From Interval Computations to Modal Mathematics: Applications and Computational Complexity

Bernadette Bouchon-Meunier[1] and Vladik Kreinovich[1,2]

[1]LAFORIA-IBP
Boite 169
University Paris VI
4, place Jussieu, 75252
Paris Cedex 05, France
email Bernadette.Bouchon-Meunier@laforia.ibp.fr

[2]Department of Computer Science
University of Texas at El Paso
El Paso, TX 79968, USA
email vladik@cs.utep.edu

1 Formulation of the Problem

1.1 Traditional Interval Mathematics: A Brief Reminder

Before we start explaining why we need to go beyond interval computations, let us briefly recall our motivation for the use of interval computations in data processing.

Traditional data processing methods of numerical mathematics are based on the assumptions that we know the exact values of the input quantities. In reality, the data come from measurements, and measurements are never 100% precise; hence, the actual value x of each input quantity may differ from its measurement result $\tilde{x}$. In some cases, we know the probabilities of different values of error $\Delta x = \tilde{x} - x$, but in most case, we only know the guaranteed upper bound Δ for the error; in these cases, the only information we have about the (unknown) actual value x is that x belongs to the *interval* $\mathbf{x} = [\tilde{x} - \Delta, \tilde{x} + \Delta]$.

One of the basic problems of interval mathematics is, therefore, as follows:

given a data processing algorithm $f(x_1, \ldots, x_n)$ and n intervals $\mathbf{x}_1, \ldots, \mathbf{x}_n$, compute the range $\mathbf{y}$ of possible values of $y = f(x_1, \ldots, x_n)$ when $x_i \in \mathbf{x}_i$.

1.2 Non-Traditional Interval Problems: Control, Design, Optimization, etc.

The above formulation makes perfect sense when we estimate the value of a certain physical quantity y that is related to the directly measured quantities x_i. In this case, the goal is to find all real numbers that *can* be the values of this quantity y (for the given measurement results).

The goal of data processing is, however, often more complicated. For example, we may want to *control* a certain system; in this case, we must find a control value that, e.g., guarantees stability of the system for all possible values of the parameters x_i (i.e., for all $x_i \in \mathbf{x}_i$). In this case, we want to find all real numbers y for which stability *must* occur. Similarly, many real-life problems of design, control, and optimization lead to complicated mathematical formulations.

1.3 Shary's Approach: Successes and Limitations

Shary's approach. For the case when the relationship between different variables is described by a system of (linear or non-linear) equations, different possible problems have been described by Shary (see, e.g., [12]). Shary distinguishes between different possible formulations by using different quantifiers for different variables: e.g., if we are interested in the set of *possible* values of y, we are interested in values y for which $\exists x_i$ such that the given equations are true; if we want a control y that leads to stability for *all* possible values x_i, we use a universal quantifier $\forall x_i$.

Successes. Shary's classification contains practically all known problems, and it seems to be sufficient for describing *objective* knowledge, that is usually described in terms of equations.

Limitations. The main limitation of Shary's approach is that an essential part of our knowledge comes from experts, and experts often describe their knowledge not in terms of equations, but in terms of logical statements (e.g., in terms of "rules" of the type "if A then B").

1.4 Modal Logic: a Way of Describing "Can" and "Must"

In case of measurement uncertainty, expert statements cannot be formulated in terms of pure logic; we also need some formalization of the words like "can" and "must" that were used in the above descriptions of our objectives. Logics that formalize these terms are called *modal logics* (see, e.g., [10, 9]); in these logics, "A can happen" is usually described as $\Diamond A$, and "A must happen" as $\Box A$.

1.5 Our Problem

Our problem is thus: *to add these modal operators to mathematics, and thus, go from interval mathematics to modal mathematics.*

Since our intended application is data processing, this generalization only makes sense while its results are still computable (and feasibly computable). So, among the first problems to handle are the problems of *computability* and *computational complexity* of the resulting modal mathematics. These problems will be handled in the paper.

Historical comment. The fact that formulas and relations of interval mathematics can be described in terms of modal logic is not new: it was first noticed in [4] (see also [6] and [7]).

2 Example and an Idea of the General Description

Example. Let us use capital letters (e.g., X_i) to describe variables that are not necessarily uniquely determined by our knowledge, i.e., that can still take different values (e.g., values from an interval). In these terms, if a measurement leads us to a conclusion that the value of this variable belongs to an interval $[x_i^-, x_i^+]$, then we can express this knowledge as: $\Box(x_i^- \le X_i \le x_i^+)$. In these terms, the basic problem of interval computations can be, crudely speaking, reformulated as follows: given the values x_i^-, x_i^+, and y, check whether the following formula is true:

$$(\Box(x_1^- \le X_1 \le x_1^+)\& \ldots \& \Box(x_n^- \le X_n \le x_n^+)) \to \Box(f(X_1, \ldots, X_n) \le y). \tag{1}$$

Actually, by checking the validity of this formula, we check whether the upper bound of the range $f([x_1^-, x_1^+], \ldots, [x_n^-, x_n^+])$ is greater than a given number y or not. If we can do that for all numbers y, then, by applying bisection, we can compute the actual upper endpoint of the range interval with greater and greater accuracy.

Similarly, we can compute the lower endpoint of the range interval if we check a formula

$$(\Box(x_1^- \le X_i \le x_1^+)\& \ldots \& \Box(x_n^- \le X_n \le x_n^+)) \to$$

$$\Box(f(X_1, \ldots, X_n) \ge y).$$

The idea of a general description. Let us denote by $\mathcal{X} \subseteq R^n$ the (unknown) set of possible values of tuples $(X_1, \ldots, X_n)$. For each tuple $(X_1, \ldots, X_n)$, it is easy to define truth values of elementary formulas (e.g., equations, inequalities, etc.), and formulas of first order logic (that are obtained from elementary ones by using propositional connectives “and”, “or”, “not”, and quantifiers). If a formula A is well defined, then:

- we say that $\Diamond A$ is true iff A is true for *at least one* tuple from $\mathcal{X}$, and
- that $\Box A$ is true if A is true for *all* tuples from $\mathcal{X}$.

We then say that a formula is true if it is true for all sets $\mathcal{X} \subseteq R^n$.

In particular, the validity of formula (1) means the following:

- The formula $\Box(x_i^- \le X_i \le x_i^+)$ means that for every tuple from $\mathcal{X}$, the value X_i belongs to the interval $[x_i^-, x_i^+]$. In other words, this formula is true iff the set $\mathcal{X}$ is a subset of the box $B = [x_1^-, x_1^+] \times \ldots \times [x_n^-, x_n^+]$.
- The formula $\Box(f(X_1, \ldots, X_n) \le y)$ means that for every tuple $(X_1, \ldots, X_n) \in \mathcal{X}$, the value $f(X_1, \ldots, X_n)$ does not exceed y. In other words, this formula means that the range $f(\mathcal{X})$ of the function f on the set $\mathcal{X}$ belongs to the semi-line $(-\infty, y]$.
- Finally, the formula (1) itself means that if $\mathcal{X} \subseteq B$, then $f(\mathcal{X}) \subseteq (-\infty, y]$. To check this implication, it is sufficient to check it for $\mathcal{X} = B$, because from $F(B) \subseteq (-\infty, y]$ and $\mathcal{X} \subseteq B$, it follows that $f(\mathcal{X}) \subseteq f(B) \subseteq (-\infty, y]$. Thus, the formula (1) is equivalent to $f(B) \subseteq (-\infty, y]$.

Comment. The possibility of quantification over all possible *sets* (not only numbers from intervals) is what distinguishes this formalism from Shary’s. In particular, this formalism leads to a new description of the so-called *Kaucher arithmetic* (see, e.g., [5]).

This description can also be generalized to describe unknown *functions* [7]: For example, we often know that a physical quantity y depend on some other quantity x (i.e., that $y = f(x)$ for an unknown function f); as a result of the measurements, we have intervals $[y_i^-, y_i^+]$ of possible values of $y_i = f(x_i)$ at given points $x_1 < \ldots < x_n$. A natural question is: is this information consistent with the assumption that the function f is monotonic?

For example, if $f(x)$ describes the dependency of the brightness y of the astronomical source on the coordinate x, then this question has a precise physical meaning: does this course constitute a single component, or it consists of several components? (There exist several other applications of this problem; these applications and a feasible algorithm for solving this problem are described in [14, 15].) In terms of modal logic, this problem can be formulated as follows: if we are *sure* that $f(x_i) \in [y_i^-, y_i^+]$ for all i, then is it *possible* that f is monotonic? In other words, is it true that

$$(\Box(y_1^- \le F(x_1) \le y_1^+) \& \ldots \& \Box(y_n^- \le F(x_n) \le y_n^+)) \to$$

$$\Diamond \forall x \forall y (x < y \to F(x) \le F(y)).$$

3 Definitions and the Main Result

Definition 1. *Let us define the language of modal mathematics. Let us first describe the alphabet of the designed language. The formulas of this language will be formed from the following symbols:*

- *constants for all rational numbers;*
- *variables $x_1, \ldots$ (denoted by small letters) that run over all real numbers;*
- *modal variables $X_1, \ldots$ (denoted by capital letters);*
- *arithmetic operations $+$, $-$, $\cdot$, and $/$; and*
- *relations $=$, $<$, $\leq$, $>$, $\geq$.*

A term is defined as follows:

- *every constant and every variable is a term;*
- *if t and t' are terms, then (t), $t + t'$, $t - t'$, $t \cdot t'$, and t/t' are terms;*
- *nothing else is a term.*

An elementary formula is defined as a formula of the type $t \circ t'$, where t and t' are terms, and $\circ$ is one of the relations (i.e., $=$, $<$, $\leq$, $>$, or $\geq$). A formula is defined as follows:

- *every elementary formula is a formula;*
- *if F and F' are formulas, then (F), $\neg F$, $F \& F'$, $F \vee F'$, and $F \rightarrow F'$ are formulas;*
- *if F is a formula, and x_i is a variable, then $\forall x_i F$ and $\exists x_i F$ are formulas;*
- *if F is a formula, then $\Diamond F$ and $\Box F$ are formulas;*
- *nothing else is a formula.*

A formula is called quantifier-free if it does not use quantifiers $\forall$ and $\exists$, and modal-free if it does not use modalities $\Diamond$ and $\Box$. A formula is called closed if every variable x_i is within the scope of some quantifier, and every modal variable X_i is within the scope of some modality.

Comment. Modal-free formulas are exactly formulas of first order theory of real numbers described in [13, 11].

Definition 2. *To define the truth value of an arbitrary formula F of modal mathematics, we must select:*

- *some values $x_1, \ldots, x_m$ for all the variables from this formulas;*

- *some values $X_1, \ldots, X_n$ for all modal variables from this formula, and*
- *a set $\mathcal{X} \subseteq R^n$ (where n is the total number of modal variables in a formula F).*

For this choice, the truth value of a formula F is defined as follows:

- *The value of a term is defined in a straightforward way (as the result of the corresponding computations).*
- *Correspondingly, the truth value of an elementary formula $t \circ t'$ depends on whether the values of the terms t and t' are indeed connected by a relation $\circ$.*
- *The truth value of a composite formula $F \& F'$, $\forall x F$, etc., is defined according to the normal understanding of the logical operations &, $\forall$, etc.*
- *The formula $\Diamond A$ is true iff there exists a tuple $(X_1, \ldots, X_n) \in \mathcal{X}$ for which A is true.*
- *The formula $\Box A$ is true iff A is true for all tuples $(X_1, \ldots, X_n) \in \mathcal{X}$.*

In particular, for closed formulas, the truth value does not depend on the choice of the values of its variables, only on the choice of the set $\mathcal{X}$.

Definition 3. *We say that a closed formula is valid (in modal mathematics) if it is true for all sets $\mathcal{X} \subseteq R^n$.*

In terms of this definition, the basic computation problem of modal mathematics is to check whether a given closed formula is true or not.

Theorem 1.

- *For closed quantifier-free formulas:*
  - *Validity of modal-free formulas can be checked in polynomial time.*
  - *Checking validity of formulas that use modalities is a decidable but NP-hard problem.*
- *For formulas with quantifiers:*
  - *Checking validity of arbitrary modal-free formulas is an algorithmically decidable problem.*
  - *Checking validity of arbitrary formulas with modality is an algorithmically undecidable problem.*

This result can be represented as a table:

	Modal-Free Formulas	Formulas With Modalities
Quantifier-Free Formulas	Polynomial time	Decidable, NP-hard
Formulas With Quantifiers	Decidable	Undecidable

4 Proof

4.1 Modal-Free Formulas: General Comment

According of our definition of a closed formula, every modal formula must be within a scope of some modality. Therefore, if a closed formula is modal-free (i.e., contains no modalities), then it does not contain any modal variables at all.

4.2 Modal-Free Quantifier-Free Formulas

If a closed modal-free formula F is also quantifier-free, this means that F contains no variables at all, only constants, and the formula itself is a propositional combination of elementary formulas of the type $t \circ t'$, where t and t' are terms made composed from constants by applying elementary arithmetic operations. Computing the values of these terms step-by-step takes linear time (i.e., time that is bounded by a linear function of the size of the formula); comparing these values and applying propositional connectives to the Boolean-valued results of this comparison is also linear-time. So, as a result, we can check whether a formula is true or not in linear (hence, in polynomial) time.

4.3 Modal-Free Formulas: General Case

In general, if a closed modal-free formula is not necessarily quantifier-free, it is, as we have mentioned, a first order formula from the theory of real numbers described in [13, 11]. For this theory, there is an algorithm testing whether a given closed formula is valid or not [13, 11], therefore, for this class, the problem of checking whether a given formula is valid or not is algorithmically decidable.

4.4 Quantifier-Free Formulas With Modalities

Let us first show that this problem is NP-hard. Indeed, it is known [2, 3] that the problem of computing the range of an interval function is NP-hard; actually, it has been proven that the problem of checking whether, say, one of the endpoints of the resulting range interval is $\geq$ a given number, is NP-hard. But this problem, as we have shown, can be reformulated as a quantifier-free formula of modal mathematics. Thus, checking validity of such formulas is an NP-hard problem.

Let us now prove that checking validity of quantifier-free closed formulas F of modal mathematics is decidable. For this, we will use the deciding algorithm proposed in [13, 11]. This algorithm transforms each first-order formula from the theory of real numbers (i.e., in our terms, each modal-free formula) into an equivalent quantifier-free modal-free formula (in particular, a closed formula is transformed into its Boolean value "true" or "false").

For each quantifier-free modal-free formula, we can describe the set of all tuples that make it true. Such a set is called *semi-algebraic*. By this definition, the union, complement and intersection of semi-algebraic sets are also semi-algebraic (because they correspond to the disjunction $\vee$, conjunction &, and the negation $\neg$ of the corresponding formulas).

We will first show that if F is a quantifier-free formula with modalities, then each subformula of the formula F is equivalent to the propositional combination of quantifier-free modal-free formulas and formulas of the type $\mathcal{X} \subseteq A$ for semi-algebraic sets A. We will show this by induction over the length of the formula:

Induction base: Elementary formulas are quantifier-free modal-free, and therefore, they are themselves of the right type.

Induction step: Let us assume that all formulas shorter than a formula G are equivalent to formulas of the desired type. This means, in particular, that all subformulas of G are equivalent to formulas of the given type. Then, depending on the structure of the formula G, we have three possible situations:

- If G is a propositional combination (i.e., if $G = G'\&G''$, $G' \vee G''$, etc.), then, since each of its subformulas G', G'' is equivalent to a propositional combination of the formulas of the right type, G is also equivalent to such a propositional combination; so, for this case, the induction step is proven.

- Let us now consider the case when G has the form $\Box H$ for some formula H. We know that H is equivalent to a propositional combination H' of quantifier-free formulas and finitely many formulas of the type $\mathcal{X} \subseteq A$; let us denote the total number of such sets A by m, and the sets themselves by $A_1, \ldots, A_m$. Depending on the whether $\mathcal{X} \subseteq A_j$ for each $j = 1, \ldots, m$, we have 2^m possible situations. Each situation will be denoted by s_k, where k is an $m-$digit number, in which 1 in $i-$th place means that $\mathcal{X} \subseteq A_i$, and 0 that $\mathcal{X} \not\subseteq A_i$. (For example, s_0 means that $\mathcal{X} \not\subseteq A_1 \& \ldots \& \mathcal{X} \not\subseteq A_p$.) For each of these situations s_k, H' can be reduced to a quantifier-free modal-free formula; we will denote such a formula by H_k. Hence, for every set $\mathcal{X}$ that is characterized by this situation s_k, the formula G of the type $\Box H$ is equivalent $\Box H'$, which, in its turn, is equivalent to $\forall X_1, \ldots, \forall X_n((X_1, \ldots, X_n) \in \mathcal{X} \to H_k)$. If we denote the set of all tuples that satisfy the condition H_k by A'_k, then this condition is equivalent to $\mathcal{X} \subseteq A'_k$. So, for each situation s_k, the formula $\Box H$ is equivalent to $\mathcal{X} \subseteq A'_k$. Therefore, in general, the formula $\Box H$ is equivalent to the following propositional combination

$$(s_1\&(\mathcal{X} \subseteq A'_1)) \vee \ldots \vee (s_{2^m-1}\&(\mathcal{X} \subseteq A'_{2^m-1})).$$

Each of the conditions s_k is already in the desired form, so H is also in the desired form.

- The case when G is of the type $\Diamond H$ can be reduced to the previous one, because, as one can easily check, $\Diamond H$ is equivalent to $\neg\Box(\neg H)$.

The statement is proven. In particular, it is applicable to the original closed formula F. Since this formula is closed, it has no modal variables left, and therefore, F is equivalent to a propositional combination of the formulas f_j of the type $\mathcal{X} \subseteq A$. This propositional combination can be reduced to a conjunctive normal form (CNF), i.e., to a formula $C_1\&\ldots\&C_p$, where each subformula C_k (called *conjunction*) is of the type $a \vee ... \vee b$, and each of the subformulas $a, \ldots, b$ is either f_j, or $\neg f_j$. So, the validity of the formula F is equivalent to the fact that for every set $\mathcal{X}$, the conjunction $C_1\&\ldots\&C_p$ is true. This means that for each set $\mathcal{X}$, each conjunction must be true. So, to check validity, it is sufficient to be able to check, for $k = 1, \ldots, p$, that each conjunction C_k is true for all sets $\mathcal{X} \subseteq R^n$.

If we take into consideration that each conjunction C_k is a conjunction of the formulas f_j and $\neg f_j$, and f_j is of the type $\mathcal{X} \subseteq A_j$, then (after, if necessary, reordering the terms inside C_k) we get the formula of the type

$$(\mathcal{X} \subseteq B_1) \vee \ldots (\mathcal{X} \subseteq B_q) \vee (\mathcal{X} \not\subseteq C_1) \vee \ldots \vee (\mathcal{X} \not\subseteq C_r)$$

for some semi-algebraic sets B_k and C_l. This formula is equivalent to

$$(\mathcal{X} \subseteq C_1\&\ldots\&\mathcal{X} \subseteq C_r) \to (\mathcal{X} \subseteq B_1 \vee \ldots \mathcal{X} \subseteq B_q).$$

The condition of this implication is that $\mathcal{X}$ is a subset of r sets $C_1, \ldots, C_r$; this is equivalent to $\mathcal{X}$ being a subset of the intersection $C_1 \cap \ldots \cap C_r$. This intersection of semi-algebraic sets (we will denote it by C) is also semi-algebraic. So, for a given set $\mathcal{X}$, the truth of the formula F is equivalent to the truth of the following formula:

$$\mathcal{X} \subseteq C \to (\mathcal{X} \subseteq B_1 \vee \ldots \mathcal{X} \subseteq B_q). \tag{2}$$

The validity of F means that this implication must be true for all sets $\mathcal{X}$. Let us show that this formula is true for all $\mathcal{X}$ iff

$$C \subseteq B_1 \vee \ldots \vee C \subseteq B_q. \tag{3}$$

Indeed:

- If (2) is true for all sets $\mathcal{X}$, it must also be true for $\mathcal{X} = C$. For this set, the condition of (2) is true, and therefore, the conclusion must be true, and this conclusion is exactly (3).
- Vice versa, let (3) be true. Then, if a set $\mathcal{X}$ satisfies the condition of the implication (2), then $\mathcal{X}$ is a subset of C, and C (by (3)) is a subset of one of the sets B_k. Hence, $\mathcal{X}$ is a subset of one of the sets B_k, which is exactly the conclusion of the formula (2). So, the implication that constitutes the formula (2) is true.

So, validity of F is equivalent to the validity of a formula (3) with semi-algebraic sets C and B_k. The fact that these sets are semi-algebraic means that each of these sets is a set of all tuples $(X_1, \ldots, X_n)$ that satisfy a certain modal-free formula; we will denote the corresponding formulas by $F_C(X_1, \ldots, X_n)$ and $F_{B_k}(X_1, \ldots, X_n)$. In terms of these formulas, the condition $C \subseteq B_k$ becomes a first order (modal-free) statement $\forall x_1 \ldots \forall x_n (F_C(x_1, \ldots, x)n \to F_{B_k}(x_1, \ldots, x_n))$, and first order modal-free statement are decidable (we can use Tarski-Seidenberg algorithm [13, 11]).

4.5 General Case: Formulas That Contain Both Quantifiers and Modalities

Let us now prove that the problem of checking validity of general formulas is algorithmically undecidable. To prove this statement, we will use the known fact that there exists no algorithm for checking whether a given *Diophantine equation* has a solution, i.e., whether for a given polynomial $P(x_1, \ldots, x_n)$ with integer coefficients, there exist natural numbers (non-negative integers) $x_1, \ldots, x_n$ for which $P(x_1, \ldots, x_n)$ [8, 1]. Let us show that checking whether such integers exist is equivalent to checking the validity of the following formula from our language:

$$(Z_1 \& \ldots Z_n \& C) \to \Diamond P(X_1, \ldots, X_n) = 0, \tag{4}$$

where Z_i stands for

$$\Diamond(X_i = 0) \& \forall x (\Diamond(X_i = x) \to \Diamond(X_i = x + 1)),$$

and C for

$$\forall x_1 \ldots \forall x_n ((\Diamond(X_1 = x_1) \& \ldots \& \Diamond(X_n = x_n)) \leftrightarrow \Diamond(X_1 = x_1 \& \ldots \& X_n = x_n)).$$

Indeed, according to our definitions, the validity of the statement Z_i means that the set $\mathcal{X}_i$ of all possible values of X_i (i.e., of all values X_i from the tuples $(X_1, \ldots, X_i, \ldots, X_n) \in \mathcal{X}$) contains 0 and contains $x+1$ with each its element x. This means, in particular, that it contains $0, 1, 2, \ldots$, i.e., all natural numbers.

The condition C means that if for each $i = 1, \ldots, n$, X_i is possible (i.e., $X_i \in \mathcal{X}_i$), then the tuple $(X_1, \ldots, X_n)$ is also possible (i.e., $(X_1, \ldots, X_n) \in \mathcal{X}$).

In particular, since each natural number x_i is a possible value of X_i, an arbitrary tuple of natural numbers $(x_1, \ldots, x_n)$ belongs to the set $\mathcal{X}$.

The conclusion of the implication (4) means that $P(X_1, \ldots, X_n) = 0$ for some tuple $(X_1, \ldots, X_n) \in \mathcal{X}$.

If the equation $P(x_1, \ldots, x_n)$ has a solution $x_1^{(0)}, \ldots, x_n^{(0)}$ in which all values $x_i^{(0)}$ are natural numbers, then for every set $\mathcal{X}$, for which the conditions of the implication (4) are satisfied, the tuple $(x_1^{(0)}, \ldots, x_n^{(0)})$ corresponding to this solution is an element of $\mathcal{X}$, and therefore, the conclusion of the implication is true. Thus, if the original Diophantine equation has a solution, then the formula (4) is valid.

Vice versa, if the formula (4) is valid, then it is valid for an arbitrary set $\mathcal{X}$, in particular, for the set $\mathcal{X} = N^n$ (where N is the set of all integers). For this set, the conditions of the implication (4) are true, and therefore, the conclusion must also be true. Hence, the equation $P(X_1, \ldots, X_n)$ must have a solution $(X_1, \ldots, X_n) \in \mathcal{X}$. Since $\mathcal{X} = N^n$, this means that the original equation has a solution in natural numbers.

So, a Diophantine equation has a solution iff the corresponding formula (4) of modal mathematics is valid. Since it is impossible to algorithmically check whether a given Diophantine equation has a solution, it is thus impossible to check whether a given formula of modal mathematics is valid or not. Hence, in the general case, the problem of checking validity of formulas of modal mathematics is algorithmically undecidable. Q.E.D.

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