Towards Biresiduated Multi-Adjoint Logic Programming

Jesús Medina,[1] Manuel Ojeda-Aciego,[1] Agustín Valverde,[1] Peter Vojtáš[2]

[1] Dept. Matemática Aplicada. Universidad de Málaga.
{jmedina,aciego,a_valverde}@ctima.uma.es
[2] Inst. of Computer Science. P.J. Šafárik University.
vojtas@kosice.upjs.sk

Abstract. Multi-adjoint logic programs were recently proposed as a generalization of monotonic and residuated logic programs, in that simultaneous use of several implications in the rules and rather general connectives in the bodies are allowed. In this work, the need of biresiduated pairs is justified through the study of a very intuitive family of operators, which turn out to be not necessarily commutative and associative and, thus, might have two different residuated implications; finally, we introduce the framework of biresiduated multi-adjoint logic programming and sketch some considerations on its fixpoint semantics.

1 Introduction

A number of logics have been introduced in the recent years motivated by the problem of reasoning in situations where information may be vague or uncertain. Such type of reasoning has been called inexact or fuzzy or approximate reasoning. Here we propose a lattice-valued logic programming paradigm that we call *biresiduated* and *multi-adjoint*, which permits the articulation of vague concepts and generalizes several approaches to the extension of logic programming techniques to the fuzzy case.

Multi-adjoint logic programming was introduced in [6] as a refinement of both initial work in [11] and residuated logic programming [2]. It allows for very general connectives in the body of the rules and, in addition, sufficient conditions for the continuity of its semantics are known. Such an approach is interesting for applications in which connectives depend on different users preferences; or in which knowledge is described by a many-valued logic program where connectives can be general aggregation operators (conjunctors, disjunctors, arithmetic mean, weighted sum, . . .), even different aggregators for different users and, moreover, the program is expected to adequately manage different implications for different purposes. The special features of multi-adjoint logic programs are the following:

1. A number of different implications are allowed in the bodies of the rules.
2. Sufficient conditions for continuity of its semantics are known.
3. The requirements on the lattice of truth-values are weaker that those for the residuated approach.

It is important to recall that many different "and" and "or" operations have been proposed for use in fuzzy logic. It is therefore important to select, for each particular application, the operations which are the best for this particular application. Several papers discuss the optimal choice of "and" and "or" operations for fuzzy control, when the main criterion is to get the most stable control. In reasoning applications, however, it is more appropriate to select operations which are the best in reflecting human reasoning, i.e., operations which are "the most logical".

In this paper, we build on the fact that conjunctors in multi-adjoint logic programs need not be either commutative or associative and, thus, consider the possibility of including a further generalisation of the framework, allowing for *biresiduation*, in the sense of [10]. This way, each conjunctor in our multi-adjoint setting may potentially have two "lateral" residuated implications. This approach has been recently used by Morsi [8] to develop a formal system based on a biresiduation construction.

Yet another reason for introducing biresiduation is that fuzzy logic in a narrow sense [4] is still an open system and thus, new connectives can and should be introduced. Note that the concept of biresiduation used in this paper is not to be confused with the generalized biimplication, which is another usual meaning of this term as used in v.gr. [7]. A natural question then arises, whether the basic syntactico-semantical properties are not harmed by this generalisation.

The structure of the paper is as follows: In Section 2, the preliminary definitions are introduced; later, in Section 3, some motivating examples towards considering biresiduated implications are presented; in Section 4, the syntax and semantics of biresiduated multi-adjoint logic programs is given; finally, in Section 5, some concluding remarks and pointers to related works and ideas for future research are presented.

2 Preliminary Definitions

The main concept in the extension to logic programming to the fuzzy case is that of *adjoint pair*, firstly introduced in a logical context by Pavelka, who interpreted the poset structure of the set of truth-values as a category, and the relation between the connectives of implication and conjunction as functors in this category. The result turned out to be another example of the well-known concept of adjunction, introduced by Kan in the general setting of category theory in 1950.

Definition 1 (Adjoint pair). *Let $\langle P, \preceq \rangle$ be a partially ordered set and a pair $(\&, \leftarrow)$, of binary operations in P, such that:*

(a1) *Operation $\&$ is increasing in both arguments, i.e. if $x_1, x_2, y \in P$ and $x_1 \preceq x_2$, then $(x_1 \& y) \preceq (x_2 \& y)$ and $(y \& x_1) \preceq (y \& x_2)$;*
(a2) *Operation $\leftarrow$ is increasing in the first argument (the consequent) and decreasing in the second argument (the antecedent), i.e. if $x_1, x_2, y \in P$ and $x_1 \preceq x_2$, then $(x_1 \leftarrow y) \preceq (x_2 \leftarrow y)$ and $(y \leftarrow x_2) \preceq (y \leftarrow x_1)$;*
(a3) *For any $x, y, z \in P$, we have that $x \preceq (y \leftarrow z)$ holds if and only if $(x \& z) \preceq y$ holds.*

Then we say that $(\&, \leftarrow)$ forms an adjoint pair *in $\langle P, \preceq \rangle$.*

The need of the monotonicity of operators $\leftarrow$ and $\&$ is clear, if they are going to be interpreted as generalised implications and conjunctions. The third property in the definition corresponds to the categorical adjointness, and can be adequately interpreted in terms of multiple-valued inference as asserting both that the truth-value of $y \leftarrow z$ is the maximal x satisfying $x \mathbin{\&} z \preceq y$, and also the validity of the following generalised modus ponens rule [4]:

> If x is a lower bound of $\psi \leftarrow \varphi$, and z is a lower bound of φ then a lower bound y of ψ is $x \mathbin{\&} z$.

In addition to $(a1)$–$(a3)$ it will be necessary to assume the existence of bottom and top elements in the poset of truth-values (the zero and one elements), and the existence of joins (suprema) for every directed subset; that is, we will assume a structure of complete lattice, but nothing about associativity, commutativity and general boundary conditions of $\&$. In particular, the requirement that $(L, \&, \top)$ has to be a commutative monoid in a residuated lattice is too restrictive, in that commutativity needn't be required in the proofs of soundness and correctness.

Extending the results in [2, 11] to a more general setting, in which different implications (Łukasiewicz, Gödel, product) and thus, several modus ponens-like inference rules are used, naturally leads to considering several *adjoint pairs* in the lattice:

Definition 2 (Multi-Adjoint Lattice). *Let $\langle L, \preceq \rangle$ be a lattice. A* multi-adjoint lattice $\mathcal{L}$ *is a tuple $(L, \preceq, \leftarrow_1, \&_1, \dots, \leftarrow_n, \&_n)$ satisfying the following items:*

(l1) *$\langle L, \preceq \rangle$ is bounded, i.e. it has bottom $(\bot)$ and top $(\top)$ elements;*
(l2) *$(\&_i, \leftarrow_i)$ is an adjoint pair in $\langle L, \preceq \rangle$ for $i = 1, \dots, n$;*
(l3) *$\top \mathbin{\&_i} \vartheta = \vartheta \mathbin{\&_i} \top = \vartheta$ for all $\vartheta \in L$ for $i = 1, \dots, n$.*

Remark 1. Note that residuated lattices are a special case of multi-adjoint lattice.

From the point of view of expressiveness, it is interesting to allow extra operators to be involved with the operators in the multi-adjoint lattice. The structure which captures this possibility is that of a multi-adjoint Ω-algebra, where Ω is the set of operators, which comprises the set of adjoint pairs and, possibly, other monotone operators such as *disjunctors* or *aggregators*.

3 Towards biresiduation

In fuzzy logic there is a well developed theory of t-norms, t-conorms and residual implications. The objective of this section is to show some interesting non-standard connectives to motivate the consideration of a more general class of connectives in fuzzy logic. The motivation is the following:

When evaluating the relevance of answers to a given query it is common to use some subjective interpretation of human preferences in a granulated way. This is, fuzzy truth-values usually describe steps in the degree of perception (numerous advocations of this phenomenon have been pointed out by Zadeh). This is connected to the well-known fact that people can only distinguish finitely many degrees of quantity or quality (closeness, cheapness, etc.). Thus, in practice, although we used the product t-norm $x\&_p y = x \cdot y$, we would be actually working with a piece-wise constant approximation of it. In this generality, it is possible to work with approximations of t-norms and/or conjunctions learnt from data by a neural net.

Regarding the use of non-standard connectives, just consider that a variable represented by x can be observed with $m+1$ different values, then surely we should be working with a regular partition of $[0,1]$ into m pieces. This means that a given value x should be fitted to this "observation" scale as the least upper bound with the form k/m (analytically, this corresponds to $\lceil m \cdot x \rceil / m$ where $\lceil \_ \rceil$ is the ceiling function). A similar consideration can be applied to both, variable y and the resulting conjunction; furthermore, it might be possible that each variable has different granularity.

Formally, assume in x-axis we have a partition into n pieces, in y-axis into m pieces, and in z-axis into k pieces. Then the approximation of the product conjunction looks like the definition below

Definition 3. *Denote* $(z)_p = \dfrac{\lceil p \cdot z \rceil}{p}$ *and define, for naturals* $n, m, k > 0$

$$C^k_{n,m}(x,y) = \big((x)_n \cdot (y)_m\big)_k$$

Note that $C^k_{n,m}$ satisfies the properties of a general *conjunctor*, that is, it is monotone in both variables and generalizes the classical conjunction.

There are connectives of the form $C^k_{n,m}(x,y)$ which are non-associative and, there are connectives of the same form which are non-commutative as well. The following example shows some of them:

Example 1.

1. The connective $C = C^{10}_{10,10}$ is not associative.

$$C(0.7, C(0.7, 0.3)) = C(0.7, (0.21)_{10}) = C(0.7, 0.3) = (0.21)_{10} = 0.3$$
$$C(C(0.7, 0.7), 0.3) = C((0.49)_{10}, 0.3) = C(0.5, 0.3) = (0.15)_{10} = 0.2$$

2. The connective $C^4_{10,5}(x,y)$ is not commutative.

$$C^4_{10,5}(0.82, 0.79) = ((0.82)_{10} \cdot (0.79)_5)_4 = (0.9 \cdot 0.8)_4 = (0.72)_4 = 0.75$$
$$C^4_{10,5}(0.79, 0.82) = ((0.79)_{10} \cdot (0.82)_5)_4 = (0.8 \cdot 1)_4 = 1$$

□

Connectives as those in the example above can be reasonably justified as follows: If we are looking for a hotel which is close to downtown, with reasonable price and being a new building, then classical fuzzy approaches would assign a user "his" particular interpretation of "close", "reasonable" and "new". As, in practice, we can only recognize finitely many degrees of being close, reasonable, new, then the corresponding fuzzy sets have a stepwise shape. This motivates the lattice-valued approach we will assume in this paper: it is just a matter of representation that the outcome is done by means of intervals of granulation and/or indistinguishability.

Although the multi-adjoint approach to logic programming has been shown to be more general than the monotonic and residuated paradigms (which, in turn, generalizes annotated LP, possibilistic LP, probabilistic LP, etc), the discussion below introduces a context which cannot be easily accommodated under multi-adjointness.

On operators $C^k_{n,m}$ and biresidua

By using operators $C^k_{n,m}$, we study an example of adjoint pair which will motivate the need of biresiduation. Firstly, let us recall the well-known *residuum* construction, which has been extensively studied in the context of continuous t-norms, see [4, 5].

Definition 4. *Given $C^k_{n,m} \colon [0,1] \times [0,1] \to [0,1]$, we define the following operator:*

$$z \swarrow^k_{n,m} y = \sup\{x \in [0,1] \mid C^k_{n,m}(x,y) \le z\}$$

for each $z, y \in [0,1]$, which is called r-adjoint implication *of $C^k_{n,m}$.*

As we have proved above, in general $C^k_{n,m}$ need not be commutative, therefore we have to show that the term *r-adjoint implication* makes sense: actually, the pair $(C^k_{n,m}, \swarrow^k_{n,m})$ satisfies the properties of adjoint pair.

We will prove a slightly more general version of the statement: that the construction above leads to an adjoint pair assuming only that the supremum in the definition is indeed a maximum, but without assuming either commutativity or continuity of the conjunctor.

Proposition 1. *Consider a conjunctor $\& \colon L \times L \to L$ in a lattice, and assume that for all $y, z \in L$ there exists a maximum of the set $\{x \in L \mid x \,\&\, y \preceq z\}$. Then $(\&, \swarrow)$ is an adjoint pair.*

Proof. Let us prove that it is increasing in the first argument. Assume $z_1 \preceq z_2$, then

$$\{x \mid x \,\&\, y \preceq z_1\} \subseteq \{x \in L \mid x \,\&\, y \preceq z_2\}$$

therefore we have

$$z_1 \swarrow y = \sup\{x \in L \mid x \,\&\, y \preceq z_1\} \preceq \sup\{x \in L \mid x \,\&\, y \preceq z_2\} = z_2 \swarrow y$$

To show that $\swarrow$ decreases in the second argument, assume $y_1 \preceq y_2$. By monotonicity in the second argument, we have $x \,\&\, y_1 \preceq x \,\&\, y_2$ for all $x \in L$, which implies the following inclusion

$$\{x \in L \mid x \,\&\, y_2 \preceq z\} \subseteq \{x \in L \mid x \,\&\, y_1 \preceq z\}$$

therefore $z \swarrow y_2 \preceq z \swarrow y_1$.

For the adjoint property, consider $x, y, z \in L$ and assume $x \,\&\, y \preceq z$, then $x \in \{x' \mid x' \,\&\, y \preceq z\}$ and, therefore

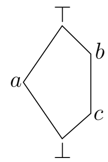
$$x \preceq \sup\{x' \mid x' \,\&\, y \preceq z\} = z \swarrow y$$

For the converse, assume that $x \preceq z \swarrow y = \sup\{x' \mid x' \,\&\, y \preceq z\}$.

By the hypothesis, we have that the supremum is indeed a maximum; let us denote it by x_0. Then we have $x \preceq x_0$ and, by monotonicity in the first argument, we have

$$x \,\&\, y \preceq x_0 \,\&\, y \preceq z$$

□



In this lattice we have

$$c \swarrow b = \sup\{x \mid x \,\&\, b \preceq c\} = \sup\{a, c, \bot\} = \top$$

so, for $x = \top$, $y = b$, and $z = c$ the adjoint property fails.

Fig. 1. Counterexample for Proposition 1.

The adjoint property might fail in the case that the supremum is not maximum, as the following example shows:

Proposition 2. $(C^k_{n,m}, \swarrow^k_{n,m})$ *is an adjoint pair.*

Proof. By the previous proposition, we have only to show that in the definition of the r-adjoint implication of $C^k_{n,m}$ the supremum is indeed a maximum.

Given $y, z, \in [0,1]$, let A denote the set $\{x \in [0,1] \mid C^k_{n,m}(x,y) \leq z\}$. The result follows from the fact that if we have $x \in A$, then we also have $(x)_n \in A$ by the definition of $C^k_{n,m}$. Thus, the only possible values for the supremum are in the set $\left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\right\}$, but suprema on a discrete subset of a linear order are always maxima. □

An interesting 'coherence' property that holds for continuous t-norms and their residua is that the truth-value of the consequent is greater or equal than the truth-value of the antecedent if and only if the truth-value of the implication is defined to be the top element. However, the property "if $z \geq y$, then $z \leftarrow y = \top$" does not hold for connectives $C^k_{n,m}$:

Example 2. If we consider the conjunctor $C^8_{4,5}\colon [0,1] \times [0,1] \to [0,1]$, we can check that $0.6 \geq 0.45$ and $0.6 \swarrow^8_{4,5} 0.45 \neq 1$. We have,

$$C^8_{4,5}(x, 0.45) = \frac{\left\lceil 8 \cdot \dfrac{\lceil 4 \cdot x \rceil}{4} \cdot \dfrac{\lceil 5 \cdot 0.45 \rceil}{5} \right\rceil}{8} = \frac{\left\lceil 8 \cdot \dfrac{\lceil 4 \cdot x \rceil}{4} \cdot 0.6 \right\rceil}{8}$$

and then

$$0.6 \swarrow^8_{4,5} 0.45 = \sup\{x \in [0,1] \mid C^8_{4,5}(x, 0.45) \leq 0.6\} = 0.75 \neq 1$$

□

In the following proposition, where we introduce a necessary and sufficient condition for the fulfillment of the coherence property for the connectives $C^k_{n,m}$, the notation $[0,1]_n$ will be used to denote the discrete set $\left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\right\}$.

Proposition 3. *Consider* $z, y \in [0,1]$ *such that* $z \geq y$ *then* $z \swarrow^k_{n,m} y = 1$ *if and only if* $y \in [0,1]_k \cap [0,1]_m$

Proof. $(\Rightarrow)$ Assume $z \swarrow^k_{n,m} y = 1$ for all $z \geq y$. In particular, for $z = y$ we have, from the hypothesis and the adjoint property, that $C^k_{n,m}(1, y) \leq y$. Moreover, we have the following chain of inequalities

$$y \leq \frac{\lceil m \cdot y \rceil}{m} \leq \frac{\left\lceil k \cdot \frac{\lceil m \cdot y \rceil}{m} \right\rceil}{k} = C^k_{n,m}(1, y) \leq y$$

Therefore, we actually have a chain of equalities. Now, considering the first equality we obtain $m \cdot y = \lceil m \cdot y \rceil$ which leads to $y \in [0, 1]_m$. Similarly, one can obtain $y \in [0, 1]_k$.

$(\Leftarrow)$ It is sufficient to show that $1 \in \{x \in [0, 1] \mid C^k_{n,m}(x, y) \leq z\}$:

$$C^k_{n,m}(1, y) = \frac{\left\lceil k \cdot \frac{\lceil n \cdot 1 \rceil}{n} \cdot \frac{\lceil m \cdot y \rceil}{m} \right\rceil}{k} = \frac{\left\lceil k \cdot \frac{\lceil m \cdot y \rceil}{m} \right\rceil}{k} \overset{(i)}{=} \frac{\lceil k \cdot y \rceil}{k} \overset{(ii)}{=} y \leq z$$

where (i) and (ii) follow from $y \in [0, 1]_m$ and $y \in [0, 1]_k$ respectively. □

Taking into account this result we obtain a condition for the fulfillment of the coherence property for the r-adjoint implication defined above in terms of the set of truth-values that can be attached to any given formula.

For example, consider that we are working on a multiple-valued logic, whose truth-values are taken on a regular partition of $[0, 1]$, then the coherence property for the conjunctor $C^k_{n,m}$ automatically holds in any finitely-valued logics on a regular partition of $\gcd(k, m) + 1$ elements.

Consider for example that we need to work with the operator $C^8_{4,5}$ in our language. By the result above, the r-adjoint implication only fulfills the coherence property in the classical case, since $\gcd(8, 5) = 1$. Could we use another implication which turns out to be coherent in a multiple-valued logic? We have an obvious choice by using the non-commutativity of the operator. This leads to what we call the *l-adjoint implication* to the conjunctor.

Definition 5. *Given the conjunctor $C^k_{n,m} \colon [0, 1] \times [0, 1] \to [0, 1]$, we define its* l-adjoint implication *as follows:*

$$z \nwarrow^k_{n,m} y = \sup\{x \in [0, 1] \mid C^k_{n,m}(y, x) \leq z\}$$

for each $z, y \in [0, 1]$.

It is easy to prove a similar result to that of Proposition 3 and obtain that the l-adjoint implication to $C^8_{4,5}$ is coherent on a 5-valued logic, for $\gcd(8, 4) = 4$.

The biresiduated structure is obtained by allowing, for each adjoint conjunctor, two "sided" adjoint implications, as detailed in the following definition:

Definition 6. *Let $\langle L, \preceq \rangle$ be a lattice. A* biresiduated multi-adjoint lattice *$\mathcal{L}$ is a tuple $(L, \preceq, \swarrow^1, \nwarrow_1, \&_1, \dots, \swarrow^n, \nwarrow_n, \&_n)$ satisfying the following items:*

1. *$\langle L, \preceq \rangle$ is bounded, i.e. it has bottom and top elements;*

2. $(\&_i, \swarrow^i, \nwarrow_i)$ *satisfies the following properties, for all* $i = 1, \dots, n$*; i.e.*
   (a) *Operation* $\&_i$ *is increasing in both arguments,*
   (b) *Operations* $\swarrow^i, \nwarrow_i$ *are increasing in the first argument and decreasing in the second argument,*
   (c) *For any* $x, y, z \in P$*, we have that*

$$\begin{array}{llll} 1. & x \preceq y \swarrow^i z & \textit{if and only if} & x \,\&_i\, z \preceq y \\ 2. & x \preceq y \nwarrow_i z & \textit{if and only if} & z \,\&_i\, x \preceq y \end{array}$$

The last conditions in (3c) make this algebraic structure a flexible and suitable tool for being used in a logical context, for they can be interpreted as a two possible multiple-valued *modus ponens*-like rules. From a categorical point of view, these conditions arise when considering the conjunctor as a bifunctor, and applying the adjointness either in its second or first argument, respectively. Also note that the requirements of the boundary conditions for the conjunctors are not necessary.

The proposition below introduces a characterization of the adjoint property for the pairs $(\&, \nwarrow)$ and $(\&, \swarrow)$.

Proposition 4. *Property (3c2) in the definition of biresiduated multi-adjoint lattice is equivalent to the following pair of conditions for all* $x, y \in P$*:*

$$\begin{array}{ll} (\mathbf{r1}')\ x \preceq ((y \,\&\, x) \nwarrow y) & (\mathbf{r1}'')\ x \preceq ((x \,\&\, y) \swarrow y) \\ (\mathbf{r2}')\ (x \,\&(y \nwarrow x)) \preceq y & (\mathbf{r2}'')\ ((y \swarrow x) \,\&\, x) \preceq y \end{array}$$

Remark 2. In the classical residuated case, conditions (r1) and (r2) are sometimes called $(\Phi 2)$ and $(\Phi 3)$, whereas (3b) is known as $(\Phi 1)$, see e.g. [3].

4 Syntax and semantics of biresiduated multi-adjoint programs

Similarly to the (mono)-residuated multi-adjoint case, the structure which allows the possibility of using additional operators is that of a *biresiduated multi-adjoint* Ω*-algebra*. In this section we introduce the definition of the syntax and semantics of biresiduated multiadjoint logic programs. The programs are defined on a language $\mathfrak{F}$ constructed on a set of propositional symbol Π and the set of operators Ω and using a biresiduated multi-adjoint lattice $\mathcal{L}$ as the set of truth-values.

Definition 7. *A* biresiduated multi-adjoint logic program *(in short* a program*) on a language* $\mathfrak{F}$ *with values in* $\mathcal{L}$ *is a set* $\mathbb{P}$ *of rules of the form* $\langle A \swarrow^i \mathcal{B}, \vartheta\rangle$ *or* $\langle A \nwarrow_i \mathcal{B}, \vartheta\rangle$ *such that:*

1. *The* head *of the rule,* A*, is a propositional symbol of* Π*;*
2. *The* body *formula,* $\mathcal{B}$*, is a formula of* $\mathfrak{F}$ *built from propositional symbols* $B_1, \dots, B_n$ *(*$n \geq 0$*) and monotone functions;*
3. Facts *are rules with empty body.*
4. *The* weight ϑ *is an element (a truth-value) of* L*.*

As usual, a query *(or* goal*) is a propositional symbol intended as a question* $?A$ *prompting the system.*

Let $\mathcal{I}_{\mathcal{L}}$ be the set of *interpretations*, that is, the unique recursive extensions of mappings $I\colon \Pi \to L$. The ordering $\preceq$ of the truth-values L can be easily extended to $\mathcal{I}_{\mathfrak{L}}$, which also inherits the structure of complete lattice. The minimum element of the lattice $\mathcal{I}_{\mathfrak{L}}$, which assigns $\bot$ to any propositional symbol, will be denoted $\triangle$.

A rule of a biresiduated multi-adjoint logic program is satisfied whenever its truth-value is greater or equal than the weight associated with the rule. The formal definition uses the following terminology: given an operator $\omega \in \Omega$, we will write $\dot{\omega}$ to denote its interpretation in $\mathcal{L}$.

Definition 8.

1. *An interpretation* $I \in \mathcal{I}_{\mathfrak{L}}$ satisfies $\langle A, \vartheta\rangle$ *iff* $\vartheta \preceq I(A)$.
2. *An interpretation* $I \in \mathcal{I}_{\mathfrak{L}}$ satisfies $\langle A \swarrow^{i} \mathcal{B}, \vartheta\rangle$ *iff* $\vartheta \mathbin{\dot{\&}_i} \hat{I}(\mathcal{B}) \preceq I(A)$.
3. *An interpretation* $I \in \mathcal{I}_{\mathfrak{L}}$ satisfies $\langle A \nwarrow_{i} \mathcal{B}, \vartheta\rangle$ *iff* $\hat{I}(\mathcal{B}) \mathbin{\dot{\&}_i} \vartheta \preceq I(A)$.
4. *An interpretation* $I \in \mathcal{I}_{\mathfrak{L}}$ *is a* model *of a program* $\mathbb{P}$ *iff all weighted rules in* $\mathbb{P}$ *are satisfied by* I.
5. *An element* $\lambda \in L$ *is a* correct answer *for a query* $?A$ *and a program* $\mathbb{P}$ *if for any interpretation* $I \in \mathcal{I}_{\mathfrak{L}}$ *which is a model of* $\mathbb{P}$ *we have* $\lambda \preceq I(A)$.

Note that, for instance, $\bot$ is always a correct answer for any query and program.

The core of the Apt-van Emden-Kowalski semantics, the immediate consequences operator, can be easily generalised to biresiduated multi-adjoint logic programs.

Definition 9. *Let* $\mathbb{P}$ *be a program, the* immediate consequences operator $T_{\mathbb{P}}$ *maps interpretations to interpretations, and for an interpretation* I *and propositional variable* A, $T_{\mathbb{P}}(I)(A)$ *is defined as the supremum of the following set*

$$\{\vartheta \mathbin{\dot{\&}_i} \hat{I}(\mathcal{B}) \mid \langle A \swarrow^{i} \mathcal{B}, \vartheta\rangle \in \mathbb{P}\} \cup \{\hat{I}(\mathcal{B}) \mathbin{\dot{\&}_i} \vartheta \mid \langle A \nwarrow_{i} \mathcal{B}, \vartheta\rangle \in \mathbb{P}\} \cup \{\vartheta \mid \langle A, \vartheta\rangle \in \mathbb{P}\}$$

As usual, the semantics of a biresiduated multi-adjoint logic program is characterised by the post-fixpoints of $T_{\mathbb{P}}$; that is:

Theorem 1. *An interpretation* I *of* $\mathcal{I}_{\mathfrak{L}}$ *is a model of a biresiduated multi-adjoint logic program* $\mathbb{P}$ *iff* $T_{\mathbb{P}}(I) \sqsubseteq I$.

Note that the result is still true even without any further assumptions on conjunctors, which definitely need not be either commutative or associative or satisfy any boundary condition.

The monotonicity of the operator $T_{\mathbb{P}}$ for biresiduated multi-adjoint logic programming follows from the monotonicity of all the involved operators. As a result of $T_{\mathbb{P}}$ being monotone, the semantics of a program $\mathbb{P}$ is given by its least model which, as shown by Knaster-Tarski's theorem, is exactly the least fixpoint of $T_{\mathbb{P}}$, which can be obtained by transfinitely iterating $T_{\mathbb{P}}$ from the least interpretation $\triangle$.

5 Related work and concluding remarks

To the best of our knowledge, there is not much work done using biresidua in the sense considered in this paper. Recently, in [10] a study of the representability of biresiduated algebras was presented. This type of algebras were introduced in a purely algebraic context, and were studied for instance in [1]. For instance, in [9] a structure of biresiduated algebra is defined together with a corresponding logical system regarding the use of fuzzy sets in the representation of multicriteria decision problems. On a different basis, Morsi [8] developed an axiom system based on a biresiduation construction and the completeness and soundness theorems were proved.

We have presented analytical evidence of reasonable non-commutative conjunctors which lead to the consideration of two biresiduated implications. Some properties have been obtained for these conjunctors and, finally, an application to the development of an extended logic programming paradigm has been presented, together with its fixpoint semantics. The introduction of a procedural semantics for the framework, together with the study of existence of a completeness theorem for the given semantics is on-going.

Although the central topics of this paper are mainly at the theoretical level, some practical applications are envisaged for the obtained results, such as the development of a generalized multiple-valued resolution. In a generalized context it is not possible to deal with Horn clauses and refutation, mainly due to the fact that $A \wedge \neg A$ can have strictly positive truth-value, but also to the fact that material implication (the truth value function of $\neg A \vee B$) has not commutative adjoint conjunctor. As our approach does not require adjoint conjunctors to be commutative, it would allow the development of a sound and complete graded resolution.

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