

LECTURE HALL THEOREMS, q -SERIES AND TRUNCATED OBJECTS

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ABSTRACT. We show here that the refined theorems for both lecture hall partitions and anti-lecture hall compositions can be obtained as straightforward consequences of two q -Chu Vandermonde identities, once an appropriate recurrence is derived. We use this approach to get new lecture hall-type theorems for truncated objects. The *truncated lecture hall partitions* are sequences $(\lambda_1, \dots, \lambda_k)$ such that

$$\frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \dots \geq \frac{\lambda_k}{n-k+1} \geq 0$$

and we show that their generating function is :

$$\sum_{m=0}^k \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\binom{m+1}{2}} \frac{(-q^{n-m+1}; q)_m}{(q^{2n-m+1}; q)_m}.$$

From this, we are able to give a combinatorial characterization of truncated lecture hall partitions and new finitizations of refinements of Euler's theorem. The *truncated anti-lecture hall compositions* are sequences $(\lambda_1, \dots, \lambda_k)$ such that

$$\frac{\lambda_1}{n-k+1} \geq \frac{\lambda_2}{n-k+2} \geq \dots \geq \frac{\lambda_k}{n} \geq 0.$$

We show that their generating function is :

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-q^{n-k+1}; q)_k}{(q^{2(n-k+1)}; q)_k},$$

giving a finitization of a well-known partition identity. We give two different multivariate refinements of these new results : the q -calculus approach gives (u, v, q) -refinements, while a completely different approach gives odd/even (x, y) -refinements.

1. INTRODUCTION

For a sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ of nonnegative integers, define the *weight* of λ to be $|\lambda| = \lambda_1 + \dots + \lambda_n$ and call each λ_i a *part* of λ . If λ has all parts nonnegative, we call it a *composition* and if, in addition, λ is a nonincreasing sequence, we call it a *partition*.

In [4], inspired by work of Eriksson and Eriksson on Coxeter groups, Bousquet-Mélou and Eriksson considered *lecture hall partitions*, specifically, the set L_n of partitions, λ , into n nonnegative parts satisfying

$$\frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \dots \geq \frac{\lambda_{n-1}}{2} \geq \frac{\lambda_n}{1} \geq 0,$$

and proved the following surprising result.

Supported by ATIP Jeune chercheur CNRS.

Research supported by NSA grant MDA 904-01-0-0083 and NSF INT-0230800.

The Lecture Hall Theorem [4] :

$$(1) \quad L_n(q) \triangleq \sum_{\lambda \in L_n} q^{|\lambda|} = \frac{1}{(q; q^2)_n}$$

with $(a; q)_n = \prod_{i=0}^{n-1} (1 - aq^i)$.

In [4], Bousquet-Mélou and Eriksson gave two proofs of the Lecture Hall Theorem : one based on Coxeter groups and one combinatorial. Subsequently, Andrews gave a proof based on partition analysis [2]. A refinement and generalizations of the identity (1) were given by Bousquet-Mélou and Eriksson in [5]. They showed that an elegant mapping between certain partitions in L_n and partitions in L_{n-1} gives a functional equation which easily implies the result. The first bijective proof of the Lecture Hall Theorem was given by Yee [11] and is close to [4]. Others followed [3, 10].

An involution in [4] gave the following nice refinement of (1).

The Odd/Even Lecture Hall Theorem [4] : Given $\lambda = (\lambda_1, \lambda_2, \dots)$, define $\lambda_o = (\lambda_1, \lambda_3, \dots)$ and $\lambda_e = (\lambda_2, \lambda_4, \dots)$. Then

$$(2) \quad L_n(x, y) \triangleq \sum_{\lambda \in L_n} x^{|\lambda_o|} y^{|\lambda_e|} = \prod_{i=1}^n \frac{1}{1 - x^i y^{i-1}}.$$

A different approach of Bousquet-Mélou and Eriksson in [6] led to the following.

The Refined Lecture Hall Theorem [6] :

$$(3) \quad L_n(u, v, q) \triangleq \sum_{\lambda \in L_n} q^{|\lambda|} u^{|\lceil \lambda \rceil} v^{o(\lceil \lambda \rceil)} = \frac{(-uvq; q)_n}{(u^2 q^{n+1}; q)_n},$$

where for a partition $\lambda = (\lambda_1, \dots, \lambda_n)$, $o(\lambda)$ is the number of odd parts of λ and $\lceil \lambda \rceil$ is the partition $(\lceil \lambda_1/n \rceil, \lceil \lambda_2/(n-1) \rceil, \dots, \lceil \lambda_{n-1}/2 \rceil, \lceil \lambda_n/1 \rceil)$.

Setting $u = v = 1$ in (3) gives (1). Yee gave a beautiful bijective proof [12] of this theorem.

In [7], we considered a new twist on these results by studying the set A_n of *compositions* into at most n parts satisfying

$$\frac{\lambda_1}{1} \geq \frac{\lambda_2}{2} \geq \dots \geq \frac{\lambda_n}{n} \geq 0.$$

We referred to these as *anti-lecture hall compositions* and showed the following with a bijective proof along the lines of Yee's proof of (3) in [12]

The Refined Anti-Lecture Hall Theorem [7] :

$$(4) \quad A_n(u, v, q) \triangleq \sum_{\lambda \in A_n} q^{|\lambda|} u^{|\lfloor \lambda \rfloor} v^{o(\lfloor \lambda \rfloor)} = \frac{(-uvq; q)_n}{(u^2 q^2; q)_n},$$

where $\lfloor \lambda \rfloor = (\lfloor \lambda_1/1 \rfloor, \lfloor \lambda_2/2 \rfloor, \dots, \lfloor \lambda_n/n \rfloor)$ and $o(\lambda)$ denotes the number of odd parts of a composition λ .

Setting $u = v = 1$ in (4) gives the following analog of (1) :

The Anti-Lecture Hall Theorem [7] :

$$(5) \quad A_n(q) \triangleq \sum_{\lambda \in A_n} q^{|\lambda|} = \frac{(-q; q)_n}{(q^2; q)_n}.$$

At the time of [7] we did not have a q -series proof of (4) and we were not able to prove an odd/even refinement of (5) analogous to (2), although we had a conjecture as to its form.

In this paper we extend all of these results. Our starting point is a proof of The Refined Lecture Hall Theorem from [6], where Bousquet-Mélou and Eriksson gave a two-step proof of (3) using basic q -series identities. They explained separately the numerator and the denominator using elementary techniques, deriving a recurrence to obtain the denominator and noting that the recurrence could be solved using a special case of the q -analogue of the Chu-Vandermonde summation ([1], 3.3.10).

In Section 2.1, we pursue their approach, but proceed directly to a recurrence for $L_n(u, v, q)$ which can be solved in a straightforward way using the identity q -Chu Vandermonde II [9] :

$$\frac{a^n (c/a; q)_n}{(c; q)_n} = \sum_{m=0}^n \frac{(a; q)_m (q^{-n}; q)_m}{(c; q)_m (q; q)_m} q^m.$$

With this modified approach, we give, in Section 2.2, a new proof of the Refined Anti-Lecture Hall Theorem. We show that it can be obtained using another q -Chu Vandermonde I [9] :

$$\frac{(c/a; q)_n}{(c; q)_n} = \sum_{m=0}^n \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(a; q)_m}{(c; q)_m} (-c/a)^m q^{\binom{m}{2}},$$

where $\begin{bmatrix} n \\ m \end{bmatrix}_q = (q^{n-m+1}; q)_m / (q; q)_m$ is the classical Gaussian polynomial, the generating function for partitions into m nonnegative parts of size at most $n - m$.

From this point on, all of our results are new. We conjectured them thanks to the Maple implementation of the generating function developed in [8]. In Section 3, we show how the q -series techniques of Section 2 can be extended to get new identities for the enumeration of *truncated objects*. For $n \geq k$, let $L_{n,k}$ be the set of partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ into k nonnegative parts satisfying

$$\frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \dots \geq \frac{\lambda_k}{n-k+1} \geq 0.$$

We refer to these as *truncated lecture hall partitions*. (Note that the case $n = k$ corresponds to the ordinary lecture hall partitions.) These objects were introduced by Eriksen in [10], where he gave a recurrence for their generating function in one variable, but no closed-form solution.

We show in Section 3.1 how to compute the three-variable generating function :

$$L_{n,k}(u, v, q) \triangleq \sum_{\lambda \in L_{n,k}} q^{|\lambda|} u^{|\lambda|} v^{o([\lambda])}$$

where $[\lambda] = ([\lambda_1/n], [\lambda_2/(n-1)], \dots, [\lambda_{k-1}/(n-k+2)], [\lambda_k/(n-k+1)])$.

The Refined Truncated Lecture Hall Theorem:

$$(6) \quad L_{n,k}(u, v, q) = \sum_{m=0}^k (uv)^m q^{\binom{m+1}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(-(u/v)q^{n-m+1}; q)_m}{(u^2 q^{2n-m+1}; q)_m}.$$

In Section 3.2 we study *truncated anti-lecture hall compositions*, defined for each $n \geq k - 1$ as the set $A_{n,k}$ of compositions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ into k nonnegative parts satisfying

$$\frac{\lambda_1}{n-k+1} \geq \frac{\lambda_2}{n-k+2} \geq \dots \geq \frac{\lambda_k}{n} \geq 0.$$

(When $n = k$, these are the ordinary anti-lecture hall compositions.) If we let $[\lambda]$ denote the partition $(\lfloor \lambda_1 / (n - k + 1) \rfloor, \lfloor \lambda_2 / (n - k + 2) \rfloor, \dots, \lfloor \lambda_{k-1} / (n - 1) \rfloor, \lfloor \lambda_n / n \rfloor)$, we get the following, for $n \geq k$:

The Refined Truncated Anti-Lecture Hall Theorem:

$$(7) \quad A_{n,k}(u, v, q) \triangleq \sum_{\lambda \in A_{n,k}} q^{|\lambda|} u^{|\lambda|} v^{o([\lambda])} = \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-uvq^{n-k+1}; q)_k}{(u^2q^{2(n-k+1)}; q)_k}.$$

We next consider odd/even refinements of the generating functions for truncated objects, analogous to (2) for lecture hall partitions. In Section 4.1, we show the following for truncated lecture hall partitions.

The Odd/Even Truncated Lecture Hall Theorem:

$$(8) \quad L_{n,k}(x, y) \triangleq \sum_{\lambda \in L_{n,k}} x^{|\lambda_o|} y^{|\lambda_e|} = \sum_{m=0}^k \frac{(x^{\lfloor m/2 \rfloor + 1} y^{\lfloor m/2 \rfloor})^{\lceil m/2 \rceil} \begin{bmatrix} n - \lceil m/2 \rceil \\ \lfloor m/2 \rfloor \end{bmatrix}_{xy}}{(x; xy)_{\lfloor m/2 \rfloor} (x^n y^{n-1}; (xy)^{-1})_{\lfloor m/2 \rfloor}}.$$

Similarly, in Section 4.2 we find the odd/even generating function for truncated anti-lecture hall compositions. Let $n \geq k - 1$.

The Odd/Even Truncated Anti-Lecture Hall Theorem:

$$(9) \quad A_{n,k}(x, y) \triangleq \sum_{\lambda \in A_{n,k}} x^{|\lambda_o|} y^{|\lambda_e|} = \frac{\begin{bmatrix} n \\ \lfloor k/2 \rfloor \end{bmatrix}_{xy}}{(x; xy)_{\lfloor k/2 \rfloor} (x^{n-k+1} y^{n-k+2}; xy)_{\lfloor k/2 \rfloor}}.$$

In particular, setting $n = k$ in (9) gives for the first time the odd/even generating function for anti-lecture hall compositions:

$$(10) \quad A_n(x, y) = A_{n,n}(x, y) = \frac{\begin{bmatrix} n \\ \lfloor n/2 \rfloor \end{bmatrix}_{xy}}{(x; xy)_{\lfloor n/2 \rfloor} (xy^2; xy)_{\lfloor n/2 \rfloor}}.$$

One of the many interesting things about lecture hall partitions is that the Lecture Hall Theorem gives them a simple interpretation in terms of partitions into odd parts: *the number of partitions of N in L_n is equal to the number of partitions of N into odd parts less than $2n$* . A similar interpretation of truncated lecture hall partitions is not so evident from their generating function in (6) or in (8). However, in Section 5 we show the following correspondence between truncated lecture hall partitions and partitions into odd parts with certain restrictions.

Characterization of Truncated Lecture Hall Partitions:

The number of truncated lecture hall partitions of N in $L_{n,k}$ is equal to the number of partitions of N into odd parts less than $2n$, with the following constraint on the parts: at most $\lfloor k/2 \rfloor$ parts can be chosen from the set

$$\{2\lfloor k/2 \rfloor + 1, 2\lfloor k/2 \rfloor + 3, \dots, 2(n - \lfloor k/2 \rfloor) - 1\}.$$

As $n \rightarrow \infty$, the set L_n of lecture hall partitions approaches the set of partitions into distinct parts. Similarly, the right-hand side of (1) approaches the set of partitions into odd parts. In this sense, the Lecture Hall Theorem is viewed as a *finitization* of Euler's Theorem, which states: *The number of partitions of an integer N into distinct parts is equal to the number of partitions of N into odd parts.* In Section 5.1, we show how our results on truncated lecture hall partitions lead to finitizations of certain refinements of Euler's Theorem that are implied by Sylvester's bijection.

We show in Section 5.2 how truncated anti-lecture hall theorems can be viewed as finitizations of another well-known identity: *The number of partitions of N into k parts is equal to the number of partitions of N with no part larger than k .*

2. THE REFINED LECTURE HALL THEOREMS

We will make use of the q -multinomial coefficient, defined for $n = n_0 + n_1 + \cdots + n_t$ by

$$\left[\begin{matrix} n \\ n_0, n_1, \dots, n_t \end{matrix} \right]_q = \frac{(q; q)_n}{(q; q)_{n_0} (q; q)_{n_1} \cdots (q; q)_{n_t}},$$

from which it follows that

$$(11) \quad \left[\begin{matrix} n \\ n_0, n_1, \dots, n_t \end{matrix} \right]_q = \left[\begin{matrix} n \\ n_0 \end{matrix} \right]_q \left[\begin{matrix} n - m_0 \\ n_1, \dots, n_t \end{matrix} \right]_q.$$

2.1. Lecture Hall Partitions. We review the proof of the Refined Lecture Hall Theorem (3) of Bousquet-Mélou and Eriksson. Most of the basic ideas of this proof come from [6]. Given a lecture hall partition $\lambda \in L_n$, denote by $[\lambda]$ the sequence $(\lceil \lambda_1/n \rceil, \lceil \lambda_2/(n-1) \rceil, \dots, \lceil \lambda_n/1 \rceil)$. We can write $\lambda_i = (n-i+1)\mu_i - r_i$, with $0 \leq r_i \leq n-i$ for $1 \leq i \leq n$. Then $(\mu_1, \dots, \mu_n) = [\lambda]$.

Proposition 1. [6] *A partition λ is in L_n if and only if*

- (1) $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n \geq 0$ and
- (2) $r_i \leq r_{i+1}$ whenever $\mu_i = \mu_{i+1}$.

The first condition implies that $[\lambda]$ is a partition into n nonnegative parts.

Let P_n be the set of partitions into n nonnegative parts. Let μ be a partition in P_n . In [6] the authors compute the generating function of the lecture hall partitions λ having $[\lambda] = \mu$. Let this generating function be $L_\mu(q)$.

Proposition 2. [6] *For $\mu \in P_n$,*

$$L_\mu(q) \triangleq \sum_{\substack{\lambda \in L_n \\ [\lambda] = \mu}} q^{|\lambda|} = q^{\sum_{i=1}^n (n-i+1)\mu_i} \left[\begin{matrix} n \\ m_0, m_1, \dots, m_{\mu_1} \end{matrix} \right]_{1/q}.$$

where m_i is the multiplicity of the part i in μ .

Proof. We sketch here the main ideas of the proof. As $|\lambda| = \sum_{i=1}^n ((n-i+1)\mu_i - r_i)$,

$$L_\mu(q) = q^{\sum_{i=1}^n (n-i+1)\mu_i} \sum_{(r_1, \dots, r_n)} q^{-\sum_{i=1}^n r_i}.$$

For $0 \leq i \leq \mu_1$, let

$$\ell_i = \begin{cases} 0 & \text{if } i = 0 \\ \sum_{j=\mu_1-i+1}^{\mu_1} m_j & \text{otherwise.} \end{cases}$$

Then the condition (2) in Proposition 1 implies that $(r_{\ell_{i+1}}, r_{\ell_{i+1}-1}, \dots, r_{\ell_i+1})$ is a partition into $\ell_{i+1} - \ell_i = m_{\mu_1-i}$ nonnegative parts and that these parts are less than or equal to $n - \ell_{i+1}$. Therefore their generating function is well-known to be a Gaussian polynomial :

$$(12) \quad \sum_{(r_{\ell_{i+1}}, \dots, r_{\ell_i+1})} q^{(r_{\ell_{i+1}} + \dots + r_{\ell_i+1})} = \begin{bmatrix} n - \ell_{i+1} + m_{\mu_1-i} \\ m_{\mu_1-i} \end{bmatrix}_q = \begin{bmatrix} n - \ell_i \\ m_{\mu_1-i} \end{bmatrix}_q.$$

Hence, the result follows from the computation below which uses (12) for the second equality and repeated application of (11) for the third.

$$\begin{aligned} \sum_{(r_1, \dots, r_n)} q^{-\sum_{i=1}^n r_i} &= \prod_{i=0}^{\mu_1} \sum_{(r_{\ell_{i+1}}, \dots, r_{\ell_i+1})} (1/q)^{(r_{\ell_{i+1}} + \dots + r_{\ell_i+1})} \\ &= \prod_{i=0}^{\mu_1} \begin{bmatrix} n - \ell_i \\ m_{\mu_1-i} \end{bmatrix}_{1/q} \\ &= \begin{bmatrix} n \\ m_0, m_1, \dots, m_{\mu_1} \end{bmatrix}_{1/q}. \end{aligned}$$

□

At this point, we take a different turn from [6], proceeding directly to enumeration of the partitions μ with all parts positive. Let $P_{n,m}$ be the set of partitions into n nonnegative parts, m of which are positive. Given μ in $P_{m,n}$, we define $\tilde{\mu}$ in P_m by $\tilde{\mu}_i = \mu_i - 1$, $1 \leq i \leq m$. Then

Proposition 3. For $\mu \in P_{m,n}$,

$$L_\mu(q) = q^{(n-m)|\tilde{\mu}| + \binom{m+1}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_q L_{\tilde{\mu}}(q).$$

Proof. Using Proposition 2, we know that if m_i is the multiplicity of i in μ , then

$$L_\mu(q) = q^{\sum_{i=1}^m (n-i+1)(\tilde{\mu}_i+1)} \begin{bmatrix} n \\ m_0, m_1, \dots, m_{\mu_1} \end{bmatrix}_{1/q},$$

and that

$$L_{\tilde{\mu}}(q) = q^{\sum_{i=1}^m (m-i+1)\tilde{\mu}_i} \begin{bmatrix} m \\ m_1, \dots, m_{\mu_1} \end{bmatrix}_{1/q}.$$

Since $m_0 = n - m$, using (11), we can write the first equation as

$$L_\mu(q) = q^{(n-m)|\tilde{\mu}| + m(n+1) - m(m+1)/2} \begin{bmatrix} n \\ m \end{bmatrix}_{1/q} q^{\sum_{i=1}^m (m-i+1)\tilde{\mu}_i} \begin{bmatrix} m \\ m_1, \dots, m_{\mu_1} \end{bmatrix}_{1/q}.$$

The result follows from the second equation and the identity

$$\begin{bmatrix} n \\ m \end{bmatrix}_{1/q} = q^{-m(n-m)} \begin{bmatrix} n \\ m \end{bmatrix}_q.$$

□

An easy consequence of Proposition 3 is that

$$(13) \quad u^{|\mu|} v^{o(\mu)} L_\mu(q) = (uv)^m u^{|\tilde{\mu}|} v^{-o(\tilde{\mu})} q^{(n-m)|\tilde{\mu}| + \binom{m+1}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_q L_{\tilde{\mu}}(q),$$

as $|\mu| = |\tilde{\mu}| + m$ and $o(\mu) = m - o(\tilde{\mu})$.

In order to prove the Refined Lecture Hall Theorem (3), the generating function we are looking for, $L_n(u, v, q) \triangleq \sum_{\lambda \in L_n} u^{|\lambda|} v^{o(\lambda)} q^{|\lambda|}$, can be rewritten as :

$$(14) \quad L_n(u, v, q) = \sum_{\mu \in P_n} u^{|\mu|} v^{o(\mu)} L_\mu(q) = \sum_{m=0}^n \sum_{\mu \in P_{n,m}} u^{|\mu|} v^{o(\mu)} L_\mu(q).$$

Then we can prove the following recurrence for $L_n(u, v, q)$.

Proposition 4. $L_0(u, v, q) = 1$ and for $n > 0$,

$$L_n(u, v, q) = \sum_{m=0}^n \begin{bmatrix} n \\ m \end{bmatrix}_q (uv)^m q^{\binom{m+1}{2}} L_m(uq^{n-m}, 1/v, q).$$

Proof.

$$\begin{aligned} L_n(u, v, q) &= \sum_{m=0}^n \sum_{\mu \in P_{n,m}} u^{|\mu|} v^{o(\mu)} L_\mu(q) \\ &= \sum_{m=0}^n \sum_{\tilde{\mu} \in P_m} (uv)^m u^{|\tilde{\mu}|} v^{-o(\tilde{\mu})} q^{(n-m)|\tilde{\mu}| + \binom{m+1}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_q L_{\tilde{\mu}}(q) \quad (\text{using (13)}) \\ &= \sum_{m=0}^n \begin{bmatrix} n \\ m \end{bmatrix}_q (uv)^m q^{\binom{m+1}{2}} \sum_{\tilde{\mu} \in P_m} (uq^{n-m})^{|\tilde{\mu}|} v^{-o(\tilde{\mu})} L_{\tilde{\mu}}(q) \end{aligned}$$

If we now apply the first equality of (14) to $L_m(uq^{n-m}, 1/v, q)$, we get the last sum in the last line above and the proposition is proved. \square

Proof of The Refined Lecture Hall Theorem (3): We solve the recurrence of Proposition 4 using the identity q -Chu Vandermonde II:

$$(15) \quad \frac{a^n (c/a; q)_n}{(c; q)_n} = \sum_{m=0}^n \frac{(a; q)_m (q^{-n}; q)_m}{(c; q)_m (q; q)_m} q^m.$$

If we set $a = -vq^{-n}/u$, $c = q^{-2n}/u^2$ in (15), we get that

$$(-vq^{-n}/u)^n \frac{(-q^{-n}/uv; q)_n}{(q^{-2n}/u^2; q)_n} = \sum_{m=0}^n \frac{(-vq^{-n}/u; q)_m (q^{-n}; q)_m}{(q^{-2n}/u^2; q)_m (q; q)_m} q^m.$$

Now on every factor above of the form $(bq^{-t}; q)_m$ we use the identity

$$(bq^{-t}; q)_m = (-bq^{-t+(m-1)/2})^m (q^{t-m+1}/b; q)_m$$

and get

$$\frac{(-uvq; q)_n}{(u^2q^{n+1}; q)_n} = \sum_{m=0}^n (uv)^m q^{\binom{m+1}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(-u/v)q^{n-m+1}; q)_m}{(u^2q^{2n-m+1}; q)_m}.$$

This shows that

$$L_n(u, v, q) = \frac{(-uvq; q)_n}{(u^2q^{n+1}; q)_n}$$

is the solution to the recurrence of Proposition 4. \square

2.2. Anti-Lecture Hall Compositions. A q -series proof of the Refined Anti-Lecture Hall Theorem (4) will follow the same approach as in the previous subsection. Given an anti-lecture hall composition $\lambda \in A_n$, define the floor of λ , as $\lfloor \lambda \rfloor = (\lfloor \lambda_1/1 \rfloor, \lfloor \lambda_2/2 \rfloor, \dots, \lfloor \lambda_n/n \rfloor)$. Then write $\lambda_i = i\mu_i + r_i$, with $0 \leq r_i \leq i-1$ for $1 \leq i \leq n$. Note that $(\mu_1, \dots, \mu_n) = \lfloor \lambda \rfloor$ and that $\lambda \in A_n$ if and only if

- (i) $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n \geq 0$ and
- (ii) $r_i \geq r_{i+1}$ whenever $\mu_i = \mu_{i+1}$.

The condition (i) implies that $\lfloor \lambda \rfloor$ is a partition in P_n . We fix $\mu \in P_n$ and compute the generating function A_μ of the anti-lecture hall compositions λ having $\lfloor \lambda \rfloor = \mu$.

Proposition 5. For $\mu \in P_n$,

$$A_\mu(q) \triangleq \sum_{\substack{\lambda \in A_n \\ \lfloor \lambda \rfloor = \mu}} q^{|\lambda|} = q^{\sum_{i=1}^n i\mu_i} \begin{bmatrix} n \\ m_0, m_1, \dots, m_{\mu_1} \end{bmatrix}_q.$$

where m_i is the multiplicity of the part i in μ .

Proof. As $\lambda_i = i\mu_i + r_i$

$$A_\mu(q) = q^{\sum_{i=1}^n i\mu_i} \sum_{(r_1, \dots, r_n)} q^{\sum_{i=1}^n r_i}.$$

For $0 \leq i \leq \mu_1 + 1$, let $\ell_i = n - \sum_{j=0}^{i-1} m_j$. Then the condition (ii) implies that $(r_{\ell_{i+1}+1}, r_{\ell_{i+1}+2}, \dots, r_{\ell_i})$ is a partition into $\ell_i - \ell_{i+1} = m_i$ nonnegative parts and that these parts are less than or equal to ℓ_{i+1} . Therefore, since $\ell_0 = n$,

$$\sum_{(r_1, \dots, r_n)} q^{\sum_{i=1}^n r_i} = \prod_{i=0}^{\mu_1} \begin{bmatrix} \ell_{i+1} + m_i \\ m_i \end{bmatrix}_q = \begin{bmatrix} n \\ m_0, m_1, \dots, m_{\mu_1} \end{bmatrix}_q.$$

\square

As before if $\mu \in P_{n,m}$ then we can get $\tilde{\mu}$ in P_m by $\tilde{\mu}_i = \mu_i - 1$, $1 \leq i \leq m$.

Proposition 6. For $\mu \in P_{n,m}$,

$$A_\mu(q) = q^{\binom{m+1}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_q A_{\tilde{\mu}}(q).$$

Proof. From Proposition 5 we know that

$$A_\mu(q) = q^{\sum_{i=1}^m i(\tilde{\mu}_i+1)} \begin{bmatrix} n \\ m_0, m_1, \dots, m_{\mu_1} \end{bmatrix}_q,$$

and that

$$A_{\tilde{\mu}}(q) = q^{\sum_{i=1}^m i\tilde{\mu}_i} \begin{bmatrix} m \\ m_1, \dots, m_{\mu_1} \end{bmatrix}_q.$$

Since $m = n - m_0$, we get the result. \square

If $\mu \in P_{n,m}$, then $|\mu| = |\tilde{\mu}| + m$ and $o(\mu) = m - o(\tilde{\mu})$, so we can conclude from Proposition 6 that :

$$(16) \quad u^{|\mu|} v^{o(\mu)} A_\mu(q) = (uv)^m u^{|\tilde{\mu}|} v^{-o(\tilde{\mu})} q^{\binom{m+1}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_q A_{\tilde{\mu}}(q).$$

In order to prove the Refined Anti-Lecture Hall Theorem (4), the generating function we are looking for, $A_n(u, v, q) \triangleq \sum_{\lambda \in A_n} u^{|\lambda|} v^{o(\lambda)} q^{|\lambda|}$ can be rewritten as

$$(17) \quad A_n(u, v, q) = \sum_{\mu \in P_n} u^{|\mu|} v^{o(\mu)} A_\mu(q) = \sum_{m=0}^n \sum_{\mu \in P_{n,m}} u^{|\mu|} v^{o(\mu)} A_\mu(q).$$

Then we can prove the following recurrence for $A_n(u, v, q)$.

Proposition 7. $A_0(u, v, q) = 1$ and for $n > 0$

$$A_n(u, v, q) = \sum_{m=0}^n \begin{bmatrix} n \\ m \end{bmatrix}_q (uv)^m q^{\binom{m+1}{2}} A_m(u, 1/v, q).$$

Proof.

$$\begin{aligned} A_n(u, v, q) &= \sum_{m=0}^n \sum_{\mu \in P_{n,m}} u^{|\mu|} v^{o(\mu)} A_\mu(q) \\ &= \sum_{m=0}^n \sum_{\tilde{\mu} \in P_m} (uv)^m u^{|\tilde{\mu}|} v^{-o(\tilde{\mu})} q^{\binom{m+1}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_q A_{\tilde{\mu}}(q) \quad (\text{using (16)}) \\ &= \sum_{m=0}^n \begin{bmatrix} n \\ m \end{bmatrix}_q (uv)^m q^{\binom{m+1}{2}} A_m(u, 1/v, q), \end{aligned}$$

where the last step follows from first equality of (17). \square

Proof of the Refined Anti-Lecture Hall Theorem (4). We solve the recurrence of Proposition 7 using the identity q -Chu Vandermonde I:

$$(18) \quad \frac{(c/a; q)_n}{(c; q)_n} = \sum_{m=0}^n \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(a; q)_m}{(c; q)_m} (-c/a)^m q^{\binom{m}{2}}.$$

Setting $a = -uq/v$ and $c = u^2q^2$ in (18) gives immediately

$$\frac{(-uvq; q)_n}{(u^2q^2; q)_n} = \sum_{m=0}^n \begin{bmatrix} n \\ m \end{bmatrix}_q (uv)^m q^{\binom{m+1}{2}} \frac{(-uq/v; q)_m}{(u^2q^2; q)_m}.$$

With the recurrence of Proposition 7, we conclude that :

$$A_n(u, v, q) = \frac{(-uvq; q)_n}{(u^2q^2; q)_n}.$$

\square

3. REFINED LECTURE HALL THEOREMS FOR TRUNCATED OBJECTS

In this section we apply the techniques used in Section 2 to derive the new refined truncated lecture hall theorems.

3.1. Truncated Lecture Hall Partitions. Recall that a truncated lecture hall partition in $L_{n,k}$ is a sequence $\lambda = (\lambda_1, \dots, \lambda_k)$ such that

$$\frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \dots \geq \frac{\lambda_k}{n-k+1} \geq 0.$$

Given a lecture hall partition $\lambda \in L_{n,k}$, we write $\lambda_i = (n-i+1)\mu_i - r_i$, with $0 \leq r_i \leq n-i$ for $1 \leq i \leq k$. Let $[\lambda] = ([\lambda_1/n], \dots, [\lambda_k/(n-k+1)])$. Then $\mu = [\lambda]$. Note that λ has k positive parts if and only if μ has k positive parts.

Theorem 1. (The Refined Truncated Lecture Hall Theorem)

$$(19) \quad L_{n,k}(u, v, q) = \sum_{m=0}^k (uv)^m q^{\binom{m+1}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(-(u/v)q^{n-m+1}; q)_m}{(u^2q^{2n-m+1}; q)_m}.$$

Proof. Let $\bar{L}_{n,k}$ be the set of lecture hall partitions in L_n with k positive parts. Then

$$\bar{L}_{n,k}(u, v, q) \triangleq \sum_{\lambda \in \bar{L}_{n,k}} u^{|\lambda|} v^{o([\lambda])} q^{|\lambda|} = \sum_{\mu \in P_{n,k}} u^{|\mu|} v^{o(\mu)} L_{\mu}.$$

It follows from the proof of Proposition 4 (ignoring the outer sum there) that

$$\bar{L}_{n,k}(u, v, q) = (uv)^k q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q L_k(uq^{n-k}, 1/v, q)$$

and then applying the Refined Lecture Hall Theorem (3) to L_k gives

$$(20) \quad \bar{L}_{n,k}(u, v, q) = (uv)^k q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-(u/v)q^{n-k+1}; q)_k}{(u^2q^{2n-k+1}; q)_k}.$$

The result follows since $L_{n,k}(u, v, q) = \sum_{m=0}^k \bar{L}_{n,m}(u, v, q)$. \square

3.2. Truncated Anti-Lecture Hall Compositions. Let $n \geq k$. A truncated anti-lecture hall composition in $A_{n,k}$ is a sequence $\lambda = (\lambda_1, \dots, \lambda_k)$ such that

$$\frac{\lambda_1}{n-k+1} \geq \frac{\lambda_2}{n-k+2} \geq \dots \geq \frac{\lambda_k}{n} \geq 0.$$

We write $\lambda_i = (n-k+i)\mu_i + r_i$, with $0 \leq r_i \leq n-k+i-1$ for $1 \leq i \leq k$ and define $[\lambda] = (\mu_1, \dots, \mu_k)$. As before, $\lambda \in A_{n,k}$ if and only if

- (i) $\mu_1 \geq \mu_2 \geq \dots \geq \mu_k \geq 0$ and
- (ii) $r_i \geq r_{i+1}$ whenever $\mu_i = \mu_{i+1}$.

For $\mu \in P_k$, we compute the generating function $A_{\mu,k}(q)$ of those $\lambda \in A_{n,k}$ having $[\lambda] = \mu$.

Proposition 8.

$$A_{\mu,k}(q) \triangleq \sum_{\substack{\lambda \in A_{n,k} \\ [\lambda] = \mu}} q^{|\lambda|} = \begin{bmatrix} n \\ k \end{bmatrix}_q q^{(n-k)|\mu|} A_{\mu}(q),$$

where $A_{\mu}(q)$ is the generating of $\lambda \in A_k$ having $[\lambda] = \mu$.

Proof.

$$A_{\mu,k}(q) = q^{\sum_{i=1}^k (n-k+i)\mu_i} \sum_{(r_1, \dots, r_k)} q^{\sum_{i=1}^k r_i}.$$

For $0 \leq i \leq \mu_1 + 1$, let $\ell_i = n - \sum_{j=0}^{i-1} m_j$. Then the condition (ii) implies that $(r_{\ell_{i+1}+1}, r_{\ell_{i+1}+2}, \dots, r_{\ell_i})$ is a partition into m_i nonnegative parts and that these parts are less than or equal to ℓ_{i+1} . Therefore, since $\ell_i - \ell_{i+1} = m_i$ and $\ell_{\mu_1+1} = n - k$,

$$\sum_{(r_1, \dots, r_k)} q^{\sum_{i=1}^k r_i} = \prod_{i=0}^{\mu_1} \begin{bmatrix} \ell_{i+1} + m_i \\ m_i \end{bmatrix}_q = \begin{bmatrix} n \\ m_0, m_1, \dots, m_{\mu_1}, n - k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} k \\ m_0, m_1, \dots, m_{\mu_1} \end{bmatrix}_q.$$

By Proposition 5,

$$A_\mu(q) = q^{\sum_{i=1}^k i\mu_i} \begin{bmatrix} k \\ m_0, m_1, \dots, m_{\mu_1} \end{bmatrix}_q,$$

and the result follows. \square

Theorem 2. (The Refined Truncated Anti-Lecture Hall Theorem).

$$(21) \quad A_{n,k}(u, v, q) \triangleq \sum_{\lambda \in A_{n,k}} q^{|\lambda|} u^{|\lambda|} v^{o(\lambda)} = \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-uvq^{n-k+1}; q)_k}{(u^2q^{2(n-k+1)}; q)_k}.$$

Proof. Applying first Proposition 8, then the definition of $A_k(u, v, q)$, and finally the Refined Anti-Lecture Hall Theorem 4, we get

$$\begin{aligned} A_{n,k}(u, v, q) &= \sum_{\mu \in P_k} u^{|\mu|} v^{o(\mu)} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{(n-k)|\mu|} A_\mu(q) \\ &= \begin{bmatrix} n \\ k \end{bmatrix}_q A_k(uq^{n-k}, v, q) \\ &= \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-uvq^{n-k+1}; q)_k}{(u^2q^{2(n-k+1)}; q)_k}. \end{aligned}$$

\square

4. ODD/EVEN GENERATING FUNCTIONS FOR TRUNCATED OBJECTS

In this section we adapt the technique introduced in [4] to get the two-variable generating functions of the truncated objects. In [4], Bousquet-Mélou and Eriksson introduce a bijection BME: $L_{n-1} \times \mathbb{N} \rightarrow L_n$ that we recall here. For $\lambda \in L_{n-1}$ and $s \in \mathbb{N}$, $\text{BME}(\lambda, s) = \mu$, where

$$\begin{aligned} \mu_1 &\leftarrow \left\lfloor \frac{n\lambda_1}{n-1} \right\rfloor + s \\ \mu_{2\ell} &\leftarrow \lambda_{2\ell-1}, \quad 1 \leq \ell \leq n/2; \\ \mu_{2\ell+1} &\leftarrow \left\lfloor \frac{(n-2\ell)\lambda_{2\ell+1}}{n-2\ell-1} \right\rfloor + \left\lfloor \frac{(n-2\ell)\lambda_{2\ell-1}}{n-2\ell+1} \right\rfloor - \lambda_{2\ell}, \quad 1 \leq \ell \leq (n-1)/2. \end{aligned}$$

It is proved in [4] that $\mu \in L_n$, that $|\mu_e| = |\lambda_o|$ and $|\mu_o| = 2|\lambda_o| - |\lambda_e| + s$ and that BME is a bijection. This implies that :

$$L_n(x, y) \triangleq \sum_{\mu \in L_n} x^{|\mu_o|} y^{|\mu_e|} = \sum_{\lambda \in L_{n-1}} \sum_{s=0}^{\infty} x^{2|\lambda_o| - |\lambda_e| + s} y^{|\lambda_o|} = \frac{1}{1-x} \sum_{\lambda \in L_{n-1}} (x^2y)^{|\lambda_o|} (1/x)^{|\lambda_e|} = \frac{L_{n-1}(x^2y, x^{-1})}{1-x},$$

giving the recurrence

$$(22) \quad L_n(x, y) = \frac{L_{n-1}(x^2 y, x^{-1})}{1-x}.$$

As $L_0(x, y) = 1$, this gives $L_n = 1/(x; xy)_n$, the Odd/Even Lecture Hall Theorem (2).

4.1. Truncated Lecture Hall Partitions. Recall that $\bar{L}_{n,k}$ for $k \leq n$ is the set of partitions in $L_{n,k}$ with k positive parts. Let

$$L_{n,k}(x, y) \triangleq \sum_{\lambda \in L_{n,k}} x^{|\lambda_o|} y^{|\lambda_e|}; \quad \bar{L}_{n,k}(x, y) \triangleq \sum_{\lambda \in \bar{L}_{n,k}} x^{|\lambda_o|} y^{|\lambda_e|}.$$

Note that $\bar{L}_{n,k}(x, y) = L_{n,k}(x, y) - L_{n,k-1}(x, y)$.

For $n \geq k \geq 1$, define a variation on the function BME,

$$\text{BME}_{n,k} : \bar{L}_{n-1,k-1} \times \mathbb{N} \rightarrow \bar{L}_{n,k},$$

by $\text{BME}_{n,k}(\lambda, s) = (\mu_1, \mu_2, \dots, \mu_k)$, where

$$\begin{aligned} \mu_1 &\leftarrow \left\lfloor \frac{n\lambda_1}{n-1} \right\rfloor + s \\ \mu_{2\ell} &\leftarrow \lambda_{2\ell-1}, \quad 1 \leq \ell \leq k/2; \\ \mu_{2\ell+1} &\leftarrow \left\lfloor \frac{(n-2\ell)\lambda_{2\ell+1}}{n-2\ell-1} \right\rfloor + \left\lfloor \frac{(n-2\ell)\lambda_{2\ell-1}}{n-2\ell+1} \right\rfloor - \lambda_{2\ell}, \quad 1 \leq \ell \leq (k-1)/2; \end{aligned}$$

and, if k is odd with $k = 2t + 1$,

$$\mu_{2t+1} \leftarrow \left\lfloor \frac{(n-2t)\lambda_{2t-1}}{n-2t+1} \right\rfloor - \lambda_{2t} + 1.$$

As was true for the function BME, $\text{BME}_{n,k}$ is one-to-one. It is straightforward to check when k is odd that the image of $\bar{L}_{n-1,k-1} \times \mathbb{N}$ under $\text{BME}_{n,k}$ is $\bar{L}_{n,k}$, and that $|\mu_e| = |\lambda_o|$ and $|\mu_o| = 2|\lambda_o| - |\lambda_e| + s + 1$. This gives a recurrence for $\bar{L}_{n,k}(x, y)$ when k is odd.

Proposition 9.

$$(23) \quad \bar{L}_{n,2k+1}(x, y) = \frac{x}{1-x} \bar{L}_{n-1,2k}(x^2 y, x^{-1}).$$

Getting a recurrence for the even case will be harder.

For $i > 0$, let $\bar{L}_{n,k,i}$ be the set of partitions in $\bar{L}_{n,k}$ whose k^{th} part is equal to i . Let $\bar{L}_{n,k,0} = \bar{L}_{n,k-1}$. We can check that when k is even, $\text{BME}_{n,k}$ gives a bijection between $\bar{L}_{n-1,k-1,i} \times \mathbb{N}$ and $\bar{L}_{n,k,i}$. Furthermore, when $\lambda \in \bar{L}_{n-1,2k-1,i}$ and $\mu = \text{BME}_{n,2k}(\lambda, s)$, then $|\mu_e| = |\lambda_o|$ and

$$|\mu_o| = 2|\lambda_o| - |\lambda_e| + s - \left\lfloor \frac{(n-2k)i}{n-2k+1} \right\rfloor = 2|\lambda_o| - |\lambda_e| + s + l - i,$$

with $l = \lfloor i/(n-2k+1) \rfloor$. This implies that :

Proposition 10. For $i > 0$,

$$(24) \quad \bar{L}_{n,2k,i}(x, y) = \frac{x^{l-i}}{1-x} \bar{L}_{n-1,2k-1,i}(x^2 y, 1/x),$$

with $l = \lfloor i/(n-2k+1) \rfloor$.

Now we will decompose the set $\bar{L}_{n,2k+1,i}$. In what follows, in order to compress the notation, we will sometimes write a function $f(x, y)$ as f , when the arguments are (x, y) .

Proposition 11. *Let $i = l(n - 2k) + r$, where $1 \leq r \leq n - 2k$.*

If $r \geq 2$, then

$$\bar{L}_{n,2k+1,i} = x\bar{L}_{n,2k+1,i-1} - \frac{x}{1-x}\bar{L}_{n-1,2k-1,i+l}(x^2y, 1/x).$$

Otherwise, $r = 1$ and

$$\bar{L}_{n,2k+1,i} = x\bar{L}_{n,2k+1,i-1} - \frac{x}{1-x}\bar{L}_{n-1,2k-1,i+l-1}(x^2y, 1/x) - \frac{x}{1-x}\bar{L}_{n-1,2k-1,i+l}(x^2y, 1/x).$$

Proof. For $i = l(n - 2k) + r$,

$$(25) \quad \bar{L}_{n,2k+1,i} = \begin{cases} x\bar{L}_{n,2k+1,i-1} - x^i\bar{L}_{n,2k,i+l} & \text{if } r \geq 2 \text{ or } i = 1 \\ x\bar{L}_{n,2k+1,i-1} - x^i\bar{L}_{n,2k,i+l-1} - x^i\bar{L}_{n,2k,i+l} & \text{if } r = 1 \text{ and } i \neq 1 \end{cases}$$

which can be seen as follows. Adding one to the $(2k + 1)^{\text{th}}$ part of a partition in $\bar{L}_{n,2k+1,i-1}$ gives a partition in $\bar{L}_{n,2k+1,i}$, except in the following two cases (i) if the $2k^{\text{th}}$ part was equal to $i + l$ and (ii) if $r = 1$ and the $2k^{\text{th}}$ part was equal to $i + l - 1$.

Now using Proposition 10, for $i = l(n - 2k) + r$ with $1 \leq r \leq n - 2k$,

$$\bar{L}_{n,2k,i+l} = \frac{x^{1-i}}{1-x}\bar{L}_{n-1,2k-1,i+l}(x^2y, 1/x),$$

and for $i = l(n - 2k) + 1$,

$$\bar{L}_{n,2k,i+l-1} = \frac{x^{1-i}}{1-x}\bar{L}_{n-1,2k-1,i+l-1}(x^2y, 1/x).$$

Combining this with the recurrence (25), we get the result. \square

We will combine the previous results to get a recurrence for $\bar{L}_{n,2k}(x, y)$.

Proposition 12.

$$(26) \quad \bar{L}_{n,2k} = \begin{cases} \bar{L}_{n-1,2k}(x^2y, 1/x) + \frac{1}{1-x}\bar{L}_{n-1,2k-1}(x^2y, 1/x) & \text{if } n > 2k \\ \frac{1}{1-x}\bar{L}_{n-1,2k-1}(x^2y, 1/x) & \text{if } n = 2k \end{cases}$$

Proof. The $n = 2k$ case is (22). Assume $n > 2k$. From Proposition 11, for $i = l(n - 2k) + r$ with $2 \leq r \leq n - 2k$, we have

$$\bar{L}_{n,2k+1,i} = x\bar{L}_{n,2k+1,i-1} - \frac{x}{1-x}\bar{L}_{n-1,2k-1,i+l}(x^2y, 1/x).$$

and for $i = l(n - 2k) + 1$,

$$\bar{L}_{n,2k+1,i} = x\bar{L}_{n,2k+1,i-1} - \frac{x}{1-x}\bar{L}_{n-1,2k-1,i+l-1}(x^2y, 1/x) - \frac{x}{1-x}\bar{L}_{n-1,2k-1,i+l}(x^2y, 1/x).$$

We sum

$$\bar{L}_{n,2k+1} = \sum_{i=1}^{\infty} \bar{L}_{n,2k+1,i} = x\bar{L}_{n,2k} + x \sum_{i=1}^{\infty} \bar{L}_{n,2k+1,i} - \frac{x}{1-x} \sum_{i=1}^{\infty} \bar{L}_{n-1,2k-1,i}(x^2y, 1/x).$$

Therefore

$$\bar{L}_{n,2k+1} = x\bar{L}_{n,2k} + x\bar{L}_{n,2k+1} - \frac{x}{1-x}\bar{L}_{n-1,2k-1}(x^2y, 1/x).$$

Using Proposition 9, that

$$(1-x)\bar{L}_{n,2k+1} = x\bar{L}_{n-1,2k}(x^2y, 1/x),$$

we get the result. \square

Now we can compute the generating function we seek.

Theorem 3. For $n \geq k \geq 0$,

$$\bar{L}_{n,k}(x, y) = \frac{(x^{\lfloor k/2 \rfloor + 1} y^{\lfloor k/2 \rfloor})^{\lfloor k/2 \rfloor} \begin{bmatrix} n - \lfloor k/2 \rfloor \\ \lfloor k/2 \rfloor \end{bmatrix}_{xy}}{(x; xy)_{\lfloor k/2 \rfloor} (x^n y^{n-1}; (xy)^{-1})_{\lfloor k/2 \rfloor}}.$$

Proof. For $k = 0$ the result holds, as $\bar{L}_{n,0} = 1$. Let $n \geq m > 0$ and assume inductively that the theorem is true for $(n, m-1)$, $(n-1, m-1)$, and, if $n > m$, for $(n-1, m)$.

For the odd case $m = 2k+1$, by Proposition 9,

$$\bar{L}_{n,2k+1} = \frac{x}{1-x} \bar{L}_{n-1,2k}(x^2y, x^{-1}).$$

By the induction hypothesis,

$$\bar{L}_{n-1,2k} = \frac{(x^{k+1}y^k)^k \begin{bmatrix} n-1-k \\ k \end{bmatrix}_{xy}}{(x; xy)_k (x^{n-1}y^{n-2}; (xy)^{-1})_k},$$

and substituting in the previous equation gives the result.

For the even case $m = 2k$, we have by Proposition 12,

$$(27) \quad \bar{L}_{n,2k} = \begin{cases} \bar{L}_{n-1,2k}(x^2y, 1/x) + \frac{1}{1-x} \bar{L}_{n-1,2k-1}(x^2y, 1/x) & \text{if } n > 2k \\ \frac{1}{1-x} \bar{L}_{n-1,2k-1}(x^2y, 1/x) & \text{if } n = 2k. \end{cases}$$

By the induction hypothesis,

$$\bar{L}_{n-1,2k-1} = \frac{(x^k y^{k-1})^k \begin{bmatrix} n-1-k \\ k-1 \end{bmatrix}_{xy}}{(x; xy)_k (x^{n-1}y^{n-2}; (xy)^{-1})_{k-1}}.$$

When $n = 2k$, substituting this in the previous equation gives the result. When $n > 2k$, by the induction hypothesis, we also have

$$\bar{L}_{n-1,2k} = \frac{(x^{k+1}y^k)^k \begin{bmatrix} n-1-k \\ k \end{bmatrix}_{xy}}{(x; xy)_k (x^{n-1}y^{n-2}; (xy)^{-1})_k}.$$

Then let

$$\bar{L}_{n,2k} = \bar{L}_{n-1,2k}(x^2y, 1/x) + \frac{1}{1-x} \bar{L}_{n-1,2k-1}(x^2y, 1/x) = \frac{N_{n,k}}{D_{n,k}}$$

with

$$D_{n,k} = (x; xy)_{k+1} (x^n y^{n-1}; (xy)^{-1})_k.$$

Then

$$N_{n,k} = (x^{k+2}y^{k+1})^k \begin{bmatrix} n-1-k \\ k \end{bmatrix}_{xy} (1-x) + (x^{k+1}y^k)^k \begin{bmatrix} n-1-k \\ k-1 \end{bmatrix}_{xy} (1-x^{n-k+1}y^{n-k}).$$

We use these identities to simplify the numerator.

$$(28) \quad \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q; \quad \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q$$

So,

$$\begin{aligned} N_{n,k} &= (x^{k+1}y^k)^k \left(\begin{bmatrix} n-1-k \\ k-1 \end{bmatrix}_{xy} + x^k y^k \begin{bmatrix} n-1-k \\ k \end{bmatrix}_{xy} \right) \\ &\quad - (x^{k+1}y^k)^k x^{k+1} y^k \left(\begin{bmatrix} n-1-k \\ k \end{bmatrix}_{xy} + x^{n-2k} y^{n-2k} \begin{bmatrix} n-1-k \\ k-1 \end{bmatrix}_{xy} \right) \\ &= (x^{k+1}y^k)^k (1 - x^{k+1}y^k) \begin{bmatrix} n-k \\ k \end{bmatrix}_{xy}. \end{aligned}$$

□

The Odd/Even Truncated Lecture Hall Theorem (8) is an immediate consequence of Theorem 3.

4.2. Truncated Anti-Lecture Hall Compositions. Recall that for $n \geq k-1$, $A_{n,k}$ is the set of compositions such that :

$$\frac{\lambda_1}{n-k+1} \geq \dots \geq \frac{\lambda_k}{n} \geq 0.$$

Our goal is to compute the generating function

$$A_{n,k}(x, y) = \sum_{\lambda \in A_{n,k}} x^{|\lambda_o|} y^{|\lambda_e|}.$$

We will again adapt the mapping of [4]. For $n \geq k \geq 1$, define

$$\Theta_{n,k} : A_{n,k-1} \times \mathbb{N} \rightarrow A_{n,k},$$

by $\Theta_{n,k}(\lambda, s) = (\mu_1, \mu_2, \dots, \mu_k) = \mu$, where

$$\begin{aligned} \mu_1 &\leftarrow \left\lfloor \frac{(n-k+1)\lambda_1}{n-k+2} \right\rfloor + s \\ \mu_{2\ell} &\leftarrow \lambda_{2\ell-1}, \quad 1 \leq \ell \leq k/2; \\ \mu_{2\ell+1} &\leftarrow \left\lfloor \frac{(n+2\ell-k+1)\lambda_{2\ell+1}}{n+2\ell-k+2} \right\rfloor + \left\lfloor \frac{(n+2\ell-k+1)\lambda_{2\ell-1}}{n+2\ell-k} \right\rfloor - \lambda_{2\ell}, \quad 1 \leq \ell \leq (k-1)/2. \end{aligned}$$

Similar to the mapping $\text{BME}_{n,k}$ of the previous section, it can be checked that $\Theta_{n,k}$ is one-to-one and $\Theta_{n,k}(A_{n,k-1} \times \mathbb{N}) \subseteq A_{n,k}$. Furthermore, when k is odd, $\Theta_{n,k}$ is onto $A_{n,k}$ and if $\mu = \Theta_{n,k}(\lambda, s)$, then $|\mu_e| = |\lambda_o|$ and $|\mu_o| = 2|\lambda_o| - |\lambda_e| + s$. This implies :

Proposition 13. For $n \geq 2k$,

$$(29) \quad A_{n,2k+1}(x, y) = \frac{1}{1-x} A_{n,2k}(x^2 y, x^{-1}).$$

Again the recurrence for even k is more difficult. For $i \geq 0$, let $A_{n,k,i}$ be the set of compositions in $A_{n,k}$ with k^{th} part equal to i . It can be checked that $\Theta_{n,2k}$ maps $A_{n,2k-1,i} \times \mathbb{N}$ bijectively to $A_{n,2k,i}$. If $\mu = \Theta_{n,2k}(\lambda, s)$ it is not too hard to see that $\mu \in A_{n,2k+1}$, $|\mu_e| = |\lambda_o|$ and $|\mu_o| = 2|\lambda_o| - |\lambda_e| + s - i - l$ with $l = \lfloor i/n \rfloor$. This implies :

Proposition 14. For $n \geq 2k$,

$$(30) \quad A_{n,2k,i}(x,y) = \frac{x^{-i-l}}{1-x} A_{n,2k-1,i}(x^2y, x^{-1}).$$

with $l = \lfloor i/n \rfloor$.

Proposition 15. For $n \geq 2k$,

$$(31) \quad (1-x)A_{n,2k+1} = A_{n-1,2k} - \frac{x}{1-x} A_{n-1,2k-1}(x^2y, 1/x),$$

Proof. Let $i = ln + r$ with $0 \leq r \leq n-1$. It follows from the definitions that $A_{n,k,0} = A_{n-1,k-1}$. Consider the case $r = 0$ and $l > 0$. If $\lambda \in A_{n,2k+1,i-1}$, then $\lambda_{2k}/(n-1) \geq (i-1)/n$, so $\lambda_{2k}/(n-1) \geq i/n$. Therefore adding 1 to the last part of λ gives a composition in $A_{n,2k+1,i}$. Thus, $A_{n,2k+1,i} = xA_{n,2k+1,i-1}$. For $r > 0$, we have

$$A_{n,2k+1,i} = xA_{n,2k+1,i-1} - x^i A_{n-1,2k,i-l-1}.$$

To see this, adding one to the $(2k+1)^{th}$ part of a composition in $A_{n,2k+1,i-1}$ gives a composition in $A_{n,2k+1,i}$ unless the $(2k)^{th}$ part was equal to $i-l-1$.

By Proposition 14, we get that

$$A_{n-1,2k,i-l-1} = \frac{x^{1-i}}{1-x} A_{n-1,2k-1,i-l-1}(x^2y, 1/x).$$

We apply this and get if $r \geq 1$,

$$A_{n,2k+1,i} = xA_{n,2k+1,i-1} - \frac{x}{1-x} A_{n-1,2k-1,i-l-1}(x^2y, 1/x);$$

if $r = 0$ and $l > 0$

$$A_{n,2k+1,i} = xA_{n,2k+1,i-1};$$

otherwise, $i = 0$ and $A_{n,k,0} = A_{n-1,k-1}$. Now we sum on i :

$$\sum_{i=1}^{\infty} A_{n,2k+1,i} = x \sum_{i=1}^{\infty} A_{n,2k+1,i-1} - \frac{x}{1-x} \sum_{r=1}^{n-1} \sum_{l=0}^{\infty} A_{n-1,2k-1,ln+r-l-1}(x^2y, 1/x).$$

This gives

$$A_{n,2k+1} - A_{n-1,2k} = xA_{n,2k+1} - \frac{x}{1-x} A_{n-1,2k-1}(x^2y, 1/x),$$

and therefore the result. \square

Now we get a recurrence for the number of even parts.

Proposition 16.

$$A_{n,2k} = \begin{cases} \frac{1}{1-x} A_{2k-1,2k-1}(y, x) & \text{if } n = 2k-1 \\ A_{n-1,2k}(1/y, xy^2) + \frac{y}{1-y} A_{n-1,2k-1} & \text{if } n \geq 2k. \end{cases}$$

Proof. For the case $n = 2k-1$. These are the objects such that

$$\frac{\lambda_1}{0} \geq \frac{\lambda_2}{1} \geq \dots \geq \frac{\lambda_{2k}}{2k-1} \geq 0.$$

As the division by 0 makes the first inequality always valid if $\lambda_1 \geq 0$, then

$$A_{2k-1,k}(x, y) = \frac{1}{1-x} A_{2k-1,2k-1}(y, x).$$

For $n \geq 2k$, we use the previous Proposition 15. From Proposition 13, we know that $(1-x)A_{n,2k+1} = A_{n,2k}(x^2y, 1/x)$. Therefore,

$$A_{n,2k}(x^2y, 1/x) = A_{n-1,2k} - \frac{x}{1-x}A_{n-1,2k-1}(x^2y, 1/x).$$

We make the substitutions $x = 1/y$ and $y = xy^2$ and get the result. \square

Remark. Note that, using Proposition 16, we get :

$$A_{2k,2k} = \frac{1}{1-y}(A_{2k-1,2k-1} - yA_{2k-1,2k-1}(xy^2, 1/y)).$$

Theorem 4. (The Odd/Even Truncated Anti-Lecture Hall Theorem) *For $n \geq k-1$ and $n, k \geq 0$,*

$$A_{n,k}(x, y) = \frac{\left[\begin{matrix} n \\ \lfloor k/2 \rfloor \end{matrix} \right]_{xy}}{(x; xy)_{\lfloor k/2 \rfloor} (x^{n-k+1}y^{n-k+2}; xy)_{\lfloor k/2 \rfloor}}.$$

Proof. We will prove the result by induction. We know that $A_{n,0} = 1$. Let $n+1 \geq m > 0$ and assume inductively that the theorem is true for $(n, m-1)$, $(n-1, m-1)$, and, if $n \geq m$, for $(n-1, m)$. For the odd case $m = 2k+1$, by the induction hypothesis,

$$A_{n,2k} = \frac{\left[\begin{matrix} n \\ k \end{matrix} \right]_{xy}}{(x; xy)_k (x^{n-2k+1}y^{n-2k+2}; xy)_k}.$$

Apply Proposition 13,

$$A_{n,2k+1} = \frac{1}{1-x}A_{n,2k}(x^2y, 1/x),$$

and get the result.

For the even case $m = 2k$, if $n = 2k-1$, by the induction hypothesis, using Proposition 16 :

$$A_{2k-1,2k}(x, y) = \frac{1}{1-x}A_{2k-1,2k-1}(y, x) = \frac{\left[\begin{matrix} 2k-1 \\ k-1 \end{matrix} \right]_{xy}}{(y; xy)_k (x; xy)_k}.$$

Now for $n \geq 2k$, by the induction hypothesis, using Proposition 16 :

$$A_{n-1,2k}(1/y, xy^2) = \frac{\left[\begin{matrix} n-1 \\ k \end{matrix} \right]_{xy}}{(1/y; xy)_k (x^{n-2k+1}y^{n-2k+2}; xy)_k}; \quad A_{n-1,2k-1} = \frac{\left[\begin{matrix} n-1 \\ k-1 \end{matrix} \right]_{xy}}{(x; xy)_k (x^{n-2k+1}y^{n-2k+2}; xy)_{k-1}}.$$

$$A_{n-1,2k}(1/y, xy^2) - \frac{1/y}{1-1/y}A_{n-1,2k-1} = \frac{N_{n,k}}{D_{n,k}}$$

with $D_{n,k} = (1/y; xy)_{k+1} (x^{n-2k+1} y^{n-2k+2}; xy)_k$. Then

$$\begin{aligned} N_{n,k} &= \begin{bmatrix} n-1 \\ k \end{bmatrix}_{xy} (1 - x^k y^{k-1}) - 1/y \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{xy} (1 - x^{n-k} y^{n-k+1}) \\ &= \begin{bmatrix} n-1 \\ k \end{bmatrix}_{xy} + x^{n-k} y^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{xy} - \frac{1}{y} \left(x^k y^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_{xy} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{xy} \right) \\ &= (1 - 1/y) \begin{bmatrix} n \\ k \end{bmatrix}_{xy}, \end{aligned}$$

where the last step follows from (28). \square

5. COMBINATORIAL CHARACTERIZATIONS AND REFINEMENTS

5.1. Characterization of Truncated Lecture Hall Partitions. In this section we characterize truncated lecture hall partitions in terms of partitions into odd parts with certain restrictions. We first need a few steps.

Proposition 17. *Given n, t, j, l , the weight generating function for the number of partitions into odd parts in $\{1, 3, \dots, 2n-1\}$ in which exactly t of the parts can be chosen from the set $\{2j+1, 2j+3, \dots, 2l-1\}$ is*

$$\frac{q^{t(2j+1)}}{(q; q^2)_j (q^{2n-1}; q^{-2})_{n-l}} \begin{bmatrix} l-j-1+t \\ t \end{bmatrix}_{q^2}.$$

Proof. The fraction, ignoring the denominator, is the generating function for the partitions into the odd parts which are not restricted. Now count partitions into exactly t parts from the set $\{2j+1, 2j+3, \dots, 2l-1\}$. Take off $2j+1$ from each part. This is counted by $q^{t(2j+1)}$. We are left with a partition into even parts in a $t \times 2(l-j-1)$ box. \square

Corollary 1. *The generating function for the number of partitions into odd parts in which exactly t of the parts are greater than or equal to $2j+1$ is*

$$\frac{q^{t(2j+1)}}{(q; q^2)_j (q^2; q^2)_t}.$$

Applying Proposition 17 gives the following.

Proposition 18. *Let $R_{n,k}$ be the set of partitions into odd parts less than or equal to $2n-1$ where at most $\lfloor k/2 \rfloor$ parts can be chosen from the set $\{2\lfloor k/2 \rfloor + 1, 2\lfloor k/2 \rfloor + 3, \dots, 2(n - \lfloor k/2 \rfloor) - 1\}$.*

$$(32) \quad R_{n,k}(q) \triangleq \sum_{\lambda \in R_{n,k}} q^{|\lambda|} = \frac{\sum_{i=0}^{\lfloor k/2 \rfloor} q^{i(2\lfloor k/2 \rfloor + 1)} \begin{bmatrix} n-k-1+i \\ i \end{bmatrix}_{q^2}}{(q; q^2)_{\lfloor k/2 \rfloor} (q^{2n-1}; q^{-2})_{\lfloor k/2 \rfloor}}.$$

\square

Let $\bar{R}_{n,k} = R_{n,k} - R_{n,k-1}$ and let $\bar{R}_{n,k}(q)$ be the corresponding generating function. Recall that $\bar{L}_{n,k}$ is the set of lecture hall partitions in $L_{n,k}$ with k positive parts.

Proposition 19. For $k > 0$,

$$\bar{L}_{n,k}(q) \triangleq \sum_{\lambda \in \bar{L}_{n,k}} q^{|\lambda|} = \frac{q^{\binom{k+1}{2}} \left[\begin{matrix} n - \lfloor k/2 \rfloor \\ \lfloor k/2 \rfloor \end{matrix} \right]_{q^2}}{(q; q^2)_{\lfloor k/2 \rfloor} (q^{2n-1}; q^{-2})_{\lfloor k/2 \rfloor}}$$

Proof. Let $x = y = q$ in Theorem 3. □

Theorem 5. (Characterization of Truncated Lecture Hall Partitions) *The number of truncated lecture partitions of N in $L_{n,k}$ is equal to the number of partitions of N into odd parts less than $2n$, with the following constraint on the parts: at most $\lfloor k/2 \rfloor$ parts can be chosen from the set*

$$\{2\lfloor k/2 \rfloor + 1, 2\lfloor k/2 \rfloor + 3, \dots, 2(n - \lfloor k/2 \rfloor) - 1\}.$$

Proof. We must show that $L_{n,k}(q) \triangleq \sum_{m=0}^k \bar{L}_{n,m} = R_{n,k}(q)$. Since $L_{n,0}(q) = R_{n,0}(q) = 1$, it suffices to show that for $k > 0$, $\bar{L}_{n,k}(q) = \bar{R}_{n,k}(q)$. We use $\bar{L}_{n,k}(q)$ from Proposition 19, and now compute $\bar{R}_{n,k}(q) = R_{n,k}(q) - R_{n,k-1}(q)$.

For $k > 0$, first look at the case $k = 2s$. Using Proposition 18,

$$\bar{R}_{n,2s}(q) = \frac{\sum_{i=0}^s q^{i(2s+1)} \left[\begin{matrix} n - 2s - 1 + i \\ i \end{matrix} \right]_{q^2} - (1 - q^{2(n-s)+1}) \sum_{i=0}^{s-1} q^{i(2s+1)} \left[\begin{matrix} n - 2s + i \\ i \end{matrix} \right]_{q^2}}{(q; q^2)_s (q^{2n-1}; q^{-2})_s}.$$

Since the denominators of $\bar{R}_{n,2s}(q)$ and $\bar{L}_{n,2s}(q)$ agree, we focus on the numerator $E(n, s)$:

$$E(n, s) = \sum_{i=0}^s q^{i(2s+1)} \left[\begin{matrix} n - 2s - 1 + i \\ i \end{matrix} \right]_{q^2} - (1 - q^{2(n-s)+1}) \sum_{i=0}^{s-1} q^{i(2s+1)} \left[\begin{matrix} n - 2s + i \\ i \end{matrix} \right]_{q^2}.$$

Therefore, using the first identity of (28),

$$\begin{aligned} \sum_{i=0}^{s-1} q^{i(2s+1)} \left(\left[\begin{matrix} n - 2s - 1 + i \\ i \end{matrix} \right]_{q^2} - \left[\begin{matrix} n - 2s + i \\ i \end{matrix} \right]_{q^2} \right) &= -q^{2(n-2s)} \sum_{i=0}^{s-1} q^{i(2s+1)} \left[\begin{matrix} n - 2s + i - 1 \\ i - 1 \end{matrix} \right]_{q^2} \\ &= -q^{2(n-s)+1} \sum_{i=0}^{s-2} q^{i(2s+1)} \left[\begin{matrix} n - 2s + i \\ i \end{matrix} \right]_{q^2}. \end{aligned}$$

We then get :

$$\begin{aligned} E(n, s) &= q^{s(2s+1)} \left[\begin{matrix} n - s - 1 \\ s \end{matrix} \right]_{q^2} + q^{2(n-s)+1} q^{(s-1)(2s+1)} \left[\begin{matrix} n - s - 1 \\ s - 1 \end{matrix} \right]_{q^2} \\ &= q^{s(2s+1)} \left[\begin{matrix} n - s \\ s \end{matrix} \right]_{q^2} = q^{\binom{k+1}{2}} \left[\begin{matrix} n - \lfloor k/2 \rfloor \\ \lfloor k/2 \rfloor \end{matrix} \right]_{q^2}. \end{aligned}$$

Now when $k = 2s + 1$, applying Proposition 18 gives

$$\bar{R}_{n,2s+1}(q) = \frac{\sum_{i=0}^s q^{i(2s+3)} \left[\begin{matrix} n - 2s - 2 + i \\ i \end{matrix} \right]_{q^2} - (1 - q^{2s+1}) \sum_{i=0}^s q^{i(2s+1)} \left[\begin{matrix} n - 2s - 1 + i \\ i \end{matrix} \right]_{q^2}}{(q; q^2)_{s+1} (q^{2n-1}; q^{-2})_s}.$$

Again we can focus on the numerator $O(n, s)$:

$$O(n, s) = \sum_{i=0}^s q^{i(2s+3)} \begin{bmatrix} n-2s-2+i \\ i \end{bmatrix}_{q^2} - (1-q^{2s+1}) \sum_{i=0}^s q^{i(2s+1)} \begin{bmatrix} n-2s-1+i \\ i \end{bmatrix}_{q^2}.$$

Then, using the second identity of (28),

$$\begin{aligned} \sum_{i=0}^s q^{i(2s+1)} \left(q^{2i} \begin{bmatrix} n-2s-2+i \\ i \end{bmatrix}_{q^2} - \begin{bmatrix} n-2s-1+i \\ i \end{bmatrix}_{q^2} \right) &= - \sum_{i=1}^s q^{i(2s+1)} \begin{bmatrix} n-2s-2+i \\ i-1 \end{bmatrix}_{q^2} \\ &= -q^{2s+1} \sum_{i=0}^{s-1} q^{i(2s+1)} \begin{bmatrix} n-2s-1+i \\ i \end{bmatrix}_{q^2}. \end{aligned}$$

Applying this to $O(n, s)$ gives the result :

$$O(n, s) = q^{2s+1} q^{s(2s+1)} \begin{bmatrix} n-s-1 \\ s \end{bmatrix}_{q^2} = q^{\binom{k+1}{2}} \begin{bmatrix} n - \lfloor k/2 \rfloor \\ \lfloor k/2 \rfloor \end{bmatrix}_{q^2}.$$

□

5.2. Finitizations of Refinements of Euler's Theorem. Euler's Theorem says that the number of partitions of N into distinct parts is equal to the number of partitions of N into odd parts. (A straightforward proof shows $(-q; q)_\infty = 1/(q; q^2)_\infty$.) The Lecture Hall Theorem is a finitization of that theorem : the number of partitions of N in L_n is equal to the number of partitions of N into odd parts less than $2n$. We will show how the truncated lecture hall results give finitizations of refinements of Euler's theorem that can be easily from (slight modifications of) Sylvester's bijection.

Finitization 1.

$$(33) \quad \sum_{m=0}^k q^{\binom{m+1}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(q^{n-m+1}; q)_m}{(q^{2n-m+1}; q)_m} = \sum_{i=0}^{\lfloor k/2 \rfloor} \frac{q^{i(2\lfloor k/2 \rfloor + 1)} \begin{bmatrix} n-k-1+i \\ i \end{bmatrix}_{q^2}}{(q; q^2)_{\lfloor k/2 \rfloor} (q^{2n-1}; q^{-2})_{\lfloor k/2 \rfloor}}.$$

This is Theorem 5, which states that $L_{n,k}(q) = R_{n,k}(q)$, where $L_{n,k}(q)$ and $R_{n,k}(q)$ are given by Theorem 19 and Proposition 18, respectively.

Taking limits as $n \rightarrow \infty$ in (33) gives

$$\sum_{m=0}^k \frac{q^{\binom{m+1}{2}}}{(q; q)_m} = \sum_{i=0}^{\lfloor k/2 \rfloor} \frac{q^{i(2\lfloor k/2 \rfloor + 1)}}{(q; q^2)_{\lfloor k/2 \rfloor} (q^2; q^2)_i}.$$

Note that $q^{\binom{m+1}{2}}/(q; q)_m$ is the generating function for partitions into m distinct parts. Applying Corollary 1 to the right hand side, then, gives the following refinement of Euler's Theorem.

Refinement 1: The number of partitions of N into at most k distinct parts is equal to the number of partitions of N into odd parts such that at most $\lfloor k/2 \rfloor$ of the parts are greater than or equal to $2\lfloor k/2 \rfloor + 1$.

Theorem 5 itself is a further refinement.

Finitization 2.

$$(34) \quad q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(q^{n-k+1}; q)_k}{(q^{2n-k+1}; q)_k} = \frac{q^{\binom{k+1}{2}} \begin{bmatrix} n - \lceil k/2 \rceil \\ \lfloor k/2 \rfloor \end{bmatrix}_{q^2}}{(q; q^2)_{\lceil k/2 \rceil} (q^{2n-1}; q^{-2})_{\lfloor k/2 \rfloor}}.$$

This is Proposition 19 combined with eq. (20), setting $u = v = 1$ in eq. (20).

Letting $n \rightarrow \infty$ in (34) gives

$$(35) \quad \frac{q^{\binom{k+1}{2}}}{(q; q^2)_k} = \frac{q^{\binom{k+1}{2}}}{(q; q^2)_{\lceil k/2 \rceil} (q^2; q^{-2})_{\lfloor k/2 \rfloor}}.$$

On the right-hand side of (35), if we use

$$q^{\binom{k+1}{2}} = q^{\lceil k/2 \rceil - \lfloor k/2 \rfloor} q^{\lfloor k/2 \rfloor (2\lceil k/2 \rceil + 1)}$$

and apply Corollary 1, (35) can be read as the following refinement of Euler's theorem.

Refinement 2: The number of partitions of N into exactly k distinct parts is equal to the number of partitions of $N - (\lceil k/2 \rceil - \lfloor k/2 \rfloor)$ into odd parts such that exactly $\lfloor k/2 \rfloor$ of the parts are greater than or equal to $2\lceil k/2 \rceil + 1$.

A combinatorial interpretation of (34) will give a further refinement of Refinement 2. To get this, note that the right-hand side of (34) can be written as

$$\left(\frac{q^{\lceil k/2 \rceil - \lfloor k/2 \rfloor}}{1 - q^{2n - 2\lfloor k/2 \rfloor + 1}} \right) \frac{q^{\lfloor k/2 \rfloor (2\lceil k/2 \rceil + 1)} \begin{bmatrix} n - \lceil k/2 \rceil \\ \lfloor k/2 \rfloor \end{bmatrix}_{q^2}}{(q; q^2)_{\lceil k/2 \rceil} (q^{2n-1}; q^{-2})_{\lfloor k/2 \rfloor - 1}}.$$

Using Proposition 17, the last quotient above is the generating function for the number of partitions into odd parts less than $2n$ with exactly $\lfloor k/2 \rfloor$ parts in the set $\{2\lceil k/2 \rceil + 1, \dots, 2n - 2\lfloor k/2 \rfloor + 1\}$. So (34) says:

Further refinement 2: The number of partitions of N in $\bar{L}_{n,k}$ is equal to the number of partitions of $N - (\lceil k/2 \rceil - \lfloor k/2 \rfloor)$ into odd parts less than $2n$ with at least $\lfloor k/2 \rfloor$ parts in $\{2\lceil k/2 \rceil + 1, \dots, 2n - 2\lfloor k/2 \rfloor + 1\}$, but at most $\lfloor k/2 \rfloor$ parts less than $2n - 2\lfloor k/2 \rfloor + 1$.

Finitization 3.

Using Theorem 3 with $x = zq$ and $y = q/z$, we get

$$\bar{L}_{n,k}(zq, q/z) = \frac{zq^{2\lceil k/2 \rceil + 1} \begin{bmatrix} n - \lceil k/2 \rceil \\ \lfloor k/2 \rfloor \end{bmatrix}_{q^2}}{(zq; q^2)_{\lceil k/2 \rceil} (zq^{2n-1}; (q^2)^{-1})_{\lfloor k/2 \rfloor}}.$$

Now let $n \rightarrow \infty$ and \mathcal{D}_k be the set of partitions into k distinct parts and get

$$\sum_{\lambda \in \mathcal{D}_{2k-1} \cup \mathcal{D}_{2k}} z^{|\lambda_o| - |\lambda_e|} q^{|\lambda|} = \frac{z^k q^{k(2k-1)}}{(zq; q^2)(q^2; q^2)_k}.$$

We say that the Durfee rectangle size of a partition λ into odd parts is k if $\lambda_k \geq 2k - 1$ and $\lambda_{k+1} \leq 2k - 1$.

Refinement 3: The number of partitions λ of N into $2k - 1$ or $2k$ distinct parts with $|\lambda_o| - |\lambda_e| = j$ is equal to the number of partitions of N into j parts and Durfee rectangle size k .

5.3. Interpretation of Anti-Lecture Hall Theorems. We consider the limiting case for truncated anti-lecture hall compositions. Note that for fixed k , as $n \rightarrow \infty$, the set $A_{n,k}$ approaches P_k , the set of partitions into k nonnegative parts. From Theorem 4 we have

$$\sum_{\lambda \in A_{n,k}} x^{|\lambda_o|} y^{|\lambda_e|} = \frac{\left[\begin{matrix} n \\ \lfloor k/2 \rfloor \end{matrix} \right]_{xy}}{(x; xy)_{\lceil k/2 \rceil} (x^{n-k+1} y^{n-k+2}; xy)_{\lfloor k/2 \rfloor}}.$$

Taking limits as $n \rightarrow \infty$ and substituing $x = zq$ and $y = q/z$, gives

$$\sum_{\lambda \in P_k} q^{|\lambda|} z^{|\lambda_o| - |\lambda_e|} = \frac{1}{(zq; q^2)_{\lceil k/2 \rceil} (q^2; q^2)_{\lfloor k/2 \rfloor}},$$

which is the well known ‘‘transpose theorem’’: The number of partitions λ of N into k nonnegative parts and $|\lambda_o| - |\lambda_e| = j$ is equal to the number of partitions of N with largest part less than or equal to k and j of the parts odd.

Similarly, consider the identity which combines Theorem 7, setting $u = v = 1$ with Theorem 4, setting $x = y = q$:

$$(36) \quad \left[\begin{matrix} n \\ k \end{matrix} \right]_q \frac{(-q^{n-k+1}; q)_k}{(q^{2(n-k+1)}; q)_k} = \frac{\left[\begin{matrix} n \\ \lfloor k/2 \rfloor \end{matrix} \right]_{q^2}}{(q; q^2)_{\lceil k/2 \rceil} (q^{2n-2k+3}; q^2)_{\lfloor k/2 \rfloor}}.$$

Taking limits as $n \rightarrow \infty$ shows that (36) can be interpreted as a finitization of the transpose theorem.

6. CONCLUSION

We hopefully have demonstrated that basic hypergeometric q -series are a good tool to give refined Lecture Hall-Type Theorem proofs. Furthermore, the BME mapping plays a significant role in the development of 2-variable generating functions for anti-Lecture Hall compositions and truncated objects. Our study of the 2-variable generating function of the truncated objects leads to the following x, y -series identity. Using Equations (2) and (8), we have

Proposition 20.

$$\sum_{m=0}^n \left(x^{\lfloor m/2 \rfloor + 1} y^{\lfloor m/2 \rfloor} \right)^{\lceil m/2 \rceil} \frac{\left[\begin{matrix} n - \lfloor m/2 \rfloor \\ \lfloor m/2 \rfloor \end{matrix} \right]_{xy}}{(x; xy)_{\lceil m/2 \rceil} (x^n y^{n-1}; (xy)^{-1})_{\lfloor m/2 \rfloor}} = \prod_{i=1}^n \frac{1}{1 - x^i y^{i-1}}.$$

As a special case, substituing $x = a$ and $y = q/a$, we get the identity :

Proposition 21.

$$\sum_{m=0}^n \left(aq^{\lfloor m/2 \rfloor} \right)^{\lceil m/2 \rceil} \frac{\left[\begin{matrix} n - \lfloor m/2 \rfloor \\ \lfloor m/2 \rfloor \end{matrix} \right]_q}{(a; q)_{\lceil m/2 \rceil} (aq^{n-1}; q^{-1})_{\lfloor m/2 \rfloor}} = 1/(a; q)_n.$$

In future work we will consider truncated versions of the (k, l) -lecture hall partitions of [5].

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Acknowledgments. The authors want to thank the Université de Versailles Saint-Quentin and in particular the laboratory PRISM for inviting Dr Savage for the month of December 2002. They also want to thank the CNRS and NSF for their joint-grant.

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