

# Simulations in Coalgebra

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## Abstract

A new approach to simulations is proposed within the theory of coalgebras by taking a notion of order on a functor as primitive. Such an order forms a basic building block for a “lax relation lifting”, or “relator” as used by other authors. Simulations appear as coalgebras of this lifted functor, and similarity as greatest simulation. Two-way similarity is then similarity in both directions. In general, it is different from bisimilarity (in the usual coalgebraic sense), but a sufficient condition is formulated (and illustrated) to ensure that bisimilarity and two-way similarity coincide. Also, suitable conditions are identified which ensures that similarity on a final coalgebra forms an (algebraic) dcpo structure. This involves a close investigation of the iterated applications  $F^n(\emptyset)$  and  $F^n(1)$  of a functor  $F$  with an order to the initial and final sets.

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## 1 Introduction

Simulations are relations between one (dynamical) system and another, expressing that if one system can do a move, then the other can do a similar move. Simulations are heavily used for transition systems and automata (see *e.g.* [13]), especially for refinement proofs. Also, they are studied in modal logic [2], domain theory [14,6], category theory [19] (using spans, following earlier, unpublished work of Claudio Hermida on modules). Here we study simulations in a purely coalgebraic context, starting from a new, elementary notion of ordering on a functor, and using familiar techniques based on “relation lifting” or “relators”. An early version appeared as [?].

The main contribution of the paper is systematisation, namely, systematisation of the definition, examples, results (for instance, about the properties of the order) and connections (*e.g.* between two-way similarity and bisimilarity). But many research issues remain.

Part of our work, especially in Sections 8–10, is also closely related to Jiří Adámek’s development in [1]. There, he defined an order on the final coalgebra  $Z$  of a functor  $F$  such that  $Z$  was the ideal completion of the initial algebra (under the same order). His order is not typically a simulation for some order  $\sqsubseteq$ , as we study here, and so his result is not subsumed by our work here. Nonetheless, his approach informed many of the choices we made, especially our emphasis of bottom elements.

The paper starts with our main definition, namely of order on a functor in Section 2. These orders are combined with ordinary relation lifting (recalled in Section 3) to form “lax relation liftings” in Section 4. Simulations then appear as coalgebras of such lax relation lifting functors. Similarity is the greatest simulation, and two-way similarity is similarity in both directions. Its relation with ordinary bisimilarity is established in Section 6. Section 7 turns the similarity order on a final coalgebra into a dcpo structure in presence of a certain distributive law, or equivalently, a preservation property. Section 8 investigates the order on sets of terms  $F^n(\emptyset)$  and observations  $F^n(1)$  that arise in the construction of initial algebras and final coalgebras as  $\omega$ -(co)limits. Section 9 establishes that the limit order on such a final coalgebra coincides with similarity (under a suitable preservation property). Section 10 then describes conditions that guarantee that the final coalgebra forms an *algebraic* cpo in which the finite elements arise from the finite elements from  $F^n(1)$ . Finally, Section 11 shows how elements of  $F^n(\emptyset)$  appear within a final coalgebra as those elements without infinite transitions.

## 2 Orders on functors

We shall write **Sets** for the category of sets and functions, and **PreOrd** for the category of preorders  $(X, \leq)$  (with  $\leq$  a reflexive and transitive relation on  $X$ ) and order-preserving (monotone) functions between them. There is an obvious forgetful functor **PreOrd**  $\rightarrow$  **Sets** sending a preorder  $(X, \leq)$  to its underlying set  $X$ . This functor will remain unnamed.

**Definition 2.1** *Let  $F: \mathbf{Sets} \rightarrow \mathbf{Sets}$  be an arbitrary endofunctor on **Sets**. We define an **order on  $F$**  to be a functor  $\sqsubseteq: \mathbf{Sets} \rightarrow \mathbf{PreOrd}$  making the*

following diagram commute.

$$\begin{array}{ccc}
 & & \mathbf{PreOrd} \\
 & \sqsubseteq & \downarrow \\
 \mathbf{Sets} & \xrightarrow{F} & \mathbf{Sets}
 \end{array}$$

In this paper our examples are of a set-theoretic nature, so we restrict the above notion to endofunctors on sets, and we do not strive for the highest level of generality. But it is very easy to generalise it to other categories  $\mathbb{C}$ . The category **PreOrd** should then be suitably replaced by a category of preorders in  $\mathbb{C}$  (or even a fibred category of preorder relations over  $\mathbb{C}$  in some logic).

In concrete terms, an order  $\sqsubseteq$  on a functor  $F$ , as just defined, assigns to each set  $X$  a preorder  $\sqsubseteq_X \subseteq F(X) \times F(X)$  such that, for any **Sets**-map  $f: X \rightarrow Y$ , the function  $Ff: FX \rightarrow FY$  is monotone with respect to  $\sqsubseteq_X$  and  $\sqsubseteq_Y$ . Preorderedness seems to be the minimal requirement that one wishes to impose on such orders in the current setting.

Often, like in [14,6], notions of simulation are studied in an ordered setting, where the functor  $F$  acts on some category of dcpos. In that case each  $X$  and  $F(X)$  is a dcpo and thus automatically carries on order. Our approach is minimal in a sense, because it only requires an order on the images  $F(X)$  of  $F$ , and not on arbitrary objects.

**Example 2.2** *We illustrate the notion of order on a functor in the following examples.*

- (1) *For each functor  $F: \mathbf{Sets} \rightarrow \mathbf{Sets}$  we have both the discrete order (only equal elements are related) and the indiscrete one (any two elements are related).*
- (2) *Consider the functor  $\mathcal{S}(X) = 1 + (A \times X)$  which adds a bottom element  $*$  to a product set  $A \times X$ , where  $A$  is an arbitrary, fixed set. The behaviours of coalgebras of this functor consist of both finite and infinite sequences of elements of  $A$ . The sets  $\mathcal{S}(X)$  carry the familiar “flat” order: for  $u, v \in \mathcal{S}(X)$ ,*

$$\begin{aligned}
 u \sqsubseteq v &\iff u \neq * \Rightarrow u = v \\
 &\iff \forall a \in A. \forall x \in X. u = (a, x) \Rightarrow v = (a, x).
 \end{aligned}$$

*(In this formulation we have left the coproduct coprojections  $1 \xrightarrow{\kappa_1} 1 + (A \times X) \xleftarrow{\kappa_2} A \times X$  implicit.)*

- (3) *Next we consider the list (or free monoid) functor  $\mathcal{L}(X) = X^*$ . A coalgebra of this functor maps an element to a finite list of successor states  $\langle x_0, \dots, x_{n-1} \rangle$ , so that order and multiplicity of such states matter. Sev-*

eral orderings on  $\mathcal{L}$  are possible, which may or may not take the order and multiplicity into account.

$$\begin{aligned} \langle x_0, \dots, x_{n-1} \rangle \sqsubseteq_1 \langle y_0, \dots, y_{m-1} \rangle \\ \iff \text{there is a strictly monotone function} \\ \varphi: \{0, 1, \dots, n-1\} \rightarrow \{0, 1, \dots, m-1\} \\ \text{with } x_i = y_{\varphi(i)}, \text{ for } i < n. \end{aligned}$$

Strict monotonicity means that  $i < j$  implies  $\varphi(i) < \varphi(j)$ . As a result,  $\varphi$  is injective, and  $n \leq m$ . This order  $\sqsubseteq_1$  basically says that the smaller sequence can be obtained by removing elements from the bigger one.

Our second ordering on  $\mathcal{L}$  is much simpler, and ignores much of the existing structure:

$$\langle x_0, \dots, x_{n-1} \rangle \sqsubseteq_2 \langle y_0, \dots, y_{m-1} \rangle \iff \forall i < n. \exists j < m. x_i = y_j.$$

Thus, for different elements  $x, y, z \in X$  we have  $\langle x, z \rangle \sqsubseteq_i \langle x, x, y, z \rangle$  for both  $i = 1, 2$ . But  $\langle y, x, x \rangle \sqsubseteq_i \langle x, y \rangle$  only holds for  $i = 2$ . Clearly,  $\sqsubseteq_1 \subseteq \sqsubseteq_2$ .

- (4) Our next example involves the related “bag” functor  $\mathcal{B}$ , capturing free commutative monoids (as algebras of the associated monad). It can be described as:

$$\mathcal{B}(X) = \{\alpha: X \rightarrow \mathbb{N} \mid \text{only finitely many } x \in X \text{ have } \alpha(x) \neq 0\}.$$

Often one says that such an  $\alpha$  has “finite support”. When using the bag instead of the list functor, we care about multiplicities  $\alpha(x)$  of elements  $x \in X$ , but not about the order in which they occur. Like before we consider two orderings on the functor  $\mathcal{B}$ . The first explicitly includes a multiplicity requirement:

$$\alpha \sqsubseteq_1 \beta \iff \forall x \in X. \alpha(x) \leq \beta(x).$$

When we wish to ignore multiplicities and only consider occurrences we order as follows:

$$\alpha \sqsubseteq_2 \beta \iff \forall x \in X. \alpha(x) \neq 0 \Rightarrow \beta(x) \neq 0.$$

This says that if  $x$  occurs in  $\alpha$ , then it should also occur in  $\beta$ , without regard to the multiplicities of each.

- (5) Our final example involves the powerset functor  $\mathcal{P}$  with a set  $A$  of “labels”, in the functor  $\mathcal{T}(X) = \mathcal{P}(X)^A \cong \mathcal{P}(A \times X)$ . As is well-known, coalgebras of this functor are labeled transition systems. The obvious order on  $\alpha, \beta \in \mathcal{T}(X)$  is pointwise inclusion:

$$\alpha \sqsubseteq \beta \iff \forall a \in A. \alpha(a) \subseteq \beta(a).$$

At the end of this section we like to point out that our general notion of order on a functor, as given in Definition 2.1, allows us to formulate general results like: given a natural transformation  $\sigma: F \Rightarrow G$ , then an order  $\sqsubseteq^G$  on  $G$  induces an order  $\sqsubseteq^F \stackrel{\text{def}}{=} \sigma^*(\sqsubseteq^G)$  on  $F$ , namely as  $u \sqsubseteq^F v \iff \sigma_X(u) \sqsubseteq^G \sigma_X(v)$ , for  $u, v \in F(X)$ . In this way one can organise orders in a category which is fibred over a category of endofunctors.

Also, for a functor  $F$  with order  $\sqsubseteq$  one can define a category  $\text{CoAlg}_{\sqsubseteq}(F)$  of  $F$ -coalgebras with “simulation mappings”: a map  $f$  from  $X \xrightarrow{c} F(X)$  to  $Y \xrightarrow{d} F(Y)$  in  $\text{CoAlg}_{\sqsubseteq}(F)$  is then a function  $f: X \rightarrow Y$  with  $F(f)(c(x)) \sqsubseteq d(f(x))$  on  $F(Y)$ . Such a category is sometimes used for transition systems, if one wants maps to only preserve (and not reflect) transitions.

### 3 A recap on relation lifting and bisimulations

We shall write **Rel** for the category of binary relations. Its objects are arbitrary relations  $R \subseteq X_1 \times X_2$ ; and its morphisms from  $R \subseteq X_1 \times X_2$  to  $S \subseteq Y_1 \times Y_2$  are pairs of functions  $f_1: X_1 \rightarrow Y_1, f_2: X_2 \rightarrow Y_2$  between the underlying sets which preserve the relation, in the sense that  $R(x_1, x_2) \implies S(f_1(x_1), f_2(x_2))$ . There is then an obvious forgetful functor  $\mathbf{Rel} \rightarrow \mathbf{Sets} \times \mathbf{Sets}$  mapping a relation to its underlying sets. Notice that there is a full and faithful embedding  $\mathbf{PreOrd} \hookrightarrow \mathbf{Rel}$ , describing preorders as a subcategory.

It is fairly standard in the theory of coalgebras [8,11] to associate with an endofunctor  $F: \mathbf{Sets} \rightarrow \mathbf{Sets}$  a **relation lifting**  $\text{Rel}(F): \mathbf{Rel} \rightarrow \mathbf{Rel}$  in a diagram:

$$\begin{array}{ccc} \mathbf{Rel} & \xrightarrow{\text{Rel}(F)} & \mathbf{Rel} \\ \downarrow & & \downarrow \\ \mathbf{Sets} \times \mathbf{Sets} & \xrightarrow{F \times F} & \mathbf{Sets} \times \mathbf{Sets} \end{array}$$

For an arbitrary functor, this relation lifting  $\text{Rel}(F)$  can be defined on a relation  $\langle r_1, r_2 \rangle: R \hookrightarrow X_1 \times X_2$  by taking the image of the pair

$$\langle F(r_1), F(r_2) \rangle: F(R) \longrightarrow F(X_1) \times F(X_2),$$

see *e.g.* [4,15]. In the language of fibred categories, then,

$$\text{Rel}(F)(R) = \coprod_{(Fr_1, Fr_2)} F(R)$$

and in set-theoretic terms,

$$\begin{aligned} \text{Rel}(F)(R) \\ = \{(u, v) \in FX_1 \times FX_2 \mid \exists w \in F(R). F(r_1)(w) = u \text{ and } F(r_2)(w) = v\}. \end{aligned}$$

For the special case of polynomially defined functors  $F$ ,  $\text{Rel}(F)$  may equivalently be defined by induction the structure of  $F$ , see *e.g.* [11].

This relation lifting is assumed to satisfy the following properties.

- (1) Equality is preserved:  $\text{Rel}(F)(=_X) = =_{F(X)}$ .
- (2) Composition is preserved: for  $R \subseteq X \times Y$  and  $S \subseteq Y \times Z$ , the relational composition  $S \circ R = \{(x, z) \mid \exists y. R(x, y) \wedge S(y, z)\}$  satisfies:

$$\text{Rel}(F)(S \circ R) = \text{Rel}(F)(S) \circ \text{Rel}(F)(R).$$

- (3) Inclusions are preserved: if  $R \subseteq S$  then  $\text{Rel}(F)(R) \subseteq \text{Rel}(F)(S)$ .
- (4) Reversals are preserved:  $\text{Rel}(F)(R^{\text{op}}) = \text{Rel}(F)(R)^{\text{op}}$ .
- (5) Inverse images (or substitution, or reindexing) is preserved: for functions  $f_1: X_1 \rightarrow Y_1$ ,  $f_2: X_2 \rightarrow Y_2$  and a relation  $S \subseteq Y_1 \times Y_2$  we have:

$$\text{Rel}(F)\left((f_1 \times f_2)^{-1}(S)\right) = (F(f_1) \times F(f_2))^{-1}\left(\text{Rel}(F)(S)\right).$$

All these properties hold for functors  $F$  that preserve weak pullbacks.

For example, as a consequence, the graph relation

$$\mathbf{Graph}(f) = (f \times id)^*(=_{Y}) \subseteq X \times Y$$

of a function  $f: X \rightarrow Y$  satisfies

$$\text{Rel}(F)\left(\mathbf{Graph}(f)\right) = \mathbf{Graph}(F(f)).$$

A **bisimulation** is then just a  $\text{Rel}(F)$ -coalgebra. It is a map in **Rel** over two maps in **Sets**, which are the underlying coalgebras. Concretely, in terms of such coalgebras  $c: X \rightarrow F(X)$  and  $d: Y \rightarrow F(Y)$  of the same functor  $F$ , a bisimulation (between  $c$  and  $d$ ) is a relation  $R \subseteq X \times Y$  satisfying for all  $x \in X$  and  $y \in Y$ ,

$$R(x, y) \implies \text{Rel}(F)(R)(c(x), d(y)).$$

Or, pictorially, as a map in **Rel**:

$$\begin{array}{ccc}
 R & \dashrightarrow & \text{Rel}(F)(R) \\
 \downarrow & & \downarrow \\
 X \times Y & \xrightarrow{c \times d} & F(X) \times F(Y)
 \end{array}$$

The next result mentions some standard properties (see *e.g.* [16]) that are relevant in the current setting. Proofs are omitted.

**Proposition 3.1** *Let  $F$  be an endofunctor on **Sets** with a relation lifting functor  $\text{Rel}(F)$  as described above. Then, with respect to coalgebras  $X \xrightarrow{c} FX$  and  $Y \xrightarrow{d} FY$  one has that:*

- (1) *Bisimulations are closed under arbitrary unions; as a result, there is a greatest bisimulation relation  $\Leftrightarrow \subseteq X \times Y$ , which is called **bisimilarity**.*
- (2) *The equality relation  $=_X \subseteq X \times X$  is a bisimulation (for the single coalgebra  $c$ ). Similarly, bisimilarity  $\Leftrightarrow \subseteq X \times X$  is an equivalence relation.*
- (3) *An arbitrary function  $f: X \rightarrow Y$  is a homomorphism of coalgebras (that is, satisfies  $d \circ f = F(f) \circ c$ ) if and only if its graph relation  $\text{Graph}(f)$  is a bisimulation.*

*Hence if  $f$  is a homomorphism, then  $x \Leftrightarrow f(x)$ .*

- (4) *For a homomorphism  $f: X \rightarrow Y$  and elements  $x, x' \in X$  one has  $x \Leftrightarrow x'$  iff  $f(x) \Leftrightarrow f(x')$ .*
- (5) *If  $F$  has a final coalgebra  $Z \xrightarrow{\cong} FZ$ , then bisimilarity on  $Z$  is equality. Hence for  $x \in X$  and  $y \in Y$  one has  $x \Leftrightarrow y$  iff  $!(x) = !(y)$ —where  $!$  is the unique homomorphism to the final coalgebra.*

**Example 3.2** *We briefly describe bisimulations for the examples from the previous section.*

- (1) *Consider two coalgebras  $X \xrightarrow{c} \mathcal{S}(X)$ ,  $Y \xrightarrow{d} \mathcal{S}(Y)$  of the sequence functor  $\mathcal{S}(X) = 1 + (A \times X)$ . A relation  $R \subseteq X \times Y$  is a bisimulation iff for all  $x \in X$  and  $y \in Y$  with  $R(x, y)$  we have either  $c(x) = d(y) = *$ , or  $c(x) = (a, x')$  and  $d(y) = (b, y')$  with  $a = b$  and  $R(x', y')$ .*
- (2) *For two list-functor coalgebras  $X \xrightarrow{c} X^*$ ,  $Y \xrightarrow{d} Y^*$  we have  $z \Leftrightarrow w$  iff there is a relation  $R \subseteq X \times Y$  with  $R(z, w)$  and for all elements  $x \in X$  and  $y \in Y$ , if  $R(x, y)$ , then if  $c(x) = \langle x_0, \dots, x_{n-1} \rangle$  and if  $d(y) = \langle y_0, \dots, y_{m-1} \rangle$ , then  $n = m$  and  $R(x_i, y_i)$  for all  $i < n$ .*
- (3) *For bag-coalgebras  $X \xrightarrow{c} \mathcal{B}(X)$ ,  $Y \xrightarrow{d} \mathcal{B}(Y)$  the situation is more complicated. A relation  $R$  is a bisimulation iff for all  $x \in X$  and  $y \in Y$  with  $R(x, y)$  there is a  $\gamma: R \rightarrow \mathbb{N}$  such that the following hold.*
  - $\gamma(x, y) = 0$  for all but finitely many  $x$  and  $y$ .
  - $c(x)(x') = \sum_{y'} \{\gamma(x', y') \mid R(x', y')\}$

- $d(y)(y') = \sum_{x'} \{\gamma(x', y') \mid R(x', y')\}$ .
- (4) Finally, for transition system coalgebras  $X \xrightarrow{c} \mathcal{P}(X)^A$ ,  $Y \xrightarrow{d} \mathcal{P}(Y)^A$ , a relation  $R \subseteq X \times Y$  is a bisimulation as defined above iff it is a (strong) bisimulation in the usual sense: if  $R(x, y)$ , then both:
- if  $x \xrightarrow{a} x'$  (i.e.,  $x' \in c(x)(a)$ ), then there is an  $y' \in Y$  with  $y \xrightarrow{a} y'$  and  $R(x', y')$ .
  - if  $y \xrightarrow{a} y'$ , then there is an  $x' \in X$  with  $x \xrightarrow{a} x'$  and  $R(x', y')$ .

#### 4 Lax relation lifting and simulations

In the previous section we have seen how bisimulations were defined as coalgebras. We shall follow the same approach in this section for simulations. They are defined as coalgebras of a “lax relation lifting” functor  $\text{Rel}_{\sqsubseteq}(F)$  which is defined as a suitable combination of an order  $\sqsubseteq$  on an endofunctor  $F$  and standard relation lifting.

**Definition 4.1** For an endofunctor  $F: \mathbf{Sets} \rightarrow \mathbf{Sets}$  carrying a relation  $\sqsubseteq$  (as in Definition 2.1) we define a lax relation operation  $\text{Rel}_{\sqsubseteq}(F)$  as:

$$\begin{aligned}
R &\longmapsto \sqsubseteq \circ \text{Rel}(F)(R) \circ \sqsubseteq \\
&= \{(u, v) \mid \exists u', v'. u \sqsubseteq u' \wedge (u', v') \in \text{Rel}(F)(R) \wedge v' \sqsubseteq v\} \\
&= \{(u, v) \mid \exists w \in F(R). u \sqsubseteq F(r_1)(w) \wedge F(r_2)(w) \sqsubseteq v\} \\
&= (Fr_2 \times id)^{-1} \sqsubseteq \circ (id \times Fr_1)^{-1} \sqsubseteq,
\end{aligned}$$

where  $R$  has projections  $\langle r_1, r_2 \rangle: R \hookrightarrow X \times Y$ .

In other terms,

$$\text{Rel}_{\sqsubseteq}(F)(R) = \prod_{(\pi_1, \pi_3)} \left( (\pi_1, F(r_1) \circ \pi_2)^{-1}(\sqsubseteq_X) \wedge (F(r_2) \circ \pi_2, \pi_3)^{-1}(\sqsubseteq_Y) \right),$$

as in the diagram below.

$$\begin{array}{ccc}
FX \times FX & \xleftarrow{(\pi_1, F(r_1) \circ \pi_2)} & FX \times FR \times FY & \xrightarrow{(F(r_2) \circ \pi_2, \pi_3)} & FY \times FY \\
& & \downarrow (\pi_1, \pi_3) & & \\
& & FX \times FY & & 
\end{array}$$

A **simulation** is then defined as a  $\text{Rel}_{\sqsubseteq}(F)$ -coalgebra.

What we call lax relation lifting is called a relational extension in [9] and a (weak) relator in [18,2].



**Lemma 4.2** For  $F$  with order  $\sqsubseteq$  as above we have:

(1)  $Rel_{\sqsubseteq}(F)$  is a functor in commuting diagram:

$$\begin{array}{ccc} \mathbf{Rel} & \xrightarrow{Rel_{\sqsubseteq}(F)} & \mathbf{Rel} \\ \downarrow & & \downarrow \\ \mathbf{Sets} \times \mathbf{Sets} & \xrightarrow{F \times F} & \mathbf{Sets} \times \mathbf{Sets} \end{array}$$

- (2)  $Rel_{\sqsubseteq}(F)(=) = \sqsubseteq$ .  
(3)  $R \subseteq S \Rightarrow Rel_{\sqsubseteq}(F)(R) \subseteq Rel_{\sqsubseteq}(F)(S)$ .  
(4)  $Rel_{\sqsubseteq}(F)(R^{op}) = Rel_{\sqsubseteq^{op}}(F)(R)^{op}$   
(5) Simulations are closed under arbitrary unions.  
(6) If  $R$  is a bisimulation, then both  $R$  and  $R^{op}$  are simulations.  
(7) For every  $f: X \rightarrow Z$  and  $g: Y \rightarrow W$ ,

$$Rel_{\sqsubseteq}(F)\left((f \times g)^{-1}(R)\right) \subseteq (Ff \times Fg)^{-1}\left(Rel_{\sqsubseteq}(F)(R)\right).$$

(8) For every  $f: X \rightarrow Z$  and  $g: Y \rightarrow W$ ,

$$\coprod_{Ff \times Fg} \left(Rel_{\sqsubseteq}(F)(R)\right) \subseteq Rel_{\sqsubseteq}(F)\left(\coprod_{f \times g} R\right).$$

**Proof.** We prove each claim in turn.

(1) Consider a morphism  $R \rightarrow S$  in  $\mathbf{Rel}$ , consisting of relations  $R \subseteq X \times Y$  and  $S \subseteq Z \times W$  with functions  $f: X \rightarrow Z$  and  $g: Y \rightarrow W$  between the underlying sets with  $R(x, y) \Rightarrow S(f(x), g(y))$ . Assuming  $(u, v) \in Rel_{\sqsubseteq}(F)(R)$  we have to prove that  $(F(f)(u), F(g)(v)) \in Rel_{\sqsubseteq}(F)(S)$ . The assumption gives us  $u' \in F(X)$  and  $v' \in F(Y)$  with  $u \sqsubseteq u'$ ,  $(u', v') \in Rel(F)(R)$  and  $v' \sqsubseteq v$ . Since  $\sqsubseteq$  and  $Rel(F)$  are functors we then get  $Ff(u) \sqsubseteq Ff(u')$ ,  $(Ff(u'), Fg(v')) \in Rel(F)(S)$  and  $Fg(v') \sqsubseteq Fg(v)$ . This establishes our goal.

(2) Because:

$$\begin{aligned} Rel_{\sqsubseteq}(F)(=) &= \sqsubseteq \circ Rel(F)(=) \circ \sqsubseteq \\ &= \sqsubseteq \circ = \circ \sqsubseteq \\ &= \sqsubseteq \circ \sqsubseteq \\ &= \sqsubseteq, \quad \text{since } \sqsubseteq \text{ is transitive.} \end{aligned}$$

(3) Obvious, because ordinary relation lifting preserves inclusions.

(4) Because:

$$\begin{aligned}
& (u, v) \in \text{Rel}_{\sqsubseteq}(F)(R^{\text{op}}) \\
& \iff \exists u', v'. u \sqsubseteq u' \wedge (u', v') \in \text{Rel}(F)(R^{\text{op}}) \wedge v' \sqsubseteq v \\
& \iff \exists u', v'. u' \sqsubseteq^{\text{op}} u \wedge (u', v') \in \text{Rel}(F)(R)^{\text{op}} \wedge v \sqsubseteq^{\text{op}} v' \\
& \iff \exists u', v'. v \sqsubseteq^{\text{op}} v' \wedge (v', u') \in \text{Rel}(F)(R) \wedge u' \sqsubseteq^{\text{op}} u \\
& \iff (v, u) \in \text{Rel}_{\sqsubseteq^{\text{op}}}(F)(R) \\
& \iff (u, v) \in \text{Rel}_{\sqsubseteq^{\text{op}}}(F)(R)^{\text{op}}.
\end{aligned}$$

(5) Since composition of relations and ordinary relation lifting preserve inclusions.

(6) If  $R$  is a bisimulation then so is  $R^{\text{op}}$ , and hence  $R$  and  $R^{\text{op}}$  are simulations because  $\sqsubseteq$  is reflexive.

(7) Suppose that  $(u, v) \in \text{Rel}_{\sqsubseteq}(F)((f \times g)^{-1}R)$ . Then, there are  $u', v'$  such that

$$u \sqsubseteq u' \wedge (u', v') \in \text{Rel}(F)((f \times g)^{-1}R) \wedge v' \sqsubseteq v.$$

Since relation lifting preserves inverse images, we see that

$$(u', v') \in (Ff \times Fg)^{-1}\text{Rel}(F)(R),$$

i.e.,  $(Ff(u'), Fg(v')) \in \text{Rel}(F)(R)$ . Thus,

$$Ff(u) \sqsubseteq Ff(u') \wedge (Ff(u'), Fg(v')) \in \text{Rel}(F)(R) \wedge Fg(v') \sqsubseteq Fg(v)$$

and so  $(u, v) \in (Ff \times Fg)^{-1}\text{Rel}_{\sqsubseteq}(F)(R)$ .

(8) By (7), we have

$$\text{Rel}_{\sqsubseteq}(F)((f \times g)^{-1} \coprod_{f \times g} R) \subseteq (Ff \times Fg)^{-1}\text{Rel}_{\sqsubseteq}(F)(\coprod_{f \times g} R),$$

and hence, since  $\coprod_{f \times g} \dashv (f \times g)^{-1}$ ,

$$\begin{aligned}
\coprod_{Ff \times Fg} \text{Rel}_{\sqsubseteq}(F)(R) & \subseteq \coprod_{Ff \times Fg} \text{Rel}_{\sqsubseteq}(F)((f \times g)^{-1} \coprod_{f \times g} (R)) \\
& \subseteq \text{Rel}_{\sqsubseteq}(F)(\coprod_{f \times g} R). \quad \square
\end{aligned}$$

**Definition 4.3** We say that  $F$  with order  $\sqsubseteq$  is **stable** if the associated lax relation lifting operation  $\text{Rel}_{\sqsubseteq}(F)$  commutes with substitution. This means that the inclusion  $\subseteq$  in Lemma 4.2 (7) is an equality.

Throughout, we will consider the following class of polynomial functors (with order) as a running example. These functors are of special interest to us, as they provide the basic examples of functors in which the final coalgebra  $Z$  is an algebraic cpo, as we will see in Section 10.

**Definition 4.4 Poly** is the least class of functors closed under the following.

- For every pre-order  $(A, \leq)_A$ , the constant functor  $X \mapsto A$  with the order given by  $\sqsubseteq_X = \leq_A$  is in **Poly**.
- The identity functor  $X \mapsto X$  with  $\sqsubseteq_X = =_X$  is in **Poly**.
- Given two polynomial functors  $F_1$  and  $F_2$ , the product functor  $F_1 \times F_2$  with componentwise order is in **Poly**.
- Given polynomial  $F$ , the functor  $F^A$  taking  $X \mapsto (FX)^A$  with order

$$\alpha \sqsubseteq \beta \iff \forall a \in A. \alpha(a) \sqsubseteq^F \beta(a)$$

is in **Poly**.

- Given  $F_1$  and  $F_2$  with orders  $\sqsubseteq^1$  and  $\sqsubseteq^2$  respectively, the functor  $F_1 + F_2$  with  $\sqsubseteq_X$  the disjoint union of  $\sqsubseteq_X^1$  and  $\sqsubseteq_X^2$  is in **Poly**.
- Consider again the functor  $F_1 + F_2$ , but with the concatenation order

$$a \sqsubseteq'_X b \text{ iff } a \sqsubseteq_X b \text{ or } (a \in F_1X \text{ and } b \in F_2X),$$

where  $\sqsubseteq_X$  is as in the previous item. This ordered functor is again in **Poly**. (We use this order for the functor  $\mathcal{S}(X) = 1 + (A \times X)$  in Example 2.2 (2).)

Every polynomial functor is stable. However, not all of our examples involve polynomial functors. We extend the result presently.

If  $F$  has a stable order  $\sqsubseteq^F$ , then the following are also stable.

- The functor  $\mathcal{P} \circ F$ , with order

$$S \sqsubseteq T \iff \forall s \in S. \exists t \in T. s \sqsubseteq^F t.$$

- The functor  $\mathcal{L} \circ F$ , where  $\mathcal{L}$  is the list functor from Example 2.2 (3). As before, there are two evident derived orders. The first is a strict order involving multiplicities.

$$\begin{aligned} \langle x_0, \dots, x_{n-1} \rangle \sqsubseteq_1 \langle y_0, \dots, y_{m-1} \rangle \\ \iff \text{there is a strictly monotone function} \\ \varphi: \{0, 1, \dots, n-1\} \rightarrow \{0, 1, \dots, m-1\} \\ \text{with } x_i \sqsubseteq^F y_{\varphi(i)}, \text{ for } i < n. \end{aligned}$$

The second is a simpler order, given by:

$$\langle x_0, \dots, x_{n-1} \rangle \sqsubseteq_2 \langle y_0, \dots, y_{m-1} \rangle \iff \forall i < n. \exists j < m. x_i \sqsubseteq^F y_j.$$

Finally, the bag functor  $\mathcal{B}$  is stable with either order  $\sqsubseteq_1$  or  $\sqsubseteq_2$  from Example 2.2 (4).

Thus, all of the functors from Example 2.2 are stable.

In fact, the orders in which we are interested satisfy a stronger condition than stability, namely: for every  $f: X \rightarrow Y$ , we have

$$(id \times Ff)^{-1} \sqsubseteq_Y \subseteq \coprod_{Ff \times id} \sqsubseteq_X. \quad (1)$$

One can show that  $F$  satisfies (1) iff (a)  $F$  is stable and (b) for every relation  $R \subseteq X \times Y$ ,

$$\text{Rel}(F)(R) \circ \sqsubseteq_X \subseteq \sqsubseteq_Y \circ \text{Rel}(F)(R).$$

One finds that checking (1) is typically easier than checking stability. All of our constructions above preserve (1), save one. The functor  $F_1 + F_2$  with concatenation order  $\sqsubseteq'$  need not satisfy (1) when  $F_1$  and  $F_2$  do. However, if  $F_1$  is constant and  $F_2$  satisfies (1), then so does  $F_1 + F_2$ , which applies to our examples.

It seems that stability is a most reasonable condition to require for an order on a functor. We shall require and use it throughout.

The condition is not trivial, however. Functors with “lexicographic” ordering need not be stable. In particular, consider the functor  $FX = 2 \times X$  with the order

$$(n, x) \sqsubseteq_X (m, y) \text{ iff } n < m \text{ or } (n = m \text{ and } x = y).$$

This order is not stable. For example, consider  $X = \{x\}$  and  $Y = \{y\}$  with the functions  $\text{in}_X: X \rightarrow X + Y$  and  $\text{in}_Y: Y \rightarrow X + Y$ . The reader may check that the pair of elements  $(0, x)$ ,  $(1, y)$  is in the relation

$$(F \text{in}_X \times F \text{in}_Y)^{-1} \text{Rel}_{\sqsubseteq}(F)(=_{X+Y})$$

but not in the relation

$$\text{Rel}_{\sqsubseteq}(F)((\text{in}_X \times \text{in}_Y)^{-1} =_{X+Y}).$$

**Example 4.5** *We describe concrete simulations using the functors described in Examples 2.2 and 3.2. Note that each of these functors is stable.*

- (1) *For two sequence coalgebras  $X \xrightarrow{c} \mathcal{S}(X)$ ,  $Y \xrightarrow{d} \mathcal{S}(Y)$  of the sequence functor  $\mathcal{S}(X) = 1 + (A \times X)$  a relation  $R \subseteq X \times Y$  is a simulation iff for all  $x \in X$  and  $y \in Y$  with  $R(x, y)$  we have  $(c(x), d(y)) \in \text{Rel}_{\sqsubseteq}(\mathcal{S})(R)$ —where the order  $\sqsubseteq$  is as described in Example 2.2 (2). This means that there are  $u, v$  with  $c(x) \sqsubseteq u$ ,  $(u, v) \in \text{Rel}(F)(R)$  and  $v \sqsubseteq d(y)$ . If  $c(x) = *$  this yields no information, but if  $c(x) = (a, x')$  we know that  $u = (a, x')$ , and so that  $v = (a, y')$  with  $R(x', y')$ . But then  $d(y) = (a, y')$ . In conclusion, if  $R(x, y)$ , then either  $x$  is an empty sequence ( $c(x) = *$ ), or  $c(x) = (a, x')$  and  $d(y) = (a, y')$  with  $R(x', y')$ .*
- (2) *For the list functor  $\mathcal{L}(X) = X^*$  we have seen two orderings  $\sqsubseteq_1$  and  $\sqsubseteq_2$  in Example 2.2 (3). Hence for two list-functor coalgebras  $X \xrightarrow{c} X^*$ ,  $Y \xrightarrow{d} Y^*$*

there are two associated notions of simulation. A relation  $R \subseteq X \times Y$  is a simulation for  $\sqsubseteq_1$  if  $R(x, y)$  implies the following: If  $c(x) = \langle x_0, \dots, x_{n-1} \rangle$  and if  $d(y) = \langle y_0, \dots, y_{m-1} \rangle$ , then there is a strictly monotone function  $\varphi: \{0, 1, \dots, n-1\} \rightarrow \{0, 1, \dots, m-1\}$  with  $R(x_i, y_{\varphi(i)})$  for each  $i < n$ .

For the second order  $\sqsubseteq_2$  we would only have:  $\forall i < n. \exists j < m. R(x_i, y_j)$ .

- (3) For the bag functor  $\mathcal{B}$  we only consider the first ordering  $\sqsubseteq_1$  from Example 2.2 (4). For two coalgebras  $X \xrightarrow{c} \mathcal{B}(X)$ ,  $Y \xrightarrow{d} \mathcal{B}(Y)$  a relation  $R$  is a simulation (wrt.  $\sqsubseteq_1$ ) iff for all  $x \in X$  and  $y \in Y$  with  $R(x, y)$ , there is a  $\gamma: R \rightarrow \mathbb{N}$  such that  $\gamma$  is zero almost everywhere and
- For each  $x' \in X$ , one has  $c(x)(x') \leq \sum_{y'} \{\gamma(x', y') \mid R(x', y')\}$
  - For each  $y' \in Y$ , one has  $d(y)(y') \geq \sum_{x'} \{\gamma(x', y') \mid R(x', y')\}$ .
- (4) Finally, for transition system coalgebras  $X \xrightarrow{c} \mathcal{P}(X)^A$ ,  $Y \xrightarrow{d} \mathcal{P}(Y)^A$ , a relation  $R \subseteq X \times Y$  is a simulation with respect to the inclusion iff it is a simulation in the usual sense: if  $R(x, y)$ , then  $x \xrightarrow{a} x'$  implies there is an  $y' \in Y$  with  $y \xrightarrow{a} y'$  and  $R(x', y')$ .

## 5 Similarity

As a result of point (5) in Lemma 4.2 we can take, for given coalgebras, the union of all simulations and obtain again a simulation, for which we shall write  $\lesssim$ . It will be called **similarity**.

As one may expect, similarity arises as a greatest fixed point for a **Rel**-functor.

**Lemma 5.1** *Let  $\alpha: A \rightarrow FA$  and  $\beta: B \rightarrow FB$  be  $F$ -coalgebras. The similarity order  $\lesssim$  between  $A$  and  $B$  is the greatest fixed point for the functor*

$$R \mapsto (\alpha \times \beta)^{-1} \text{Rel}_{\sqsubseteq}(F)(R).$$

**Proof.** It is clear that  $\lesssim$  contains any fixed point for this functor, so it is sufficient to show that  $\lesssim$  itself is a fixed point.

Clearly,  $\lesssim \subseteq (\alpha \times \beta)^{-1} \text{Rel}_{\sqsubseteq}(F)(\lesssim)$ . For the other direction, suppose that we have  $\alpha(a) \text{Rel}_{\sqsubseteq}(F)(\lesssim) \beta(b)$ . Then  $\lesssim \cup \{(a, b)\}$  is a simulation, and hence  $(a, b) \in \lesssim$ .  $\square$

Since the equality relation is a bisimulation, it is included in similarity. Hence similarity is a reflexive relation. In this section we shall look at properties (especially related to transitivity) and examples of similarity. The next section will concentrate on “two-way similarity”, i.e., on  $\lesssim \cap \lesssim^{\text{op}}$ .

**Example 5.2** *Transition system simulations, see Example 4.5 (4), are related to trace inclusions in the following (standard) way. For a state  $x$  in a transition*

system with label set  $A$  we define:

$$\begin{aligned} \mathit{trace}(x) &= \{ \langle (x_0, a_0), (x_1, a_1), \dots \rangle \in (X \times A)^\infty \mid x_0 = x \wedge \forall i \in \mathbb{N}. x_i \xrightarrow{a_i} x_{i+1} \} \\ \mathit{behtrace}(x) &= \{ \pi_2^\infty(\sigma) \in A^\infty \mid \sigma \in \mathit{trace}(x) \}. \end{aligned}$$

Thus, the elements of  $\mathit{behtrace}(x)$  are the (finite or infinite) sequences of labels that may occur via transitions out of  $x$ .

Given a simulation  $R$  with  $R(x, y)$ , for each trace

$$\sigma = \langle (x_0, a_0), (x_1, a_1), \dots \rangle \in \mathit{trace}(x)$$

there is a  $\tau = \langle (y_0, a_0), (y_1, a_1), \dots \rangle \in \mathit{trace}(y)$  with  $R(x_i, y_i)$ . We thus see that

$$\begin{aligned} x \lesssim y &\implies \mathit{trace}(x) \subseteq \mathit{trace}(y) \\ &\implies \mathit{behtrace}(x) \subseteq \mathit{behtrace}(y). \end{aligned}$$

For this reason simulations form a standard ingredient of proofs of refinement (i.e., behaviour trace inclusion), where  $x$  is an initial state of an implementation, and  $y$  is an initial state of an abstract system (the specification) describing the appropriate behaviour.

What is special about the approach in this paper is that we take orderings on functors as primitive, and define lax relation lifting in terms of this order (and ordinary relation lifting, which is seen as canonical and taken for granted). In [9] such a lifting (or relational extension, as it is called there) is taken as primitive, subject to certain requirements. For a comparison we recall this approach. A relational extension (for a given endofunctor  $F$ ) is a mapping  $G$  sending a relation  $R \subseteq X \times Y$  to a relation  $GR \subseteq FX \times FY$  such that:

- (1)  $=_{FX} \subseteq G(=_X)$
- (2)  $R \subseteq S \implies GR \subseteq GS$
- (3)  $GR \circ GS = G(R \circ S)$
- (4) “functoriality”

This last requirement is written out in detail, but amounts to the property that  $G$  is a functor  $\mathbf{Rel} \rightarrow \mathbf{Rel}$  as in Lemma 4.2 (1). Interestingly, a “normal form” is proven in [9] (Lemma 1) showing that each relator can be described as a composite like in Definition 4.1, where the order  $\sqsubseteq$  is  $G(=)$ . This shows that our approach—with a defined operation  $\mathit{Rel}_{\sqsubseteq}(F)$  instead of an assumed  $G$ —is more primitive.

However, the third condition about preservation of composition requires some attention in our approach. It follows from stability, as shown in [18, Theorem 2.2.2].

**Lemma 5.3** *For a functor  $F$  with stable ordering  $\sqsubseteq$ , lax relation lifting preserves composition of relations:*

$$\text{Rel}_{\sqsubseteq}(F)(R \circ S) = \text{Rel}_{\sqsubseteq}(F)(R) \circ \text{Rel}_{\sqsubseteq}(F)(S).$$

(The inclusion  $\subseteq$  always holds, because ordinary relation lifting preserves compositions, and  $\sqsubseteq$  is reflexive.)

**Proof.** We need to prove  $\supseteq$ . Assume  $\langle s_1, s_2 \rangle : S \hookrightarrow X \times Y$  and  $\langle r_1, r_2 \rangle : R \hookrightarrow Y \times Z$ . Then:

$$\begin{aligned} & \text{Rel}_{\sqsubseteq}(F)(R) \circ \text{Rel}_{\sqsubseteq}(F)(S) \\ &= (Fr_2 \times id)^{-1} \sqsubseteq \circ (id \times Fr_1)^{-1} \sqsubseteq \circ (Fs_2 \times id)^{-1} \sqsubseteq \circ (id \times Fs_1)^{-1} \sqsubseteq \\ & \quad \text{by Definition 4.1} \\ &= (Fr_2 \times id)^{-1} \sqsubseteq \circ (Fs_2 \times Fr_1)^{-1} (\sqsubseteq \circ \sqsubseteq) \circ (id \times Fs_1)^{-1} \sqsubseteq \\ &= (Fr_2 \times id)^{-1} \sqsubseteq \circ (Fs_2 \times Fr_1)^{-1} \text{Rel}_{\sqsubseteq}(F)(=_Y) \circ (id \times Fs_1)^{-1} \sqsubseteq \\ &= (Fr_2 \times id)^{-1} \sqsubseteq \circ \text{Rel}_{\sqsubseteq}(F)((s_2 \times r_1)^{-1} =_Y) \circ (id \times Fs_1)^{-1} \sqsubseteq \\ & \quad \text{by stability} \\ &= \sqsubseteq \circ \coprod_{Fs_1 \times Fr_2} \text{Rel}_{\sqsubseteq}(F)((s_2 \times r_1)^{-1} =_Y) \circ \sqsubseteq \\ & \subseteq \sqsubseteq \circ \text{Rel}_{\sqsubseteq}(F)\left(\coprod_{s_1 \times r_2} (s_2 \times r_1)^{-1} =_Y\right) \circ \sqsubseteq \\ & \quad \text{by Lemma 4.2 (8)} \\ &= \text{Rel}_{\sqsubseteq}(F)(R \circ S) \quad \square \end{aligned}$$

Here are some consequences of the preservation property of this lemma.

**Proposition 5.4** *Let  $F$  be a functor with a stable ordering  $\sqsubseteq$ . Then:*

- (1) *Simulations are closed under composition.*
- (2) *Similarity is a transitive relation.*
- (3) *For homomorphisms  $f, g$  between coalgebras,*

$$x \lesssim y \iff f(x) \lesssim g(y).$$

- (4) *Similarity  $\lesssim$  on the final coalgebra is the final  $\text{Rel}_{\sqsubseteq}(F)$ -coalgebra.*

**Proof.** We prove each in turn.

- (1) Obvious, because relation composition preserves inclusions.

- (2) Suppose  $x \lesssim y$  and  $y \lesssim z$ . Then there are simulations  $R, S$  with  $R(x, y)$  and  $S(y, z)$ . Hence  $(S \circ R)(x, z)$ , and so  $x \lesssim z$  because  $S \circ R$  is a simulation by (1).
- (3) Since  $f$  is a homomorphism of coalgebras, its graph relation  $\mathbf{Graph}(f)$  is a bisimulation. Hence both  $\mathbf{Graph}(f)$  and  $\mathbf{Graph}(f)^{\text{op}}$  are simulations. This means that both  $x \lesssim f(x)$  and  $f(x) \lesssim x$ . Similarly,  $y \lesssim g(y)$  and  $g(y) \lesssim y$ . Hence we can easily prove the third point in the proposition, using the second:
- ( $\Rightarrow$ ) If  $x \lesssim y$ , then  $f(x) \lesssim x \lesssim y \lesssim g(y)$  so that  $f(x) \lesssim g(y)$ .
- ( $\Leftarrow$ ) If  $f(x) \lesssim g(y)$ , then  $x \lesssim f(x) \lesssim g(y) \lesssim y$ , so that  $x \lesssim y$ .
- (4) Let  $R$  be a simulation over  $A \xrightarrow{\alpha} F(A)$  and  $B \xrightarrow{\beta} F(B)$  and let  $!_A: A \rightarrow Z$  and  $!_B: B \rightarrow Z$  be the unique homomorphisms into the final  $F$ -coalgebra  $\zeta: Z \xrightarrow{\cong} F(Z)$  and consider the diagram in Figure 1. By (3),

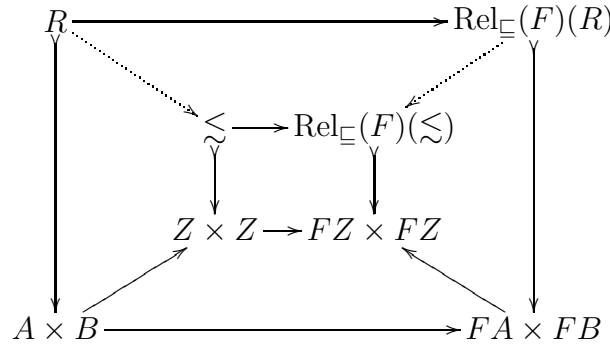


Fig. 1. Similarity on the final coalgebra is the final  $\text{Rel}_{\sqsubseteq}(F)$ -coalgebra

there is a (necessarily unique) arrow  $R \rightarrow \lesssim$  in  $\mathbf{Rel}$ , as shown on the left. By functoriality of  $\text{Rel}_{\sqsubseteq}(F)$  we  $\text{Rel}_{\sqsubseteq}(F)(R) \rightarrow \text{Rel}_{\sqsubseteq}(F)(\lesssim)$  on the right. One must show that this  $R \rightarrow \lesssim$  is a  $\text{Rel}_{\sqsubseteq}(F)$ -homomorphism, i.e., that the top trapezoid commutes. This follows by the fact that  $\text{Rel}_{\sqsubseteq}(F)(\lesssim) \rightarrow FZ \times FZ$  is monic.  $\square$

Here is another consequence, that will be generalised in a subsequent definition.

**Lemma 5.5** *Let  $F: \mathbf{Sets} \rightarrow \mathbf{Sets}$  have a stable order  $\sqsubseteq$ . Then  $F$  extends to  $F: \mathbf{PreOrd} \rightarrow \mathbf{PreOrd}$  by  $\langle X, \leq \rangle \mapsto \langle FX, \text{Rel}_{\sqsubseteq}(F)(\leq) \rangle$ .*

**Proof.** We need to show that  $\text{Rel}_{\sqsubseteq}(F)(\leq)$  is reflexive and transitive. Reflex-



ivity is easy, because  $=_X \sqsubseteq \leq$  implies

$$\begin{aligned}
&=_{F(X)} \sqsubseteq \sqsubseteq \\
&= \sqsubseteq \circ =_{F(X)} \circ \sqsubseteq \\
&= \sqsubseteq \circ \text{Rel}(F)(=_X) \circ \sqsubseteq \\
&\subseteq \sqsubseteq \circ \text{Rel}(F)(\leq) \circ \sqsubseteq \\
&= \text{Rel}_{\sqsubseteq}(F)(\leq).
\end{aligned}$$

For transitivity we use Lemma 5.3:

$$\begin{aligned}
\text{Rel}_{\sqsubseteq}(F)(\leq) \circ \text{Rel}_{\sqsubseteq}(F)(\leq) &= \text{Rel}_{\sqsubseteq}(F)(\leq \circ \leq) \\
&= \text{Rel}_{\sqsubseteq}(F)(\leq). \quad \square
\end{aligned}$$

**Definition 5.6** For a subcategory  $\mathbb{C} \hookrightarrow \mathbf{PreOrd}$  we say that  $F$  with stable  $\sqsubseteq$  preserves  $\mathbb{C}$  if  $F$  from the previous lemma restricts to  $\mathbb{C}$  as in:

$$\begin{array}{ccc}
\mathbb{C} & \xrightarrow[\langle FX, \text{Rel}_{\sqsubseteq}(F)(\leq) \rangle]{\langle X, \leq \rangle \mapsto} & \mathbb{C} \\
\downarrow & & \downarrow \\
\mathbf{PreOrd} & \xrightarrow{F} & \mathbf{PreOrd}
\end{array}$$

Later we shall use this definition especially when  $\mathbb{C}$  is the category of dcpo's or of algebraic cpo's.

**Example 5.7** We recall that the final coalgebra for the sequence functor

$$\mathcal{S}(X) = 1 + (A \times X)$$

is the set  $A^\infty$  of finite and infinite sequences with coalgebra structure  $A^\infty \xrightarrow{\cong} 1 + (A \times A^\infty)$  given by

$$\sigma \longmapsto \begin{cases} * & \text{if } \sigma \text{ is the empty sequence } \langle \rangle \\ (a, \sigma') & \text{if } \sigma = a \cdot \sigma' \text{ with head } a \text{ and tail } \sigma'. \end{cases}$$

This set of sequences  $A^\infty$  carries the usual “prefix” order:

$$\sigma \leq \tau \iff \sigma \cdot \rho = \tau, \text{ for some } \rho \in A^\infty.$$

We claim that this prefix order is the same as similarity.

The inclusion  $\leq \subseteq \lesssim$  is easy, because  $\leq$  is a simulation: if  $\sigma \leq \tau$ , say via  $\sigma \cdot \rho = \tau$ , and  $\sigma = a \cdot \sigma'$ , then  $\tau = a \cdot \tau'$  where  $\sigma' \cdot \rho = \tau'$ . This shows  $\sigma' \leq \tau'$ .

For the reverse inclusion  $\lesssim \subseteq \leq$  we assume  $\sigma \lesssim \tau$ , say via a simulation  $R \subseteq A^\infty \times A^\infty$  with  $R(\sigma, \tau)$ . We determine elements  $a_0, a_1, \dots \in A$  and  $\sigma_0, \sigma_1, \dots \in A^\infty$  with for each  $n$ ,  $\sigma = a_0 \cdot a_1 \cdots a_n \cdot \sigma_n$ . By induction we find  $\tau_0, \tau_1, \dots \in A^\infty$  with for each  $n$ ,  $\tau = a_0 \cdot a_1 \cdots a_n \cdot \tau_n$ . There are two cases:

- $\sigma$  is finite, say,  $\sigma = a_0 \cdots a_n$ . Then  $\tau = \sigma \cdot \tau_n$ , so that  $\sigma \leq \tau$ .
- $\sigma$  is infinite. Then  $\sigma = \tau$ , and thus also  $\sigma \leq \tau$ .

As a consequence of Proposition 5.4 (3) we now have for arbitrary sequence coalgebras  $X \xrightarrow{c} \mathcal{S}(X)$ ,  $Y \xrightarrow{d} \mathcal{S}(Y)$  and elements  $x \in X, y \in Y$ ,

$$x \lesssim y \iff !(x) \leq !(y),$$

where  $!$  is the unique homomorphism to the final coalgebra and  $\leq$  is its prefix order.

## 6 Two-way similarity

Having seen similarity  $\lesssim$ , we define **two-way similarity** as  $\approx = \lesssim \cap \lesssim^{\text{op}}$ , i.e., as:

$$x \approx y \stackrel{\text{def}}{\iff} x \lesssim y \text{ and } y \lesssim x.$$

An immediate consequence of Lemma 4.2 (6) is that bisimilarity implies two-way similarity:  $\Leftrightarrow \subseteq \approx$ . In this section we are interested in the converse, i.e., in whether or not  $\approx \subseteq \Leftrightarrow$ . The next examples show that this may or may not be the case.

**Example 6.1** We give an example in which  $\approx \subseteq \Leftrightarrow$ , and one in which the inclusion fails.

(1) Let's consider the sequence example, with two coalgebras  $X \xrightarrow{c} 1 + (A \times X)$  and  $Y \xrightarrow{d} 1 + (A \times Y)$ . Assume  $x \approx y$ . Then there are simulations  $R \subseteq X \times Y$  and  $S \subseteq Y \times X$  with  $R(x, y)$  and  $S(y, x)$ . The fact that  $R$  and  $S$  are simulations means that for all  $z \in X, w \in Y$ :

- (a)  $R(z, w)$  and  $c(z) = (a, z')$  implies  $d(w) = (a, w')$  with  $R(z', w')$ .
- (b)  $S(w, z)$  and  $d(w) = (a, w')$  implies  $c(z) = (a, z')$  with  $S(w', z')$ .

We claim that  $T = (R \cap S^{\text{op}}) \subseteq X \times Y$  is a bisimulation with  $T(x, y)$ . The last point is obvious. In order to show that  $T$  is a bisimulation, assume  $T(z, w)$ . Then:

- If  $c(z) = *$  but  $d(w) = (a, w')$ , then we get a contradiction by (1b) above. Hence  $d(w) = *$ . The reverse implication is obtained similarly.

- If  $c(z) = (a, z')$ , then  $d(w) = (a, w')$  with  $R(z', w')$ , by (1). Applying (1a) yields that  $c(z) = (a, z'')$  with  $S(w', z'')$ . But then we get  $z' = z''$ , so that  $T(z', w')$ , as required.
- (2) Here is a simple variation on the previous example. Let  $F(X) = X + X$  with order  $\sqsubseteq$  given by:

$$u \sqsubseteq v \iff \forall x \in X. u = \kappa_2(x) \Rightarrow v = \kappa_2(x).$$

Notice that no relation is required in case  $u$  is in the first (left) component of  $X + X$ .

The associated notion of similarity says, for given coalgebras  $c: X \rightarrow X + X$  and  $d: Y \rightarrow Y + Y$ , that  $R \subseteq X \times Y$  is a simulation if for each  $x, y$  with  $R(x, y)$  one has that if  $c(x) = \kappa_2(x')$ , then  $d(y)$  must be of the form  $\kappa_2(y')$  with  $R(x', y')$ . In case we have a two-way similarity there must also be a relation  $S$  with  $S(y, x)$  implies that  $d(y) = \kappa_2(y')$ , then  $c(x) = \kappa_2(x')$  with  $S(y', x')$ .

But this is not the same as bisimilarity for this functor, because then we must also have a relation in the first components of the coproduct  $+$ :  $R \subseteq X \times Y$  is a bisimulation if  $R(x, y)$  implies both:

- if  $c(x) = \kappa_1(x')$ , then  $d(y) = \kappa_1(y')$  with  $R(x', y')$ ;
- if  $c(x) = \kappa_2(x')$ , then  $d(y) = \kappa_2(y')$  with  $R(x', y')$ .

In the second example we see that there is something missing from the relation  $\sqsubseteq$  that ensures that two-way similarity implies bisimilarity. The following result gives a sufficient condition.

**Theorem 6.2** *Let  $F$  be a functor with a relation  $\sqsubseteq$  such that the associated relation liftings satisfy the condition:*

$$Rel_{\sqsubseteq}(F)(R_1) \cap Rel_{\sqsubseteq^{op}}(F)(R_2) \subseteq Rel(F)(R_1 \cap R_2).$$

*Then two-way similarity (for coalgebras of this functor) is the same as bisimilarity:*

$$x \xleftrightarrow{\sim} y \iff x \approx y.$$

**Proof.** We only need to prove the direction  $(\Leftarrow)$ , and so we assume  $x \approx y$ , say via simulations  $R, S$  with  $R(x, y)$  and  $S(y, x)$ . The fact that  $R, S$  are simulations says that  $R \subseteq (c \times d)^{-1}(Rel_{\sqsubseteq}(F)(R))$  and  $S \subseteq (d \times c)^{-1}(Rel_{\sqsubseteq}(F)(S))$ . We take as new relation  $T = (R \cap S^{op})$ , like in Example 6.1 (1). Clearly,  $T(x, y)$ . We are done if we can show that  $T$  is a bisimulation, i.e., satisfies

$T \subseteq (c \times d)^{-1}(\text{Rel}(F)(T))$ . But:

$$\begin{aligned} S^{\text{op}} &\subseteq \left( (d \times c)^{-1} \text{Rel}_{\sqsubseteq}(F)(S) \right)^{\text{op}} \\ &= (c \times d)^{-1} \left( \text{Rel}_{\sqsubseteq}(F)(S)^{\text{op}} \right) \\ &= (c \times d)^{-1} \left( \text{Rel}_{\sqsubseteq^{\text{op}}}(F)(S^{\text{op}}) \right) \quad \text{by Lemma 4.2 (4)}. \end{aligned}$$

Hence:

$$\begin{aligned} T &= (R \cap S^{\text{op}}) \\ &\subseteq (c \times d)^{-1} \left( \text{Rel}_{\sqsubseteq}(F)(R) \right) \cap (c \times d)^{-1} \left( \text{Rel}_{\sqsubseteq^{\text{op}}}(F)(S^{\text{op}}) \right) \\ &= (c \times d)^{-1} \left( \text{Rel}_{\sqsubseteq}(F)(R) \cap \text{Rel}_{\sqsubseteq^{\text{op}}}(F)(S^{\text{op}}) \right) \\ &\subseteq (c \times d)^{-1} \left( \text{Rel}(F)(T) \right). \end{aligned}$$

The last step uses the condition of the theorem.  $\square$

Notice that the condition in this theorem can be formulated because we take an order  $\sqsubseteq$  on a functor as primitive (and not the resulting relation lifting). This allows us to change the order (by taking the opposite  $\sqsubseteq^{\text{op}}$ ) and consider the associated lifting.

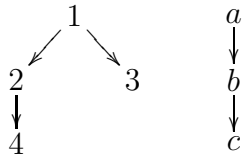
**Example 6.3** *In this example we show that the first ordering  $\sqsubseteq_1$  for the list functor  $\mathcal{L}$  in Example 2.2 satisfies the condition of the previous theorem.*

Assume two sequences  $u = \langle x_0, \dots, x_{n-1} \rangle$  and  $v = \langle y_0, \dots, y_{m-1} \rangle$  satisfy  $(u, v) \in \text{Rel}_{\sqsubseteq_1}(\mathcal{L})(R_1) \cap \text{Rel}_{\sqsubseteq_1^{\text{op}}}(\mathcal{L})(R_2)$ . This means that there are strictly monotone functions

$$\begin{aligned} \varphi: \{0, 1, \dots, n-1\} &\rightarrow \{0, 1, \dots, m-1\}, \\ \psi: \{0, 1, \dots, m-1\} &\rightarrow \{0, 1, \dots, n-1\} \end{aligned}$$

with  $R_1(x_i, y_{\varphi(i)})$  and  $R_2(x_{\psi(j)}, y_j)$ . But this can only happen if  $n = m$  and  $\varphi = \psi = \text{id}$ . Hence  $(R_1 \cap R_2)(x_i, y_i)$ , so that  $(u, v) \in \text{Rel}(\mathcal{L})(R_1 \cap R_2)$ .

**Example 6.4** *For (labeled) transition systems it is not the case that two-way similarity is the same as bisimilarity. Here is a simple (unlabeled) example.*



The following is a simulation from left to right:

$$R = \{(1, a), (2, b), (3, b), (4, c)\}.$$

Indeed,  $R(x, y)$  and  $x \longrightarrow x'$  implies  $y \longrightarrow y'$  for some  $y'$  with  $R(x', y')$ .

And a simulation from right to left is:

$$S = \{(a, 1), (b, 2), (c, 4)\}.$$

This shows that  $1 \approx a$ . But we do not have  $1 \Leftrightarrow a$ .

## 7 Dcpo structure by finality

In Example 5.7 we have seen that similarity on the final coalgebra of sequences coincides with the prefix order. The latter happens to provide a dcpo structure: every directed subset has a join. Such a dcpo structure can be used in a denotational semantics of a programming language, to give meaning to constructs like loops or recursion.

In this section we shall see that this dcpo structure results from a distributive law between the sequence functor and the free dcpo monad on preorders. Moreover, the presence of such a distributive law is equivalent to requiring that the functor  $\text{Rel}_{\sqsubseteq}(F)$  preserves dcpos. We begin with some rudimentary facts about dcpos.

We write **Dcpo** for the category of directed complete preorders. It comes with a forgetful functor  $\mathcal{U}: \mathbf{Dcpo} \rightarrow \mathbf{PreOrd}$ . This functor has a left adjoint, for which we write  $\mathcal{D}$ . It maps a preorder to its directed downsets, ordered by inclusion. The join in  $\mathcal{D}(X)$  of a directed collection  $(U_i)_{i \in I}$  of directed downsets  $U_i$  is then simply their union  $\bigcup_{i \in I} U_i$ . The adjunction induces a monad on **PreOrd**, for which we shall also write  $\mathcal{D}$ , with:

$$\begin{aligned} \text{unit:} \quad & \eta_X: X \rightarrow \mathcal{D}(X) & x & \longmapsto \downarrow x \\ \text{multiplication:} \quad & \mu_X: \mathcal{D}^2(X) \rightarrow \mathcal{D}(X) & (U_i)_{i \in I} & \longmapsto \bigcup_{i \in I} U_i. \end{aligned}$$

The following result is standard.

**Lemma 7.1** *For a preorder  $X$  the following are equivalent.*

- (1)  $X$  is a dcpo;

- (2)  $X$  carries an (Eilenberg-Moore) algebra structure for the monad  $\mathcal{D}$ ;  
(3) the unit  $\eta_X: X \rightarrow \mathcal{D}(X)$  has a left adjoint.

The structure map in (2) and (3) is of course the join operation

$$\bigvee: \mathcal{D}(X) \longrightarrow X.$$

Successive left adjoints to the unit are studied in [10] and describe continuity and algebraicity in the dcpo.

**Lemma 7.2** *A functor  $F$  with stable order  $\sqsubseteq$  preserves dcpos if and only if there is a **distributive law***

$$\begin{array}{ccc} \mathbf{PreOrd} & \xrightarrow{\mathcal{D}} & \mathbf{PreOrd} \\ F \downarrow & \xrightarrow{\tau} & \downarrow F \\ \mathbf{PreOrd} & \xrightarrow{\mathcal{D}} & \mathbf{PreOrd} \end{array}$$

consisting of monotone functions  $\tau_X: \mathcal{D}(FX) \rightarrow F(\mathcal{D}(X))$ , where  $\mathcal{D}(FX)$  carries the inclusion order  $\subseteq$  on the completion  $\mathcal{D}(FX) = \mathcal{D}(FX, \text{Rel}_{\sqsubseteq}(F)(\leq))$ , and  $F(\mathcal{D}(X))$  carries the lifting  $\text{Rel}_{\sqsubseteq}(F)(\leq)$  of the inclusion order  $\subseteq$  on  $\mathcal{D}(X) = \mathcal{D}(X, \leq)$ . Such a distributive law is required to make the following two diagrams commute.

$$\begin{array}{ccc} F(X) \xrightarrow{\eta_{F(X)}} \mathcal{D}(F(X)) & \mathcal{D}^2(F(X)) \xrightarrow{\mathcal{D}(\tau_X)} \mathcal{D}(F(\mathcal{D}(X))) \xrightarrow{\tau_{\mathcal{D}(X)}} F(\mathcal{D}^2(X)) & \\ \searrow F(\eta_X) & \downarrow \mu_{F(X)} & \downarrow F(\mu_X) \\ & \mathcal{D}(F(X)) \xrightarrow{\tau_X} F(\mathcal{D}(X)) & \end{array}$$

**Proof.** Suppose that  $F$  preserves dcpos and let  $(X, \leq_X)$  be a preorder. As in the statement of the theorem, we abuse notation by simply writing  $FX$  for  $(FX, \text{Rel}_{\sqsubseteq}(F)(\leq_X))$ , and similarly for maps.

By assumption, the preorder  $F\mathcal{D}X$  is a dcpo. Let  $\tau_X: \mathcal{D}FX \rightarrow F\mathcal{D}X$  be the adjoint transpose of  $F\eta_X: FX \rightarrow F\mathcal{D}X$ , so that  $\tau_X \circ \eta_{FX} = F\eta_X$ . We claim that  $\tau$  thus defined is the desired distributive law. We omit the easy proof that  $\tau$  is natural.

Clearly,  $\tau$  satisfies the unit law. For the distributive law, we note

$$\begin{aligned} F\mu_X \circ \tau_{\mathcal{D}X} \circ \mathcal{D}\tau_X \circ \eta_{\mathcal{D}FX} &= F\mu_X \circ \tau_{\mathcal{D}X} \circ \eta_{F\mathcal{D}X} \circ \tau_X \\ &= F\mu_X \circ F\eta_{\mathcal{D}X} \circ \tau_X \\ &= \tau_X \\ &= \tau_X \circ \mu_{FX} \circ \eta_{\mathcal{D}FX}. \end{aligned}$$

Since  $\tau_X \circ \mu_{FX}$  and  $F\mu_X \circ \tau_{\mathcal{D}X} \circ \mathcal{D}\tau_X$  are continuous, it follows that they are indeed equal.

For the other direction, suppose we have a distributive law  $\tau$  and that  $(X, \leq_X)$  is a dcpo. Let  $\bigvee_{FX}: \mathcal{D}FX \rightarrow FX$  be the composite

$$\mathcal{D}FX \xrightarrow{\tau_X} F\mathcal{D}X \xrightarrow[\cong]{F(\bigvee_X)} FX$$

We will check that it satisfies the laws for Eilenberg-Moore algebras.

For the unit law, we see

$$\begin{aligned} \bigvee_{FX} \circ \eta_{FX} &= F\bigvee_X \circ \tau_X \circ \eta_{FX} \\ &= F\bigvee_X \circ F\eta_X \\ &= id_{FX}. \end{aligned}$$

For the multiplication,

$$\begin{aligned} \bigvee_{FX} \circ \mathcal{D}\bigvee_{FX} &= F\bigvee_X \circ \tau_X \circ \mathcal{D}F\bigvee_X \circ \mathcal{D}\tau_X \\ &= F\bigvee_X \circ F\mathcal{D}\bigvee_X \circ \tau_{\mathcal{D}X} \circ \mathcal{D}\tau_X \\ &= F(\bigvee_X \circ \mathcal{D}\bigvee_X) \circ \tau_{\mathcal{D}X} \circ \mathcal{D}\tau_X \\ &= F(\bigvee_X \circ \mu_X) \circ \tau_{\mathcal{D}X} \circ \mathcal{D}\tau_X \\ &= F\bigvee_X \circ (F\mu_X \circ \tau_{\mathcal{D}X} \circ \mathcal{D}\tau_X) \\ &= F\bigvee_X \circ (\tau_X \circ \mu_{FX}) \\ &= \bigvee_{FX} \circ \mu_{FX}. \end{aligned} \quad \square$$

**Theorem 7.3** *Let  $F: \mathbf{Sets} \rightarrow \mathbf{Sets}$  with a stable order  $\sqsubseteq$  preserve dcpos. If  $F$  has a final coalgebra, then it forms with its similarity order a dcpo.*

**Proof.** Let  $\zeta: Z \xrightarrow{\cong} F(Z)$  be the final coalgebra. We may assume a distributive law  $\tau$ , like in the previous result. We then define an (Eilenberg-Moore) algebra structure  $\bigvee: \mathcal{D}(Z) \rightarrow Z$  by finality, in:

$$\begin{array}{ccc} F(\mathcal{D}(Z)) & \xrightarrow{F(\bigvee)} & F(Z) \\ \tau_Z \uparrow & & \uparrow \cong \zeta \\ \mathcal{D}(F(Z)) & & \\ \mathcal{D}(\zeta) \uparrow & & \\ \mathcal{D}(Z) & \xrightarrow{\bigvee} & Z \end{array}$$

Note that  $\bigvee$  is monotone by Proposition 5.4. We have to verify the laws for Eilenberg-Moore algebras:  $\bigvee \circ \eta_Z = id$  and  $\bigvee \circ \mu_Z = \bigvee \circ \mathcal{D}(\bigvee)$ . Both equations follow from uniqueness. The first one holds because the unit is a homomorphism of coalgebras:

$$\begin{array}{ccc}
 F(Z) & \xrightarrow{F(\eta_Z)} & F(\mathcal{D}(Z)) \\
 & \searrow \eta_{F(Z)} & \uparrow \tau_Z \\
 & & \mathcal{D}(F(Z)) \\
 \zeta \uparrow & & \uparrow \mathcal{D}(\zeta) \\
 Z & \xrightarrow{\eta_Z} & \mathcal{D}(Z)
 \end{array}$$

The lower-left diagram commutes by naturality, and the upper-right one is the first distributivity requirement mentioned above. The composite  $\bigvee \circ \eta_Z$  is then a homomorphism  $\zeta \rightarrow \zeta$ , and must thus be the identity.

In a similar way one proves that both  $\bigvee \circ \mu_Z$  and  $\bigvee \circ \mathcal{D}(\bigvee)$  are coalgebra homomorphisms from the coalgebra  $\tau_{F(Z)} \circ \mathcal{D}(\tau_Z) \circ \mathcal{D}^2(\zeta): \mathcal{D}^2(Z) \rightarrow F(\mathcal{D}^2(Z))$  to the final coalgebra.  $\square$

The dcpo structure on sequences in Example 5.7 indeed follows from Theorem 7.3. Soon, we will show that all of our ordered functors in **Poly** (assuming the constant functors involve dcpos) preserve dcpos, by explicitly exhibiting a distributive law for each. Here, we confirm directly that  $\mathcal{S}(X) = 1 + (A \times X)$  preserves dcpos.

Let  $(X, \leq_X)$  be a dcpo and let  $D \subseteq 1 + (A \times X)$  be directed with respect to  $\text{Rel}_{\sqsubseteq}(F)(\leq_X)$ . If  $D \subseteq 1$ , then clearly  $\bigvee D = *$ . Otherwise, since  $*$  is a bottom element for  $\text{Rel}_{\sqsubseteq}(F)(\leq_X)$ , we have  $\bigvee D = \bigvee (D \cap (A \times X))$ . The order  $\text{Rel}_{\sqsubseteq}(F)(\leq_X)$  restricted to  $A \times X$  is given by

$$(a, x) \text{Rel}_{\sqsubseteq}(F)(\leq_X)(b, y) \iff a = b \text{ and } x \leq_X y,$$

i.e., the componentwise order for  $A \times -$ . Because  $D$  is directed, one can write

$$D \cap (A \times X) = \{(a, x) \mid x \in D'\}$$

for some  $a \in A$  and directed set  $D' \subseteq X$ . Hence,

$$\bigvee D = (a, \bigvee D').$$

Actually, the definition via finality of the join for sequences occurs already in [7], but here we put this definition in a wider context via distributive laws.



We consider another such example.

**Example 7.4** We fix a set  $V$ , and think of its elements as variables. We use  $V$  in the functor  $\mathcal{T}_V: \mathbf{Sets} \rightarrow \mathbf{Sets}$  given by

$$\mathcal{T}_V(X) = 1 + (V^* \times V \times X^*)$$

We shall write the final coalgebra as  $\zeta: \mathbf{BT} \xrightarrow{\cong} 1 + (V^* \times V \times \mathbf{BT}^*)$ . Its elements will be considered as (abstract) Böhm trees, see [3]. For  $A \in \mathbf{BT}$  we can write:

$$\zeta(A) = * \quad \text{or} \quad \zeta(A) = \left( \begin{array}{c} \lambda x_1 \dots x_n. y \\ \zeta(A_1) \quad \dots \quad \zeta(A_m) \end{array} \right)$$

where, on the right,  $\zeta(A) = (\langle x_1, \dots, x_n \rangle, y, \langle A_1, \dots, A_m \rangle)$ . The ‘ $\lambda$ ’ is just syntactic sugar, used to suggest the analogy with the standard notation for Böhm trees [3]. The elements of  $\mathbf{BT}$  are thus finitely branching, possibly infinite rooted trees, with labels of the form  $\lambda x_1 \dots x_n. y$ , for variables  $x_i, y \in V$ .

The order considered on Böhm trees as formulated in [3, §10.2] is:

$$A \subseteq B \iff A \text{ results from } B \text{ by cutting of some subtrees}$$

This description is fairly informal. The question is how to make it precise, via an order on the functor  $\mathcal{T}_V$ . Two possible orders come to mind: the flat order from Example 2.2 (2) or the precise order  $\sqsubseteq_1$  on the list functor from Example 2.2 (3). The following illustrations from [3, §10.2] help.

$$\begin{array}{ccc} \lambda x. x & \subseteq & \lambda x. x \\ \swarrow \searrow & & \swarrow \searrow \\ & x & x \\ & | & \\ & x & \end{array} \quad \lambda x. x \not\subseteq \lambda x. x$$

These pictures show that “cutting off subtrees” should be interpreted as: replacing a node by  $*$ . Thus, the order  $\subseteq$  that we consider on the functor  $\mathcal{T}_V$  is simply the flat order, like for sequences in Example 2.2 (2):  $u \subseteq v$  iff  $u \neq * \Rightarrow u = v$ .

The induced similarity order  $\lesssim$  on  $\mathbf{BT}$  is then the above order  $\subseteq$ . The previous theorem allows us to conclude that it is a dcpo.

Other examples can be readily constructed for polynomial functors **Poly**, defined in Section 4, provided that the constant functors  $X \mapsto A$  are restricted to  $\text{depos } A$ . It is sufficient, of course, just to confirm that these functors with order

preserve dcpos. Nonetheless, we give here the explicit associated distributive laws, which can be found via the proof of Lemma 7.2. The distributive laws are constructed by induction on the structure of the polynomial functor (and its associated order) as follows.

- For any dcpo  $(A, \leq_A)$ , the constant functor  $F X = A$  with order  $\sqsubseteq_X = \leq_A$  has a distributive law given by  $\bigvee: \mathcal{D}A \rightarrow A$ . In particular, this applies when we take  $\leq_A$  to be  $=_A$ .
- For the identity functor  $F X = X$  with the discrete order, the identity transformation  $\mathcal{D}X \rightarrow \mathcal{D}X$  is a distributive law.
- Suppose that functors  $F_1$  and  $F_2$  have distributive laws  $\tau_1$  and  $\tau_2$ , and define an order  $\sqsubseteq$  on  $F_1 \times F_2$  by taking the orders on  $F_1$  and  $F_2$  component-wise. Then

$$((\tau_1)_X \circ \mathcal{D}\pi_1, (\tau_2)_X \circ \mathcal{D}\pi_2): \mathcal{D}(F_1 \times F_2)X \rightarrow (F_1 \times F_2)\mathcal{D}X$$

is a distributive law.

- Let  $F$  have distributive law  $\tau$  and for each  $a \in A$ , let  $\text{ev}_a: F^A X \rightarrow F X$  denote evaluation at  $a$ , i.e.,  $\text{ev}_a(f) = f(a)$ . Then

$$\begin{aligned} \mathcal{D}F^A X &\longrightarrow F^A \mathcal{D}X \\ S &\longmapsto \lambda a. \tau_X(\coprod_{\text{ev}_a} S) \end{aligned}$$

is a distributive law for  $F^A$ .

- Let  $F_1$  and  $F_2$  be as above, and let  $\sqsubseteq$  be the disjoint order for  $F_1 + F_2$ . Then,

$$\begin{aligned} \mathcal{D}(F_1 + F_2)X &\longrightarrow (F_1 + F_2)\mathcal{D}X \\ S &\longmapsto \begin{cases} (\tau_1)_X S & \text{if } S \subseteq F_1 X \\ (\tau_2)_X S & \text{else.} \end{cases} \end{aligned}$$

defines a distributive law for  $F_1 + F_2$  with the given order.

- For the concatenation order  $\sqsubseteq'_X$  given by

$$a \sqsubseteq'_X b \text{ iff } a \sqsubseteq_X b \text{ or } (a \in F_1 X \text{ and } b \in F_2 X),$$

there is a related distributive law given by

$$\begin{aligned} \mathcal{D}(F_1 + F_2)X &\longrightarrow (F_1 + F_2)\mathcal{D}X \\ S &\longmapsto \begin{cases} (\tau_1)_X S & \text{if } S \subseteq F_1 X \\ (\tau_2)_X (S \cap F_2 X) & \text{else.} \end{cases} \end{aligned}$$

The functors  $\mathcal{L} \circ F$  and  $\mathcal{B}$  (with either of their respective orders) do *not* preserve dcpos. (This does not contradict Example 7.4 — there, the ordering is the flat ordering, so it does not involve our orders for  $\mathcal{L}$  in Section 4.) The powerset functor has no final coalgebra, so Theorem 7.3 does not apply to

it. Bounded versions of the powerset functor (finite powerset, etc.) do not preserve dcpos.

## 8 Terms and observations

The situation that we shall investigate in this section is described in Figure 2. It is obtained by repeated application of a functor  $F$  to the initial  $\emptyset$  and final  $1$  objects. What we have not included is that if  $F$  carries an order, all the objects in this diagram carry a derived order. Some of the arrows in this figure only exist if  $F$  satisfies certain properties. The aim of the figure is to give an overview of the structure that will be analysed below.

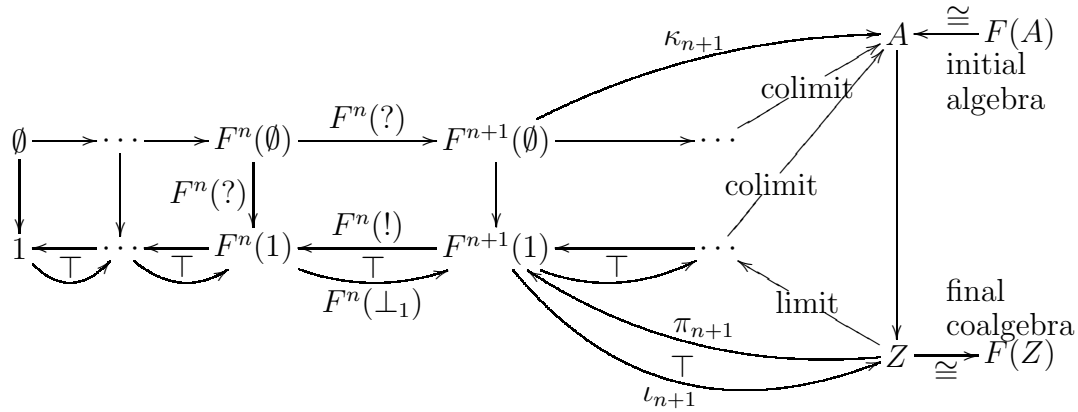


Fig. 2. Terms  $F^n(\emptyset)$  and observations  $F^n(1)$  with their colimit and limit

### 8.1 Ordering terms

For an endofunctor  $F$  the inhabitants of the sets  $F^n(\emptyset)$ , for  $n \in \mathbb{N}$ , are usually called terms. There are obvious inclusion maps  $F^n(?_{F(\emptyset)}): F^n(\emptyset) \rightarrow F^{n+1}(\emptyset)$ . Zooming in on the upper row in Figure 2, we get for  $m \leq n$  the following commuting diagrams.

$$\begin{array}{ccccccc}
 F^m(\emptyset) & \xrightarrow{F^m(?)} & F^{m+1}(\emptyset) & \xrightarrow{F^{m+1}(?)} & \dots & \xrightarrow{F^{n-1}(?)} & F^n(\emptyset) \\
 & & & & & & \searrow \\
 & & & & & & F^m(?_{F^{n-m}(\emptyset)})
 \end{array} \tag{2}$$

In this section we assume that our functor  $F$  carries a stable order  $\sqsubseteq$ . It

induces for each  $n \in \mathbb{N}$  an order  $\sqsubseteq_n$  on the set  $F^n(\emptyset)$  of terms, namely via:

$$\sqsubseteq_0 \stackrel{\text{def}}{=} =_{\emptyset} \quad \text{and} \quad \sqsubseteq_{n+1} \stackrel{\text{def}}{=} \text{Rel}_{\sqsubseteq}(F)(\sqsubseteq_n).$$

Each  $\sqsubseteq_n$  is then a preorder by Lemma 5.5.

Next we assume that our functor is pointed, i.e., comes with a point  $\perp : 1 \Rightarrow F$  such that each  $\perp_X$  is a bottom element<sup>1</sup> for the order  $\sqsubseteq_X$  on  $F(X)$ . We note that a natural transformation  $1 \Rightarrow F$  corresponds to an element in  $F(\emptyset)$ , as is demanded in [1].

Given such a point  $\perp$ , we define, for  $n \geq 1$ ,  $\perp_n \in F^n(\emptyset)$  to be the distinguished bottom element  $\perp_{F^{n-1}(\emptyset)}$  for  $\sqsubseteq$ .

**Lemma 8.1** (1) *Each function  $F^n(?) : F^n(\emptyset) \rightarrow F^{n+1}(\emptyset)$  preserves  $\perp_n$ , and is monotone, i.e., satisfies:*

$$\sqsubseteq_n \subseteq \left( F^n(?) \times F^n(?) \right)^{-1}(\sqsubseteq_{n+1}).$$

(2) *Each  $\perp_n$  is a bottom element for  $\sqsubseteq_n$ .*

**Proof.** (1) Preservation of  $\perp_n$  is immediate from the naturality of  $\perp : 1 \Rightarrow F$ . Preservation of the order is proved by induction on  $n$ . If  $n = 0$ , then the claim is trivially true. For the inductive case, suppose that

$$F^n(?) : (F^n(\emptyset), \sqsubseteq_n) \rightarrow (F^{n+1}(\emptyset), \sqsubseteq_{n+1})$$

is monotone. Then

$$F^{n+1}(?) : (F^{n+1}(\emptyset), \text{Rel}_{\sqsubseteq}(F)(\sqsubseteq_n)) \rightarrow (F^{n+2}(\emptyset), \text{Rel}_{\sqsubseteq}(F)(\sqsubseteq_{n+1}))$$

is monotone by definition of  $\sqsubseteq$ . Of course,  $\sqsubseteq_{n+1} = \text{Rel}_{\sqsubseteq}(F)(\sqsubseteq_n)$  and  $\sqsubseteq_{n+2} = \text{Rel}_{\sqsubseteq}(F)(\sqsubseteq_{n+1})$ , so the result is proved.

(2) By assumption,  $\perp_n$  is a bottom element for  $\sqsubseteq$ . Because each  $\sqsubseteq_n$  is reflexive, so is  $\text{Rel}(F)(\sqsubseteq_n)$ . Hence for  $t \in F^n(\emptyset)$  we have

$$\perp_n \sqsubseteq t \text{Rel}(F)(\sqsubseteq_{n-1}) t \sqsubseteq t,$$

and so  $\perp_n \sqsubseteq_n t$ . □

---

<sup>1</sup> We do not assume that  $\perp_X$  is the only bottom element for the preorder  $\sqsubseteq_X$ . It is merely a distinguished bottom.

Let, like in Figure 2,  $A$  be the colimit in **Sets** of the  $\omega$ -chain,

$$\begin{array}{ccccccc}
 & & & & & & A \\
 & & & & & & \nearrow \\
 & & & & & & \nearrow \\
 & & & & & & \nearrow \\
 \emptyset & \xrightarrow{?} & F(\emptyset) & \xrightarrow{F(?)} & F^2(\emptyset) & \xrightarrow{F^2(?)} & \dots \\
 & & \nearrow^{\kappa_0} & \nearrow^{\kappa_1} & \nearrow^{\kappa_2} & & \\
 & & & & & & 
 \end{array}$$

with coprojections  $\kappa_n$  satisfying  $\kappa_{n+1} \circ F^n(?) = \kappa_n$ . We can then order the elements of the colimit in the following standard manner<sup>2</sup>. For  $x, y \in A$ ,

$$\begin{aligned}
 x \leq y &\stackrel{\text{def}}{\iff} \exists m, n \in \mathbb{N}. \exists x' \in F^m(\emptyset). \exists y' \in F^n(\emptyset). \\
 &x = \kappa_m x' \wedge y = \kappa_n y' \wedge m \leq n \wedge F^m(?)(x') \sqsubseteq_n y'.
 \end{aligned}$$

Then it is easy to see that  $(A, \leq)$  is the colimit of the  $\omega$ -chain  $(F^n(\emptyset), \sqsubseteq_n)$  in **PreOrd**.

Further, since  $\perp \stackrel{\text{def}}{=} \kappa_1(\perp_1) \in A$  is the bottom element with respect to this order we even get a colimit in the category **PreOrd**<sub>⊥</sub> of preorders with bottom element (preserved by homomorphisms). For this to work we need to drop the empty set  $\emptyset$  as starting point of the  $\omega$ -chain.

A standard trick in this setting is to consider the cocone  $F^{n+1}(\emptyset) \xrightarrow{F(\kappa_n)} F(A)$  in **PreOrd**<sub>⊥</sub>, where  $F(A)$  is equipped with the order  $\text{Rel}_{\sqsubseteq}(F)(\leq)$  and bottom element  $\perp_A$ . The fact that  $A$  is a colimit yields a unique monotone, bottom-preserving map  $\alpha: A \rightarrow F(A)$  with  $\alpha \circ \kappa_{n+1} = F(\kappa_n)$ . It is well-known (going back to [17]) that if  $F$  preserves colimits of  $\omega$ -chains, then  $\alpha$  is an isomorphism and its inverse  $\alpha^{-1}: F(A) \xrightarrow{\cong} A$  is the initial algebra “of terms” for  $F$ . Note that at this stage we only know for  $\alpha$ , and not for the initial algebra  $\alpha^{-1}$ , that it is monotone.

## 8.2 Ordering observations

In this section we shift our attention from the sets  $F^n(\emptyset)$  of terms to the sets  $F^n(1)$  of observations in Figure 2. Between these sets of observations there are obvious maps  $F^n(!): F^{n+1}(1) \rightarrow F^n(1)$ , satisfying the analogue of (2). Moreover, there are maps  $F^n(?): F^n(\emptyset) \rightarrow F^n(1)$  between terms and observations,

<sup>2</sup> Using that the forgetful functor **PreOrd**  $\rightarrow$  **Sets** creates colimits.

making the following diagram commute.

$$\begin{array}{ccc}
F^n(\emptyset) & \xrightarrow{F^n(?)} & F^{n+1}(\emptyset) \\
F^n(?) \downarrow & & \downarrow F^{n+1}(?) \\
F^n(1) & \xleftarrow{F^n(!)} & F^{n+1}(1)
\end{array}$$

Each set of observations  $F^n(1)$  carries a preorder  $\sqsubseteq_n$  with a bottom element  $\perp_n$ , via the definitions:

$$\sqsubseteq_n \stackrel{\text{def}}{=} \text{Rel}_{\sqsubseteq}(F)^n(=1) \quad \text{and} \quad \perp_n \stackrel{\text{def}}{=} \begin{cases} * & \text{if } n = 0 \\ \perp_{F^{n-1}(1)} & \text{otherwise} \end{cases}$$

We thus use  $*$  for the sole element of the singleton set 1. It is easy to see that  $\perp_n$  is the bottom element of  $\langle F^n(1), \sqsubseteq_n \rangle$ . Notice that we overload the notation  $\sqsubseteq_n, \perp_n$  for the preorder and bottom element on terms  $F^n(\emptyset)$  and on observations  $F^n(1)$ .

**Lemma 8.2** (1) *Each  $F^n(!): F^{n+1}(1) \rightarrow F^n(1)$  is a map in  $\mathbf{PreOrd}_{\perp}$ , that is, it preserves  $\sqsubseteq_n$  and  $\perp_n$ .*

(2) *These maps  $F^n(!): F^{n+1}(1) \rightarrow F^n(1)$  have a left adjoint*

$$F^n(\perp_1): F^n(1) \rightarrow F^{n+1}(1)$$

*with  $F^n(!) \circ F^n(\perp_1) = id$ .*

(3) *Each  $\perp_n$  is a bottom element for  $\sqsubseteq_n$ .*

(4) *Each  $F^n(?): F^n(\emptyset) \rightarrow F^n(1)$  preserves  $\sqsubseteq_n$  and (if  $n \geq 1$ ) also  $\perp_n$ .*

**Proof.** (1) Preservation of bottom elements is easy, and preservation of the order follows by induction, much like in the proof of Lemma 8.1 (1).

(2) We first note that each  $F^n(\perp_1)$  is indeed a map in  $\mathbf{PreOrd}_{\perp}$ . That this holds for  $n = 0$  is trivial. For the inductive step:  $F^{n+1}(\perp_1)$  preserves  $\perp_{n+1}$  by naturality of  $\perp: 1 \Rightarrow F$  and definition of  $\perp_{n+1}$ . Monotonicity follows the proof of Lemma 8.1 (1).

We proceed by induction on  $n$ , with the claim obvious for  $n = 0$ . Suppose that the claim holds for  $n$ . The adjunction  $F^n(\perp_1) \dashv F^n(!)$  can be explicitly stated

$$(id \times F^n(!))^{-1}(\sqsubseteq_n) = (F^n(\perp_1) \times id)^{-1}(\sqsubseteq_{n+1}).$$

We will show the same equation holds for  $n + 1$ . Here, we use stability of

the order.

$$\begin{aligned}
(id \times F^{n+1}(!))^{-1}(\sqsubseteq_{n+1}) &= (id \times F^{n+1}(!))^{-1} \text{Rel}_{\sqsubseteq}(F)(\sqsubseteq_n) \\
&= \text{Rel}_{\sqsubseteq}(F)((id \times F^n!)^{-1}(\sqsubseteq_n)) \\
&\stackrel{\text{(IH)}}{=} \text{Rel}_{\sqsubseteq}(F)((F^n(\perp_1) \times id)^{-1}(\sqsubseteq_{n+1})) \\
&= (F^{n+1}(\perp_1) \times id)^{-1}(\text{Rel}_{\sqsubseteq}(F)(\sqsubseteq_{n+1})) \\
&= (F^{n+1}(\perp_1) \times id)^{-1}(\sqsubseteq_{n+2}).
\end{aligned}$$

(3) The same proof as Lemma 8.1 (2).

(4) Recall from Lemma 5.5 that  $F$  restricts to a functor  $\mathbf{PreOrd} \rightarrow \mathbf{PreOrd}$ . Clearly,  $?: (\emptyset, =_\emptyset) \rightarrow (1, =_1)$  is a map of preorders. Therefore  $F^n(?)$  is a map of preorders from  $F^n(\emptyset)$  with order  $\sqsubseteq_n = \text{Rel}_{\sqsubseteq}(F)^n(=_\emptyset)$  to  $F^n(1)$  with order  $\sqsubseteq_n = \text{Rel}_{\sqsubseteq}(F)^n(=_1)$ . Preservation of the bottom elements (for  $n \geq 1$ ) is trivial.  $\square$

We shall write  $Z$  for the limit in  $\mathbf{Sets}$  of the  $\omega$ -chain  $1 \leftarrow F(1) \leftarrow F^2(1) \leftarrow \dots$ , with projections  $\pi_n: Z \rightarrow F^n(1)$  satisfying  $F^n(!) \circ \pi_{n+1} = \pi_n$ , like in Figure 2. This limit can also be understood as a limit in  $\mathbf{PreOrd}_\perp$  via the following order and bottom element on  $Z$ .

$$x \leq y \stackrel{\text{def}}{\iff} \forall n \in \mathbb{N}. \pi_n x \sqsubseteq_n \pi_n y \quad \text{and} \quad \perp \stackrel{\text{def}}{=} \langle \perp_n \rangle_{n \in \mathbb{N}}. \quad (3)$$

The object<sup>3</sup>  $\langle F(Z), \text{Rel}_{\sqsubseteq}(F)(\leq), \perp_Z \rangle$  in  $\mathbf{PreOrd}_\perp$  carries a cone structure with maps  $F(\pi_n): F(Z) \rightarrow F^{n+1}(1)$  and  $!: F(Z) \rightarrow 1$ . It yields a (monotone and bottom-preserving) mediating map  $\xi: F(Z) \rightarrow Z$  in  $\mathbf{PreOrd}_\perp$ . If  $F: \mathbf{Sets} \rightarrow \mathbf{Sets}$  preserves limits of  $\omega$ -cochains,  $\xi$  is an isomorphism, and  $\zeta = \xi^{-1}$  a final coalgebra. Like for algebras, we do not know yet that  $\zeta$  is monotone and bottom-preserving. It will be shown at the end of the next section, when we give sufficient conditions that  $\leq$  and  $\lesssim$  coincide.

The polynomial functors  $\mathbf{Poly}$  preserve limits of  $\omega$ -cochains.

Assuming finality we obtain for each  $n \in \mathbb{N}$  a coalgebra homomorphism  $\iota_n: F^n(1) \rightarrow Z$  in:

$$\begin{array}{ccc}
F^{n+1}(1) & \xrightarrow{F(\iota_n)} & F(Z) \\
F^n(\perp_1) \uparrow & & \cong \uparrow \zeta \\
F^n(1) & \xrightarrow{\iota_n} & Z
\end{array}$$

By uniqueness we then get  $\iota_{n+1} \circ F^n(\perp_1) = \iota_n$ . This allows us to prove the

<sup>3</sup> Note  $\perp_Z$  is the bottom element of  $\text{Rel}_{\sqsubseteq}(F)(\leq)$  and not the bottom element of  $\text{Rel}_{\sqsubseteq}(F)(\lesssim)$ . That is,  $\perp_Z = \perp_{(Z, \leq)}$  in this section.

following alternative description.

$$\iota_0 = \zeta^{-1} \circ \perp_Z \quad \text{and} \quad \iota_{n+1} = \zeta^{-1} \circ F(\iota_n). \quad (4)$$

Note that each  $\iota_n$  is monotone and bottom-preserving, as the composition of monotone, bottom-preserving maps. The main result about these  $\iota_n$ 's is the following.

**Lemma 8.3** *The limit projections  $\pi_n: Z \rightarrow F^n(1)$  have  $\iota_n$  as left adjoint with  $\pi_n \circ \iota_n = id_{F^n(1)}$ .*

**Proof.** We first prove the equation  $\pi_n \circ \iota_n = id$ , by induction on  $n$ , using the formulation (4). The base case  $n = 0$  is trivial. And:

$$\begin{aligned} \pi_{n+1} \circ \iota_{n+1} &= \pi_{n+1} \circ \zeta^{-1} \circ F(\iota_n) \\ &= F(\pi_n) \circ F(\iota_n) \\ &= F(\pi_n \circ \iota_n) \\ &\stackrel{\text{(IH)}}{=} id. \end{aligned}$$

This equation can be used to prove the  $\subseteq$  part of the claimed adjunction  $(\iota_n \times id)^{-1}(\leq) = (id \times \pi_n)^{-1}(\sqsubseteq_n)$ . The proof is by induction, and the base case is again trivial. The induction step uses that  $\pi_{n+1}$  is monotone:

$$\begin{aligned} (\iota_{n+1} \times id)^{-1}(\leq) &\subseteq (\iota_{n+1} \times id)^{-1}(\pi_{n+1} \times \pi_{n+1})^{-1}(\sqsubseteq_{n+1}) \\ &= ((\pi_{n+1} \circ \iota_{n+1}) \times \pi_{n+1})^{-1}(\sqsubseteq_{n+1}) \\ &= (id \times \pi_{n+1})^{-1}(\sqsubseteq_{n+1}). \end{aligned}$$

The proof of the reverse inclusion uses that  $\zeta^{-1}: F(Z) \rightarrow Z$  is monotone (by construction). Specifically, it means that  $(\zeta(x), \zeta(y)) \in \text{Rel}_{\sqsubseteq}(F)(\leq) \Rightarrow x \leq y$ . This is used in the last (inclusion) step in:

$$\begin{aligned} &(id \times \pi_{n+1})^{-1}(\sqsubseteq_{n+1}) \\ &= (id \times \pi_{n+1})^{-1} \text{Rel}_{\sqsubseteq}(F)(\sqsubseteq_n) \\ &= (id \times \zeta)^{-1} (id \times F(\pi_n))^{-1} \text{Rel}_{\sqsubseteq}(F)(\sqsubseteq_n) \\ &= (id \times \zeta)^{-1} \text{Rel}_{\sqsubseteq}(F)((id \times \pi_n)^{-1}(\sqsubseteq_n)) \\ &\stackrel{\text{IH}}{\subseteq} (id \times \zeta)^{-1} \text{Rel}_{\sqsubseteq}(F)((\iota_n \times id)^{-1}(\leq)) \\ &= (id \times \zeta)^{-1} (F(\iota_n) \times id)^{-1} \text{Rel}_{\sqsubseteq}(F)(\leq) \\ &= (\iota_{n+1} \times id)^{-1} (\zeta \times \zeta)^{-1} \text{Rel}_{\sqsubseteq}(F)(\leq) \\ &\subseteq (\iota_{n+1} \times id)^{-1}(\leq). \quad \square \end{aligned}$$



## 9 Similarity $\lesssim$ as an $\omega$ -limit

Throughout this section, we assume that  $F: \mathbf{Sets} \rightarrow \mathbf{Sets}$  preserves limits of  $\omega$ -cochains, so the carrier of the final coalgebra  $(Z, \zeta)$  is given as the limit of the  $\omega$ -cochain

$$1 \xleftarrow{!} F1 \xleftarrow{F!} F^2 1 \xleftarrow{\dots} \dots \quad (5)$$

from Subsection 8.2. There we have seen that  $Z$  with order  $\leq$  and bottom  $\perp$  is the limit of  $1 \xleftarrow{!} F1 \xleftarrow{F!} F^2 1 \xleftarrow{\dots} \dots$  in  $\mathbf{PreOrd}_\perp$ . In this section, we will give sufficient conditions that  $\lesssim = \leq$ .

**Remark 9.1** *In fact, in what follows, we do not make any especial use of the bottom element  $\perp$ . The same arguments would show that  $\lesssim = \leq$  in  $\mathbf{PreOrd}$ , without alteration. Since we're interested in algebraic cpos hereafter, we do the proofs in  $\mathbf{PreOrd}_\perp$  for convenience.*

First, we show that the greatest simulation  $\lesssim$  is always contained in  $\leq$ . For this, we do not require any assumptions aside from those listed above.

**Lemma 9.2**  $\lesssim \subseteq \leq$ .

**Proof.** Recall  $\leq = \bigcap_{n \in \mathbb{N}} (\pi_n \times \pi_n)^{-1} \sqsubseteq_n$ . We will show that, for each  $n$ , we have  $\lesssim \subseteq (\pi_n \times \pi_n)^{-1} \sqsubseteq_n$ . We proceed by induction, with the base case obvious.

$$\begin{aligned} \lesssim &= (\zeta \times \zeta)^{-1} \text{Rel}_{\sqsubseteq}(F)(\lesssim) \quad \text{by Lemma 5.1} \\ &\stackrel{\text{IH}}{\subseteq} (\zeta \times \zeta)^{-1} \text{Rel}_{\sqsubseteq}(F)((\pi_n \times \pi_n)^{-1} \sqsubseteq_n) \\ &= (\zeta \times \zeta)^{-1} (F\pi_n \times F\pi_n)^{-1} \text{Rel}_{\sqsubseteq}(F)(\sqsubseteq_n) \\ &= (\pi_{n+1} \times \pi_{n+1})^{-1} \text{Rel}_{\sqsubseteq}(F)(\sqsubseteq_n) \\ &= (\pi_{n+1} \times \pi_{n+1})^{-1} \sqsubseteq_{n+1}. \quad \square \end{aligned}$$

Thus, to complete the proof that  $\leq = \lesssim$ , we must show  $\leq \subseteq \lesssim$ . Since  $\lesssim$  is the greatest simulation, it suffices to show that  $\leq$  is a simulation, too. For this, we impose an additional condition on the functor  $F$ .

**Definition 9.3** *We say that a functor  $F$  with order  $\sqsubseteq$  preserves intersections of reflexive relations if, given a set  $\langle R_i \mid i \in I \rangle$  of reflexive relations over  $X$  and  $Y$ , we have*

$$\bigcap_{i \in I} \text{Rel}_{\sqsubseteq}(F)(R_i) = \text{Rel}_{\sqsubseteq}(F)\left(\bigcap_{i \in I} R_i\right).$$

This property is used at a critical step in the following proof that  $\leq \subseteq \lesssim$ .

The polynomial functors **Poly** defined in Section 4 preserve intersections of reflexive relations. In fact, with the exception of the order  $\sqsubseteq'_{F_1+F_2}$ , these functors preserve intersections of *arbitrary* (not just reflexive) relations.

**Theorem 9.4** *Suppose that  $F$  has a stable order and preserves intersections of reflexive relations. Then  $\lesssim = \leq$ .*

**Proof.** It suffices to show that  $\leq$  is a simulation on the final coalgebra, i.e., that  $\leq \subseteq (\zeta \times \zeta)^{-1} \text{Rel}_{\sqsubseteq}(F)(\leq)$ .

$$\begin{aligned}
\leq &= \bigcap_{n \in \mathbb{N}} (\pi_n \times \pi_n)^{-1} \sqsubseteq_n \\
&= \bigcap_{n \in \mathbb{N}} (\pi_{n+1} \times \pi_{n+1})^{-1} \sqsubseteq_{n+1} \\
&= \bigcap_{n \in \mathbb{N}} (F\pi_n \circ \zeta \times F\pi_n \circ \zeta)^{-1} \text{Rel}_{\sqsubseteq}(F)(\sqsubseteq_n) \\
&= \bigcap_{n \in \mathbb{N}} (\zeta \times \zeta)^{-1} (F\pi_n \times F\pi_n)^{-1} \text{Rel}_{\sqsubseteq}(F)(\sqsubseteq_n) \\
&= \bigcap_{n \in \mathbb{N}} (\zeta \times \zeta)^{-1} \text{Rel}_{\sqsubseteq}(F) ((\pi_n \times \pi_n)^{-1} \sqsubseteq_n) \\
&= (\zeta \times \zeta)^{-1} \bigcap_{n \in \mathbb{N}} \text{Rel}_{\sqsubseteq}(F) ((\pi_n \times \pi_n)^{-1} \sqsubseteq_n) \\
&= (\zeta \times \zeta)^{-1} \text{Rel}_{\sqsubseteq}(F) \left( \bigcap_{n \in \mathbb{N}} (\pi_n \times \pi_n)^{-1} \sqsubseteq_n \right) \\
&= (\zeta \times \zeta)^{-1} \text{Rel}_{\sqsubseteq}(F)(\leq). \quad \square
\end{aligned}$$

In Section 8.2, we saw that  $(Z, \leq, \perp)$  is the limit of the  $\omega$ -cochain (5), although we could not prove at that point (even assuming that  $F$  preserves limits of  $\omega$ -cochains) that the structure map  $\zeta: Z \rightarrow FZ$  is monotone. Of course,  $\zeta$  is monotone with respect to  $\lesssim$ , and hence, as corollary to Theorem 9.4, it is monotone with respect to  $\leq$ .

## 10 Algebraic cpo structure on final coalgebras

In this section, we will investigate sufficient conditions that the final coalgebra  $(Z, \zeta)$ , together with similarity order  $\lesssim$  and bottom element  $\perp = \langle \perp_n \rangle$  forms an algebraic cpo. We begin by reviewing some terminology and stating the assumptions which we impose hereafter.

Let **Cpo** denote the category of complete pre-orders (directed complete pre-orders with bottom) and continuous, bottom-preserving maps. Note that the final coalgebra  $(Z, \zeta)$  with similarity order  $\lesssim$  and bottom element  $\zeta^{-1}(\perp_Z)$  is a cpo.

**Definition 10.1** Let  $(X, \leq_X)$  be a cpo. An element  $x$  of  $X$  is finite if, for every directed set  $D \subseteq X$ , we have

$$x \leq_X \bigsqcup D \quad \Rightarrow \quad \exists d \in D. x \leq_X d.$$

Let  $K: \mathbf{Cpo} \rightarrow \mathbf{PreOrd}_\perp$  be the operator taking a cpo  $(X, \leq_X)$  to its suborder  $KX$  of finite elements.

**Definition 10.2** A cpo  $(X, \leq_X)$  is algebraic if the following hold.

- (1) For each  $x \in X$ , the set  $\{d \in KX \mid d \leq_X x\}$  is directed.
- (2) Furthermore,  $x = \bigsqcup \{d \in KX \mid d \leq_X x\}$  (up to isomorphism).

The following lemma gives a sufficient condition that a morphism  $l: X \rightarrow Y$  between algebraic cpos  $X$  and  $Y$  preserves the finite elements of  $X$ . We use it in constructing a colimit of  $KF^n 1$  hereafter, and also in showing that the constructed colimit consists of finite elements of the final coalgebra  $Z$ .

**Lemma 10.3** Let  $(X, \leq_X)$  and  $(Y, \leq_Y)$  be algebraic cpos and let  $l: X \rightarrow Y$ ,  $r: Y \rightarrow X$  be monotone, maps satisfying the following:

- (1)  $r$  is continuous;
- (2)  $l \dashv r$ ;

Then  $l$  preserves finite elements, i.e., restricts to a map  $KX \rightarrow KY$ , as in the commutative diagram below.

$$\begin{array}{ccc} X & \begin{array}{c} \xrightarrow{l} \\ \xleftarrow{r} \end{array} & Y \\ \uparrow & & \uparrow \\ KX & \xrightarrow{l} & KY \end{array}$$

**Proof.** Let  $x \in X$  be finite and we will show that  $lx$  is also finite. Suppose that  $lx \leq_Y \bigsqcup D$  for some directed  $D \subseteq Y$ . Then  $x \leq_X r \bigsqcup D = \bigsqcup \prod_r D$ . Consequently, there is a  $y \in D$  such that  $x \leq_X ry$  and hence  $lx \leq_Y y$ .  $\square$

Hereafter, we assume that the pointed functor  $F$  with stable order  $\sqsubseteq$  preserves dcpos (as in Definition 5.6), so that the final coalgebra  $(Z, \zeta)$  together with  $\lesssim$  forms a dcpo, as in Section 7. We also assume that  $F$  preserves algebraic cpos. In fact, we use this assumption only to ensure that each preorder in the cochain

$$1 \longleftarrow ! F 1 \longleftarrow F! F^2 1 \longleftarrow \dots$$

is an algebraic cpo (with order  $\sqsubseteq_n$ ), so we could have simply assumed the algebraicity of each  $F^n 1$  instead. In any case, the polynomial functors **Poly** from Section 4 all preserve algebraic cpos, with some caveats. The constant

functors  $X \mapsto A$  preserve algebraic cpos iff  $A$  is an algebraic cpo. Also, the disjoint order on  $F_1 + F_2$  must be altered so that it has a bottom element, either by introducing a new  $\perp$  or by identifying  $\perp_{F_1}$  and  $\perp_{F_2}$ .

We also continue our assumption from Section 9 that  $F$  preserves intersections of reflexive relations. Thus, the carrier  $Z$  of the final coalgebra with similarity order  $\lesssim$  is the limit in  $\mathbf{PreOrd}_\perp$  of the cochain

$$1 \longleftarrow \text{!} F1 \longleftarrow \text{!} F! F^2 1 \longleftarrow \dots$$

Each of the horizontal arrows is clearly continuous. Hence, since the forgetful functor  $\mathbf{Dcpo} \rightarrow \mathbf{PreOrd}_\perp$  creates limits,  $(Z, \lesssim)$  is also the limit of the same chain in  $\mathbf{Dcpo}$ . In particular, this entails that the projections  $\pi: Z \rightarrow F^n 1$  and both  $\zeta$  and  $\zeta^{-1}$  are continuous.

Summing up, we assume

- (1)  $F$  has stable order  $\sqsubseteq$  and bottom  $\perp: 1 \Rightarrow F$ ;
- (2)  $F$  preserves dcpo;
- (3)  $F$  preserves algebraic cpos;
- (4)  $F$  preserves intersections of reflexive relations;
- (5)  $F$  preserves limits of  $\omega$ -cochains.

The polynomial functors  $\mathbf{Poly}$  satisfy these conditions, given that the constant functors involve algebraic cpos and the functor has a bottom element.

Recall that  $\perp_1$  is the bottom element in the preorder  $(F1, \sqsubseteq_1)$ . For each  $n$ ,  $F^n(\perp_1)$  is an injection from  $F^n 1$  to  $F^{n+1} 1$ . Intuitively, the  $F^n 1$ 's are finite approximations of the final coalgebra  $Z$  and  $F^n(\perp_1)$  is the ‘inclusion’ of the  $n$ th approximation into the  $(n+1)$ st. The following lemma ensures that these inclusions preserve finite elements. We aim to show that the union of the finite elements appearing in the cochain (5) is exactly the set  $KZ$  of finite elements of the final coalgebra  $(Z, \zeta)$ .

**Lemma 10.4** *For each  $n$ , the function  $F^n(\perp_1)$  preserves finite elements, so that  $F^n(\perp_1)$  restricts to a function  $KF^n 1 \rightarrow KF^{n+1} 1$ .*

**Proof.** We wish to apply Lemma 10.3. Thus, it suffices to show that each  $F^n(\perp_1)$  has a continuous right adjoint. By Lemma 8.2,  $F^n(!)$  is right adjoint to  $F^n(\perp_1)$ . Moreover,  $F^n(!)$  is trivially continuous for  $n = 0$ , and is continuous for  $n > 0$  by the assumption that  $F$  preserves dcpo.  $\square$

Let  $A_K$  denote the colimit (in  $\mathbf{PreOrd}_\perp$ ) of the  $\omega$ -chain

$$K1 \xrightarrow{\perp_1} KF1 \xrightarrow{F\perp_1} KF^2 1 \xrightarrow{F^2\perp_1} \dots \quad (6)$$



$z$ . From this, the result easily follows.

**Lemma 10.6** *Let  $z \in Z$ . The sequence  $\langle (\iota_n \circ \pi_n)(z) \mid n \in \mathbb{N} \rangle$  is a  $\lesssim$ -chain.*

**Proof.** We use the fact that  $F^n(!) \circ \pi_{n+1} = \pi_n$  as starting point to derive the required result:

$$\frac{\frac{\pi_n z \sqsubseteq_n (F^n(!) \circ \pi_{n+1})z}{(F^n(\perp_1) \circ \pi_n)z \sqsubseteq_{n+1} \pi_{n+1}z} \text{ (Lemma 8.2)}}{\frac{(\iota_{n+1} \circ F^n(\perp_1) \circ \pi_n)z \lesssim (\iota_{n+1} \circ \pi_{n+1})z}{(\iota_n \circ \pi_n)z \lesssim (\iota_{n+1} \circ \pi_{n+1})z} \text{ (Monotonicity)}} \text{ } (\iota_{n+1} \circ F^n(\perp_1) = \iota_n)$$

□

The following theorem shows that each  $z \in Z$  is determined by the chain constructed above, in the usual algebraic sense. In other words: each  $z$  is the join of the chain

$$(\iota_0 \circ \pi_0)(z) \lesssim (\iota_1 \circ \pi_1)(z) \lesssim (\iota_2 \circ \pi_2)(z) \lesssim \dots$$

Of course, in a complete pre-order, such joins are determined only up to isomorphism, i.e., two-way similarity. (In the case that  $\approx = \Leftrightarrow = =_Z$ , as in Theorem 6.2, then  $Z$  is a complete partial order and the stronger result attains.)

**Lemma 10.7** *For each  $z \in Z$ , we have  $z \approx \bigsqcup_{n \in \mathbb{N}} (\iota_n \circ \pi_n)(z)$ .*

**Proof.** Clearly,  $\pi_n z \sqsubseteq_n \pi_n$ , and so  $(\iota_n \circ \pi_n)(z) \leq z$ , by  $\iota_n \dashv \pi_n$ . Since  $\leq = \lesssim$  by Theorem 9.4 we get  $(\iota_n \circ \pi_n)(z) \lesssim z$ , and thus  $\bigsqcup_{n \in \mathbb{N}} (\iota_n \circ \pi_n)(z) \lesssim z$ . For the other direction, we note that, for every  $n$ , we have  $(\iota_n \circ \pi_n)(z) \lesssim \bigsqcup_{n \in \mathbb{N}} (\iota_n \circ \pi_n)(z)$ . Hence, for every  $n$ ,

$$\pi_n(z) \sqsubseteq_n \pi_n \left( \bigsqcup_{n \in \mathbb{N}} (\iota_n \circ \pi_n)(z) \right).$$

In other words,  $(z, \bigsqcup_{n \in \mathbb{N}} (\iota_n \circ \pi_n)(z)) \in \bigcap_{n \in \mathbb{N}} (\pi_n \times \pi_n)^{-1}(\sqsubseteq_n) = \leq$ . Theorem 9.4 (proving  $leq = \lesssim$ ) completes the proof. □

The following corollary expresses the relationship between  $KZ$  and  $\text{Im}(m)$  in a general case. If bisimilarity is not the same as two-way similarity, then the best one can do is: each finite element of  $z$  is two-way similar to an element of  $\text{Im}(m)$ . If the  $\Leftrightarrow$  and  $\approx$  are equal relations, then one can do better. In that case, since  $Z$  is final, we have  $\text{Im}(m) = KZ$ .

**Corollary 10.8**  *$\text{Im}(m: A_K \rightarrow Z) = KZ$  up to two-way similarity. In other*

words,

$$\{x \mid \exists x' \in \text{Im}(m). x \approx x'\} = KZ.$$

**Proof.** We already have  $\subseteq$  from Lemma 10.5. Thus, we wish to show, for each  $z \in KZ$ , there is an  $n \in \mathbb{N}$  and  $x \in KF^n1$  such that  $z \approx \iota_n x$ . Let finite  $z \in Z$  be given. Since  $z \lesssim \bigsqcup (\iota_n \circ \pi_n)(z)$ , we see that  $z \lesssim (\iota_n \circ \pi_n)(z)$  for some  $n$ . But, by the adjunction  $\iota_n \dashv \pi_n$ , we also have  $(\iota_n \circ \pi_n)(z) \lesssim z$  and hence  $z \approx (\iota_n \circ \pi_n)(z)$ .  $\square$

We have now characterized  $KZ$  in terms of the finite elements of the finite approximations  $F^n1$ . We use that characterization to show that the set of finite elements below a given element of  $Z$  is directed. This is the last “big” step in showing that  $(Z, \lesssim)$  is an algebraic cpo.

**Lemma 10.9** *For every  $z \in Z$ , the set*

$$\{v \in KZ \mid v \lesssim z\}$$

*is directed.*

**Proof.** Let  $y$  and  $y'$  be finite elements of  $Z$  such that  $y, y' \lesssim z$ . Without loss of generality, we may assume that  $y$  and  $y'$  are in  $\text{Im}(m)$ . Then there are  $k, k'$  such that  $y = (\iota_k \circ \pi_k)(y)$  and  $y' = (\iota_{k'} \circ \pi_{k'})(y')$  and such that  $\pi_k y$  and  $\pi_{k'} y'$  are finite in  $F^k1$  and  $F^{k'}1$ , respectively.

Note that, for all  $n \in \mathbb{N}$  and  $x \in Z$ , if  $(\iota_n \circ \pi_n)(x) = x$ , then

$$\begin{aligned} (\iota_{n+1} \circ \pi_{n+1})(x) &= (\zeta^{-1} \circ F(\iota_n) \circ F(\pi_n) \circ \zeta)(x) \\ &= (\zeta^{-1} \circ F(\text{id}) \circ \zeta)(x) \\ &= x. \end{aligned}$$

Also, if  $x = (\iota_n \circ \pi_n)(x)$  where  $\pi_n(x)$  is finite in  $F^n1$ , then

$$\begin{aligned} \pi_{n+1}(x) &= (\pi_{n+1} \circ \iota_n \circ \pi_n)(x) \\ &= (\pi_{n+1} \circ \iota_{n+1} \circ F^n(\perp_1) \circ \pi_n)(x) \\ &= (F^n(\perp_1) \circ \pi_n)(x). \end{aligned}$$

Since  $F^n(\perp_1)$  preserves finite elements (Lemma 10.4), we see that  $\pi_{n+1}(x)$  is also finite.

Suppose that  $k' \leq k$ . We may conclude that  $y' = (\iota_k \circ \pi_k)(y')$  and  $\pi_k(y')$  is finite. Hence, without loss of generality, we may assume that  $k = k'$ .

By the adjunction  $\iota_k \dashv \pi_k$ , we see that  $\pi_k(y), \pi_k(y') \sqsubseteq_k \pi_k(z)$ . Since both  $\pi_k(y)$  and  $\pi_k(y')$  are finite, there is a finite  $x \in F^k(1)$  such that  $\pi_k(y), \pi_k(y') \sqsubseteq_k x$

and  $x \sqsubseteq_k \pi_k(z)$ . Hence,  $\iota_k(x)$  is finite (in  $Z$ ) and by the adjunction again,  $\iota_k(x) \lesssim z$ . Since  $\iota_k$  is monotone, we also have  $y = (\iota_k \circ \pi_k)(y) \lesssim \iota_k(x)$  and similarly  $y' \lesssim \iota_k(x)$ .  $\square$

**Theorem 10.10** *Let  $F$  with an order satisfy conditions (1)–(5) from the beginning of this section. Then the final coalgebra  $(Z, \zeta)$  with similarity order  $\lesssim$  is an algebraic cpo.*

**Proof.** Theorem 10.9 establishes that each  $\{v \in KZ \mid v \lesssim z\}$  is directed and Theorem 10.7 yields

$$z \lesssim \bigsqcup_{n \in \mathbb{N}} (\iota_n \circ \pi_n)(z) \lesssim \bigsqcup \{v \in KZ \mid v \lesssim z\}. \quad \square$$

## 11 Terms and finite behaviour

In the previous section we have seen how the (finite elements from the) sets  $F^n(1)$  play a role as finite approximations of elements in the final coalgebra. This section concentrates on sets  $F^n(\emptyset)$ , and shows that its elements correspond to the elements of the final coalgebra with “finite behaviour”, i.e., with only finite transition sequences. In order to be able to express such a result we first describe transitions in a general coalgebraic sense, using temporal logic [12].

So far we have made extensive use of the relation lifting  $\text{Rel}(F): \mathbf{Rel} \rightarrow \mathbf{Rel}$  of a functor  $F: \mathbf{Sets} \rightarrow \mathbf{Sets}$ . There is also a useful “predicate” lifting functor, which lifts  $F$  to an endofunctor on the category  $\mathbf{Pred}$  of predicates. Its objects are predicate ( $P \subseteq X$ ) on an underlying set. And its morphisms  $f: (P \subseteq X) \rightarrow (Q \subseteq Y)$  are functions  $f: X \rightarrow Y$  which restrict to the predicates: if  $P(x)$ , also written frequently as  $x \in P$ , then  $Q(f(x))$ . For an arbitrary category  $\mathbb{C}$  one sees the notation  $\text{Sub}(\mathbb{C})$  for the suitably generalized version of this category  $\mathbf{Pred}$ .

For an arbitrary functor  $F: \mathbf{Sets} \rightarrow \mathbf{Sets}$  one can define predicate lifting  $\text{Pred}(F): \mathbf{Pred} \rightarrow \mathbf{Pred}$  on a predicate (or subset)  $m: P \hookrightarrow X$  by taking the image of  $F(m)$ , as in:

$$\begin{array}{ccc} F(P) & \xrightarrow{\quad} & \text{Pred}(F)(P) \\ & \searrow_{F(m)} & \downarrow \\ & & F(X) \end{array} \quad \text{i.e., as} \quad \text{Pred}(F)(P) = \coprod_{F(m)} F(P)$$

For many of our examples the functor  $F$  preserves inclusions (monomorphisms) so that we simply have  $\text{Pred}(F)(P) = F(P)$ . This is for instance the case when



$F$  preserves weak pullbacks. But it is conceptually clearer to make a distinction between  $F$  and its lifting to predicates.

We shall use the following preservation properties of predicate lifting.

- (1) Inclusions:  $P \subseteq Q$  implies  $\text{Pred}(F)(P) \subseteq \text{Pred}(F)(Q)$ .
- (2) Arbitrary intersections:  $\text{Pred}(F)(\bigcap_{i \in I} P_i) = \bigcap_{i \in I} \text{Pred}(F)(P_i)$ .
- (3) Inverse images:  $\text{Pred}(F)(f^{-1}(Q)) = F(f)^{-1}(\text{Pred}(F)(Q))$ , for  $f: X \rightarrow Y$  and  $Q \subseteq Y$ .

The first point is automatic. The third one follows if the functor  $F$  preserves weak (binary) pullbacks, and the second one if it preserves arbitrary pullbacks.

Given a coalgebra  $c: X \rightarrow F(X)$  we can define associated temporal operators in terms of predicate lifting (following [12]). The most important operator that we shall use is “nexttime”  $\bigcirc$ . It is defined on a predicate  $P \subseteq X$  on the coalgebra’s state space as a new predicate  $\bigcirc P \subseteq X$ , namely as

$$\begin{aligned} \bigcirc P &\stackrel{\text{def}}{=} c^{-1}(\text{Pred}(F)(P)) \\ &= \{x \in X \mid c(x) \in \text{Pred}(F)(P)\}. \end{aligned}$$

Intuitively,  $\bigcirc P$  contains those states  $x$  such that  $P$  holds for all of the successors of  $x$ , if any. This intuition will be made precise below. Notice that the coalgebra  $c$  is left implicit in the notation  $\bigcirc P$ .

Once we have nexttime  $\bigcirc$  we can set up an extensive temporal machinery, see [12]. For instance,  $P$  is called an invariant if  $P \subseteq \bigcirc P$ . And  $\square P$  may be defined as the as the greatest fixed point of  $Q \mapsto P \wedge \bigcirc Q$ . This  $\square P$  is then the greatest invariant contained in  $P$ .

Our next step is to associate an unlabeled transition system with an arbitrary coalgebra  $c: X \rightarrow F(X)$ . For states  $x, x' \in X$  we define

$$\begin{aligned} x \longrightarrow x' &\stackrel{\text{def}}{\iff} x \in (\neg \bigcirc \neg)(\{y \mid y = x\}) \\ &\iff x \notin \bigcirc(\{y \mid y \neq x'\}) \\ &\iff c(x) \notin \text{Pred}(F)(\{y \mid y \neq x'\}). \end{aligned} \tag{7}$$

We need the following two basic results about this induced transition relation.

**Proposition 11.1** *Let  $c: X \rightarrow F(X)$  be a coalgebra with induced transition relation  $\longrightarrow \subseteq X \times X$  as defined above. Then:*

(1) For a predicate  $P \subseteq X$ ,

$$\bigcirc P = \{x \in X \mid \forall x'. x \longrightarrow x' \implies P(x')\}.$$

(2) If the functor  $F$  carries an order  $\sqsubseteq$  such that predicate lifting is down-closed (i.e.,  $u \sqsubseteq v \in \text{Pred}(F)(P)$  implies  $u \in \text{Pred}(F)(P)$ , for all  $P \subseteq X$  and  $u, v \in F(X)$ ), then for all  $x, x' \in X$ ,

$$x \lesssim y \text{ and } x \longrightarrow x' \implies \exists y'. x' \lesssim y' \text{ and } y \longrightarrow y'.$$

where we assume a second coalgebra  $d: Y \rightarrow F(Y)$  with  $y \in Y$ .

**Proof.** (1) For the inclusion ( $\subseteq$ ), assume  $x \in \bigcirc P$ , and let  $x \longrightarrow x'$  but  $\neg P(x')$ . The latter gives  $P \subseteq \{y \mid y \neq x'\}$  and so we get a contradiction from  $x \in \bigcirc P \subseteq \bigcirc(\{y \mid y \neq x'\}) = \{z \mid \neg(z \longrightarrow x')\}$ . For the reverse inclusion ( $\supseteq$ ), assume  $x \longrightarrow x' \implies P(x')$ , for all  $x'$ . Then  $\neg P(x') \implies c(x) \in \text{Pred}(F)(\{y \mid y \neq x'\})$ , and so:

$$\begin{aligned} c(x) &\in \bigcap_{x' \notin P} \text{Pred}(F)(\{y \mid y \neq x'\}) \\ &= \text{Pred}(F)\left(\bigcap_{x' \notin P} \{y \mid y \neq x'\}\right) \\ &\subseteq \text{Pred}(F)(P). \end{aligned}$$

The latter inclusion follows from  $\bigcap_{x' \notin P} \{y \mid y \neq x'\} \subseteq P$ . Hence we have  $\bigcirc P(x)$ .

(2) From  $x \lesssim y$  we obtain a simulation  $\langle r_1, r_2 \rangle: R \hookrightarrow X \times Y$  with  $R(x, y)$ . Then  $(c(x), c(y)) \in \text{Rel}_{\sqsubseteq}(F)(R)$ , which means that there is a  $w \in F(R)$  with  $c(x) \sqsubseteq u \stackrel{\text{def}}{=} F(r_1)(w)$  and  $v \stackrel{\text{def}}{=} F(r_2)(w) \sqsubseteq c(y)$ . We then reason as follows.

$$\begin{aligned} x \longrightarrow x' &\iff c(x) \notin \text{Pred}(F)(\{z \mid z \neq x'\}) \\ &\implies u \notin \text{Pred}(F)(\{z \mid z \neq x'\}) \\ &\iff w \notin F(r_1)^{-1} \text{Pred}(F)(\{z \mid z \neq x'\}) \\ &= \text{Pred}(F)(r_1^{-1}(\{z \mid z \neq x'\})) \\ &= \text{Pred}(F)(\{(a, b) \in R \mid a \neq x'\}) \\ &\implies w \notin \text{Pred}(F)(\{(a, b) \in R \mid \neg R(x', b)\}) \\ &= \text{Pred}(F)(r_2^{-1}(\neg R(x', -))) \\ &= F(r_2)^{-1}(\text{Pred}(F)(\neg R(x', -))) \\ &\iff v \notin \text{Pred}(F)(\neg R(x', -)) \\ &\implies c(y) \notin \text{Pred}(F)(\neg R(x', -)) \\ &\iff y \notin \bigcirc(\neg R(x', -)) \\ &\stackrel{(1)}{\iff} \exists y'. R(x', y') \wedge y \longrightarrow y'. \end{aligned}$$

Notice that downclosure is used twice, for the first and third implication ‘ $\implies$ ’.  $\square$

Our next step is to consider for an arbitrary coalgebra those states that have only finitely many successor states. We introduce this predicate as a least fixed point of nexttime, following [11, Section 8]. Hence, for a coalgebra with state space  $X$ ,

$$\text{FMS} \stackrel{\text{def}}{=} \bigcap \{P \subseteq X \mid \bigcirc P \subseteq P\}.$$

By construction, the predicate FMS is the least one with  $\text{FMS} = \bigcirc \text{FMS} = \{x \mid \forall x'. x \longrightarrow x' \implies x' \in \text{FMS}\}$ . We claim that it contains those states with only finitely many successors w.r.t. the transition relation  $\longrightarrow$ .

**Lemma 11.2** (1) *If  $x \longrightarrow x'$  and  $x \in \text{FMS}$  then  $x' \in \text{FMS}$ .*

(2)  *$x \in \text{FMS} \iff \neg \exists (x_n)_{n \in \mathbb{N}}. x_0 = x \wedge \forall n. x_n \longrightarrow x_{n+1}$ .*

(3) *If  $x \lesssim y$  and  $y \in \text{FMS}$  then  $x \in \text{FMS}$ .*

The second point expresses our intuition: elements in FMS are the states that do not have infinitely many successors.

**Proof.** (1) If  $x \longrightarrow x'$  and  $x' \notin \text{FMS}$ , then  $x \notin \bigcirc \text{FMS} = \text{FMS}$ .

(2) ( $\implies$ ) Suppose there is an infinite sequence  $(x_n)_{n \in \mathbb{N}}$  of successors with  $x_0 = x$  and  $x_n \longrightarrow x_{n+1}$ . Take  $P = \text{FMS} - \{x_0, x_1, x_2, \dots\}$ . We claim that  $\bigcirc P \subseteq P$ . If this holds we are done, because then  $\text{FMS} \subseteq P$ , and thus  $x = x_0 \notin \text{FMS}$ .

Clearly,  $\bigcirc P \subseteq \bigcirc \text{FMS} \subseteq \text{FMS}$ . Hence it suffices to show that  $y \in \bigcirc P$  implies that  $y \neq x_n$ , for some  $n$ . Well, suppose we do have  $x_n \in \bigcirc P$ . Then  $x_{n+1} \in P$ , which gives a contradiction.

( $\impliedby$ ) Suppose  $x \notin \text{FMS}$ . Then we can choose an infinite sequence  $(x_n)_{n \in \mathbb{N}}$  as follows.

(a) Take  $x_0 = x$ .

(b) Since  $x_0 \notin \text{FMS} = \bigcirc \text{FMS}$ , there is an  $x_1$  with  $x_0 \longrightarrow x_1$  and  $x_1 \notin \text{FMS}$ .

(c) Since  $x_1 \notin \text{FMS} = \bigcirc \text{FMS}$ , there is an  $x_2$  with  $x_1 \longrightarrow x_2$  and  $x_2 \notin \text{FMS}$ .

(d) *Etcetera.*

(3) Suppose  $x \lesssim y$  and  $x \notin \text{FMS}$ . By the previous point there is then an infinite sequence  $x = x_0 \longrightarrow x_1 \longrightarrow x_2 \cdots$ . By Proposition 11.1 (2) we then also get an infinite sequence  $y = y_0 \longrightarrow y_1 \longrightarrow y_2 \cdots$  where  $x_n \lesssim y_n$ . This means  $y \notin \text{FMS}$ .  $\square$

For an arbitrary coalgebra  $c: X \rightarrow F(X)$  we define for  $n \in \mathbb{N}$  a function

$c^{(n)}: X \rightarrow F^n(X)$  by induction:

$$c^{(0)} = id \quad \text{and} \quad c^{(n+1)} = F(c^{(n)}) \circ c \\ = F^n(c) \circ c^{(n)}.$$

For the final coalgebra  $\zeta: Z \xrightarrow{\cong} F(Z)$ , if any, we have that each  $\zeta^{(n)}$  is an isomorphism. Hence we can define for each  $n \in \mathbb{N}$  a function  $\uparrow_n: F^n(\emptyset) \rightarrow Z$  by

$$\uparrow_n \stackrel{\text{def}}{=} \left( F^n(\emptyset) \xrightarrow{F^n(?)} F^n(Z) \xrightarrow[\cong]{(\zeta^{(n)})^{-1}} Z \right)$$

There are various alternative ways to describe these maps  $\uparrow_n$ . For instance as unique map to the final coalgebra from  $F^n(\emptyset)$ —with coalgebra structure  $F^n(?)$ . Alternatively as:

$$\uparrow_n = \left( F^n(\emptyset) \xrightarrow{F^n(?)} F^n(1) \xrightarrow{\iota_n} Z \right)$$

or as:

$$\uparrow_0 = ?: \emptyset \rightarrow Z \quad \text{and} \quad \uparrow_{n+1} = \zeta^{-1} \circ F(\uparrow_n).$$

Via this inclusion we may consider the sets of “terms”  $F^n(\emptyset) \hookrightarrow Z$  as subsets of the final coalgebra.

**Theorem 11.3** *Call a coalgebra  $c: X \rightarrow F(X)$  **finitely branching** if for each state  $x \in X$ , the set  $\{x' \mid x \rightarrow x'\}$  of successors is finite. This means that the induced transition system  $(\gamma)$  is of the form  $X \rightarrow \mathcal{P}_{\text{fin}}(X)$ .*

(1) *For such a finitely branching coalgebra  $c$  one has:*

$$x \in FMS \iff \exists n \in \mathbb{N}. c^{(n)}(x) \in F^n(\emptyset) \\ \iff \exists n \in \mathbb{N}. \exists y \in F^n(\emptyset). F^n(?)(y) = c^{(n)}(x) \\ \text{(more formally.)}$$

(2) *For the special case when  $c$  is a final coalgebra this becomes:*

$$FMS = \bigcup_{n \in \mathbb{N}} F^n(\emptyset) \\ = \bigcup_{n \in \mathbb{N}} \prod_{\uparrow_n} (\top) \quad \text{(more formally.)}$$

**Proof.** (1) Let  $\perp$  be the predicate false (or empty subset  $\emptyset$ ). Then  $\bigcirc^n(\perp) = \{x \mid \neg \exists x_1, \dots, x_n. x \rightarrow x_1 \rightarrow \dots \rightarrow x_n\}$ . We also have that  $\bigcirc^n(\perp) = (c^{(n)})^{-1} F^n(\emptyset)$ . Hence we must prove  $FMS = \bigcup_{n \in \mathbb{N}} \bigcirc^n(\perp)$ .

- ( $\supseteq$ ) Assume  $x \in \bigcirc^n(\perp)$ , but  $x \notin \text{FMS}$ . The latter means by Lemma 11.2 (2) that there is an infinite sequence  $x = x_0 \longrightarrow x_1 \longrightarrow \cdots \longrightarrow x_n \longrightarrow \cdots$ . But this contradicts  $x \in \bigcirc^n(\perp)$ .
- ( $\subseteq$ ) Suppose now  $x \in \text{FMS}$ , but  $x \notin \bigcup_{n \in \mathbb{N}} \bigcirc^n(\perp)$ . Then  $x \in \neg \perp \supseteq \neg \bigcirc^1(\perp) \supseteq \neg \bigcirc^2(\perp) \supseteq \cdots$ . The tree of transitions out of  $x$  is thus infinite, and it is finitely branching, by assumption. Hence there is by König's Lemma an infinite path  $x = x_0 \longrightarrow x_1 \longrightarrow \cdots$ , contradicting that  $x \in \text{FMS}$ .
- (2) Because

$$\begin{aligned}
x \in \bigcup_{n \in \mathbb{N}} \prod_{\uparrow_n} (\top) &\iff \exists n \in \mathbb{N}. \exists y \in F^n(\emptyset). \uparrow_n(y) = x \\
&\iff \exists n \in \mathbb{N}. \exists y \in F^n(\emptyset). F^n(?) (y) = \zeta^{(n)}(x) \\
&\iff x \in \text{FMS}. \quad \square
\end{aligned}$$

This last result  $\text{FMS} = \bigcup_{n \in \mathbb{N}} F^n(\emptyset)$  shows that the elements of the sets  $F^n(\emptyset)$  appear within a final coalgebra as those with only finitely many outgoing transitions. Notice that orders on functors do not play a role here.

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