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BAER INVARIANTS IN SEMI-ABELIAN CATEGORIES II: HOMOLOGY

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ABSTRACT. This article treats the problem of deriving the reflector of a semi-abelian category $\mathcal{A}$ onto a Birkhoff subcategory $\mathcal{B}$ of $\mathcal{A}$. Basing ourselves on Carrasco, Cegarra and Grandjeán's homology theory for crossed modules, we establish a connection between our theory of Baer invariants with a generalization—to semi-abelian categories—of Barr and Beck's cotriple homology theory. This results in a semi-abelian version of Hopf's formula and the Stallings-Stammbach sequence from group homology.

Contents

1. Introduction

1.1. A semi-abelian category is a category with binary coproducts which is pointed, Barr exact and Bourn protomodular. In [18] we show the following.

1.2. THEOREM. *Let $\mathcal{A}$ be a pointed, Barr exact and Bourn protomodular category with enough projectives. Consider a short exact sequence*

$$0 \longrightarrow K \rightarrowtail A \xrightarrow{f} B \longrightarrow 0$$

in $\mathcal{A}$. If V is a Birkhoff subfunctor of $\mathcal{A}$, then an exact sequence

$$\Delta V(A) \xrightarrow{\Delta Lf} \Delta V(B) \longrightarrow \frac{K}{V_1 f} \longrightarrow U(A) \xrightarrow{Uf} U(B) \longrightarrow 0 \qquad (\mathbf{A})$$

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exists, which depends naturally on the given short exact sequence.

The aim of this article is to interpret sequence **A** as a generalization of the Stallings-Stammbach sequence from integral homology of groups [29], [30]. We do this by proving the following semi-abelian version of Hopf's formula [21]:

$$H_2(X,U)_{\mathbb{G}} \cong \Delta V(X). \qquad (\mathbf{B})$$

1.3. Let us start by explaining the right hand side of this formula, briefly recalling the main concepts from [18].

A *presentation* of an object A in a category $\mathcal{A}$ is a regular epimorphism (a coequalizer) $p : A_0 \longrightarrow A$. The category of presentations of objects of $\mathcal{A}$—a morphism $\boldsymbol{f} = (f_0, f) : p \longrightarrow q$ being a commutative square

$$\begin{array}{ccc} A_0 & \xrightarrow{f_0} & B_0 \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ A & \xrightarrow[f]{} & B \end{array}$$

—is denoted by $\mathsf{Pr}\mathcal{A}$. $\mathrm{pr} : \mathsf{Pr}\mathcal{A} \longrightarrow \mathcal{A}$ denotes the forgetful functor which maps a presentation to the object presented, sending a morphism of presentations $\boldsymbol{f} = (f', f)$ to f. Two morphisms of presentations $\boldsymbol{f}, \boldsymbol{g} : p \longrightarrow q$ are called *isomorphic*, notation $\boldsymbol{f} \simeq \boldsymbol{g}$, if $\mathrm{pr}\,\boldsymbol{f} = \mathrm{pr}\,\boldsymbol{g}$ (or $f = g$). A functor $B : \mathsf{Pr}\mathcal{A} \longrightarrow \mathcal{A}$ is called a *Baer invariant* if $\boldsymbol{f} \simeq \boldsymbol{g}$ implies that $B\boldsymbol{f} = B\boldsymbol{g}$.

If $\mathcal{A}$ has sufficiently many projective objects, then any Baer invariant $B : \mathsf{Pr}\mathcal{A} \longrightarrow \mathcal{A}$ induces a functor $\mathcal{A} \longrightarrow \mathcal{A}$; for instance the functor $\Delta V : \mathcal{A} \longrightarrow \mathcal{A}$ arises this way. Denoting $\mathcal{W}_{\mathsf{proj}}$ the full subcategory of all projective objects of $\mathcal{A}$ (and $i : \mathcal{W}_{\mathsf{proj}} \longrightarrow \mathsf{Pr}\mathcal{A}$ the canonical inclusion), we call *choice of projective presentations in* $\mathcal{A}$ any graph morphism $c : \mathcal{A} \longrightarrow \mathcal{W}_{\mathsf{proj}}$ such that $\mathrm{pr} \circ i \circ c = 1_{\mathcal{A}}$. Then, for any choice of projective presentations $c : \mathcal{A} \longrightarrow \mathcal{W}_{\mathsf{proj}}$ in $\mathcal{A}$, $B \circ i \circ c$ is a functor. Moreover, for any other choice of projective presentations $c' : \mathcal{A} \longrightarrow \mathcal{W}_{\mathsf{proj}}$, a natural isomorphism $B \circ i \circ c \Longrightarrow C \circ i \circ c'$ exists.

1.4. For the larger part of our theory of Baer invariants we need the category $\mathcal{A}$ to be *semi-abelian*. This means that $\mathcal{A}$ is pointed, Barr exact and Bourn protomodular with binary coproducts [25]. The reason we work in this context is that it is natural for the classical theorems of homological algebra—as the Snake Lemma, the 3×3 Lemma and Noether's Isomorphism Theorems—to hold: see, e.g. Bourn [10] or Borceux and Bourn [5] for a new approach to these results, on which our paper [18] strongly depends.

A category $\mathcal{A}$ is *regular* [2] when it has finite limits and coequalizers of kernel pairs (i.e. the two projections $k_0, k_1 : R[f] \longrightarrow A$ of the pullback of an arrow $f : A \longrightarrow B$ along itself), and when a pullback of a regular epimorphism along any morphism is again a regular epimorphism. In this case, every regular epimorphism is the coequalizer of its kernel pair, and every morphism $f : A \longrightarrow B$ has an *image factorization* $f = \mathsf{Im}\, f \circ p$, unique up to isomorphism, where $p : A \longrightarrow I[f]$ is regular epi and the image $\mathsf{Im}\, f : I[f]$

$\longrightarrow B$ of f is mono. Taking images is functorial. Moreover, in a regular category, regular epimorphism are stable under composition, and if a composition $f\circ g$ is regular epi, then so is f. A regular category in which every equivalence relation is a kernel pair is called *Barr exact*.

A pointed category with pullbacks $\mathcal{A}$ is *Bourn protomodular* [7] as soon as the *Split Short Five-Lemma* holds. This means that for any commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & K' & \overset{k'}{\rhd\!\!\longrightarrow} & A' & \underset{s'}{\overset{f'}{\rightleftarrows}} & B' \\ & & \downarrow u & & \downarrow v & & \downarrow w \\ 0 & \longrightarrow & K & \underset{k}{\rhd\!\!\longrightarrow} & A & \underset{s}{\overset{f}{\rightleftarrows}} & B \end{array}$$

such that f and f' are split epimorphisms (with resp. splittings s and s') and such that $k = \mathsf{Ker}\, f$ and $k' = \mathsf{Ker}\, f'$, u and w being isomorphisms implies that v is an isomorphism.

A sequence

$$K \xrightarrow{k} A \xrightarrow{f} B \qquad (\mathbf{C})$$

in a pointed category is called *short exact* if $k = \mathsf{Ker}\, f$ and $f = \mathsf{Coker}\, k$. We denote this situation

$$0 \longrightarrow K \overset{k}{\rhd\!\!\longrightarrow} A \overset{f}{\longrightarrow\!\!\rhd} B \longrightarrow 0.$$

In a pointed, regular and protomodular category the exactness of sequence $\mathbf{C}$ is equivalent to demanding that $k = \mathsf{Ker}\, f$ and f is a regular epimorphism. Thus, a pointed, regular and protomodular category has all cokernels of kernels. A sequence of morphisms

$$\cdots \longrightarrow A_{i+1} \xrightarrow{f_{i+1}} A_i \xrightarrow{f_i} A_{i-1} \longrightarrow \cdots$$

in pointed, regular and protomodular category is called *exact* if, for any i, $\mathsf{Im}\, f_{i+1} = \mathsf{Ker}\, f_i$.

1.5. Let $\mathcal{A}$ be a semi-abelian category. A normal subfunctor V of $1_{\mathcal{A}}$ (i.e. a kernel $V \Longrightarrow 1_{\mathcal{A}}$) is called *Birkhoff subfunctor of* $\mathcal{A}$ if V preserves regular epimorphisms. Recall from Janelidze and Kelly [24] that a Birkhoff subcategory of $\mathcal{A}$ is a reflective subcategory $\mathcal{B}$ of $\mathcal{A}$ which is full and closed in $\mathcal{A}$ under subobjects and quotient objects. In [18], we show that Birkhoff subfunctors correspond bijectively to the Birkhoff subcategories of $\mathcal{A}$: assuming that, for any $A \in \mathcal{A}$, the sequence

$$0 \longrightarrow V(A) \overset{\mu_A}{\rhd\!\!\longrightarrow} A \overset{\eta_A}{\longrightarrow\!\!\rhd} U(A) \longrightarrow 0$$

is exact, V is a Birkhoff subfunctor if and only if U reflects $\mathcal{A}$ onto a Birkhoff subcategory. A Birkhoff subcategory of a semi-abelian category is semi-abelian.

Any Birkhoff subfunctor V of $\mathcal{A}$ induces a functor $V_1 : \mathrm{Pr}\,\mathcal{A} \longrightarrow \mathcal{A}$, constructed as follows. Let $p : A_0 \longrightarrow A$ be a presentation in $\mathcal{A}$ and $(R[p], k_0, k_1)$ its kernel pair. Applying V, next taking the coequalizer of Vk_0 and Vk_1 and then the kernel of $\mathrm{Coeq}(Vk_0, Vk_1)$, one gets $V_1 p$.

$$\begin{array}{ccccccccc} & & & & V(R[p]) & & & & \\ & & & & \Big\downarrow\Big\downarrow {\scriptstyle Vk_0,\, Vk_1} & & & & \\ 0 & \longrightarrow & V_1 p & \rightarrowtail & V(A_0) & \xrightarrow[\mathsf{Coeq}\,(Vk_0,Vk_1)]{} & \mathrm{Coeq}[Vk_0, Vk_1] & \longrightarrow & 0 \end{array}$$

Given a Birkhoff subfunctor V of $\mathcal{A}$ and a presentation $p : A_0 \longrightarrow A$, ΔV is the functor induced by the Baer invariant $\mathsf{Pr}\mathcal{A} \longrightarrow \mathcal{A}$ which maps p to

$$\frac{K[p] \cap V(A_0)}{V_1 p}.$$

Consider the category Gp of groups and its Birkhoff subcategory $\mathsf{Ab} = \mathsf{Gp}_{\mathsf{Ab}}$ of abelian groups. The associated Birkhoff subfunctor V sends G to $[G, G]$, the commutator subgroup of G. It is indeed well known that the abelianization of a group G, i.e. the reflection of G along the inclusion $\mathsf{Ab} \longrightarrow \mathsf{Gp}$, is just $G/[G, G]$. For groups $R \triangleleft F$, $[R, F]$ denotes the subgroup of F generated by the elements $rfr^{-1}f^{-1}$, with $r \in R$ and $f \in F$. If p denotes the quotient $F \longrightarrow F/R$, then $V_1 p = [R, F]$. Now let

$$0 \longrightarrow R \rightarrowtail F \xrightarrow{p} G \longrightarrow 0$$

be a presentation of a group G by a "group of generators" F and a "group of relations" R, i.e. a short exact sequence with F a free (or, equivalently, projective) group. Then it follows that

$$\Delta V(G) = \frac{R \cap [F, F]}{[R, F]}.$$

1.6. Hopf's formula [21] is the isomorphism

$$H_2(G, \mathbb{Z}) \cong \frac{R \cap [F, F]}{[R, F]}; \qquad \mathbf{(D)}$$

here, $H_2(G, \mathbb{Z})$ is the second integral homology group of G. It is clear that the left hand side of the formula **B** should, in a way, generalize this homology group.

That such a generalization is possible comes as no surprise, as Carrasco, Cegarra and Grandjeán (in their article [16]) already prove a generalized Hopf formula in the category CM of crossed modules. Furthermore, they obtain a five term exact sequence which generalizes the Stallings-Stammbach sequence [29], [30]. Their five term exact sequence becomes a particular case of ours; see Corollary 6.10.

Since the right hand side of **B** is defined relative to a Birkhoff subcategory $\mathcal{B}$ of a semi-abelian category $\mathcal{A}$, so must be the left-hand side. We restrict ourselves to the following situation: $\mathcal{A}$ is a semi-abelian category, monadic over Set; $U : \mathcal{A} \longrightarrow \mathcal{B}$ is a reflector onto a Birkhoff subcategory $\mathcal{B}$ of $\mathcal{A}$; $\mathbb{G}$ is the comonad on $\mathcal{A}$, defined by the adjunction

$\mathsf{Set} \rightharpoonup \mathcal{A}$. In this situation, the formula **B** holds, for $p : GX \longrightarrow X$ the standard "$\mathbb{G}$-free" projective presentation of an object X.

Carrasco, Cegarra and Grandjeán define their homology of crossed modules by deriving the functor $\mathrm{ab} : \mathsf{CM} \longrightarrow \mathsf{CM}_{\mathsf{Ab}}$, which sends a crossed module (T, G, ∂) to its abelianization $(T, G, \partial)_{\mathrm{ab}}$—an object of the abelian category $\mathsf{CM}_{\mathsf{Ab}}$. More precisely, using the monadicity of the forgetful functor $\mathcal{U} : \mathsf{CM} \longrightarrow \mathsf{Set}$, they obtain a comonad $\mathbb{G}$ on CM. This, for any crossed module (T, G, ∂), yields a canonical simplicial object $\mathbb{G}(T, G, \partial)$ in CM. The n-th homology object (an abelian crossed module) of a crossed module (T, G, ∂) is then defined as $H_{n-1}C\mathbb{G}(T, G, \partial)_{\mathrm{ab}}$, the $(n-1)$-th homology object of the unnormalized chain complex associated with the simplicial object $\mathbb{G}(T, G, \partial)_{\mathrm{ab}}$. This is an application of Barr and Beck's cotriple homology theory [1], which gives a way of deriving any functor $U : \mathcal{A} \longrightarrow \mathcal{B}$ from an arbitrary category $\mathcal{A}$ equipped with a comonad $\mathbb{G}$ to an abelian category $\mathcal{B}$.

To prove our Hopf formula, we use methods similar to those of Carrasco, Cegarra and Grandjeán—in fact, our proof of Theorem 6.9 is a modification of their [16, Theorem 12]. Therefore, a semi-abelian notion of homology must be introduced. This is done in Section 6. In Sections 2 through 5 the necessary theory is developed.

1.7. Since chain complexes are crucial in any homology theory, in Section 2, we consider them in a semi-abelian context. More precisely, in categories that are pointed, regular and protomodular. A morphism in such a category is *proper* [10] when its image is a kernel. We call a chain complex *proper* whenever all its differentials are. As in the abelian case, the n-th homology object of a proper chain complex C with differentials d_n is said to be $H_nC = \mathrm{Cok}[C_{n+1} \longrightarrow K[d_n]]$. We prove that this equals the dual $K_nC = K[\mathrm{Cok}[d_{n+1}] \longrightarrow C_{n-1}]$. Moreover, any short exact sequence of proper chain complexes gives rise to a long exact sequence of homology objects.

In Section 3 we extend the homology theory of Section 2 to simplicial objects. Therefore, we consider the *Moore functor* $N : \mathcal{S}\mathcal{A} \longrightarrow \mathsf{Ch}\mathcal{A}$. Suppose that $\mathcal{A}$ is a pointed category with pullbacks. Let us write ∂_i for the face operators of a simplicial object A in $\mathcal{A}$. The *normalized chain complex* $N(A)$ of A is the chain complex with $N_0A = A_0$,

$$N_nA = \bigcap_{i=0}^{n-1} K[\partial_i : A_n \longrightarrow A_{n-1}]$$

and differentials $d_n = \partial_n \circ \bigcap_i \mathsf{Ker}\, \partial_i : N_nA \longrightarrow N_{n-1}A$, for $n \geq 1$, and $A_n = 0$, for $n < 0$. When $\mathcal{A}$ is pointed, exact and protomodular, we prove that the Moore functor maps simplicial objects to proper chain complexes. This allows us to define the *n-th homology object* of A as $H_nA = H_nN(A)$. Furthermore, we show that, if $\epsilon : A \longrightarrow A_{-1}$ is a contractible augmented simplicial object, then $H_0A = A_{-1}$ and, for $n \geq 1$, $H_nA = 0$.

Recall from [15] and [14] that a finitely complete category is called *Mal'cev*, if every reflexive relation is an equivalence relation. A regular category is Mal'cev if and only if the composition of equivalence relations is commutative. A finitely complete protomodular category is always Mal'cev [8].

The validity of our generalized Hopf formula depends strongly on the fact that the Moore functor $N : \mathcal{SA} \longrightarrow \mathsf{Ch}\mathcal{A}$ is exact. This essentially amounts to the fact that, in a regular Mal'cev category, any regular epimorphism of simplicial objects is a Kan fibration. We prove this in Section 4, generalizing Carboni, Kelly and Pedicchio's result [14] that in a regular Mal'cev category, every simplicial object is Kan. The exactness of N is shown in Section 5. We get that any short exact sequence of simplicial objects induces a long exact sequence of homology objects. In Section 5 we moreover prove Dominique Bourn's conjecture that, for $n \geq 1$, the homology H_nA of a simplicial object A in a semi-abelian category $\mathcal{A}$ is an abelian object of $\mathcal{A}$.

Finally, in Section 6, we generalize Barr and Beck's notion of cotriple homology [1] to the situation where $\mathcal{B}$ is a pointed, regular and protomodular category. Their definition is modified to the following. Let $\mathcal{A}$ be a category, $\mathbb{G}$ a comonad on $\mathcal{A}$ and $U : \mathcal{A} \longrightarrow \mathcal{B}$ a functor. The *n-th homology object* $H_n(X, U)_{\mathbb{G}}$ *of* X *with coefficients in* U *relative to the cotriple* $\mathbb{G}$ is the object $H_{n-1}NU(\mathbb{G}X)$, the $(n-1)$-th homology object of the normalized chain complex associated with the simplicial object $U(\mathbb{G}X)$ of $\mathcal{B}$. We show that $H_1(X, U)_{\mathbb{G}} = U(X)$, give a proof of formula **B** and obtain a version of the Stallings-Stammbach sequence.

1.8. For the basic theory of semi-abelian categories we refer to the Borceux's survey [3] and Borceux and Bourn's book [5]. For general category theory we used Borceux [4] and Mac Lane [27]. Weibel's book [31] provides an excellent introduction to homological algebra. For the theory of model categories the reader is referred to Quillen [28] and Hovey [22].

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2. Chain complexes

2.1. Notation. Given a morphism $f : A \longrightarrow B$ in $\mathcal{A}$, (if it exists) its kernel is denoted by $\mathsf{Ker}\, f : K[f] \longrightarrow A$, its image by $\mathsf{Im}\, f : I[f] \longrightarrow B$ and its cokernel by $\mathsf{Coker}\, f : B \longrightarrow \mathrm{Cok}[f]$. In a diagram, the forms $A \rightarrowtail B$, $A \triangleright\!\!\longrightarrow B$ and $A \longrightarrow\!\!\triangleright B$ signify that the arrow is, respectively, a monomorphism, a normal monomorphism and a regular epimorphism.

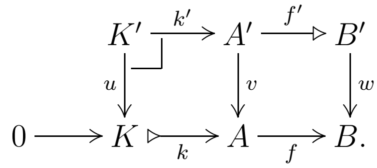
Recall that a chain complex C is a collection of morphisms $(d_n : C_n \longrightarrow C_{n-1})_{n \in \mathbb{Z}}$ such that $d_n \circ d_{n+1} = 0$, for all $n \in \mathbb{Z}$. Although usually considered in an abelian context, chain complexes of course make sense in any pointed category $\mathcal{A}$. Obtaining a good notion of homology objects H_nC of a chain complex C, however, demands a stronger assumption on $\mathcal{A}$.

When $\mathcal{A}$ is an abelian category, H_nC is $K[d_n]/I[d_{n+1}]$ (see, for example, [31]). Since this is just $\mathrm{Cok}[C_{n+1} \longrightarrow K[d_n]]$, it seems reasonable to define H_nC this way, supposed that the considered kernels and cokernels exist in $\mathcal{A}$. Yet, one could also suggest the dual $K_nC = K[\mathrm{Cok}[d_{n+1}] \longrightarrow C_{n-1}]$, since, in the abelian case, this equals H_nC. Let $\mathcal{A}$ be a pointed, regular and protomodular category. Recall that a morphism is proper if its image is a kernel. We call a chain complex C in $\mathcal{A}$ *proper* whenever all its differentials are. We will prove in Proposition 2.3 that for any proper complex C, the homology objects H_nC and K_nC are isomorphic. Furthermore, as the Snake Lemma holds in $\mathcal{A}$—see Bourn [10, Theorem 14]—we get Proposition 2.4: any short exact sequence of proper chain complexes induces a long exact sequence of homology objects.

Let $\mathsf{Ch}\mathcal{A}$ be the category of chain complexes in $\mathcal{A}$, morphisms being commutative ladders, and let $\mathsf{PCh}\mathcal{A}$ be the full subcategory of proper chain complexes. For a complex $C \in \mathsf{PCh}\mathcal{A}$ and $n \in \mathbb{Z}$, let H_nC be its *n-th homology object*, and K_nC its dual, as defined above. Note that H_nC and K_nC exist, as $\mathcal{A}$ has all cokernels of kernels. Further remark that, like $\mathcal{A}$, the category $\mathsf{Ch}\mathcal{A}$ is pointed, regular and protomodular. This is not the case for $\mathsf{PCh}\mathcal{A}$, since $\mathsf{PCh}\mathcal{A}$ e.g. need not have kernels. By an exact sequence of proper chain complexes, we mean an exact sequence in $\mathsf{Ch}\mathcal{A}$ such that the objects are proper chain complexes.

We need the following

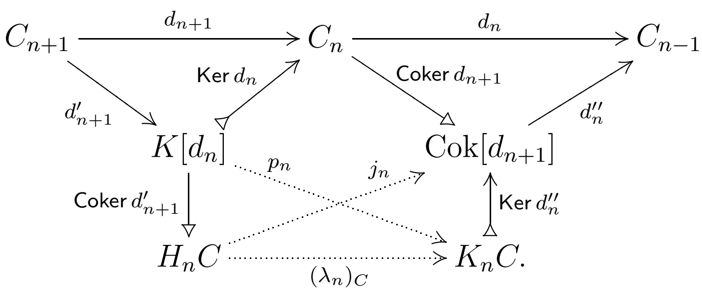
2.2. Lemma. [10, 5] *Let $\mathcal{A}$ be a pointed, regular and protomodular category. Consider the following commutative diagram, where $k = \mathsf{Ker}\, f$, f' is regular epi and the left hand square a pullback:*



If $k' = \mathsf{Ker}\, f'$, then w is a monomorphism. ∎

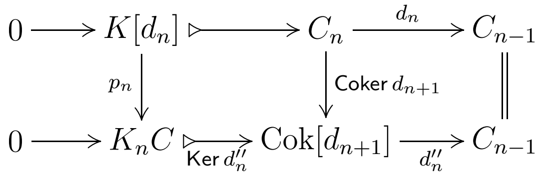
2.3. Proposition. *Let $\mathcal{A}$ be a pointed, regular and protomodular category. For any $n \in \mathbb{Z}$, H_n and K_n are naturally isomorphic functors $\mathsf{PCh}\mathcal{A} \longrightarrow \mathcal{A}$.*

Proof. Consider the commutative diagram of solid arrows



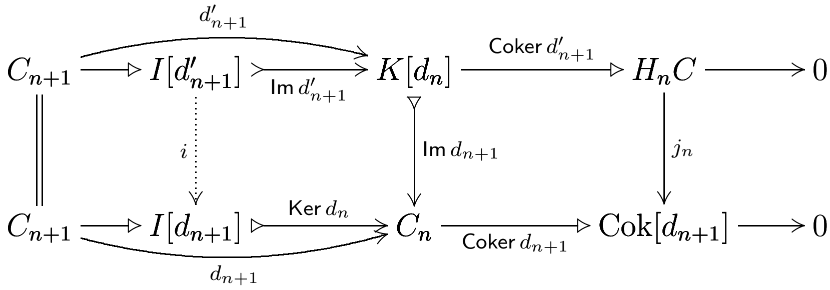
(E)

Note that all cokernels exist, because d_{n+1} and d'_{n+1} are proper. Since H_nC is a cokernel and K_nC is a kernel, unique morphisms j_n and p_n exist that keep the diagram commutative. A natural transformation $\lambda : H_n \Longrightarrow K_n$ is defined by the resulting unique $(\lambda_n)_C$. To prove it an isomorphism, first consider the following diagram with exact rows.



$1_{C_{n-1}}$ being a monomorphism, the left hand square is a pullback, and p_n is a regular epimorphism.

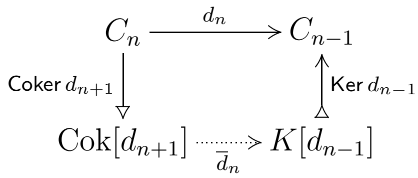
Considering the image factorizations of d'_{n+1} and d_{n+1}, there is a morphism i such that the diagram with exact rows



commutes. Clearly, it is both a monomorphism and a regular epimorphism, thus an isomorphism. Because d_{n+1} is proper, $\mathsf{Im}\, d_{n+1}$ is a kernel. We get that also $\mathsf{Im}\, d'_{n+1}$ is a kernel. Now the middle square is a pullback, so Lemma 2.2 implies that j_n is a monomorphism.

Accordingly, since $\mathbf{E}$ commutes, $(\lambda_n)_C$ is both regular epi and mono, hence it is an isomorphism. ■

For $n \in \mathbb{Z}$, let $\overline{d}_n$ denote the unique map such that the diagram



commutes. Since d_n is proper, so is $\overline{d}_n$.

The following is a straightforward generalization of the abelian case—see, for instance, Theorem 1.3.1 in Weibel [31].

2.4. Proposition. *Let $\mathcal{A}$ be a pointed, regular and protomodular category. Any short exact sequence of proper chain compexes*

$$0 \longrightarrow C'' \rightarrowtail C' \twoheadrightarrow C \longrightarrow 0$$

gives rise to a long exact sequence of homology objects

$$\cdots \longrightarrow H_{n+1}C \xrightarrow{\delta_{n+1}} H_nC'' \longrightarrow H_nC' \longrightarrow H_nC \xrightarrow{\delta_n} H_{n-1}C'' \longrightarrow \cdots \qquad (\mathbf{F})$$

which depends naturally on the given short exact sequence.

Proof. Since the d_n and $\overline{d}_n$ are proper, mimicking the abelian proof—using the Snake Lemma twice—we get an exact sequence

$$K_nC'' \longrightarrow K_nC' \longrightarrow K_nC \longrightarrow H_{n-1}C'' \longrightarrow H_{n-1}C' \longrightarrow H_{n-1}C$$

for every $n \in \mathbb{Z}$. By Proposition 2.3 we can paste these together to $\mathbf{F}$. The naturality follows from the naturality of the Snake Lemma. ■

Note that Bourn's version of the Snake Lemma [10, Theorem 14] states only the existence and the exactness of the sequence. However, it is quite clear from the construction of the connecting morphism that the sequence is, moreover, natural.

3. Simplicial objects

In this section we extend the homology theory of Section 2 to simplicial objects. We start by considering the *Moore functor* $N : \mathcal{SA} \longrightarrow \mathsf{Ch}\mathcal{A}$, which maps a simplicial object A in a pointed category with pullbacks $\mathcal{A}$ to the *normalized* chain complex $N(A)$. We prove that, when $\mathcal{A}$ is, moreover, exact and protomodular, $N(A)$ is always proper, and then define the *n-th homology object* of A as $H_nA = H_nN(A)$. Furthermore, we show that, if $\epsilon : A \longrightarrow A_{-1}$ is a contractible augmented simplicial object, then $H_0A = A_{-1}$ and, for $n \geq 1$, $H_nA = 0$.

When working with simplicial objects in a category $\mathcal{A}$, we will use the notations of [31]. The *simplicial category* Δ has, as objects, finite ordinals $[n] = \{0, \ldots, n\}$, for $n \in \mathbb{N}$ and, as morphisms, monotone functions. The category $\mathcal{SA}$ of *simplicial objects* and *simplicial maps* of $\mathcal{A}$ is the functor category $\mathsf{Fun}(\Delta^{\mathrm{op}}, \mathcal{A})$. Thus a simplicial object $A : \Delta^{\mathrm{op}} \longrightarrow \mathcal{A}$ corresponds to the following data: a sequence of objects $(A_n)_{n\in\mathbb{N}}$, *face operators* $\partial_i : A_n \longrightarrow A_{n-1}$ and *degeneracy operators* $\sigma_i : A_n \longrightarrow A_{n+1}$, for $i \in [n]$ and $n \in \mathbb{N}$, subject to the *simplicial identities*

$$\begin{aligned} \partial_i \circ \partial_j &= \partial_{j-1} \circ \partial_i \qquad \text{if } i < j \\ \sigma_i \circ \sigma_j &= \sigma_{j+1} \circ \sigma_i \qquad \text{if } i \leq j \\ \partial_i \circ \sigma_j &= \begin{cases} \sigma_{j-1} \circ \partial_i & \text{if } i < j \\ 1 & \text{if } i = j \text{ or } i = j+1 \\ \sigma_j \circ \partial_{i-1} & \text{if } i > j+1. \end{cases} \end{aligned}$$

An *augmented* simplicial object $\epsilon : A \longrightarrow A_{-1}$ consists of a simplicial object A and a map $\epsilon : A_0 \longrightarrow A_{-1}$ with $\epsilon \circ \partial_0 = \epsilon \circ \partial_1$. It is *contractible* if there exist maps $f_n : A_n \longrightarrow A_{n+1}$, $n \geq -1$, with $\epsilon \circ f_{-1} = 1_{A_{-1}}$, $\partial_0 \circ f_0 = f_{-1} \circ \epsilon$, $\partial_{n+1} \circ f_n = 1_{A_n}$ and $\partial_i \circ f_n = f_{n-1} \circ \partial_i$, for $0 \leq i \leq n$ and $n \in \mathbb{N}$.

3.1. Definition. *Let A be a simplicial object in a pointed category $\mathcal{A}$ with pullbacks. The* normalized*, or* Moore*, chain complex $N(A)$ is the chain complex with $N_0A = A_0$,*

$$N_nA = \bigcap_{i=0}^{n-1} K[\partial_i : A_n \longrightarrow A_{n-1}]$$

and differential $d_n = \partial_n \circ \bigcap_i \mathsf{Ker}\, \partial_i : N_nA \longrightarrow N_{n-1}A$, for $n \geq 1$, and $A_n = 0$, for $n < 0$. This gives rise to a functor $N : \mathcal{SA} \longrightarrow \mathsf{Ch}\mathcal{A}$.

3.2. Remark. Note that, in the above definition, $\partial_n \circ \bigcap_i \mathsf{Ker}\, \partial_i : N_nA \longrightarrow A_{n-1}$ may indeed be considered as an arrow $d_n : N_nA \longrightarrow N_{n-1}A$: the map clearly factors over $\bigcap_i \mathsf{Ker}\, \partial_i : N_{n-1}A \longrightarrow A_{n-1}$.

3.3. Remark. Obviously, for $n \geq 1$, the object of *n-cycles* $Z_nA = K[d_n]$ of a simplicial object A of $\mathcal{A}$ is equal to $\bigcap_{i=0}^{n} K[\partial_i : A_n \longrightarrow A_{n-1}]$.

3.4. Remark. The functor $N : \mathcal{SA} \longrightarrow \mathsf{Ch}\mathcal{A}$ preserves limits. Indeed, limits in $\mathcal{SA}$ and $\mathsf{Ch}\mathcal{A}$ are computed degreewise, and taking kernels and intersections (pulling back), as occurs in the construction of N, commutes with taking arbitrary limits in $\mathcal{A}$. In Section 5 we shall prove that N, moreover, preserves regular epimorphism, hence is exact.

In a protomodular category $\mathcal{A}$, an intrinsic notion of normal monomorphism exists (see Bourn [9]). We will, however, not introduce this notion here. It will be sufficient to note that, if $\mathcal{A}$ is moreover exact, the normal monomorphisms are just the kernels. To prove Theorem 3.6, we need the following

3.5. Lemma. [Non-Effective Trace of the 3×3 Lemma [11, Theorem 4.1]] *Consider, in a regular and protomodular category, a commutative square with horizontal regular epimorphisms*

$$\begin{array}{ccc} A' & \xrightarrow{f'} & B' \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle w} \\ A & \xrightarrow[f]{} & B. \end{array}$$

If w is a monomorphism and v a normal monomorphism, then w is a normal monomorphism. ■

3.6. Theorem. *Let $\mathcal{A}$ be a pointed, exact and protomodular category and A a simplicial object in $\mathcal{A}$. Then $N(A)$ is a proper chain complex of $\mathcal{A}$.*

Proof. Any differential d_n, when viewed as an arrow to A_{n-1}, is a composition of a normal monomorphism (an intersection of kernels), and a regular epimorphism (a split epimorphism, by the simplicial identities). Hence, the Non-Effective Trace of the 3×3 Lemma 3.5 implies that $d_n : N_nA \longrightarrow A_{n-1}$ is proper. This clearly remains true when, following Remark 3.2, we consider d_n as an arrow to $N_{n-1}A$. ■

3.7. Definition. *Suppose $\mathcal{A}$ is pointed, exact and protomodular. The object $H_n A = H_n N(A)$ will be called the n-th* homology object of *A, and the resulting functor $H_n : \mathcal{SA} \longrightarrow \mathcal{A}$ the n-th* homology functor, *for $n \in \mathbb{N}$.*

In order to compute, in Proposition 3.11, the homology of a contractible augmented simplicial object, we first make the following, purely categorical, observations. We start by recalling a result due to Dominique Bourn.

3.8. Lemma. [7, Proposition 14] *If, in a protomodular category, a square with vertical regular epimorphisms*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

is a pullback, it is also a pushout. ∎

A *fork* is a diagram such as $\mathbf{G}$ below where $e \circ \partial_0 = e \circ \partial_1$.

3.9. Proposition. [(cf. 10, Corollary 6)] *In a pointed and protomodular category $\mathcal{A}$, let*

$$A \underset{\partial_0}{\overset{\partial_1}{\rightrightarrows}} B \xrightarrow{\;e\;} C \qquad (\mathbf{G})$$

be a fork and $t : B \longrightarrow A$ a map with $\partial_0 \circ t = \partial_1 \circ t = 1_B$. Then the following are equivalent:

1. *e is a coequalizer of ∂_0 and ∂_1;*
2. *the square*

$$\begin{array}{ccc} A & \xrightarrow{\partial_1} & B \\ {\scriptstyle \partial_0}\downarrow & & \downarrow{\scriptstyle e} \\ B & \xrightarrow[e]{} & C \end{array}$$

   is a pushout;

3. *e is a cokernel of $\partial_1 \circ \mathsf{Ker}\, \partial_0$.*

Proof. The equivalence of 1. and 2. is obvious. In the diagram

$$\begin{array}{ccccc} K[\partial_0] & \xrightarrow{\mathsf{Ker}\,\partial_0} & A & \xrightarrow{\partial_1} & B \\ \downarrow & & {\scriptstyle \partial_0}\downarrow & & \downarrow{\scriptstyle e} \\ 0 & \longrightarrow & B & \xrightarrow[e]{} & C, \end{array}$$

the left hand side square is a pushout: indeed, Lemma 3.8 applies, since it is a pullback along the split, hence regular, epimorphism ∂_0. Consequently, the outer rectangle is a pushout if and only if the right square is, which means that 2. and 3. are equivalent. ∎

3.10. Corollary. *If $\mathcal{A}$ is a pointed, exact and protomodular category and A a simplicial object of $\mathcal{A}$ (with face operators $\partial_0, \partial_1 : A_1 \longrightarrow A_0$), then $H_0A = \mathrm{Coeq}[\partial_0, \partial_1]$.* ∎

A fork is *split* if there are two more arrows $s : C \longrightarrow B$ and $t : B \longrightarrow A$ such that $e \circ s = 1_C$, $\partial_1 \circ t = 1_B$ and $\partial_0 \circ t = s \circ e$. Every split fork is a coequalizer diagram.

3.11. Proposition. *If $\epsilon : A \longrightarrow A_{-1}$ is a contractible augmented simplicial object in a pointed, exact and protomodular category $\mathcal{A}$, then $H_0A = A_{-1}$ and, for $n \geq 1$, $H_nA = 0$.*

Proof. The contractibility of $\epsilon : A \longrightarrow A_{-1}$ implies that the fork

$$A_1 \underset{\partial_0}{\overset{\partial_1}{\rightrightarrows}} A_0 \xrightarrow{\ \epsilon\ } A_{-1}$$

is split (by the arrows $f_{-1} : A_{-1} \longrightarrow A_0$ and $f_0 : A_0 \longrightarrow A_1$). We get that it is a coequalizer diagram. The first equality now follows from Corollary 3.10.

In order to prove the other equalities, first recall from Remark 3.3 that, for $n \geq 1$,

$$Z_nA = K[d_n : N_nA \longrightarrow N_{n-1}A] = \bigcap_{i=0}^{n} K[\partial_n : A_n \longrightarrow A_{n-1}].$$

We are to show that the image of $d_{n+1} : N_{n+1}A \longrightarrow N_nA$ is $\mathsf{Ker}\, d_n : K[d_n] \longrightarrow N_nA$. But, for any $i \leq n$, the left hand downward-pointing arrow in the diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K[\partial_i] & \xrightarrow{\mathsf{Ker}\,\partial_i} & A_{n+1} & \xrightarrow{\partial_i} & A_n & \longrightarrow & 0 \\
 & & \downarrow\uparrow & & {\scriptstyle \partial_{n+1}}\downarrow\uparrow{\scriptstyle f_n} & & {\scriptstyle \partial_n}\downarrow\uparrow{\scriptstyle f_{n-1}} & & \\
0 & \longrightarrow & K[\partial_i] & \xrightarrow[\mathsf{Ker}\,\partial_i]{} & A_n & \xrightarrow[\partial_i]{} & A_{n-1} & \longrightarrow & 0
\end{array}$$

is a split epimorphism, because both its upward and downward pointing squares commute. It follows that the intersection $N_{n+1}A = \bigcap_{i \leq n} K[\partial_i] \longrightarrow \bigcap_{i \leq n} K[\partial_i] = K[d_n]$ is a split, hence a regular, epimorphism, and $\mathsf{Im}\, d_{n+1} = \mathsf{Ker}\, d_n$. ∎

4. The Kan condition

The results in this section will allow us to prove, in Section 5, two important facts concerning the functors $H_n : \mathsf{PCh}\mathcal{A} \longrightarrow \mathcal{A}$ and $N : \mathcal{SA} \longrightarrow \mathsf{PCh}\mathcal{A}$: Theorem 5.5 and Proposition 5.6. We recall the result from Carboni, Kelly and Pedicchio [14] that every simplicial object in a regular Mal'cev category is Kan. We add that every regular epimorphism between simplicial objects of a regular Mal'cev category is a Kan fibration.

Kan complexes and *Kan fibrations* are very important in the homotopy theory of simplicial sets (or simplicial objects in a variety over Set). In their article [14], Carboni, Kelly and Pedicchio extend the notion of Kan complex to an arbitrary category $\mathcal{A}$. When $\mathcal{A}$ is regular, their definition amounts to the one stated in Definition 4.1. (They consider only horns with $n \geq 2$; indeed, any simplicial object fulfils the Kan condition for $n = 1$.) In the same spirit, we propose an extension of the notion of Kan fibration to a regular category $\mathcal{A}$.

4.1. Definition. *Consider a simplicial object K in a regular category $\mathcal{A}$. For $n \geq 1$ and $k \in [n]$, a family*

$$x = (x_i : X \longrightarrow K_{n-1})_{i\in[n], i\neq k}$$

*is called an (n,k)-*horn *of K if it satisfies $\partial_i \circ x_j = \partial_{j-1} \circ x_i$, for $i < j$ and $i, j \neq k$.*

We say that K is Kan *if, for every (n,k)-horn $x = (x_i : X \longrightarrow K_{n-1})$ of K, there is a regular epimorphism $p : Y \longrightarrow X$ and a map $y : Y \longrightarrow K_n$ such that $\partial_i \circ y = x_i \circ p$ for $i \neq k$.*

A map $f : A \longrightarrow B$ of simplicial objects is said to be a Kan fibration *if, for every (n,k)-horn $x = (x_i : X \longrightarrow A_{n-1})$ of A and every $b : X \longrightarrow B_n$ with $\partial_i \circ b = f_{n-1} \circ x_i$ for all $i \neq k$, there is a regular epimorphism $p : Y \longrightarrow X$ and a map $a : Y \longrightarrow A_n$ such that $f_n \circ a = b \circ p$ and $\partial_i \circ a = x_i \circ p$ for all $i \neq k$.*

Obviously, a simplicial object K is Kan if and only if the unique map $K \longrightarrow *$ from K to a terminal object $*$ of $\mathcal{SA}$ is a Kan fibration.

In case $\mathcal{A}$ is Set, these notions have an equivalent formulation in which X and Y are both equal to a terminal object $*$. These equivalent formulations are the classical definitions of Kan simplicial set and Kan fibration—see, for instance, Weibel [31]. There is also the following connection between Definition 4.1 and the set-theoretic notions.

4.2. Proposition. *Let $\mathcal{A}$ be a regular category and $\Upsilon : \mathcal{A} \longrightarrow \mathsf{Set}$ a functor which preserves regular epimorphisms and has a left adjoint $\Phi : \mathsf{Set} \longrightarrow \mathcal{A}$. Then*

1. *for any Kan simplicial object K of $\mathcal{A}$, the simplicial set $\Upsilon(K)$ is Kan;*
2. *for any Kan fibration $f : A \longrightarrow B$ of simplicial objects in $\mathcal{A}$, the simplicial map $\Upsilon f : \Upsilon(A) \longrightarrow \Upsilon(B)$ is a Kan fibration of simplicial sets.*

Proof. Let $\zeta : 1_{\mathsf{Set}} \Longrightarrow \Upsilon \circ \Phi$ denote the unit of the adjunction and

$$\varphi_{X,A} : \hom_{\mathcal{A}}(\Phi(X), A) \cong \hom_{\mathsf{Set}}(X, \Upsilon(A))$$

the canonical isomorphism, natural in $X \in \mathsf{Set}^{\mathrm{op}}$ and $A \in \mathcal{A}$. Suppose that $f : A \longrightarrow B$ is a Kan fibration; consider an (n,k)-horn $x = (x_i : X \longrightarrow \Upsilon(A)_{n-1})_{i\in[n], i\neq k}$ of $\Upsilon(A)$ and a map $b : X \longrightarrow \Upsilon(B_n)$ with $\Upsilon\partial_i \circ b = \Upsilon f_{n-1} \circ x_i$ for all $i \neq k$. We give a proof of 2.

Note that the collection $(\varphi^{-1}(x_i) : \Phi(X) \longrightarrow A_{n-1})_{i\in[n], i\neq k}$ is an (n,k)-horn of A such that $\partial_i \circ \varphi^{-1}(b) = f_{n-1} \circ \varphi^{-1}(x_i)$: indeed, by the naturality of φ^{-1},

$$\partial_i \circ \varphi^{-1}(b) = \varphi^{-1}(\Upsilon\partial_i \circ b) = \varphi^{-1}(\Upsilon f_{n-1} \circ x_i) = f_{n-1} \circ \varphi^{-1}(x_i).$$

Since f is Kan, we get a regular epimorphism $p_1 : Y_1 \longrightarrow \Phi(X)$ and a map $a_1 : Y_1 \longrightarrow A_n$ such that $f_n \circ a_1 = \varphi^{-1}(b) \circ p_1$ and $\partial_i \circ a_1 = \varphi^{-1}(x_i) \circ p_1$, for $i \neq k$. Now consider the pullback square

$$\begin{array}{ccc} Y & \xrightarrow{\;p\;} & X \\ {\scriptstyle z}\downarrow & & \downarrow{\scriptstyle \zeta_{\Phi(X)}} \\ \Upsilon(Y_1) & \xrightarrow[\Upsilon p_1]{} & \Upsilon\Phi(X). \end{array}$$

Because Set is regular, p is a regular epimorphism. Put $a = \Upsilon a_1 \circ z : Y \longrightarrow \Upsilon(A_n)$; then p and a are the required maps:

$$\begin{aligned} \Upsilon f_n \circ a &= \Upsilon f_n \circ \Upsilon a_1 \circ z \\ &= \Upsilon(f_n \circ a_1) \circ z \\ &= \Upsilon(\varphi^{-1}(b) \circ p_1) \circ z \\ &= \Upsilon \varphi^{-1}(b) \circ \Upsilon p_1 \circ z \\ &= \Upsilon \varphi^{-1}(b) \circ \zeta_{\Phi(X)} \circ p \\ &= \varphi \varphi^{-1}(b) \circ p \\ &= b \circ p \end{aligned}$$

and, similarly, $\Upsilon \partial_i \circ a = \Upsilon x_i \circ p$, for all $i \neq k$. ■

We recall some basic definitions and observations from [14]. In a category with finite limits, a relation $R : A \longrightarrow B$ from A to B is a subobject $(d_0, d_1) : R \longrightarrow A \times B$. If a map $(f, g) : X \longrightarrow A \times B$ factorizes through (d_0, d_1), then the map $h : X \longrightarrow R$ with $(f, g) = (d_0, d_1) \circ h$ is necessarily unique; we will denote the situation by $g(R)f$. $SR : A \longrightarrow C$ denotes the composition of a relation $R : A \longrightarrow B$ with a relation $S : B \longrightarrow C$.

4.3. Proposition. [14, Proposition 2.1] *Let $\mathcal{A}$ be a regular category.*

1. *A map $b : X \longrightarrow B$ factorizes through the image of a map $f : A \longrightarrow B$ if and only if there is a regular epimorphism $p : Y \longrightarrow X$ and a map $a : Y \longrightarrow A$ with $b \circ p = f \circ a$;*

2. *given relations $R : A \longrightarrow B$ and $S : B \longrightarrow C$ and maps $a : X \longrightarrow A$ and $c : X \longrightarrow C$, $c(SR)a$ if and only if there is a regular epimorphism $p : Y \longrightarrow X$ and a map $b : Y \longrightarrow B$ with $b(R)a \circ p$ and $c \circ p(S)b$.* ■

Recall from [15] and [14] that a finitely complete category is called *Mal'cev*, if every reflexive relation is an equivalence relation. It follows from 2. that in a regular category, the composition of relations is associative. A regular category is Mal'cev if and only if the composition of equivalence relations is commutative [14], i.e. when for any two equivalence relations S and R on an object X the equality $SR = RS$ holds. Every finitely complete protomodular (hence, *a fortiori*, every semi-abelian) category is Mal'cev [8]. In a regular Mal'cev category, the equivalence relations on a given object X constitute a lattice, the join of two equivalence relations being their composition.

Carboni, Kelly and Pedicchio extend the classical result of Moore that any simplicial group is Kan: in [14], it is proven that any simplicial object in a regular Mal'cev category is Kan. This property is even seen to characterize the Mal'cev categories among the regular ones. We will add that any regular epimorphism between simplicial objects in a regular Mal'cev category is a Kan fibration.

4.4. Proposition. [14, Theorem 4.2] *Let $\mathcal{A}$ be a regular Mal'cev category. Then*

1. *every simplicial object K of $\mathcal{A}$ is Kan;*
2. *if $f : A \longrightarrow B$ is a regular epimorphism between simplicial objects of $\mathcal{A}$, then it is a Kan fibration.*

Proof. In our proof of 2. we will repeatedly use Proposition 4.3. For $n \geq 1$ and $k \in [n]$, let

$$x = (x_i : X \longrightarrow A_{n-1})_{i\in[n],i\neq k}$$

be an (n,k)-horn of A and let $b : X \longrightarrow B_n$ be a map with $\partial_i \circ b = f_{n-1} \circ x_i$, for $i \in [n]$ and $i \neq k$. Because f_n is regular epi, there is a regular epimorphism $p_1 : Y_1 \longrightarrow X$ and a map $c : Y_1 \longrightarrow A_n$ with $f_n \circ c = b \circ p_1$. For $i \in [n]$ and $i \neq k$, put $c_i = \partial_i \circ c : Y_1 \longrightarrow A_{n-1}$. Let $R[f]$ denote the kernel relation of f. Now

$$f_{n-1} \circ c_i = f_{n-1} \circ \partial_i \circ c = \partial_i \circ f_n \circ c = \partial_i \circ b \circ p_1 = f_{n-1} \circ x_i \circ p_1,$$

and, consequently, $c_i(R[f_{n-1}])x_i \circ p_1$. By the simplicial identities, this defines an (n,k)-horn

$$((c_i, x_i \circ p_1) : Y_1 \longrightarrow A_{n-1})_{i\in[n],i\neq k}$$

of $R[f]$. This simplicial object being Kan yields a regular epimorphism $p_2 : Y_2 \longrightarrow Y_1$ and a $(d,e) : Y_2 \longrightarrow R[f]_n$ such that $\partial_i \circ d = c_i \circ p_2$ and $\partial_i \circ e = x_i \circ p_1 \circ p_2$, for $i \in [n]$ and $i \neq k$, and $f_n \circ d = f_n \circ e$. Thus, $c \circ p_2(D)d$, where D is the equivalence relation $\bigwedge_{i\in[n],i\neq k} D_i$ and D_i is the kernel relation of $\partial_i : A_n \longrightarrow A_{n-1}$. It follows that $c \circ p_2(DR[f_n])e$. By the Mal'cev property, $DR[f_n]$ is equal to $R[f_n]D$. This, in turn, implies that there exists a regular epimorphism $p_3 : Y \longrightarrow Y_2$ and a map $a : Y \longrightarrow A_n$ such that $a(D)e \circ p_3$ and $c \circ p_2 \circ p_3(R[f_n])a$. The required maps are now a and $p = p_1 \circ p_2 \circ p_3 : Y \longrightarrow X$: indeed, $f_n \circ a = f_n \circ c \circ p_2 \circ p_3 = b \circ p$ and $\partial_i \circ a = \partial_i \circ e \circ p_3 = x_i \circ p$. ∎

5. Implications of the Kan condition

In this section we consider two important implications of Proposition 4.4. First we show that for a simplicial object A of $\mathcal{A}$, being Kan implies that $H_n A$ is abelian ($n \geq 1$). Next we prove that $N : \mathcal{SA} \longrightarrow \mathsf{PCh}\mathcal{A}$ is an exact functor. This is an implication of the fact that every regular epimorphism between simplicial objects is a Kan fibration. Finally, using the exactness of N, we prove that every short exact sequence of simplicial objects induces a long exact homology sequence (Corollary 5.7). We obtain it as an immediate consequence of Proposition 2.4.

Let A be a Kan simplicial object in a regular category $\mathcal{A}$ and $n \geq 1$. Recall from Remark 3.3 the notation $Z_n A = K[d_n] = \bigcap_{i=0}^{n} K[\partial_i]$. We write $z_n : Z_n A \longrightarrow A_n$ for the inclusion $\bigcap_{i=0}^{n} \mathsf{Ker}\, \partial_i$. Basing ourselves on Weibel [31], we say that morphisms $x : X \longrightarrow Z_n A$ and $x' : X' \longrightarrow Z_n A$ are *homotopic*, and write $x \sim x'$, if there is an arrow

$y : Y \longrightarrow A_{n+1}$ (called a *homotopy from x to x'*) and regular epimorphisms $p : Y \longrightarrow X$ and $p' : Y \longrightarrow X'$ such that

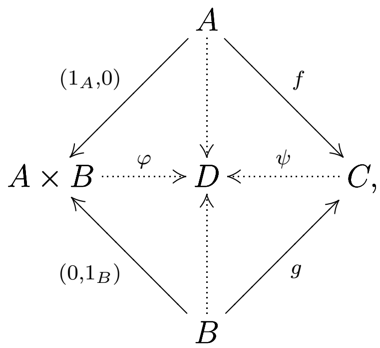
$$\partial_i \circ y = \begin{cases} 0, & \text{if } i < n; \\ z_n \circ x \circ p, & \text{if } i = n; \\ z_n \circ x' \circ p', & \text{if } i = n+1. \end{cases}$$

5.1. Proposition. *Let A be a Kan simplicial object in a regular category $\mathcal{A}$. For $n \geq 1$, $\sim$ defines an equivalence relation on the class of arrows $x : X \longrightarrow Z_n A$ in $\mathcal{A}$ with codomain $Z_n A$.*

Proof. Due to to fact that A is Kan, the proof of 8.3.1 in Weibel [31] may be copied: one just considers arrows with codomain A_n instead of elements of A_n and reads "0" instead of "$*$". ■

In Borceux and Bourn [5] and Borceux [3] a notion of commutator is introduced which generalizes a definition by Huq [23]. It is based on a construction due to Bourn [6]. Other notions of commutator exist; we use this one because it allows us to take the commutator of subobjects which are not necessarily kernels.

5.2. Definition. [6, Proposition 1.9, 3, Definition 6.4] *In a semi-abelian category $\mathcal{A}$, let $f : A \longrightarrow C$ and $g : B \longrightarrow C$ be two morphisms with the same codomain. Then the arrow ψ, obtained by taking the colimit of the diagram of solid arrows*



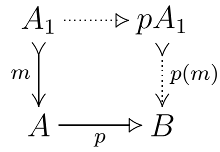
(**H**)

is a regular epimorphism. The kernel of ψ is denoted $[f, g]$ and called the commutator of f and g. *An object A of $\mathcal{A}$ is called* abelian *if* $[1_A, 1_A] = 0$.

Note that the above colimit exists, as a semi-abelian category has all finite limits. Recall, furthermore, that, in a semi-abelian category, all regular epimorphisms are cokernels, hence $D = C/[f, g]$.

Proposition 9 of [9] states that an object is abelian if and only if it can be provided with the structure of an internal abelian group. The full subcategory $\mathcal{A}_{\mathsf{Ab}}$ of all abelian objects is a Birkhoff subcategory of $\mathcal{A}$ [3, Theorem 7.1 and 7.2]. The component at an object A of $\mathcal{A}$ of the unit of the adjunction is given by $\psi : A \longrightarrow D = A/[1_A, 1_A]$.

5.3. Notation. In a regular category $\mathcal{A}$, consider a monomorphism $m : A_1 \longrightarrow A$ and a regular epimorphism $p : A \longrightarrow B$. Taking the image factorization

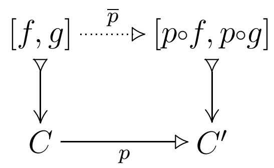


of $p \circ m$ yields a monomorphism $p(m) : pA_1 \longrightarrow B$ called the *direct image of* m *along* p.

In our proof of Theorem 5.5 we need the following properties of the commutator $[f, g]$.

5.4. Proposition. *In a semi-abelian category* $\mathcal{A}$*, let* $f : A \longrightarrow C$ *and* $g : B \longrightarrow C$ *be two morphisms with the same codomain.*

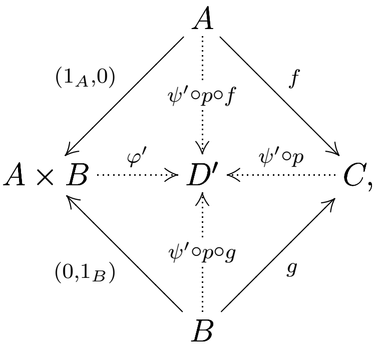
1. *If* f *or* g *is* 0 *then* $[f, g] = 0$.
2. *For any regular epimorphism* $p : C \longrightarrow C'$, $p[f, g] = [p \circ f, p \circ g]$. *This means that there exists a regular epimorphism* $\overline{p}$ *such that the square*



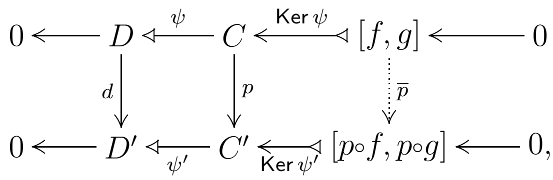
commutes.

3. *If* $k : A \longrightarrow C$ *is a kernel, then* $[k, k]$ *factors over* A*, and* $A/[k, k]$ *is an abelian object of* $\mathcal{A}$.

Proof. 1. is obvious. The rest of the proof is based on Huq [23, Proposition 4.1.4]. As in Definition 5.2, let $\psi : C \longrightarrow D$ and $\varphi : A \times B \longrightarrow D$, resp. $\psi' : C' \longrightarrow D'$ and $\varphi' : A \times B \longrightarrow D'$, denote the couniversal arrows obtained from the construction of $[f, g]$ and $[p \circ f, p \circ g]$. Then



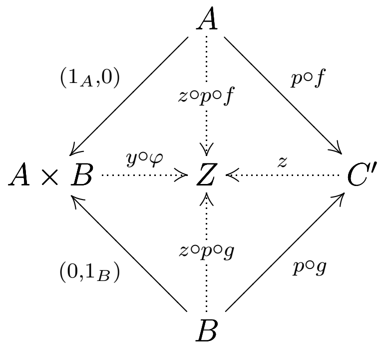
is a cocone on the diagram of solid arrows **H**. The couniversal property of colimits yields a unique map $d : D \longrightarrow D'$. In the commutative diagram of solid arrows



there exits a unique map $\overline{p} : [f,g] \longrightarrow [p{\circ}f, p{\circ}g]$ such that the right hand square commutes.

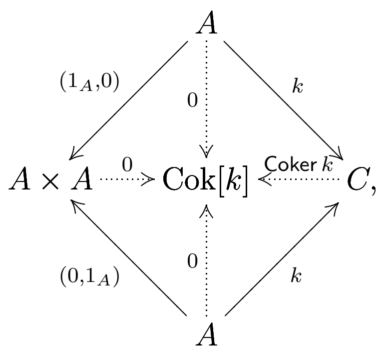
For 2. we must show that if p is a regular epimorphism, then so is $\overline{p}$. To do so, we prove that $k = \mathsf{Ker}\,\psi'{\circ}\mathsf{Im}\,\overline{p}$ is a kernel of ψ'. By the Non-Effective Trace of the 3×3 Lemma 3.5, k is a kernel; hence, it is sufficient that ψ' be a cokernel of k.

Let $z : C' \longrightarrow Z$ be a map such that $z{\circ}k = 0$. Then $z{\circ}p{\circ}\mathsf{Ker}\,\psi = 0$, which yields a map $y : D \longrightarrow Z$ with $y{\circ}\psi = z{\circ}p$. We get the following cocone.



Thus we acquire an arrow $x : D' \longrightarrow Z$ such that $x{\circ}\psi' = z$.

$[1_A, 1_A]$ is a subobject of $[k,k]$: take $p = k$ and $f = g = 1_A$ in the discussion above. Hence, using that $A/[1_A, 1_A]$ is abelian and that $\mathcal{A}_{\mathsf{Ab}}$ is closed under quotients (by definition of a Birkhoff subcategory), the first statement of 3. implies the second one. This first statement follows from the fact that



is a cocone. Thus a map may be found such that the right hand square in

$$\begin{array}{ccccccccc} 0 & \longrightarrow & [k,k] & \rightarrowtail & C & \overset{\psi}{\twoheadrightarrow} & \frac{C}{[k,k]} & \longrightarrow & 0 \\ & & {\scriptstyle i}\downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & A & \underset{k}{\rightarrowtail} & C & \underset{\mathsf{Coker}\,k}{\twoheadrightarrow} & \mathrm{Cok}[k] & \longrightarrow & 0 \end{array}$$

commutes, whence the map i. ■

The following was suggested to us by Dominique Bourn. A more conceptual proof of this theorem may be found in his forthcoming paper [12]. It is well-known to hold in case $\mathcal{A}$ is the category Gp of groups.

5.5. Theorem. *Let A be a simplicial object in a semi-abelian category $\mathcal{A}$. For all $n \geq 1$, H_nA is an abelian object of $\mathcal{A}$.*

Proof. Let $n \geq 1$. Consider the subobjects $k_n : K_n = [z_n, z_n] \longrightarrow A_n$ of A_n and

$$s_n : S_n = [\sigma_{n-1}\circ z_n, \sigma_n\circ z_n] \longrightarrow A_{n+1}$$

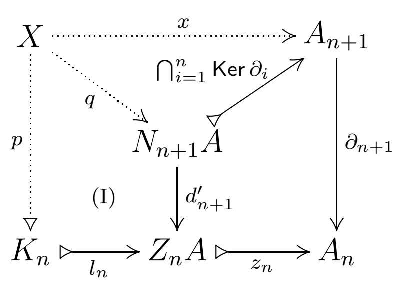
of A_{n+1}. By the second statement of Proposition 5.4,

$$\partial_i S_n = [\partial_i\circ\sigma_{n-1}\circ z_n, \partial_i\circ\sigma_n\circ z_n],$$

for $0 \leq i \leq n+1$. Hence, by the simplicial identities and the first statement of Proposition 5.4, $\partial_i S_n = 0$, for $i \neq n$, and $\partial_n S_n = [z_n, z_n] = K_n$. This last equality means that there exists a regular epimorphism filling the square

$$\begin{array}{ccc} S_n & \twoheadrightarrow & K_n \\ {\scriptstyle s_n}\downarrow & & \downarrow{\scriptstyle k_n} \\ A_{n+1} & \underset{\partial_n}{\twoheadrightarrow} & A_n. \end{array}$$

By the third statement of Proposition 5.4, there is a map $l_n : K_n \longrightarrow Z_nA$ such that $z_n\circ l_n = k_n$. It follows that $l_n \sim 0$. Now, by Proposition 4.4, A is Kan; hence, Proposition 5.1 implies that $0 \sim l_n$. Thus there exists a morphism $x : X \longrightarrow A_{n+1}$ and a regular epimorphism $p : X \longrightarrow K_n$ such that $\partial_{n+1}\circ x = z_n\circ l_n\circ p$, i.e. the outer rectangle in



commutes, and such that, moreover, $\partial_i \circ x = 0$, for $i \neq n+1$. It follows that an arrow q exists such that, in the diagram above, the triangle commutes. As z_n is a monomorphism, we get the commutativity of the trapezium (I). Now, in the diagram with exact rows

$$\begin{array}{ccccccccc} X & \xrightarrow{p} & K_n & \xrightarrow{l_n} & Z_nA & \longrightarrow & \frac{Z_nA}{K_n} & \longrightarrow & 0 \\ {\scriptstyle q}\downarrow & & \text{(I)} & & \| & & \vdots\, {\scriptstyle r} & & \\ N_{n+1}A & & \xrightarrow[d'_{n+1}]{} & & Z_nA & \longrightarrow & H_nA & \longrightarrow & 0, \end{array}$$

an arrow r exists such that the right hand square commutes. Indeed, as p is an epimorphism, $\mathsf{Coker}\,(l_n \circ p) = \mathsf{Coker}\, l_n$. This map r is a regular epimorphism. Since, by Proposition 5.4, $Z_nA/K_n = Z_nA/[z_n, z_n]$ is an abelian object, and since a quotient of an abelian object of $\mathcal{A}$ is abelian, so is the object H_nA. ∎

Recall from Remark 3.4 that N preserves kernels. If the category $\mathcal{A}$ is regular Mal'cev, Proposition 4.4 implies that N preserves regular epimorphisms. If, moreover, $\mathcal{A}$ is protomodular, all regular epimorphisms are cokernels, and we obtain

5.6. PROPOSITION. *If $\mathcal{A}$ is a pointed, regular and protomodular category, then the functor $N : \mathcal{SA} \longrightarrow \mathsf{PCh}\mathcal{A}$ is exact.*

PROOF. We prove that N preserves regular epimorphisms. Therefore, let $f : A \longrightarrow B$ be a regular epimorphism between simplicial objects of $\mathcal{A}$. Then $N_0 f = f_0$ is regular epi, as well as $N_n f = 0 : 0 \longrightarrow 0$, for $n < 0$. For $n \geq 1$, let $b : X \longrightarrow N_nB$ be a map. We are to show—see Proposition 4.3—that there is a regular epimorphism $p : Y \longrightarrow X$ and a map $a : Y \longrightarrow N_nA$ with $b \circ p = N_n f \circ a$. Now $(x_i = 0 : X \longrightarrow A_{n-1})_{i \in [n-1]}$ is an (n, n)-horn of A with $\partial_i \circ \bigcap_{i \in [n-1]} \mathsf{Ker}\, \partial_i \circ b = 0 = f_{n-1} \circ x_i$, for $i \in [n-1]$. By Proposition 4.4, f is a Kan fibration, which implies that there is a regular epimorphism $p : Y \longrightarrow X$ and a map $a' : Y \longrightarrow A_n$ such that $f_n \circ a' = \bigcap_{i \in [n-1]} \mathsf{Ker}\, \partial_i \circ b \circ p$ and $\partial_i \circ a' = x_i \circ p$. Then the unique factorization $a : Y \longrightarrow N_nA$ of a' over $\bigcap_{i \in [n-1]} \mathsf{Ker}\, \partial_i : N_nA \longrightarrow A_n$ is the required map. ∎

Proposition 2.4 may now immediately be extended to simplicial objects.

5.7. COROLLARY. *Let $\mathcal{A}$ be a pointed, exact and protomodular category. Any short exact sequence of simplicial objects*

$$0 \longrightarrow A'' \longrightarrow A' \longrightarrow A \longrightarrow 0$$

gives rise to a long exact sequence of homology objects

$$\cdots \longrightarrow H_{n+1}A \xrightarrow{\delta_{n+1}} H_nA'' \longrightarrow H_nA' \longrightarrow H_nA \xrightarrow{\delta_n} H_{n-1}A'' \longrightarrow \cdots$$

$$\cdots \longrightarrow H_1A \xrightarrow[\delta_1]{} H_0A'' \longrightarrow H_0A' \longrightarrow H_0A \xrightarrow[\delta_0]{} 0$$

which depends naturally on the given short exact sequence of simplicial objects. If $\mathcal{A}$ is semi-abelian then, except for the lowest three terms H_0A'', H_0A' and H_0A, all of its terms are abelian objects of $\mathcal{A}$. ∎

Although we shall not use it, we think it worth mentioning that this result can be formulated in terms of homological δ-functors. By a *(universal) homological δ-functor* between pointed, exact and protomodular categories $\mathcal{A}$ and $\mathcal{B}$ we mean a collection of functors $(T_n : \mathcal{A} \longrightarrow \mathcal{B})_{n\in\mathbb{N}}$ which preserve binary products, together with a collection of connecting morphisms $(\delta_n)_{n\in\mathbb{N}}$ as in [31, Definition 2.1.1 and 2.1.4].

5.8. Notation. For any simplicial object A in $\mathcal{A}$, let us denote by A^- the simplicial object defined by $A^-_n = A_{n+1}$, $\partial^-_i = \partial_i : A_{n+1} \longrightarrow A_n$, and $\sigma^-_i = \sigma_i : A_{n+1} \longrightarrow A_{n+2}$, for $i \in [n]$, $n \in \mathbb{N}$. This is the simplicial object obtained from A by leaving out A_0 and, for $n \in \mathbb{N}$, all $\partial_n : A_n \longrightarrow A_{n-1}$ and $\sigma_n : A_n \longrightarrow A_{n+1}$. Observe that $\partial = (\partial_{n+1})_n$ defines a simplicial morphism from A^- to A. Furthermore, remark that the augmented simplicial object $\partial_0 : A^- \longrightarrow A_0$ is contractible—for $n \geq -1$, put $f_n = \sigma_{n+1} : A_{n+1} \longrightarrow A_{n+2}$.

5.9. Proposition. *Let $\mathcal{A}$ be a pointed, exact and protomodular category. The sequence of functors $(H_n : \mathcal{S}\mathcal{A} \longrightarrow \mathcal{A})_{n\in\mathbb{N}}$, together with the connecting morphisms $(\delta_n)_{n\in\mathbb{N}}$, form a universal homological δ-functor.*

Proof. To prove that a functor $H_n : \mathcal{S}\mathcal{A} \longrightarrow \mathcal{A}$ preserves binary products, it suffices that, for proper complexes C and C' in $\mathcal{A}$, $H_n(C \times C') = H_nC \times H_nC'$. One shows this by using that for f and f' proper, $\mathsf{Coker}\,(f \times f') = \mathsf{Coker}\, f \times \mathsf{Coker}\, f'$. This follows from the fact that in any regular category, a product of two regular epimorphisms is regular epi. The universality is proven by modifying the proof of Theorem 2.4.7 in [31], replacing, for a simplicial object A, the projective object P by A^-. ∎

6. Cotriple homology and Hopf's formula

The aim of this section is to prove our generalized Hopf formula (Theorem 6.9) and our version of the Stallings-Stammbach sequence (Corollary 6.10).

We recall from Barr and Beck [1] (or Weibel [31]) the definition of cotriple homology, and slightly generalize it to categories that are pointed, exact and protomodular. Let $\mathcal{A}$ be an arbitrary category. A comonad $\mathbb{G}$ on $\mathcal{A}$ will be denoted by

$$\mathbb{G} = (G : \mathcal{A} \longrightarrow \mathcal{A}, \epsilon : G \Longrightarrow 1_{\mathcal{A}}, \delta : G \Longrightarrow G^2).$$

For any object X of $\mathcal{A}$, recall that the axioms of comonad state that $\epsilon_{GX} \circ \delta_X = G\epsilon_X \circ \delta_X = 1_{GX}$ and $\delta_{GX} \circ \delta_X = G\delta_X \circ \delta_X$. Putting

$$\partial_i = G^i\epsilon_{G^{n-i}X} : G^{n+1}X \longrightarrow G^nX \qquad \text{and} \qquad \sigma_i = G^i\delta_{G^{n-i}X} : G^{n+1}X \longrightarrow G^{n+2}X,$$

for $0 \leq i \leq n$, makes the sequence $(G^{n+1}X)_{n\in\mathbb{N}}$ a simplicial object $\mathbb{G}X$ of $\mathcal{A}$. This induces a functor $\mathcal{A} \longrightarrow \mathcal{S}\mathcal{A}$, which, when confusion is unlikely, will be denoted by $\mathbb{G}$.

6.1. Definition. *Let $\mathbb{G}$ be a comonad on a category $\mathcal{A}$. Let $\mathcal{B}$ be a pointed, exact and protomodular category and $U : \mathcal{A} \longrightarrow \mathcal{B}$ a functor. We say that the object*

$$H_n(X, U)_{\mathbb{G}} = H_{n-1}NU(\mathbb{G}X)$$

is the n-th homology object of X with coefficients in U relative to the cotriple $\mathbb{G}$*. This defines a functor $H_n(\cdot, U)_{\mathbb{G}} : \mathcal{A} \longrightarrow \mathcal{B}$, for any $n \in \mathbb{N}_0$.*

We now make the following assumptions:

1. U is the reflector of a semi-abelian category $\mathcal{A}$ onto a Birkhoff subcategory $\mathcal{B}$ (which is itself semi-abelian—see [18]);
2. $\mathcal{A}$ is monadic over Set;
3. $\mathbb{G} = (G : \mathcal{A} \longrightarrow \mathcal{A}, \epsilon : G \Longrightarrow 1_{\mathcal{A}}, \delta : G \Longrightarrow G^2)$ is the resulting comonad on $\mathcal{A}$.

Let $\Upsilon : \mathcal{A} \longrightarrow \mathsf{Set}$ and $\Phi : \mathsf{Set} \longrightarrow \mathcal{A}$ denote the respective right and left adjoint functors and $\epsilon : \Phi \circ \Upsilon \Longrightarrow 1_{\mathcal{A}}$ and $\zeta : 1_{\mathsf{Set}} \Longrightarrow \Upsilon \circ \Phi$ the counit and unit. Then $G = \Phi \circ \Upsilon$, ϵ is just the counit and δ is the natural transformation defined by $\delta_X = \Phi \zeta_{\Upsilon(X)}$, for $X \in \mathcal{A}$.

6.2. Remark. The requirement that $\mathcal{A}$ be monadic over Set implies that $\mathcal{A}$ is complete, cocomplete and exact (see e.g. Borceux [4, Theorem II.4.3.5]); hence, if $\mathcal{A}$ is, moreover, pointed and protomodular, it is semi-abelian.

Reciprocally, any variety of algebras over Set is monadic—see e.g. Cohn [17], Borceux [4] or Mac Lane [27]—and thus semi-abelian varieties form an example of the situation considered. A characterisation of such varieties of algebras over Set is given by Bourn and Janelidze in their paper [13]. More generally, in [19], Gran and Rosický characterize semi-abelian categories, monadic over Set.

6.3. Remark. Note that, by Beck's Theorem, the monadicity of Υ implies that for any object X of $\mathcal{A}$, the diagram

$$G^2X \overset{G\epsilon_X}{\underset{\epsilon_{GX}}{\rightrightarrows}} GX \xrightarrow{\epsilon_X} X \tag{I}$$

is a coequalizer (see, for instance, the proof of Theorem VI.7.1 in [27], or Lemma II.4.3.3 in [4]).

6.4. Remark. Because the forgetful functor $\Upsilon : \mathcal{A} \longrightarrow \mathsf{Set}$ preserves regular epimorphisms (again Borceux [4, Theorem II.4.3.5]) and because, in Set, every object is projective, $\mathcal{A}$ is easily seen to have enough projectives. In particular, any GX is projective. Accordingly, for any object X of $\mathcal{A}$, the map $\epsilon_X : GX \longrightarrow X$ is a projective presentation, the *"$\mathbb{G}$-free"* presentation of X.

Recalling Corollary 4.7 in [18], we get that $\Delta V(GX) = 0$.

The following characterization of $H_1(X, U)_{\mathbb{G}}$ is an immediate consequence of Proposition 3.9.

6.5. Proposition. *For any object X of $\mathcal{A}$, $H_1(X,U)_{\mathbb{G}} \cong U(X)$.*

Proof. On one hand, $H_1(X,U)_{\mathbb{G}} = H_0NU(\mathbb{G}X)$ is a cokernel of $UG\epsilon_X \circ \mathsf{Ker}\, U\epsilon_{GX}$. On the other hand, $\mathbf{I}$ is a coequalizer diagram, and U preserves coequalizers. Since, moreover, the map $U\delta_X : UGX \longrightarrow UG^2X$ is a splitting for both $UG\epsilon_X$ and $U\epsilon_{GX}$, Proposition 3.9 applies, and $U(X)$ is a cokernel of $UG\epsilon_X \circ \mathsf{Ker}\, U\epsilon_{GX}$ as well. ∎

In the proof of Theorem 6.9, we will need some basic facts concerning the Quillen model structure $(\mathcal{S}\mathsf{Set}, \mathrm{fib}, \mathrm{cof}, \mathrm{we})$ on the category of simplicial sets and simplicial maps. For a complete description of this model category and its properties we refer the reader to Quillen [28] and Hovey [22]. We shall, however, only use the following. fib, cof and we are classes of morphisms of $\mathcal{S}\mathsf{Set}$, respectively called *fibrations*, *cofibrations* and *weak equivalences*, subject to certain axioms. Any simplicial set X is *cofibrant*, meaning that the unique map $\varnothing \longrightarrow X$ from the initial object to X is in cof. Dually, a simplicial set X is *fibrant*, meaning that the unique map $X \longrightarrow *$ from X to the terminal object is in fib, if and only if it is Kan. More generally, a simplicial map is a fibration precisely when it is a Kan fibration. Any contractible augmented simplicial set $\epsilon : A \longrightarrow A_{-1}$ gives rise to a weak equivalence $\overline{\epsilon} : A \longrightarrow \overline{A_{-1}}$. Here $\overline{A_{-1}}$ denotes the constant functor $\Delta^{\mathrm{op}} \longrightarrow \mathsf{Set}$ mapping every object of Δ^{op} to A_{-1} and every morphism to $1_{A_{-1}}$. Conversely, if $\overline{\epsilon} : A \longrightarrow \overline{A_{-1}}$ is a weak equivalence, then $\epsilon : A \longrightarrow A_{-1}$ is contractible. The category of simplicial objects in the category Set_* of pointed sets and basepoint-preserving maps has a model structure, induced by the one on $\mathcal{S}\mathsf{Set}$, as follows: a pointed simplicial map is a fibration, cofibration or weak equivalence if and only if it is in $\mathcal{S}\mathsf{Set}$. Finally, we need the following

6.6. Lemma. [28, Proposition I.3.5] *Let $(\mathcal{C}, \mathrm{fib}, \mathrm{cof}, \mathrm{we})$ be a pointed model category. If, in the diagram*

$$\begin{array}{ccccc} K[p] & \xrightarrow{\mathsf{Ker}\, p} & E & \xrightarrow{p} & B \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\alpha} \\ K[p'] & \xrightarrow[\mathsf{Ker}\, p']{} & E' & \xrightarrow[p']{} & B' \end{array}$$

of $\mathcal{C}$, every object is both fibrant and cofibrant, p and p' are fibrations, and α and β are weak equivalences, then the induced map γ is a weak equivalence. ∎

Keeping in mind that a functor category $\mathsf{Fun}(\mathcal{C}, \mathcal{A})$ has the limits and colimits of $\mathcal{A}$, computed pointwise, the following follows immediately from the definitions.

6.7. Lemma. *Let $\mathcal{C}$ be a small category, $\mathcal{A}$ a semi-abelian category and $U : \mathcal{A} \longrightarrow \mathcal{B}$ a reflector onto a Birkhoff subcategory $\mathcal{B}$ of $\mathcal{A}$. Then*

1. *the functor category $\mathsf{Fun}(\mathcal{C}, \mathcal{A})$ is semi-abelian;*
2. *$\mathsf{Fun}(\mathcal{C}, \mathsf{Pr}\mathcal{A}) = \mathsf{PrFun}(\mathcal{C}, \mathcal{A})$;*
3. *$\mathsf{Fun}(\mathcal{C}, \mathcal{B})$ is a Birkhoff subcategory of $\mathsf{Fun}(\mathcal{C}, \mathcal{A})$;*

4. *the functor* $\mathsf{Fun}(\mathcal{C}, U) = U \circ (\cdot) : \mathsf{Fun}(\mathcal{C}, \mathcal{A}) \longrightarrow \mathsf{Fun}(\mathcal{C}, \mathcal{B})$ *is its reflector;*

5. $V_1^{\mathsf{Fun}(\mathcal{C},\mathcal{B})} = \mathsf{Fun}\big(\mathcal{C}, V_1^{\mathcal{B}}\big) : \mathsf{Fun}(\mathcal{C}, \mathsf{Pr}\mathcal{A}) \longrightarrow \mathsf{Fun}(\mathcal{C}, \mathcal{A})$*, where* $V^{\mathsf{Fun}(\mathcal{C},\mathcal{B})}$ *and* $V^{\mathcal{B}}$ *are the Birkhoff subfunctors associated with* $\mathsf{Fun}(\mathcal{C}, \mathcal{B})$ *and* $\mathcal{B}$*, respectively.*

These properties hold, in particular, when $\mathcal{C}$ *is* Δ^{op}*, i.e. when all functor categories are categories of simplicial objects and simplicial maps.* ∎

6.8. Remark. By the way of obtaining the sequence **A** from the given short exact sequence—applying the Snake Lemma, see [18]—we get that the square in

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_1 f & \rightarrowtail & K & \twoheadrightarrow & \frac{K}{V_1 f} & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & A & \xrightarrow[\eta_A]{} & U(A) & & \end{array}$$

is commutative.

6.9. Theorem. *For any object* X *of* $\mathcal{A}$*,* $H_2(X, U)_{\mathbb{G}} \cong \Delta V(X)$*.*

Proof. By Remark 6.4, the tail of the exact sequence **A**, induced by the short exact sequence

$$0 \longrightarrow K[\epsilon_X] \xrightarrow{\mathsf{Ker}\,\epsilon_X} GX \xrightarrow{\epsilon_X} X \longrightarrow 0,$$

becomes

$$0 \longrightarrow \Delta V(X) \rightarrowtail \frac{K[\epsilon_X]}{V_1 \epsilon_X} \xrightarrow{\psi} UGX.$$

Keeping in mind Lemma 6.7, note that $U\mathbb{G}\epsilon_X : U\mathbb{G}GX \longrightarrow U\mathbb{G}X$ is a presentation in $\mathcal{SA}$. By Remark 6.4, $\Delta V \mathbb{G}X$ is zero: it can be computed pointwise and every $(\mathbb{G}X)_n$ is projective. Hence, Theorem 1.2 induces the exact sequence

$$0 \longrightarrow \frac{K[\mathbb{G}\epsilon_X]}{V_1 \mathbb{G}\epsilon_X} \xrightarrow{\mathsf{Ker}\, U\mathbb{G}\epsilon_X} U\mathbb{G}GX \xrightarrow{U\mathbb{G}\epsilon_X} U\mathbb{G}X \longrightarrow 0$$

of simplicial objects in $\mathcal{B}$. Now Corollary 5.7 implies that

$$0 \longrightarrow H_2(X, U)_{\mathbb{G}} \rightarrowtail H_0 \frac{K[\mathbb{G}\epsilon_X]}{V_1 \mathbb{G}\epsilon_X} \xrightarrow{\varphi} UGX \qquad (\mathbf{J})$$

is an exact sequence in $\mathcal{B}$. Indeed, recalling the notation from 5.8, as $\mathbb{G}GX = (\mathbb{G}X)^-$, $\epsilon_{GX} : \mathbb{G}GX \longrightarrow GX$ is a contractible augmented simplicial object. Proposition 3.11 implies that $H_1 U\mathbb{G}GX = 0$ and $H_0 U\mathbb{G}GX = UGX$.

Accordingly, $\Delta V(X)$ is a kernel of ψ and $H_2(X, U)_{\mathbb{G}}$ is a kernel of φ. We prove that ψ and φ are equal.

Consider the following diagram, where the κ_i are the face operators of $K[\mathbb{G}\epsilon_X]$ and κ is defined as $K(\epsilon_{GX})$, as pictured in diagram **L** below.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V_1G^2\epsilon_X & \rightarrowtriangle & K[G^2\epsilon_X] & \rightarrowtriangle & \frac{K[G^2\epsilon_X]}{V_1G^2\epsilon_X} & \longrightarrow & 0 \\
 & & {\scriptstyle V_1(\epsilon_{G^2X},\epsilon_{GX})}\downdownarrows{\scriptstyle V_1(G\epsilon_{GX},G\epsilon_X)} & & {\scriptstyle\kappa_0}\downdownarrows{\scriptstyle\kappa_1} & & {\scriptstyle\frac{\kappa_0}{V_1(\epsilon_{G^2X},\epsilon_{GX})}}\downdownarrows{\scriptstyle\frac{\kappa_1}{V_1(G\epsilon_{GX},G\epsilon_X)}} & & \\
0 & \longrightarrow & V_1G\epsilon_X & \rightarrowtriangle & K[G\epsilon_X] & \xrightarrow{p} & \frac{K[G\epsilon_X]}{V_1G\epsilon_X} & \longrightarrow & 0 \\
 & & {\scriptstyle V_1(\epsilon_{GX},\epsilon_X)}\downarrow & & {\scriptstyle\kappa}\downarrow & \text{(I)} & \downarrow{\scriptstyle\frac{\kappa}{V_1(\epsilon_{GX},\epsilon_X)}} & & \\
0 & \longrightarrow & V_1\epsilon_X & \rightarrowtriangle & K[\epsilon_X] & \xrightarrow[q]{} & \frac{K[\epsilon_X]}{V_1\epsilon_X} & \longrightarrow & 0
\end{array} \qquad (\mathbf{K})$$

We claim that, if the middle fork in **K** is a coequalizer diagram, then φ equals ψ. Indeed, by Remark 6.3 and Proposition 3.9, the square

$$\begin{array}{ccc}
G^2X & \xrightarrow{G\epsilon_X} & GX \\
{\scriptstyle\epsilon_{GX}}\downarrow & & \downarrow{\scriptstyle\epsilon_X} \\
GX & \xrightarrow[\epsilon_X]{} & X
\end{array}$$

is a pushout; Proposition 5.1 and 5.2 of [18] now imply that $V_1(\epsilon_{GX}, \epsilon_X)$ is a regular epimorphism. We get that (I) is a pushout. Hence, if the middle fork is a coequalizer diagram, so is the right fork, and then Corollary 3.10 implies that

$$\frac{K[\epsilon_X]}{V_1\epsilon_X} = H_0\frac{K[\mathbb{G}\epsilon_X]}{V_1\mathbb{G}\epsilon_X}.$$

Consequently, using the functoriality of H_0, we get that φ is the unique morphism such that the right hand square (III) in the diagram

$$\begin{array}{ccccc}
K[G\epsilon_X] & \xrightarrow{p} & \frac{K[G\epsilon_X]}{V_1G\epsilon_X} & \xrightarrow{\frac{\kappa}{V_1(\epsilon_{GX},\epsilon_X)}} & \frac{K[\epsilon_X]}{V_1\epsilon_X} \\
{\scriptstyle\mathsf{Ker}\,G\epsilon_X}\downarrow & \text{(II)} & {\scriptstyle\mathsf{Ker}\,UG\epsilon_X}\downarrow & \text{(III)} & \vdots{\scriptstyle\varphi} \\
G^2X & \xrightarrow[\eta_{G^2X}]{} & UG^2X & \xrightarrow[U\epsilon_{GX}]{} & UGX
\end{array}$$

commutes. By Remark 6.8, also the square (II) is commutative. Further note—again using Remark 6.8—that ψ is unique in making

$$\begin{array}{ccc}
K[\epsilon_X] & \xrightarrow{q} & \frac{K[\epsilon_X]}{V_1\epsilon_X} \\
{\scriptstyle\mathsf{Ker}\,\epsilon_X}\downarrow & & \downarrow{\scriptstyle\psi} \\
GX & \xrightarrow[\eta_{GX}]{} & UGX
\end{array}$$

commute. Now

$$\begin{aligned}\varphi\circ\overline{\tfrac{\kappa}{V_1(\epsilon_{GX},\epsilon_X)}}\circ p &= U\epsilon_{GX}\circ\eta_{G^2X}\circ\mathsf{Ker}\,G\epsilon_X\\ &= \eta_{GX}\circ\epsilon_{GX}\circ\mathsf{Ker}\,G\epsilon_X\\ &= \eta_{GX}\circ\mathsf{Ker}\,\epsilon_X\circ\kappa\\ &= \psi\circ q\circ\kappa\\ &= \psi\circ\overline{\tfrac{\kappa}{V_1(\epsilon_{GX},\epsilon_X)}}\circ p,\end{aligned}$$

where the second equality follows from the naturality of η, and the third one holds by definition of κ. This proves our claim that $\varphi = \psi$.

To see that the middle fork in diagram **K** is indeed a coequalizer diagram, note that it is defined by the exactness of the rows in the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K[\mathbb{G}\epsilon_X] & \rightarrowtail & \mathbb{G}GX & \xrightarrow{\mathbb{G}\epsilon_X} & \mathbb{G}X & \longrightarrow & 0\\ & & {\scriptstyle\kappa}\vdots & & \downarrow{\scriptstyle\epsilon_{GX}} & & \downarrow{\scriptstyle\epsilon_X} & & \\ 0 & \longrightarrow & K[\epsilon_X] & \rightarrowtail & GX & \xrightarrow[\epsilon_X]{} & X & \longrightarrow & 0. \end{array} \qquad (\mathbf{L})$$

(Read it as a diagram in $\mathcal{A}$.) Since $\mathcal{A}$ is a pointed category, we get a diagram of pointed simplicial sets

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Upsilon K[\mathbb{G}\epsilon_X] & \rightarrowtail & \Upsilon\mathbb{G}GX & \xrightarrow{\Upsilon\mathbb{G}\epsilon_X} & \Upsilon\mathbb{G}X\\ & & \downarrow{\scriptstyle\Upsilon\overline{\kappa}} & & \downarrow{\scriptstyle\Upsilon\overline{\epsilon_{GX}}} & & \downarrow{\scriptstyle\Upsilon\overline{\epsilon_X}}\\ 0 & \longrightarrow & \Upsilon\overline{K[\epsilon_X]} & \rightarrowtail & \Upsilon\overline{GX} & \xrightarrow[\Upsilon\epsilon_X]{} & \Upsilon\overline{X}. \end{array} \qquad (\mathbf{M})$$

Indeed, as a right adjoint functor, Υ preserves limits, so $\Upsilon(0)$ is a terminal object $*$ of Set. Moreover, the rows of Diagram **M** are exact. Considered as unpointed augmented simplicial sets, $\Upsilon\epsilon_X : \Upsilon\mathbb{G}X \longrightarrow \Upsilon X$ and $\Upsilon\epsilon_{GX} : \Upsilon\mathbb{G}GX \longrightarrow \Upsilon GX$ are contractible: e.g. for the first one, a contraction is defined by $f_n = \zeta_{\Upsilon G^{n+1}X}$, where $\zeta : 1_{\mathcal{A}} \Longrightarrow \Phi\circ\Upsilon$ is the unit of the adjunction $\mathsf{Set} \rightharpoonup \mathcal{A}$. (This is Proposition 5.3 of Barr and Beck [1].) It follows that $\Upsilon\overline{\epsilon_X}$ and $\Upsilon\overline{\epsilon_{GX}}$ are weak equivalences of $\mathcal{S}\mathsf{Set}$ and, consequently, also of $\mathcal{S}\mathsf{Set}_*$. Furthermore, Proposition 4.4 and Proposition 4.2 imply that $\Upsilon\mathbb{G}\epsilon_X$ and $\Upsilon\epsilon_X$ are fibrations between fibrant objects. Now Lemma 6.6 applies, and $\Upsilon\overline{\kappa}$ is a weak equivalence. But this means that $\Upsilon\kappa : \Upsilon K[\mathbb{G}\epsilon_X] \longrightarrow \Upsilon K[\epsilon_X]$ is a contractible augmented simplicial object, and then Beck's Theorem implies (see, again, the proof of Theorem VI.7.1 in Mac Lane [27] or Lemma II.4.3.3 in Borceux [4]) that the middle fork in **K** is a coequalizer. ■

Combining Theorems 1.2 and 6.9 yields the following generalization of the Stallings-Stammbach sequence.

6.10. Corollary. *Let*

$$0 \longrightarrow K \rightarrowtail A \xrightarrow{f} B \longrightarrow 0$$

be a short exact sequence in $\mathcal{A}$. There exists an exact sequence

$$H_2(A,U)_{\mathbb{G}} \xrightarrow{H_2(f,U)_{\mathbb{G}}} H_2(B,U)_{\mathbb{G}} \longrightarrow \frac{K}{V_1 f} \longrightarrow H_1(A,U)_{\mathbb{G}} \xrightarrow{H_1(f,U)_{\mathbb{G}}} H_1(B,U)_{\mathbb{G}} \longrightarrow 0$$

in $\mathcal{B}$ which depends naturally on the given short exact sequence. ∎

6.11. Example. [Crossed modules] Recall that a *crossed module* (T,G,∂) is a group homomorphism $\partial : T \longrightarrow G$ together with an action of G on T (mapping a couple $(g,t) \in G \times T$ to ${}^g t \in T$) satisfying

1. $\partial({}^g t) = g\partial t g^{-1}$, for all $g \in G$, $t \in T$;
2. ${}^{\partial t}s = tst^{-1}$, for all $s,t \in T$.

A *morphism of crossed modules* $(f,\phi) : (T,G,\partial) \longrightarrow (T',G',\partial')$ is a pair of group homomorphisms $f : T \longrightarrow T'$, $\phi : G \longrightarrow G'$ with

1. $\partial' \circ f = \phi \circ \partial$;
2. $f({}^g t) = {}^{\phi(g)} f(t)$, for all $g \in G$, $t \in T$.

It is well known that CM is equivalent to a variety of Ω-groups, namely to the variety of 1-categorical groups (see Loday [26]). Hence, it is semi-abelian [25]. Moreover, under this equivalence, a crossed module (T,G,∂) corresponds to the semidirect product $G \ltimes T$ equipped with the two appropriate homomorphisms. $G \ltimes T$ and $G \times T$ have the same underlying set; thus the forgetful functor $\mathcal{U} : \mathsf{CM} \longrightarrow \mathsf{Set}$ sends a crossed module (T,G,∂) to the product $T \times G$ of its underlying sets. This determines a comonad $\mathbb{G}$ on CM. We get the cotriple homology of crossed modules described by Carrasco, Cegarra and Grandjeán in [16] as a particular case of Definition 6.1 by putting U the usual abelianization functor $\mathrm{ab} : \mathsf{CM} \longrightarrow \mathsf{CM}_{\mathsf{Ab}}$ (as defined for Ω-groups, see Higgins [20]).

In [16], Carrasco, Cegarra and Grandjeán give an explicit proof that

$$\big[(T,G,\partial),(T,G,\partial)\big] = V(T,G,\partial)$$

equals $([T,G],[G,G],\partial)$. For any crossed module (T,G,∂), and any two normal subgroups $K \lhd G$, $S \lhd T$, let $[K,S]$ denote the (normal) subgroup of T generated by the elements $({}^k s)s^{-1}$, for $k \in K$, $s \in S$. It may be shown that, for $(N,R,\partial) \lhd (Q,F,\partial)$,

$$\big[(N,R,\partial),(Q,F,\partial)\big] = V_1\Big((Q,F,\partial) \longrightarrow \tfrac{(Q,F,\partial)}{(N,R,\partial)}\Big)$$

is equal to $([R,Q][F,N],[R,F],\partial)$, but the proof involves somewhat fussy calculations, so will be omitted.

We recall the definition from [16]. Let (T,G,∂) be a crossed module and $n \geq 1$. The n-th homology object $H_n(T,G,\partial)$ of (T,G,∂) is

$$H_{n-1}C\mathbb{G}(T,G,\partial)_{\mathrm{ab}}.$$

Here $C\mathbb{G}(T,G,\partial)_{\mathrm{ab}}$ is the *unnormalized chain complex* associated with $\mathbb{G}(T,G,\partial)_{\mathrm{ab}}$, defined by $(C\mathbb{G}(T,G,\partial)_{\mathrm{ab}})_n = (\mathbb{G}(T,G,\partial)_{\mathrm{ab}})_n$ and

$$d_n = \partial_0^{\mathrm{ab}} - \partial_1^{\mathrm{ab}} + \ldots + (-1)^n \partial_n^{\mathrm{ab}}.$$

$\mathbb{G}(T,G,\partial)_{\mathrm{ab}}$ denotes the simplicial abelian crossed module $\mathrm{ab}\circ\mathbb{G}(T,G,\partial)$. The ∂_i^{ab} are its face operators.

$C\mathbb{G}(T,G,\partial)_{\mathrm{ab}}$ need not be the same as $N\mathbb{G}(T,G,\partial)_{\mathrm{ab}}$. However, their homology objects are equal, since $\mathbb{G}(T,G,\partial)_{\mathrm{ab}}$ is a simplicial object in the abelian category $\mathsf{CM}_{\mathsf{Ab}}$ (see, for instance, Weibel [31, Theorem 8.3.8]). Hence $H_n(T,G,\partial) = H_n((T,G,\partial),\mathrm{ab})_{\mathbb{G}}$.

The Hopf formula obtained in [16, Theorem 13 and above] is a particularization of our Theorem 6.9. In this particular case, the exact sequence of Corollary 6.10 becomes the exact sequence of [16, Theorem 12 (i)].

6.12. Example. [Groups] It is shown in [16, Theorem 10] that cotriple homology of crossed modules encompasses classical group homology; hence, so does our theory. If

$$0 \longrightarrow R \rightarrowtail F \twoheadrightarrow G \longrightarrow 0$$

is a presentation of a group G by generators and relations, $U : \mathsf{Gp} \longrightarrow \mathsf{Gp}_{\mathsf{Ab}} = \mathsf{Ab}$ is the abelianization functor and $\mathbb{G}$ the "free group on a set"-monad, the sequence in Corollary 6.10 becomes the Stallings-Stammbach sequence in integral homology of groups [29], [30]. The isomorphism in Theorem 6.9 is nothing but Hopf's formula [21]

$$H_2(G,U)_{\mathbb{G}} \cong \frac{R \cap [F,F]}{[R,F]}.$$

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