of this basis function. To this end we put an arbitrary point D in the triangle (see Figure 2).

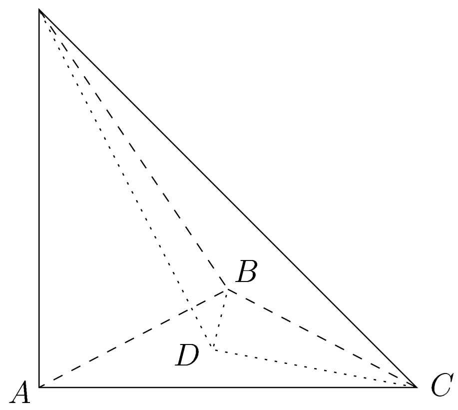

Figure 2: Family of functions used to approximate the minimum energy extension $\mathcal{E}\theta$.

Next we choose that function from the set Λ that minimizes energy norm. Our developments here differs from the original paper [1]. There a point $(1/\kappa, 1/\kappa)$ is introduced in the *reference* triangle and D is the image of this point after the corresponding affine transformation. The corresponding function does not necessarily minimize the energy over Λ but it is shown to be sufficiently accurate. For us, however *this is not sufficient* and we instead consider the point D to be in the *actual* triangle in order to obtain the optimal position of this point.

Introduce a local coordinate system such that the vertex A coincides with the origin and the edge AC lies on the axis Ox. Let $D = (a, b)$, $C = (h_1, 0)$ and $B = (h_3, h_2)$ (see Figure 3).

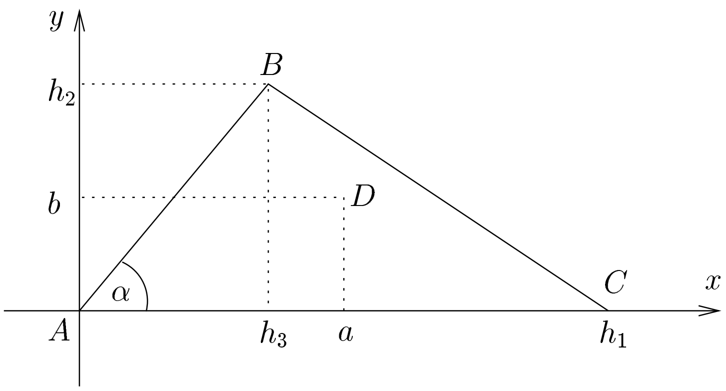

Figure 3: Notations for the parameters of an arbitrary triangle.

Let $\Theta(a, b, x, y) \in \Lambda$ be an admissible function, see Figure 2. The squared energy norm of this function is

$$\begin{aligned}\Phi(a,b) &= B_K(\Theta(a,b,x,y),\Theta(a,b,x,y))\\ &= \frac{\kappa^2}{12}(b(h_1-h_3)+ah_2)+\frac{2h_1h_2h_3-ah_1h_2-bh_1h_3+bh_3^2}{2h_1h_3^2}\\ &\quad+\frac{(h_2^2+h_3^2)(a-h_3)^2}{2h_3^2(ah_2-bh_3)}+\frac{(a-h_1)^2}{2h_1b}\end{aligned}$$

For given parameters h_1, h_2, h_3 we want to minimize $\Phi(a,b)$with respect to a and b. A number of calculations leads to a stationary point of this function

$$\begin{cases}\overline{a}^* &= \dfrac{\sqrt{6h_1}(h_3+\sqrt{h_2^2+h_3^2})}{\sqrt{12h_3+12\sqrt{h_2^2+h_3^2}+\kappa^2h_2^2h_1}}\\ \overline{b}^* &= \left(-\dfrac{h_3}{h_2}+\sqrt{\left(\dfrac{h_3}{h_2}\right)^2+1}\right)\overline{a}^*.\end{cases}$$

To prove that this solution is a minimum point it is sufficient to show convexity of the function. Therefore consider the Hessian matrix $D^2\Phi$ of the second-order derivatives. By direct computations one obtains

$$\begin{aligned}\frac{\partial^2\Phi}{\partial a^2} &= \frac{1}{bh_1}+\frac{(h_2^2+h_3^2)(b-h_2)^2}{(h_2a-h_3b)^3} &\geq 0,\\ \det D^2\Phi &= \frac{(h_2^2+h_3^2)(h_1b-h_3b+h_2a-h_1h_2)^2}{(h_2a-h_3b)^3b^3h_1} &\geq 0.\end{aligned}$$

Hence $\Phi(a,b)$ is convex and $(\overline{a}^*,\overline{b}^*)$ is the unique minimum.

We have found the function $\Theta^*(\overline{a}^*,\overline{b}^*,x,y)$ which minimizes the energy norm over the set Λ. However, in practice it is sufficient to take not the exact values of $\overline{a}^*$ and $\overline{b}^*$, but some values a^*, b^* that are equivalent for $\kappa\to\infty$, namely

$$\begin{cases}a^* &= \dfrac{\sqrt{6}}{\kappa}\dfrac{\left(h_3+\sqrt{h_2^2+h_3^2}\right)}{h_2}\\ b^* &= \dfrac{\sqrt{6}}{\kappa}.\end{cases}$$

Note that the corresponding point $D=(a^*,b^*)$ lies on the bisector of the angle $\angle BAC =: \alpha$ and $|AD| = \frac{\sqrt{6}}{\kappa\sin\alpha/2}$. The analysis given neglects the fact that (a,b) should be contained in K. Therefore, we construct the function θ^* as follows

$$\theta^* := \begin{cases}\Theta(a^*,b^*,x,y), & \text{if } (a^*,b^*)\in K,\\ \theta, & otherwise.\end{cases}$$

Lemma 3.2. *Under the above notations and assumptions the following holds*

$$\|\theta^*\|^2_{L_2(K)} \preceq |K| \min(1, h^{-1}_{min,K}\kappa^{-1}) \sim \text{meas}(\partial K) \min(h_{min,K}, \kappa^{-1}),$$

where $h_{min,K}$ is the height corresponding to the largest edge of the triangle K.

Proof. Consider K for which $(a^*, b^*) \in K$. A short calculation yields $\kappa^{-1} \preceq h_{min,K}$ and $\min(h_{min,K}, \kappa^{-1}) \sim \kappa^{-1}$. Furthermore one obtains

$$\|\theta^*\|^2_{L_2(K)} = \frac{\sqrt{6}\left(h_1 + \sqrt{h_2^2 + h_3^2}\right)}{12\kappa} \preceq \text{meas}(\partial K)\kappa^{-1} \sim \text{meas}(\partial K) \min(h_{min,K}, \kappa^{-1}).$$

It remains to consider the case $\kappa^{-1} \gg h_{min,K}$. In this case $\min(1, h^{-1}_{min,K}\kappa^{-1}) = 1$ and θ^* coincides with θ. The estimate

$$\|\theta^*\|^2_{L_2(K)} = \|\theta\|^2_{L_2(K)} \sim |K| = |K| \min(1, h^{-1}_{min,K}\kappa^{-1})$$

completes the proof. □

4 Theoretical background of the equilibrated residual method in anisotropic case

4.1 Notation of the triangle

Let a triangulation $\mathcal{T}$ be given which satisfies the usual conformity condition (see [9], Chapter 2). Following the notation of Kunert [12], the three vertices of an arbitrary triangle $K \in \mathcal{T}$ are denoted by P_0, P_1, P_2 such that P_0P_1 is the longest edge of K.

Additionally define two orthogonal vectors p_i with lengths $h_{i,K} := |p_i|$, see Figure 4. Observe that $h_{1,K} > h_{2,K}$ and set $h_{min,K} := h_{2,K}$, $h_{max,K} := h_{1,K}$.

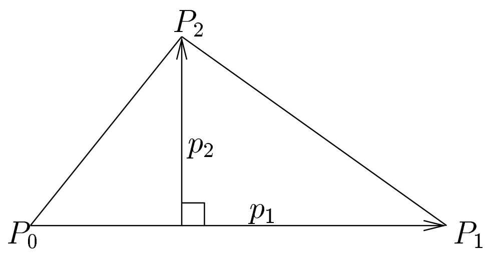

Figure 4: Notation of a triangle K.

In addition to the usual conformity conditions of the mesh we assume that the following two properties hold.

1. The number of triangles containing a node x_n is bounded uniformly.

2. The dimensions of adjacent triangles must not change rapidly, i.e.

$$h_{i,K'} \sim h_{i,K} \quad \forall K, K' \text{ with } \overline{K} \cap \overline{K'} \neq \emptyset,\ i = 1, 2.$$

Define the matrices A_K and $C_K \in \mathbb{R}^{2\times 2}$ by

$$A_K := (\overrightarrow{P_0P_1}, \overrightarrow{P_0P_2}) \quad \text{and} \quad C_K := (p_1, p_2)$$

and introduce affine linear mappings

$$F_A(\mu) := A_K \cdot \mu + \overrightarrow{P}_0 \quad \text{and} \quad F_C(\mu) := C_K \cdot \mu + \overrightarrow{P}_0,\ \mu \in \mathbb{R}^2.$$

These mappings implicitly define the so-called *standard triangle* $\overline{\overline{K}} := F_A^{-1}(K)$ and the *reference triangle* $\hat{K} := F_C^{-1}(K)$.

Variables that are related to the standard triangle $\overline{\overline{K}}$ and to the reference triangle $\hat{K}$ are referred to with a double bar and a hat, respectively (e.g. $\overline{\overline{\nabla}}, \hat{v}$).

4.2 Some basic inequalities

This paragraph provides some facts which will be useful in the subsequent analysis for obtaining the lower error bound. The following two lemmata are extracted from [14].

Lemma 4.1. ***(Inverse inequalities for bubble functions).*** *Assume that $\varphi_K \in \mathbb{P}^0(K)$ and $\varphi_\gamma \in \mathbb{P}^0(\gamma)$. Then*

$$\|b_K^{1/2} \cdot \varphi_K\|_{L_2(K)} \sim \|\varphi_K\|_{L_2(K)} \tag{4.1}$$

$$\|\nabla(b_K \cdot \varphi_K)\|_{L_2(K)} \preceq h_{min,K}^{-1} \cdot \|\varphi_K\|_{L_2(K)} \tag{4.2}$$

$$\|b_\gamma^{1/2} \cdot \varphi_\gamma\|_{L_2(\gamma)} \sim \|\varphi_\gamma\|_{L_2(\gamma)}. \tag{4.3}$$

Lemma 4.2. ***(Anisotropic trace inequality).*** *Let K be an arbitrary triangle and γ be an edge of it. For $v \in H^1(\Omega)$ the following trace inequality holds:*

$$\|v\|_{L_2(\gamma)}^2 \preceq \frac{|\gamma|}{|K|}\|v\|_{L_2(K)} \left(\|v\|_{L_2(K)} + \|C_K^\top \nabla v\|_{L_2(K)}\right).$$

Again following [14], we define special face bubble functions and state the corresponding inverse inequalities. The definition is given first for the standard triangle $\overline{\overline{K}}$ and then for the actual triangle K.

Consider the standard triangle $\overline{\overline{K}}$ and a face $\overline{\overline{\gamma}}$ thereof. Without loss of generality, assume that it lies on the axis $O\overline{\overline{y}}$. By γ we denote the corresponding face on the boundary of actual triangle K. For a real number $\delta \in (0,1]$ define a linear mapping $F_\delta : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F_\delta(\mathbf{x}) := (\delta \cdot x, y)^\top = B_\delta \cdot \mathbf{x} \quad \text{with } B_\delta = \operatorname{diag}\{\delta, 1\} \in \mathbb{R}^{2\times 2}.$$

Obviously this yields

$$|\det B_\delta| = \delta \quad \text{and } \|B_\delta^{-1}\|_{\mathbb{R}^{2\times 2}} = \delta^{-1}.$$

Set $\overline{\overline{K}}_\delta := F_\delta(\overline{\overline{K}})$, i.e it is the triangle with the face $\overline{\overline{\gamma}}$ and a vertex at $\delta \cdot \mathbf{e_1}$ (see Figure 5).

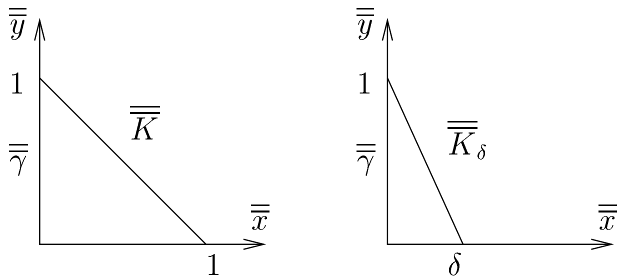

Figure 5: Special face bubble functions definition.

Let $b_{\overline{\overline{\gamma}}}$ be the usual face bubble function of $\overline{\overline{\gamma}}$ on $\overline{\overline{K}}$. Define the special bubble function $\overline{\overline{b}}_{\overline{\overline{\gamma}},\delta}$ by $\overline{\overline{b}}_{\overline{\overline{\gamma}},\delta} := b_{\overline{\overline{\gamma}}} \circ F_\delta^{-1}$, i.e. $\overline{\overline{b}}_{\overline{\overline{\gamma}},\delta}$ is the usual face bubble function of $\overline{\overline{\gamma}}$ on the triangle $\overline{\overline{K}}_\delta$. For clarity we recall that $\overline{\overline{b}}_{\overline{\overline{\gamma}},\delta} = 0$ on $\overline{\overline{K}} \setminus \overline{\overline{K}}_\delta$.

Consider now an actual triangle K. The special face bubble function $b_{\gamma,\delta} \in H^1(K)$ of a face γ of K is defined by $b_{\gamma,\delta} := \overline{\overline{b}}_{\overline{\overline{\gamma}},\delta} \circ F_A^{-1}$. The actual value of parameter δ will be specified later.

Lemma 4.3. ***(Inverse inequalities for special bubble functions).*** *Let γ be an arbitrary face of K. Assume that $\varphi_\gamma \in \mathbb{P}^0(\gamma)$. Then the following inverse inequalities hold:*

$$\|F_{ext}(\varphi_\gamma) \cdot b_{\delta,\gamma}\|_{L_2(K)} \preceq \left(\frac{|K|}{|\gamma|}\right)^{1/2} \cdot \delta^{1/2} \cdot \|\varphi_\gamma\|_{L_2(\gamma)} \tag{4.4}$$

$$\|\nabla(F_{ext}(\varphi_\gamma) \cdot b_{\delta,\gamma})\|_{L_2(K)} \preceq \left(\frac{|K|}{|\gamma|}\right)^{1/2} \cdot \delta^{1/2} \cdot \min\left\{\delta \cdot \frac{|K|}{|\gamma|}, h_{min,K}\right\}^{-1} \cdot \|\varphi_\gamma\|_{L_2(\gamma)} \tag{4.5}$$

Proof. See [14]. □

Further we will need some more facts concerning approximation properties on an anisotropic triangle.

Lemma 4.4. ***(Anisotropic approximation properties I).*** *Let K be any triangle and $v \in H^1(K)$. Denote by $\overline{v} = \frac{1}{|K|}\int_K v$ the mean value of v over an element K. Then*

1. $\|v - \overline{v}\|_{L_2(K)} \preceq \|v\|_{L_2(K)}$

2. $\|v - \overline{v}\|_{L_2(K)} \preceq \|C_K^\top \nabla v\|_{L_2(K)}$

3. Let $\gamma \subset \partial K$ be one of the faces of K. Then for each $v \in H^1(K)$ the following inequality holds:

$$\|v - \overline{v}\|_{L_2(\gamma)} \preceq \left(\frac{|\gamma|}{|K|}\right)^{1/2} \|C_K^\top \nabla v\|_{L_2(K)}.$$

Proof. Estimate 1 is obvious. For the estimate 2 see for instance [13]. Estimate 3 follows from Lemma 4.2 and the estimate 1 of the current lemma observing that the following inequality holds: $\|v\|_{L_2(K)}\|C_K^\top \nabla v\|_{L_2(K)} \preceq \|v\|^2_{L_2(K)} + \|C_K^\top \nabla v\|^2_{L_2(K)}$. □

From a heuristic point of view one should stretch the triangle in that direction where the (directional) derivative of the function shows little change. The better the anisotropic mesh $\mathcal{T}$ is aligned with the anisotropic function v, the more accurate one would expect the error estimates to be. In order to measure the alignement of $\mathcal{T}$ with v, Kunert [12, 13] has introduced the *matching function* $m_1(v, \mathcal{T})$ which is defined as follows.

Definition 4.5. (Matching function $\mathbf{m_1}$). Let $v \in H^1(\Omega)$ be an arbitrary non-constant function, and $\mathcal{F}$ be a family of triangulations of Ω. Define the matching function $m_1(\cdot,\cdot) : H^1(\Omega) \times \mathcal{F} \mapsto \mathbb{R}$ by

$$m_1(v, \mathcal{T}) := \frac{\left(\sum_{K\in\mathcal{T}} h^{-2}_{min,K} \cdot \|C_K^\top \nabla v\|^2_{L_2(K)}\right)^{1/2}}{\|\nabla v\|}. \tag{4.6}$$

Furthermore the *local* matching function $m_1(\cdot,\cdot) : H^1(\Omega) \times \mathcal{T} \mapsto \mathbb{R}$ is obviously defined by

$$m_1(v, K) := h^{-1}_{min,K} \frac{\|C_K^\top \nabla v\|_{L_2(K)}}{\|\nabla v\|_{L_2(K)}}$$

□

The matching function satisfies the following property:

$$1 \le m_1(v, \mathcal{T}) \le C \max_{K\in\mathcal{T}} \frac{h_{max,K}}{h_{min,K}}$$

The definition implies that a mesh $\mathcal{T}$ which is well aligned with an anisotropic function v, results in a small matching number $m_1(v, \mathcal{T})$. The crude upper bound of m_1 confirms that (4.6) is a natural extension of isotropic meshes.

Lemma 4.6. ***(Anisotropic approximation properties II).*** *Let K be any triangle and $v \in H^1(K)$. Let $\gamma \subset \partial K$ be one of the faces of K. Then for each $v \in H^1(K)$ the following inequalities hold:*

$$\begin{aligned}\|v-\overline{v}\|_{L_2(\gamma)} &\preceq \left(\frac{|\gamma|}{|K|}\right)^{1/2} h_{min,K}^{1/2}\min(h_{min,K},\kappa^{-1})^{1/2}\left(h_{min,K}^{-2}\|C_K^\top\nabla v\|_{L_2(K)}^2+\kappa^2\|v\|_{L_2(K)}^2\right)^{1/2}\\ &\preceq \left(\frac{|\gamma|}{|K|}\right)^{1/2} h_{min,K}^{1/2}\min(h_{min,K},\kappa^{-1})^{1/2} m_1(v,K)|||v|||_K,\end{aligned}$$

$$\begin{aligned}\left(h_{min,K}^{-1}+\kappa\right)\|v-\overline{v}\|_{L_2(K)} &\preceq \left(h_{min,K}^{-2}\|C_K^\top\nabla v\|_{L_2(K)}^2+\kappa^2\|v\|_{L_2(K)}^2\right)^{1/2}\\ &\preceq m_1(v,K)|||v|||_K.\end{aligned}$$

Proof. Using Lemma 4.4 (3) we obtain

$$\begin{aligned}\|v-\overline{v}\|_{L_2(\gamma)}^2 &\preceq \frac{|\gamma|}{|K|}\|C_K^\top\nabla v\|_{L_2(K)}^2\\ &\preceq \frac{|\gamma|}{|K|}h_{min,K}^2\left(h_{min,K}^{-2}\|C_K^\top\nabla v\|_{L_2(K)}^2+\kappa^2\|v\|_{L_2(K)}^2\right).\end{aligned}$$

Furthermore, with the aid of Lemma 4.2 and Lemma 4.4 (1,2) we get the following

$$\begin{aligned}\|v-\overline{v}\|_{L_2(\gamma)}^2 &\preceq \frac{|\gamma|}{|K|}\|v-\overline{v}\|_{L_2(K)}\left(\|v-\overline{v}\|_{L_2(K)}+\|C_K^\top\nabla v\|_{L_2(K)}\right)\\ &\preceq \frac{|\gamma|}{|K|}\kappa^{-1}\sqrt{\kappa^2\|v\|_{L_2(K)}^2}\,h_{min,K}\sqrt{h_{min,K}^{-2}\|C_K^\top\nabla v\|_{L_2(K)}^2}\\ &\preceq \frac{|\gamma|}{|K|}h_{min,K}\kappa^{-1}\left(h_{min,K}^{-2}\|C_K^\top\nabla v\|_{L_2(K)}^2+\kappa^2\|v\|_{L_2(K)}^2\right).\end{aligned}$$

Combining the two previous estimates we get the result claimed.

The second statement of the current lemma can be verified using Lemma 4.4 (1,2) and the definition of the matching function. □

4.3 Estimates for element and face residuals in the anisotropic case

In this section we prove two lemmas which we will need later. Namely, we derive the upper bounds for interior and face residuals. The jump discontinuity in the approximation of the normal flux at an interelement boundary is defined by

$$\left[\frac{\partial u_X}{\partial n}\right] := n_K\cdot(\nabla u_X)_K + n_{K'}\cdot(\nabla u_X)_{K'},$$

and the usual interior and boundary residuals r and R are given by

$$r := f + \Delta u_X - \kappa^2 u_X$$

and

$$R := \begin{cases} -\left[\frac{\partial u_X}{\partial n}\right] & \text{on } \partial K \cap \partial K' \\ 0 & \text{on } \partial K \cap \partial\Omega \end{cases}$$

Lemma 4.7. ***(Interior residual).*** *Let $K \in \mathcal{T}$. Then*

$$\|r\|_{L_2(K)} \preceq h_{min,K}^{-1} |||e|||_K + \|r - \overline{r}\|_{L_2(K)}$$

Proof. Let $v \in H_0^1(\Omega)$. Integrating by parts on each element yields

$$B(e, v) = \sum_{K \in \mathcal{T}} \int_K rv\, dx - \sum_{\gamma \in \partial\mathcal{T}} \int_\gamma Rv\, ds, \tag{4.7}$$

where $\partial\mathcal{T}$ denotes the collection of interelement faces. Hence for any $v \in H_0^1(\Omega)$

$$B(e, v) = \sum_{K \in \mathcal{T}} \int_K \overline{r}v\, dx - \sum_{\gamma \in \partial\mathcal{T}} \int_\gamma Rv\, ds + \sum_{K \in \mathcal{T}} \int_K (r - \overline{r})v\, dx.$$

Now, choosing $v := b_K \overline{r}$ in the previous equality gives

$$\int_K b_K \overline{r}^2\, dx = B_K(e, b_K\overline{r}) - \int_K (r - \overline{r}) b_K \overline{r}\, dx.$$

Using (4.1), with the aid of Cauchy-Schwarz inequality we obtain

$$\|\overline{r}\|_{L_2(K)}^2 \preceq |||e|||_K |||b_K\overline{r}|||_K + \|r - \overline{r}\|_{L_2(K)} \|b_K\overline{r}\|_{L_2(K)}.$$

Now we use (4.2) for $|||b_K\overline{r}|||_K$ as follows

$$|||b_K\overline{r}|||_K^2 = \|\nabla(b_K\overline{r})\|_{L_2(K)}^2 + \kappa^2 \|b_K\overline{r}\|_{L_2(K)}^2 \preceq h_{min,K}^{-2} \|b_K\overline{r}\|_{L_2(K)}^2 \preceq h_{min,K}^{-2} \|\overline{r}\|_{L_2(K)}^2$$

Hence,

$$\|\overline{r}\|_{L_2(K)} \preceq h_{min,K}^{-1} |||e|||_K + \|r - \overline{r}\|_{L_2(K)},$$

and the claimed result follows from the triangle inequality

$$\|r\|_{L_2(K)} \le \|\overline{r}\|_{L_2(K)} + \|r - \overline{r}\|_{L_2(K)} \preceq h_{min,K}^{-1} |||e|||_K + \|r - \overline{r}\|_{L_2(K)}.$$

□

We are now in a position to specify our parameter δ used in the definition of the special bubble function. From now on

$$\delta := \frac{1}{2} \frac{|\gamma|}{|K|} \min(h_{min,K}, \kappa^{-1}).$$

Lemma 4.8. ***(Face residual).*** *Let γ be any interior interface. Then,*

$$\begin{aligned}\|R\|_{L_2(\gamma)} &\preceq \sum_{K'\in\widetilde{\gamma}} \left\{ \left(\frac{|K'|}{|\gamma|}\right)^{1/2} h_{min,K'}^{-1/2} \min(h_{min,K'},\kappa^{-1})^{-1/2} |||e|||_{K'} \right. \\ &+ \left. \left(\frac{|K'|}{|\gamma|}\right)^{1/2} \delta^{1/2}\|r-\overline{r}\|_{L_2(K')} \right\}.\end{aligned}$$

Proof. Let $\gamma \in \partial\mathcal{T}$. Suppose that $\gamma = \overline{K}_1 \cap \overline{K}_2$. Then $\widetilde{\gamma} = \mathrm{int}(\overline{K}_1 \cup \overline{K}_2)$. Choosing $v := F_{ext}(R)b_{\delta,\gamma} \in H_0^1(\Omega)$ in (4.7) implies

$$\int_\gamma b_{\delta,\gamma} R^2\, ds = \sum_{K\subset\widetilde{\gamma}} \int_K rF_{ext}(R)b_{\delta,\gamma}\, dx - B_{\widetilde{\gamma}}(e, F_{ext}(R)b_{\delta,\gamma}).$$

Furthermore, applying the Cauchy-Schwarz inequality, one obtains

$$|B_K(e, F_{ext}(R)b_{\delta,\gamma})| \le |||e|||_K |||F_{ext}(R)b_{\delta,\gamma}|||_K.$$

Using (4.4) and (4.5) one estimates the second factor as follows:

$$\begin{aligned}|||F_{ext}(R)b_{\delta,\gamma}|||_K^2 &= \|\nabla(F_{ext}(R)b_{\delta,\gamma})\|_{L_2(K)}^2 + \kappa^2\|F_{ext}(R)b_{\delta,\gamma}\|_{L_2(K)}^2 \\ &\preceq \left(\min\left\{\delta\cdot\frac{|K|}{|\gamma|}, h_{min,K}\right\}^{-2} \delta\frac{|K|}{|\gamma|} + \kappa^2\delta\frac{|K|}{|\gamma|}\right) \|R\|_{L_2(\gamma)}^2.\end{aligned}$$

Thus, we have

$$|B_K(e, F_{ext}(R)b_{\delta,\gamma})| \preceq \left(\frac{|K|}{|\gamma|}\right)^{1/2} \left(\min\left\{\delta\cdot\frac{|K|}{|\gamma|}, h_{min,K}\right\}^{-1} \delta^{1/2} + \kappa\delta^{1/2}\right) |||e|||_K \|R\|_{L_2(\gamma)}.$$

Applying the Cauchy-Schwarz inequality, Lemma 4.7 and (4.4) to the second term we have

$$\begin{aligned}\left|\int_K rF_{ext}(R)b_{\delta,\gamma}\, dx\right| &\le \|r\|_{L_2(K)}\|F_{ext}(R)b_{\delta,\gamma}\|_{L_2(K)} \\ &\preceq \left[\left(h_{min,K}^{-1}+\kappa\right)|||e|||_K + \|r-\overline{r}\|_{L_2(K)}\right] \delta^{1/2} \left(\frac{|K|}{|\gamma|}\right)^{1/2} \|R\|_{L_2(\gamma)}.\end{aligned}$$

Combining (4.3) and two previous estimates we get

$$\begin{aligned}\|R\|_{L_2(\gamma)} &\preceq \sum_{K'\in\widetilde{\gamma}} \left\{ \left(\frac{|K'|}{|\gamma|}\right)^{1/2} \left(\min\left\{\delta\cdot\frac{|K|}{|\gamma|}, h_{min,K}\right\}^{-1} \delta^{1/2} + h_{min,K}^{-1}\delta^{1/2} + \kappa\delta^{1/2}\right) |||e|||_{K'} \right. \\ &+ \left. \left(\frac{|K'|}{|\gamma|}\right)^{1/2} \delta^{1/2}\|r-\overline{r}\|_{L_2(K')} \right\}.\end{aligned}$$

It only remains to note that

$$\min\left\{\delta\cdot\frac{|K|}{|\gamma|},h_{min,K}\right\}^{-1}\delta^{1/2}+h_{min,K}^{-1}\delta^{1/2}+\kappa\delta^{1/2}\le 4h_{min,K}^{-1/2}\min(h_{min,K},\kappa^{-1})^{-1/2},$$

with δ from (4.3). By simple manipulations we get,

$$\min\left\{\delta\cdot\frac{|K|}{|\gamma|},h_{min,K}\right\}^{-1}\delta^{1/2}+h_{min,K}^{-1}\delta^{1/2}+\kappa\delta^{1/2}\le 4\delta^{1/2}\min(h_{min,K},\kappa^{-1})^{-1}.$$

Noting that $\frac{1}{2}\frac{|\gamma|}{|K|}\le h_{min,K}^{-1}$ we finish the proof. □

4.4 Stability of the approximate fluxes in the anisotropic singularly perturbed case

Recall that we use the procedure for finding approximate fluxes described in §3.2 with the functions θ_n replaced by θ_n^* in the system (3.13). In the singularly perturbed case and using anisotropic elements we have the following theorem.

Theorem 4.9. *Suppose that the finite element subspace X is constructed using first-order (linear) elements on a partition $\mathcal{T}$ (not necessary isotropic) of the domain Ω into triangular elements. Let $\{g_K\}$ be the set of approximate fluxes, produced by the algorithm described in §3.2 with the functions θ_n replaced by θ_n^*, $n\in\mathcal{N}$. Then, for each edge γ of any element K,*

$$\begin{aligned}\left\|g_K-\left\langle\frac{\partial u_X}{\partial n_K}\right\rangle\right\|_{L_2(\gamma)} &\preceq \sum_{K'\in\widetilde{K}}\left\{\left(\frac{|K'|}{|\gamma|}\right)^{1/2}h_{min,K'}^{-1/2}\min(h_{min,K'},\kappa^{-1})^{-1/2}|||e|||_{K'}\right.\\ &+\left.\left(\frac{|K'|}{|\gamma|}\right)^{1/2}\delta^{1/2}\|r-\overline{r}\|_{L_2(K')}\right\}.\end{aligned}$$

Proof. Let $K\in\mathcal{T}$ be a fixed element and $\gamma\subset K$ be an edge thereof. Then

$$\left(g_K-\left\langle\frac{\partial u_X}{\partial n_K}\right\rangle\right)\bigg|_\gamma\in\mathbb{P}_1(\gamma).$$

Following §3.2.1 the moments of this quantity are

$$\overset{*}{\mu}{}^{\gamma}_{K,n}=\int_\gamma\left(g_K-\left\langle\frac{\partial u_X}{\partial n_K}\right\rangle\right)\theta_n\,ds.$$

By analogy with (3.7),

$$\left(g_K-\left\langle\frac{\partial u_X}{\partial n_K}\right\rangle\right)\bigg|_\gamma=\overset{*}{\mu}{}^{\gamma}_{K,l}\,\psi_l+\overset{*}{\mu}{}^{\gamma}_{K,r}\,\psi_r.$$

Therefore,

$$\left\| g_K - \left\langle \frac{\partial u_X}{\partial n_K} \right\rangle \right\|_{L_2(\gamma)} \le \left| \overset{*}{\mu}{}^{\gamma}_{K,l} \right| \|\psi_l\|_{L_2(\gamma)} + \left| \overset{*}{\mu}{}^{\gamma}_{K,r} \right| \|\psi_r\|_{L_2(\gamma)}$$

and since

$$\|\psi_l\|^2_{L_2(\gamma)} = \|\psi_r\|^2_{L_2(\gamma)} = C|\gamma|^{-1},$$

it follows that

$$\left\| g_K - \left\langle \frac{\partial u_X}{\partial n_K} \right\rangle \right\|^2_{L_2(\gamma)} \preceq |\gamma|^{-1} \sum_{n \in \mathcal{N}(\gamma)} \left| \overset{*}{\mu}{}^{\gamma}_{K,n} \right|^2. \tag{4.8}$$

With the aid of (3.15), we conclude that

$$\int_\gamma \left\langle \frac{\partial u_X}{\partial n_K} \right\rangle \theta_n \, ds = \begin{cases} \frac{1}{2}\left(\widetilde{\mu}^{\gamma}_{K,n} - \widetilde{\mu}^{\gamma}_{K',n}\right) & \text{on } \gamma = \partial K \cap \partial K' \\ \widetilde{\mu}^{\gamma}_{K,n} & \text{on } \gamma = \partial K \cap \partial \Omega \end{cases}$$

and hence, thanks to (3.12),

$$\overset{*}{\mu}{}^{\gamma}_{K,n} = \begin{cases} \frac{1}{2}\left(\sigma_{K,n} - \sigma_{K',n}\right) & \text{on } \gamma = \partial K \cap \partial K' \\ \sigma_{K,n} & \text{on } \gamma = \partial K \cap \partial \Omega \end{cases}$$

where the unknowns $\{\sigma_{K,n}\}$ are determined from conditions (3.13) and satisfy (3.16). It follows that

$$\left| \overset{*}{\mu}{}^{\gamma}_{K,n} \right|^2 \preceq \sum_{K' \in \widetilde{x}_n} \sigma^2_{K',n} \preceq \sum_{K' \in \widetilde{x}_n} \widetilde{\Delta}_{K'}(\theta^*_n)^2. \tag{4.9}$$

The terms appearing on the right-hand side may be bounded by first recalling (3.14),

$$\widetilde{\Delta}_{K'}(\theta^*_n) = B_{K'}(u_X, \theta^*_n) - (f, \theta^*_n)_{K'} - \int_{\partial K'} \left\langle \frac{\partial u_X}{\partial n_{K'}} \right\rangle \theta^*_n \, ds;$$

then, integrating by parts reveals that

$$\widetilde{\Delta}_{K'}(\theta^*_n) = -(r, \theta^*_n)_{K'} - \int_{\partial K'} R\theta^*_n \, ds.$$

Applying the Cauchy-Schwarz inequality and using Lemma 4.7, Lemma 4.8 and Lemma 3.2, it follows that

$$\begin{aligned}
\left| \widetilde{\Delta}_{K'}(\theta^*_n) \right| &\le \|r\|_{L_2(K')} \|\theta^*_n\|_{L_2(K')} + \sum_{\gamma' \subset \partial K' \cap \mathcal{E}_n} \|R\|_{L_2(\gamma')} \|\theta^*_n\|_{L_2(\gamma')} \\
&\preceq \left\{ \left(h^{-1}_{min,K'} + \kappa \right) |||e|||_{K'} + \|r - \overline{r}\|_{L_2(K')} \right\} \cdot |K'|^{1/2} \delta^{1/2} \\
&+ \sum_{\gamma' \subset \partial K' \cap \mathcal{E}_n} \sum_{K'' \subset \widetilde{\gamma}'} \left[\left(\frac{|K''|}{|\gamma'|} \right)^{1/2} h^{-1/2}_{min,K''} \min(h_{min,K''}, \kappa^{-1})^{-1/2} |||e|||_{K''} \right. \\
&+ \left. \left(\frac{|K''|}{|\gamma'|} \right)^{1/2} \delta^{1/2} \|r - \overline{r}\|_{L_2(K'')} \right] \cdot |\gamma'|^{1/2} \\
&\preceq \sum_{K'' \in \widetilde{x}_n} \left(|K''|^{1/2} h^{-1/2}_{min,K''} \min(h_{min,K''}, \kappa^{-1})^{-1/2} |||e|||_{K'} \right. \\
&+ \left. |K''|^{1/2} \delta^{1/2} \|r - \overline{r}\|_{L_2(K'')} \right),
\end{aligned}$$

where the inequality

$$\delta^{1/2}\left(h_{min,K}^{-1}+\kappa\right)\preceq h_{min,K}^{-1/2}\min(h_{min,K},\kappa^{-1})^{-1/2}$$

has been used. Hence,

$$\sum_{K'\in\widetilde{x}_n}\left|\widetilde{\Delta}_{K'}(\theta_n^*)\right|^2\preceq\sum_{K'\in\widetilde{x}_n}\left(|K'|h_{min,K''}^{-1}\min(h_{min,K'},\kappa^{-1})^{-1}|||e|||_{K'}^2+|K'|\delta\|r-\overline{r}\|_{L_2(K')}^2\right) \tag{4.10}$$

Combining (4.8), (4.9) and (4.10) leads to the result claimed. □

5 Lower error bound of the original Ainsworth-Babuška estimator in the anisotropic singularly perturbed case

Describing in §3 the equilibrated residual method, we derived the upper error bound. The original analysis of the lower error bound for *isotropic* triangle dates back to the work by Ainsworth and Babuška [1]. Here we analyse the *anisotropic* case. It turns out that the original error estimator described in [1] has degenerating lower error bound.

The next lemma states some stability properties of the estimator.

Lemma 5.1. *Let ϕ_K denote the solution of the local residual problem (3.4) for the error estimator on element K. Then, for any $v\in H^1(K)$,*

$$|B_K(\phi_K,v-\overline{v})|\preceq m_1(v,K)\left\{|||e|||_{\widetilde{K}}+\min(h_{min,K},\kappa^{-1})\|r-\overline{r}\|_{L_2(\widetilde{K})}\right\}|||v|||_K.$$

Furthermore, if $\kappa\succeq h_K^{-1}$, then

$$|\overline{\phi}_K B_K(\phi_K,1)|\preceq m_1(\phi_K,K)\left\{|||e|||_{\widetilde{K}}+\kappa^{-1}\|r-\overline{r}\|_{L_2(\widetilde{K})}\right\}|||\phi_K|||_K.$$

Proof. 1. Integrating by parts yields

$$\begin{aligned}B_K(\phi_K,v-\overline{v}) &= \int_K r(v-\overline{v})\,dx+\frac{1}{2}\int_{\partial K}R(v-\overline{v})\,ds\\ &+ \int_{\partial K}\left(g_K-\left\langle\frac{\partial u_X}{\partial n_K}\right\rangle\right)(v-\overline{v})\,ds.\end{aligned}$$

and it therefore follows that

$$\begin{aligned}|B_K(\phi_K,v-\overline{v})| &\leq \|r\|_{L_2(K)}\|v-\overline{v}\|_{L_2(K)}+\frac{1}{2}\sum_{\gamma\in\partial K}\|R\|_{L_2(\gamma)}\|v-\overline{v}\|_{L_2(\gamma)}\\ &+ \sum_{\gamma\in\partial K}\left\|g_K-\left\langle\frac{\partial u_X}{\partial n_K}\right\rangle\right\|_{L_2(\gamma)}\|v-\overline{v}\|_{L_2(\gamma)}.\end{aligned} \tag{5.1}$$

Combining results from Lemma 4.7, Lemma 4.8, Theorem 4.9 and Lemma 4.6 we get

$$
\begin{aligned}
&|B_K(\phi_K, v-\overline{v})| \preceq \left(\left(h_{min,K}^{-1}+\kappa\right) |||e|||_K + \|r-\overline{r}\|_{L_2(K)}\right) \|v-\overline{v}\|_{L_2(K)} \\
&+ \sum_{\gamma\subset\partial K}\sum_{K'\subset\widetilde{\gamma}} \left(\left(\frac{|K'|}{|\gamma|}\right)^{1/2} h_{min,K'}^{-1/2}\min(h_{min,K'},\kappa^{-1})^{-1/2}|||e|||_{K'} + \left(\frac{|K'|}{|\gamma|}\right)^{1/2}\delta^{1/2}\|r-\overline{r}\|_{L_2(K')}\right) \\
&\times \left(\left(\frac{|\gamma|}{|K|}\right)^{1/2} h_{min,K}^{1/2}\min(h_{min,K},\kappa^{-1})^{1/2} m_1(v,K)|||v|||_K\right) \\
&+ \sum_{\gamma\subset\partial K}\sum_{K'\subset\widetilde{K}} \left(\left(\frac{|K'|}{|\gamma|}\right)^{1/2} h_{min,K'}^{-1/2}\min(h_{min,K'},\kappa^{-1})^{-1/2}|||e|||_{K'} + \left(\frac{|K'|}{|\gamma|}\right)^{1/2}\delta^{1/2}\|r-\overline{r}\|_{L_2(K')}\right) \\
&\times \left(\left(\frac{|\gamma|}{|K|}\right)^{1/2} h_{min,K}^{1/2}\min(h_{min,K},\kappa^{-1})^{1/2} m_1(v,K)|||v|||_K\right) \\
&\preceq m_1(v,K)|||v|||_K \left(|||e|||_{\widetilde{K}} + \min(h_{min,K},\kappa^{-1})\|r-\overline{r}\|_{L_2(\widetilde{K})}\right),
\end{aligned}
$$

where the inequalities

$$\delta^{1/2}\cdot h_{min,K}^{1/2}\min(h_{min,K},\kappa^{-1})^{1/2} \le \min(h_{min,K},\kappa^{-1}),$$

$$h_{min,K}^{-1}+\kappa \le C\min(h_{min,K},\kappa^{-1})^{-1}$$

have been used.

2. Suppose that $\kappa \succeq h_{min,K}^{-1}$. Then

$$B_K(\phi_K,1) = (f,1)_K - B_K(u_X,1) + \int_{\partial K} g_K\, ds.$$

Integrating by parts, applying the Cauchy-Schwarz inequality, and estimating each term using Lemmas 4.7, 4.8 and Theorem 4.9 yield

$$
\begin{aligned}
|B_K(\phi_K,1)| &\le |K|^{1/2}\|r\|_{L_2(K)} + \frac{1}{2}\sum_{\gamma\in\partial K}|\gamma|^{1/2}\|R\|_{L_2(\gamma)} + \sum_{\gamma\in\partial K}|\gamma|^{1/2}\left\|g_K - \left\langle\frac{\partial u_X}{\partial n_K}\right\rangle\right\|_{L_2(\gamma)} \\
&\preceq \kappa|K|^{1/2}\left\{|||e|||_{\widetilde{K}} + \kappa^{-1}\|r-\overline{r}\|_{L_2(\widetilde{K})}\right\},
\end{aligned}
$$

where the inequality $\min(h_{min,K},\kappa^{-1}) \preceq \kappa^{-1}$ has been used. The result now can be easily obtained

$$
\begin{aligned}
|\overline{\phi}_K B_K(\phi_K,1)| &\preceq \kappa|K|^{1/2}|\overline{\phi}_K|\left\{|||e|||_{\widetilde{K}} + \kappa^{-1}\|r-\overline{r}\|_{L_2(\widetilde{K})}\right\} \\
&\preceq \kappa\|\phi_K\|_{L_2(K)}\left\{|||e|||_{\widetilde{K}} + \kappa^{-1}\|r-\overline{r}\|_{L_2(\widetilde{K})}\right\} \\
&\preceq \left\{|||e|||_{\widetilde{K}} + \kappa^{-1}\|r-\overline{r}\|_{L_2(\widetilde{K})}\right\}|||\phi_K|||_K.
\end{aligned}
$$

□

For the lower bound we have the following result.

Theorem 5.2. ***(Lower error bound).*** *Let g_K be the set of fluxes produced by the algorithm described in Section 3.2 with the functions θ_n replaced by θ_n^*, and let $\phi_K \in V_K$ denote the solution of the local residual problem (3.4). Then,*

$$|||\phi_K|||_K \preceq m_1(\phi_K, K) \left(|||e|||_{\widetilde{K}} + \min(h_{min,K}, \kappa^{-1}) \|r - \overline{r}\|_{L_2(\widetilde{K})} \right).$$

If κ vanishes, then $\min(h_{min,K}, \kappa^{-1})$ is replaced by $h_{min,K}$.

Proof. Observe that for any $v \in V_K$,

$$B_K(\phi_K, v) = B_K(\phi_K, v - \overline{v}) + \overline{v} B_K(\phi_K, 1). \tag{5.2}$$

First, suppose $\kappa h_{min,K} \succeq 1$ so that, in particular, κ is positive and $\min(h_{min,K}, \kappa^{-1})^{-1} \sim \kappa$. Therefore, with the aid of Lemma 5.1,

$$|B_K(\phi_K, \phi_K - \overline{\phi}_K)| \preceq m_1(\phi_K, K) \left\{ |||e|||_{\widetilde{K}} + \min(h_{min,K}, \kappa^{-1}) \|r - \overline{r}\|_{L_2(\widetilde{K})} \right\} |||\phi_K|||_K.$$

Choosing v to be equal to ϕ_K in (5.2), together with the above estimate, proves that the result holds for all elements K satisfying $\kappa h_{min,K} \succeq 1$.

The remaining elements satisfy $\kappa h_{min,K} \ll 1$. Thanks to the assumptions on the partition, the condition $\kappa h_{K'} \ll 1$ is satisfied by all elements K' contained in the patch $\widetilde{K}$. Therefore, Lemma 3.2 reveals that the modified basis functions reduce to the standard basis functions on the patch. Consequently, the approximate fluxes will actually satisfy the equilibration conditions (3.6) exactly. Moreover, since

$$B_K(\phi_K, 1) = (f, 1)_K - B_K(u_X, 1) + \int_{\partial K} g_K \, ds = 0,$$

the second term in (5.2) vanishes. The first estimate in Lemma 5.1 then completes the proof. □

Theorem 5.2 gives the lower error bound of the true error. The main danger for reliability of the estimator is the function $m_1(\phi_K, K)$ presented on the right hand side. One cannot guarantee that the approximation for the error ϕ_K is aligned as well as the true error e. Unfortunately, it may happen so that the alignment of the approximation ϕ_K on the element K is much worse then e: $m_1(\phi_K, K) \gg m_1(e, K)$. To avoid this problem a modification is proposed in the next paragraph.

6 Modified equilibrated residual method

For finding the equilibrated fluxes we use again the equilibrated residual method described in Section 3.2. In this paragraph, we propose an alternative method by changing the local problem, namely, instead of (3.4) we use

$$_mB_K(\phi_K, v) = (f, v)_K - B_K(u_X, v) + \int_{\partial K} g_K v \, ds \quad \forall v \in V_K \tag{6.1}$$

This local problem differs from the original local problem (3.4) only in the scalar product ${}_mB_K(u,v)$ on the left hand side. It is defined as follows.

Definition 6.1. (Mesh dependent energy scalar product). Let $K \in \mathcal{T}$ be any triangle, $u \in H_0^1(\Omega)$ and $v \in H^1(K)$, then we define mesh-dependent energy scalar product and norms by

$$ {}_mB_K(u,v) := h_{min,K}^{-2} \left(C_K^\top \nabla u, C_K^\top \nabla v\right)_K + \kappa^2 \left(u,v\right)_K , $$

$$ {}_m|||u|||_K := \left({}_mB_K(u,u)\right)^{1/2} , $$

$$ {}_m|||u||| := \left(\sum_{K\in\mathcal{T}} {}_m|||u|||_K^2\right)^{1/2} . $$

□

The local mesh-dependent energy norm satisfies the following property

$$ |||u|||_K \leq {}_m|||u|||_K \leq \frac{h_{max,K}}{h_{min,K}} |||u|||_K. $$

Note that it is equivalent to the standard energy norm in the case of isotropic element.

The quantity ϕ_K is then not equivalent to the error e, but we will show that the ${}_m|||\phi_K|||_K$ is related to $|||e|||_K$. The following two theorems give upper and lower bounds for the error.

Theorem 6.2. ***(Reliability).*** *Let $\{g_K : K \in \mathcal{T}\}$ be any set of boundary fluxes satisfying condition (3.2). In addition, if the absolute term κ vanishes, then it is assumed that the fluxes satisfy the equilibration condition (3.5) on all elements that do not abut the boundary $\partial\Omega$. Then, the global error residual may be decomposed into local contributions*

$$ B(e,v) = L(v) - B(u_X,v) = \sum_{K\in\mathcal{T}} {}_mB_K(\phi_K, v) \quad v \in H^1(K) $$

where $\phi_K \in V_K$ is the solution of the local problem (6.1). The global error in the finite element approximation may be bounded by

$$ |||e|||^2 \leq m_1(e,\mathcal{T})^2 \sum_{K\in\mathcal{T}} {}_m|||\phi_K|||_K^2, $$

where $m_1(e,\mathcal{T})$ is the matching function introduced by (4.6)*, page 14.*

Proof. Using the representation of $B(e,v)$ in the local terms and subsequently applying the Cauchy-Schwarz inequality and the definition of the matching function, we have:

$$
\begin{aligned}
|B(e,v)| &= \left| \sum_{K\in\mathcal{T}} \{(f,v)_K - B_K(u_X,v)\} \right| \\
&= \left| \sum_{K\in\mathcal{T}} \left\{(f,v)_K - B_K(u_X,v) + \int_{\partial K} g_K v\, ds\right\} \right| \\
&= \left| \sum_{K\in\mathcal{T}} {}_m B_K(\phi_K, v) \right| \\
&\leq \sum_{K\in\mathcal{T}} {}_m|||\phi_K|||_{K\,m}|||v|||_K \\
&\leq \sqrt{\sum_{K\in\mathcal{T}} {}_m|||\phi_K|||_K^2} \cdot \sqrt{\sum_{K\in\mathcal{T}} \left(h_{min,K}^{-2}\|C_K^\top \nabla v\|_{L_2(K)}^2 + \kappa^2\|v\|_{L_2(K)}^2\right)} \\
&\leq \sqrt{\sum_{K\in\mathcal{T}} {}_m|||\phi_K|||_K^2} \cdot \sqrt{m_1(v,\mathcal{T})^2\|\nabla v\|_{L_2(\Omega)}^2 + \kappa^2\|v\|_{L_2(\Omega)}^2} \\
&\leq m_1(v,\mathcal{T})|||v|||\sqrt{\sum_{K\in\mathcal{T}} {}_m|||\phi_K|||_K^2}.
\end{aligned}
$$

Substitution $v := e$ completes the proof. □

The theorem 6.2 gives the usual result for anisotropic error estimators. See for instance [14, 13].

Theorem 6.3. ***(Efficiency).*** *Let g_K be the set of approximate fluxes produced by the algorithm described in Section 3.2 with the functions θ replaced by θ^*, and let $\phi_K \in V_K$ denote the solution of the local residual problem (6.1). Then,*

$$
{}_m|||\phi_K|||_K \preceq |||e|||_{\widetilde{K}} + \min(h_{min,K}, \kappa^{-1})\|r - \overline{r}\|_{L_2(\widetilde{K})}.
$$

Proof. The proof follows the same lines as the proof of the Theorem 5.2. □

These two theorems are the main results of this work and guarantee the reliability and efficiency of the estimator.

7 Numerical experiments

One should note that up to this time we considered the infinite dimensional local problems (3.4) and (6.1). However, we have good experience in solving this problem with a finite element method, by dividing the triangles into n^2 parts as one can see in Figure 6.

Let us consider the 2D model problem

$$
-\Delta u + \kappa^2 u = 0 \text{ in } \Omega := [0,1]^2, \quad u = u_0 \text{ on } \partial\Omega.
$$

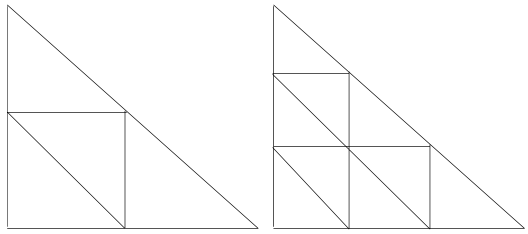

Figure 6: Triangle subdivisions. $n = 2$ and $n = 3$ respectively.

Prescribe the exact solution

$$u = e^{-\kappa x} + e^{-\kappa y}$$

which displays typical boundary layers along the sides $x = 0$ and $y = 0$. The Dirichlet boundary data u_0 are chosen accordingly.

We use a sequence of finite element meshes generated by the algorithm described in [7]. The idea of adaptive procedure is that the choice of a refinement direction is done according to the components of energy norm of an error $\left\|\frac{\partial e}{\partial x}\right\|_{L_2(K)}$, $\left\|\frac{\partial e}{\partial y}\right\|_{L_2(K)}$, and $\kappa^2\|e\|_{L_2(K)}$. One of the resulting meshes of this program is displayed in Figure 7.

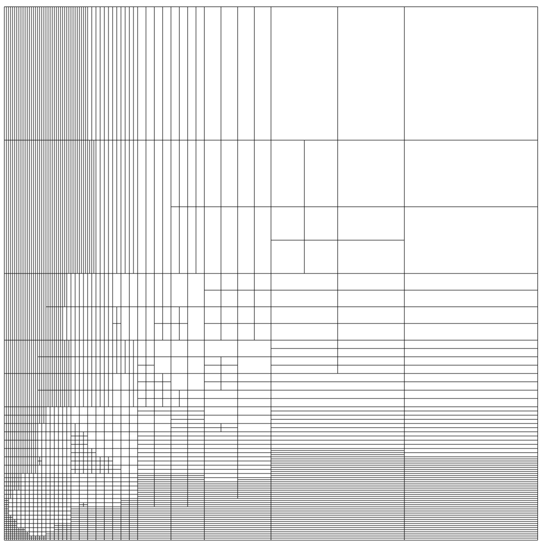

Figure 7: Mesh refinement.

The two tables below show the behavior of the estimators in the singularly perturbed case on anisotropic meshes. We observe that the new error estimator is robust while the original one overestimates the true error when the aspect ratio is large enough.

Iteration N	Unknowns N	Maximal aspect ratio	$\frac{\text{AB error}}{\text{exact err}}$	$\frac{\text{AB er(mod)}}{\text{exact err}}$
1	25	71	1.392	1.102
2	51	71	1.301	0.896
3	97	71	1.565	0.962
4	116	142	1.903	1.032
5	157	285	2.457	1.094
6	217	571	3.151	1.153
7	382	1142	4.264	1.169
8	714	2284	5.955	1.167
9	1481	4568	8.903	1.157
10	3274	9137	14.475	1.130
11	6847	18273	23.967	1.112
12	15187	36547	44.111	1.090
13	35536	73095	81.237	1.062
14	106819	146191	138.711	1.005

Table 1: Results for $n = 4$, $\kappa = 1000$. The fourth column represents the ratio between the Ainsworth and Babuška estimator and the energy norm of the true solution, while the fifth column represents the similar ratio for the estimator defined in the current work.

Iteration N	Unknowns N	Maximal aspect ratio	$\frac{\text{AB error}}{\text{exact err}}$	$\frac{\text{AB er(mod)}}{\text{exact err}}$
1	25	541	2.665	2.150
2	51	541	2.177	1.587
3	120	541	1.802	1.198
4	143	541	1.995	1.187
5	192	1083	2.410	1.243
6	217	2167	3.062	1.302
7	283	4335	4.273	1.392
8	446	8669	6.207	1.436
9	814	17339	9.748	1.405
10	1553	34679	16.646	1.411
11	3053	69359	30.149	1.433
12	5809	138718	54.363	1.429
13	11357	277436	101.47	1.420
14	23376	554873	211.02	1.407
15	104916	1109745	423.10	1.383

Table 2: Results for $n = 4$, $\kappa = 10000$.

8 Summary and additional remarks

We consider the singularly perturbed reaction-diffusion equation $-\Delta u + \kappa^2 u = f$. This work has been aiming at *a posteriori* equilibrated residual-like error estimators suitable for anisotropic triangular grids.

The Ainsworth-Babuška estimator is shown to be reliable in the anisotropic case. Unfortunately, the lower error bound fails on anisotropic meshes.

The introduced modification leads to an estimator which is robust with respect to the anisotropy of the mesh as well as to the singular perturbation. Upper and lower error bounds are proved. The factor which made the original error esimator fail does not appear in the lower bound any more, which leads to the efficiency of the modified estimator. The upper error bound of the modified estimator contains the factor $m_1(e, \mathcal{T})$ which is in accordance with the results made by Kunert in [14].

The numerical experiments verify the theory. The modified estimator yields a useful and reliable error bound not only in an asymptotic sense but also for meshes with moderate number of elements.

Remark 8.1. All the proofs are suitable for 3D case. The only two questions we should answer are about the topology matrices and the minimum energy extension of the first-order basis function. For the minimum energy extension of the first-order basis function we construct the approximation by analogy with Section 3.3. A point D in this case may be chosen on the intersection line of the bisection planes of the corresponding cone with the distance $1/\kappa$ from each face. The topology matrices are constructed in the way analogous to Section 3, but seem to be much more complicated. □

Remark 8.2. Neumann boundary conditions can be also considered as well as quadrilateral elements. In both cases an additional term corresponding to the face residual appears in the lower bound for the error:

$$|||\phi_K|||_K \preceq |||e|||_{\widetilde{K}} + \min(h_{min,K}, \kappa^{-1})\|r - \overline{r}\|_{L_2(\widetilde{K})} + \sum_{\gamma \subset \widetilde{K}} \min(h_{min,K}, \kappa^{-1})^{1/2}\|R - \overline{R}\|_{L_2(\gamma)}$$

□

Remark 8.3. If we solve our FEM problem with polynomials of p-th order then we have to talk about the p-th order equilibration. For details see [3]. □

Remark 8.4. Consider the problem in another formulation used by some authors (see [6, 8, 11, 14]), namely

$$-\varepsilon^2 \Delta u + u = f \text{ in } \Omega \subset \mathbb{R}^2, \quad u = 0 \text{ on } \partial\Omega.$$

The estimator remains the same as well as its upper error bound:

$$|||e|||^2 \leq m_1(e, \mathcal{T})^2 \sum_{K \in \mathcal{T}} {}_m|||\phi_K|||_K^2,$$

The lower error bound changes to the following:

$$_m|||\phi_K|||_K \preceq |||e|||_{\widetilde{K}} + \min(h_{min,K}\varepsilon^{-1}, 1)\|r - \overline{r}\|_{L_2(\widetilde{K})}.$$

We should note that this is exactly the same estimate as for another error estimator introduced by Kunert in [14].

□

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