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MULTILEVEL ADDITIVE SCHWARZ PRECONDITIONER FOR NONCONFORMING MORTAR FINITE ELEMENT METHODS

MAKSYMILIAN DRYJA*, ANDREAS GANTNER†, OLOF B. WIDLUND‡, AND BARBARA I. WOHLMUTH§

Abstract. Mortar elements form a family of special non-overlapping domain decomposition methods which allows the coupling of different triangulations across subdomain boundaries. We discuss and analyze a multilevel preconditioner for mortar finite elements on nonmatching triangulations. The analysis is carried out within the abstract framework of additive Schwarz methods. Numerical results show a performance of our preconditioner as predicted by the theory. Our condition number estimate depends quadratically on the number of refinement levels.

Key words. domain decomposition, elliptic mortar finite element method, non-matching triangulations, preconditioned conjugate gradients, additive Schwarz methods.

1. Introduction. Domain decomposition techniques for the numerical solution of partial differential equations have been analyzed extensively and used successfully. Large problems are decomposed into a set of smaller ones by breaking up the given domain into subdomains. Then, with parallel computation in mind, flexible methods are established by using characteristic properties of the given partial differential equation. In this paper, we consider two different aspects of domain decomposition. Within the framework of mortar finite element methods, we use a discretization scheme based on the coupling of nonmatching triangulations along the interfaces. Then in order to construct an efficient solver for the resulting linear system of equations, we introduce and analyze a special multilevel Schwarz method.

We will use the mortar finite elements introduced by Bernardi, et al. in [3, 4]. A characteristic feature of mortar methods is that subdomain meshes may be constructed separately in each subdomain and are, in general, nonmatching along the interfaces. This is in contrast to traditional domain decomposition methods, where a globally conforming triangulation is used. Mortar finite elements therefore provide a more flexible approach than standard conforming formulations. Often, mortar methods are recommended when the splitting into subdomains is somehow prescribed for physical or geometrical reasons. Then, for each subdomain an optimal approximation scheme can be chosen, involving the choice of the triangulation as well as the discretization. The strong condition of pointwise continuity across the interfaces is replaced by a weaker one resulting in a nonconforming approximation scheme. In spite thereof, mortar finite elements provide the same accuracy as standard conforming finite elements.

In Section 2, we briefly recall the standard mortar formulation. An introduction of our multilevel preconditioner can be found in Section 3. In Section 4, we establish our main result, which is an upper bound for the condition number of the additive Schwarz operator in terms of the refinement level. In Section 5, we discuss some aspects of the implementation of the method. Finally in Section 6, we present some numerical results illustrating the performance of the method.

2. Mortar spaces. Let the polygonal domain Ω be decomposed into K nonoverlapping polygonal subdomains Ω_k, $1 \leq k \leq K$. We assume that the diameter of Ω is of order one, and that the diameter of the subdomains is of order H. For simplicity, we restrict ourselves to the case of geometrically conforming decompositions. For each interface γ of the decomposition, we introduce a *master* subdomain $\Omega_{m(\gamma)}$ and a *slave* subdomain $\Omega_{s(\gamma)}$ with respect to γ such that

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$\gamma = \partial\Omega_{s(\gamma)} \cap \partial\Omega_{m(\gamma)}$. The choice which subdomain is the master subdomain is arbitrary but should be fixed.

We consider the following selfadjoint second order model problem: Find $u^* \in H_0^1(\Omega)$ such that

$$a(u^*, v) = f(v), \qquad v \in H_0^1(\Omega), \tag{2.1}$$

where

$$a(u,v) := \sum_{k=1}^{K} a_k(u,v) := \sum_{k=1}^{K} (\nabla u, \nabla v)_{L^2(\Omega_k)}, \quad f(v) := (f,v)_{L^2(\Omega)}.$$

Each subdomain Ω_k is associated with an initial shape regular triangulation $\mathcal{T}_k^{(0)}$. The meshsize is denoted by $h_k^{(0)}$ and the triangulations are, in general, nonmatching across the common interfaces. For simplicity we assume that $h_k^{(0)}$ is on the order of H_k, the diameter of Ω_k which itself is supposed to be on the order of H. Using uniform refinement, we obtain a sequence of nested triangulations in each subdomain Ω_k:

$$\mathcal{T}_k^{(0)} \subset \mathcal{T}_k^{(1)} \subset \cdots \subset \mathcal{T}_k^{(L)},$$

with the meshsizes at level l given by $h_k^{(l)} = 2^{-l} h_k^{(0)}$. We denote by $\mathcal{N}_k^{(l)}$ the set of nodal points in $\bar{\Omega}_k \setminus \partial\Omega$ of the triangulation $\mathcal{T}_k^{(l)}$, and we set $\mathcal{N}_k := \mathcal{N}_k^{(L)}$ and $n_k := |\mathcal{N}_k|$. Consequently, the corresponding finite element spaces $\mathcal{X}_k^{(l)}$ of continuous and piecewise linear functions satisfying homogeneous boundary conditions on $\partial\Omega_k \cap \partial\Omega$ are nested on each subdomain:

$$\mathcal{X}_k^{(0)} \subset \mathcal{X}_k^{(1)} \subset \cdots \subset \mathcal{X}_k^{(L)}.$$

We can now define the unconstrained product space on level l, $l = 0, \ldots, L$, by

$$\mathcal{X}_h^{(l)} := \prod_{k=1}^{K} \mathcal{X}_k^{(l)}, \quad \mathcal{X}_h := \mathcal{X}_h^{(L)}.$$

One of the main ideas of the mortar finite element method is to replace the pointwise continuity across the interfaces by a weaker one. To specify this interface condition, we introduce some trace spaces:

$$\begin{aligned} \mathcal{W}_h^{(l)}(\gamma) &:= \left\{ v_{|\gamma} \;\middle|\; v \in \mathcal{X}_{s(\gamma)}^{(l)} \right\}, & \mathcal{W}_h(\gamma) &:= \mathcal{W}_h^{(L)}(\gamma), \\ \mathcal{W}_{h;0}^{(l)}(\gamma) &:= \mathcal{W}_h^{(l)}(\gamma) \cap H_0^1(\gamma), & \mathcal{W}_{h;0}(\gamma) &:= \mathcal{W}_{h;0}^{(L)}(\gamma). \end{aligned}$$

We say that a function $v \in \mathcal{X}_h$ satisfies the *mortar condition* if its restrictions $v_{m(\gamma)}$ and $v_{s(\gamma)}$ to the master and the slave side, respectively, satisfy

$$\int_\gamma v_{m(\gamma)} \eta \, \mathrm{d}\sigma = \int_\gamma v_{s(\gamma)} \eta \, \mathrm{d}\sigma, \qquad \eta \in \mathcal{M}_h(\gamma), \tag{2.2}$$

where $\mathcal{M}_h(\gamma)$ is an appropriate Lagrange multiplier space. The Lagrange multiplier space is associated with the mesh on the interface inherited from the triangulation $\mathcal{T}_{s(\gamma)}^{(L)}$ on the slave subdomain. We denote this one dimensional mesh on γ by Σ_γ and its elements by e. Optimal a priori estimates can be obtained for several different Lagrange multiplier spaces, see, e.g. [9]. Here, we restrict ourselves to the standard Lagrange multiplier space which is a subspace of the trace space $\mathcal{W}_h(\gamma)$:

$$\mathcal{M}_h(\gamma) := \left\{ \; \eta \in \mathcal{W}_h(\gamma) \;\middle|\; \quad \eta_{|e} = \text{const. if the element } e \in \Sigma_\gamma \text{ touches } \partial\gamma \right\}. \tag{2.3}$$

If we denote by $\mathcal{N}_{s(\gamma)}$ the interior nodes of Σ_γ, a nodal basis of $\mathcal{M}_h(\gamma)$ is provided by $\{\psi_p\}_{p\in\mathcal{N}_{s(\gamma)}}$, where

$$\psi_p(q) = \delta_{pq} \qquad p, q \in \mathcal{N}_{s(\gamma)}.$$

We introduce the *mortar projection* $\Pi_\gamma : L^2(\gamma) \longrightarrow \mathcal{W}_{0;h}(\gamma)$

$$\int_\gamma \Pi_\gamma v\eta \,\mathrm{d}\sigma := \int_\gamma v\eta \,\mathrm{d}\sigma, \qquad \eta \in \mathcal{M}_h(\gamma).$$

We note that dim $\mathcal{M}_h(\gamma) =$ dim $\mathcal{W}_{0;h}(\gamma)$ and that Π_γ is $H_{00}^{1/2}$-stable, i.e.,

$$\|\Pi_\gamma v\|_{H_{00}^{1/2}(\gamma)} \le C\|v\|_{H_{00}^{1/2}(\gamma)}. \tag{2.4}$$

The constrained mortar space $\mathcal{V}_h$ is defined in terms of the mortar condition (2.2)

$$\mathcal{V}_h := \{v \in \mathcal{X}_h \mid \Pi_\gamma[v] = 0, \text{ for all interfaces } \gamma\},$$

where the jump across an interface is set to $[v] := v_{m(\gamma)} - v_{s(\gamma)}$. Now, the discrete problem for (2.1) in $\mathcal{V}_h$ reads as follows: Find $u_h \in \mathcal{V}_h$ such that

$$a(u_h, v_h) = f(v_h), \qquad v_h \in \mathcal{V}_h. \tag{2.5}$$

The discrete problem has a unique solution, and the rate of convergence is comparable to that of conforming discretizations; see, e.g., [1, 2, 4].

In contrast to the unconstrained product space $\mathcal{X}_h$, the mortar finite element space $\mathcal{V}_h$ cannot be written as the product of spaces locally defined on the $\bar{\Omega}_k$. According to (2.2), each basis function ϕ_p of $\mathcal{V}_h$ is associated with one of the following sets of degrees of freedom:

- all nodes interior to the subdomains,
- all nodes interior to the master sides of the interfaces,
- all subdomain vertices (multiple values) except those on $\partial\Omega$.

The third set of the nodal points is denoted $\mathcal{N}_\mathcal{V}$. We note that the nodal basis functions of $\mathcal{V}_h$ can be obtained in terms of the nodal basis functions of $\mathcal{X}_h$ by applying the mortar projection. Figure 2.1 shows a basis function associated with a node on the interior of a master side and the resulting basis function in $\mathcal{V}_h$.

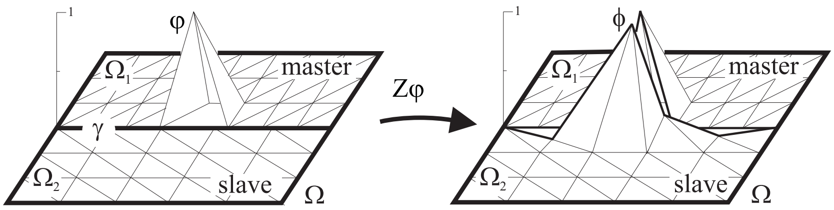


FIGURE 2.1. *Construction of a basis function of* $\mathcal{V}_h$

In the following, we will denote functions and operators using italics, e.g., v, A, and their discrete algebraic representations, i.e., vectors and matrices, using boldface, e.g., $\mathbf{v}, \mathbf{A}$. Each function $v_k \in \mathcal{X}_k$ can be represented uniquely as a vector $\mathbf{v}_k := (v_k(p))_{p\in\mathcal{N}_k} \in \mathbb{R}^{n_k}$ in terms of the nodal basis $\{\varphi_p\}_{p\in\mathcal{N}_k}$. Then, an element $v \in \mathcal{X}_h$ is given by $\mathbf{v} = (\mathbf{v}_k)_{k=1}^K \in \mathbb{R}^n$, $n := \sum_{k=1}^K n_k$. Matrices $\mathbf{A}_k \in \mathbb{R}^{n_k\times n_k}$ are now associated with the bilinear forms $a_k(\cdot,\cdot)$:

$$a_k(v, w) = \mathbf{w}_k^T \mathbf{A}_k \mathbf{v}_k \qquad v, w \in \mathcal{X}_k.$$

These $\mathbf{A}_k$ are symmetric and positive semidefinite. The matrix representation of $a(\cdot,\cdot)$ on $\mathcal{X}_h \times \mathcal{X}_h$ is then given by $\mathbf{A} =$diag$(\mathbf{A}_1, \mathbf{A}_2, \dots \mathbf{A}_K)$. To give a matrix representation of $a(\cdot,\cdot)$ on $\mathcal{V}_h \times \mathcal{V}_h$, we use interface matrices. We denote by $s_0, s_1, \dots s_{n_s+1}$ the nodes on the slave side and by m_0, m_1, $\dots m_{n_m+1}$ the nodes on the master side of the interface $\bar{\gamma}$. Both sets of nodes are assumed to be ordered lexicographically. For a function $v \in \mathcal{X}_h$, its vector $\mathbf{v}$ can now be written, after a permutation, as

$$\mathbf{v} = (\nu_s, \nu_c, \nu_m, \nu)^T ,$$

with entries given by $\nu_s = (v(s_i))_{i=1}^{n_s}$, $\nu_c = (v(s_0), v(s_{n_s+1}))^T$ on the slave side, $\nu_m = (v(m_i))_{i=0}^{n_m+1}$ on the master side, and where ν represents the values of v at the nodes that do not lie on the interface $\bar{\gamma}$. To compute ν_s for $\mathbf{v} \in \mathcal{V}_h$, we introduce a *master matrix*

$$\mathbf{M}_\gamma := \left(\left(\psi_{s_i}, \varphi_{m_j}\right)_{L^2(\gamma)} \right)_{i=1;j=0}^{n_s \quad n_m+1} \in \mathbb{R}^{n_s \times (n_m+2)}, \tag{2.6}$$

a *slave matrix*

$$\mathbf{S}_\gamma := \left(\left(\psi_{s_i}, \varphi_{s_j}\right)_{L^2(\gamma)} \right)_{i=1;j=1}^{n_s \quad n_s} \in \mathbb{R}^{n_s \times n_s}, \tag{2.7}$$

and a *corner slave matrix*

$$\mathbf{C}_\gamma := \left(\left(\psi_{s_i}, \varphi_{s_j}\right)_{L^2(\gamma)} \right)_{i=1;j=0,n_s+1}^{n_s} \in \mathbb{R}^{n_s \times 2}.$$

The slave matrix $\mathbf{S}_\gamma$ can be computed easily since $\mathrm{supp}\psi_{s_i} = \mathrm{supp}\varphi_{s_i}$. We note that in the case of standard Lagrange multipliers the matrix $\mathbf{S}_\gamma$ is tridiagonal, and in the case of dual Lagrange multipliers it is diagonal, see, e.g., [9]. In both cases, the matrix $\mathbf{C}_\gamma$ is almost empty; it has $(n_s - 2)$ zero rows and only two non-zero entries. The computation of the master matrix $\mathbf{M}_\gamma$ is more difficult because the structure of $\mathrm{supp}\psi_{s_i} \cap \mathrm{supp}\varphi_{m_j}$ depends on the triangulation. The mortar condition (2.2) can now be written in terms of these interface matrices

$$\nu_s = \mathbf{S}_\gamma^{-1} \left(\mathbf{M}_\gamma \nu_m - \mathbf{C}_\gamma \nu_c \right). \tag{2.8}$$

By applying this condition, we are able to eliminate the values ν_s from the vector $\mathbf{v}$. Each $v \in \mathcal{V}_h$ can now be represented by a *shorter* vector $\mathbf{v}_\mathcal{V}$, and the corresponding long vector can be obtained in terms of a global 'switching' matrix $\mathbf{Q}$:

$$\mathbf{v} = \mathbf{Q}\mathbf{v}_\mathcal{V}.$$

Defining $\mathbf{A}_\mathcal{V} := \mathbf{Q}^T \mathbf{A} \mathbf{Q}$ and $\mathbf{f}_\mathcal{V} := \mathbf{Q}^T \mathbf{f}$, the mortar matrix formulation on the constrained space is given by

$$\mathbf{A}_\mathcal{V} \mathbf{u}_\mathcal{V}^* = \mathbf{f}_\mathcal{V}. \tag{2.9}$$

3. A multilevel mortar preconditioner. Additive Schwarz methods provide a new operator equation which can be much better conditioned than the original problem. They are defined in terms of a suitable decomposition of $\mathcal{V}_h$ and bilinear forms. Before we introduce our multilevel decomposition of $\mathcal{V}_h$, we define an extension operator $Z_\gamma : L^2(\gamma) \longrightarrow \mathcal{X}_{s(\gamma)}$ on each interface γ

$$Z_\gamma v := \sum_{l=0}^{L} E_\gamma^{(l)} \left(P_\gamma^{(l)} - P_\gamma^{(l-1)} \right) \Pi_\gamma v.$$

Here, $E_\gamma^{(l)} : \mathcal{W}_{h;0}^{(l)}(\gamma) \longrightarrow \mathcal{X}_{s(\gamma)}^{(l)}$, is the zero extension operator on level l associated with the slave subdomain, and $P_\gamma^{(l)}$ is the L^2-projection $P_\gamma^{(l)} : \mathcal{W}_{h;0}(\gamma) \longrightarrow \mathcal{W}_{h;0}^{(l)}(\gamma)$, where we set $P_\gamma^{(-1)} := 0$. We note that $E_\gamma^{(l)} w$, $w \in \mathcal{W}_{h;0}^{(l)}(\gamma)$, is non-zero only in a strip of width $h_{s(\gamma)}^{(l)}$.

In a next step, we decompose the boundary $\partial\Omega_k \backslash \partial\Omega$ into two disjoint sets Γ_k^s, Γ_k^m, of interfaces γ such that

$$\partial\Omega_k \setminus \partial\Omega = \bigcup_{\gamma\in\Gamma_k^s} \bar{\gamma} \cup \bigcup_{\gamma\in\Gamma_k^m} \bar{\gamma},$$

where $\gamma \in \Gamma_k^s$ if Ω_k is the slave subdomain of the interface γ, and $\gamma \in \Gamma_k^m$ if Ω_k is the master subdomain of the interface γ. We now define our operator $Z_k : \mathcal{X}_k \to \mathcal{V}_h$ in terms of Z_γ and the two interface sets Γ_k^s and Γ_k^m

$$Z_k v := \begin{cases} v - \sum_{\gamma\in\Gamma_k^s} Z_\gamma v_{|\bar{\Omega}_k\cap\gamma}, & \text{on } \bar{\Omega}_k, \\ Z_\gamma v_{|\bar{\Omega}_k\cap\gamma}, & \text{on } \bar{\Omega}_{s(\gamma)}, \gamma \subset \Gamma_k^m, \\ 0, & \text{elsewhere.} \end{cases} \tag{3.1}$$

We note that $Z_k v$ is, in general, a fine level function even if v is a coarse level function, and that it is supported in Ω_k and its adjacent subdomains. Moreover, $E_\gamma^{(l)}$, restricted to the interface, is the identity and thus $(Z_\gamma v)_{|\gamma} = \Pi_\gamma v$. Now, it is easy to verify that $Z_k v \in \mathcal{V}_h$. We point out that the mortar condition with respect to the finest level holds. Figure 3.1 illustrates the mapping Z_k.

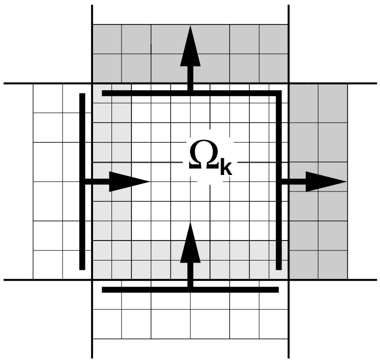


FIGURE 3.1. *Action of Z_k indicated by the arrows leading from the mortar (the dark thick lines) to the slave subdomain.*

Lemma 3.1 *The spaces $\mathcal{V}_k^{(l)} := Z_k\mathcal{X}_k^{(l)} \subset \mathcal{V}_h$, $1 \le k \le K$, $0 \le l \le L$, provide a decomposition of $\mathcal{V}_h$, i.e.,*

$$\mathcal{V}_h = \sum_{l=0}^{L}\sum_{k=1}^{K} Z_k\mathcal{X}_k^{(l)}.$$

Proof. It is sufficient to show that $Z := \sum_{k=1}^{K} Z_k$ restricted to $\mathcal{V}_h$ is the identity. We restrict ourselves to a subdomain Ω_j

$$(Zv)_{|\bar{\Omega}_j} = v_{|\bar{\Omega}_j} - \sum_{\gamma\in\Gamma_j^s} Z_\gamma v_{|\bar{\Omega}_j\cap\gamma} + \sum_{\gamma\in\Gamma_j^s} Z_\gamma v_{|\bar{\Omega}_{m(\gamma)}\cap\gamma} = v_{|\bar{\Omega}_j} + \sum_{\gamma\in\Gamma_j^s} Z_\gamma[v] = v_{|\bar{\Omega}_j}.$$

Here, we have used the fact that the mortar projection applied to $[v]$, $v \in \mathcal{V}_h$, is zero. □

The second main ingredient in the construction of additive Schwarz methods, in the nonnested setting, are bilinear forms $b_k^{(l)}(\cdot,\cdot) : \mathcal{X}_k^{(l)} \times \mathcal{X}_k^{(l)} \longrightarrow \mathbb{R}$, see Section 5.1 in [8]. We use inexact solvers and define for $x_k^{(l)}, y_k^{(l)} \in \mathcal{X}_k^{(l)}$

$$b_k^{(l)}(x_k^{(l)}, y_k^{(l)}) := \sum_{p\in\mathcal{N}_k^{(l)}} x_k^{(l)}(p) y_k^{(l)}(p). \tag{3.2}$$

The corresponding projection-like operators $\tilde{T}_k^{(l)} : \mathcal{V}_h \longrightarrow \mathcal{X}_k^{(l)}$, are then given by

$$b_k^{(l)}(\tilde{T}_k^{(l)} v, x_k^{(l)}) := a(v, Z_k x_k^{(l)}), \qquad x_k^{(l)} \in \mathcal{X}_k^{(l)}.$$

We now define the operators $T_k^{(l)} := Z_k \tilde{T}_k^{(l)} : \mathcal{V}_h \longrightarrow \mathcal{V}_k^{(l)} \subset \mathcal{V}_h$. The algebraic representation $\mathbf{T}_k^{(l)} \in \mathbb{R}^{|\mathcal{N}_\mathcal{V}|\times|\mathcal{N}_\mathcal{V}|}$ of $T_k^{(l)}$ can easily be obtained in terms of the algebraic representation $\mathbf{Z}_k \in \mathbb{R}^{|\mathcal{N}_\mathcal{V}|\times n_k}$ of Z_k. Denoting by $\mathbf{R}_k^{(l)}$ the prolongation matrix from $\mathcal{X}_k^{(l)}$ onto $\mathcal{X}_k^{(L)}$, we find

$$\mathbf{T}_k^{(l)} = \mathbf{Z}_k \mathbf{R}_k^{(l)} \left(\mathbf{Z}_k \mathbf{R}_k^{(l)}\right)^T \mathbf{A}_\mathcal{V}. \tag{3.3}$$

According to the additive Schwarz framework, we can now define a preconditioner $\mathbf{C}_\mathcal{V}$ for the linear system $\mathbf{A}_\mathcal{V}\mathbf{u}_\mathcal{V}^* = \mathbf{f}_\mathcal{V}$ by

$$\mathbf{C}_\mathcal{V} := \sum_{k=1}^{K}\sum_{l=0}^{L} \mathbf{Z}_k \mathbf{R}_k^{(l)} \left(\mathbf{Z}_k \mathbf{R}_k^{(l)}\right)^T. \tag{3.4}$$

4. Upper bound for the condition number. In this section, we establish an upper bound for the condition number of $\mathbf{C}_\mathcal{V}\mathbf{A}_\mathcal{V}$. Before we start to state the estimates, we briefly recall some important results for standard multilevel methods, which we will need in the sequel.
Let $P_k^{(l)}$ be the L^2-projection from $\mathcal{X}_k$ into $\mathcal{X}_k^{(l)}$ and let $P_k^{(-1)} := 0$. For $x = \{x_k\}_{k=1}^K \in \mathcal{X}_h$, we set

$$x_k^{(l)} := \left(P_k^{(l)} - P_k^{(l-1)}\right) x_k, \tag{4.1}$$

and use the weighted norm

$$\|u\|^2_{H^1(\Omega_k)} := |u|^2_{H^1(\Omega_k)} + \frac{1}{H^2}\|u\|^2_{L^2(\Omega_k)}.$$

Then, the following well-known result holds, see, e.g., [7][Theorem 15]:

Lemma 4.1

$$\sum_{l=0}^{L} \frac{\|x_k^{(l)}\|^2_{L^2(\Omega_k)}}{(h_k^{(l)})^2} \le C\|x_k\|^2_{H^1(\Omega_k)}.$$

We get a similar result if we consider functions on the interfaces: Let $P_\gamma^{(l)}$, as before, be the L^2-projection from $\mathcal{W}_{h;0}(\gamma)$ into $\mathcal{W}_{h;0}^{(l)}(\gamma)$ and let $P_\gamma^{(-1)} := 0$. Then, for $w \in \mathcal{W}_{h;0}(\gamma)$ we set $w_\gamma^{(l)} := \left(P_\gamma^{(l)} - P_\gamma^{(l-1)}\right) w$ and get, see [7][Theorem 15]:

Lemma 4.2

$$\sum_{l=0}^{L} \frac{\|w_\gamma^{(l)}\|^2_{L^2(\gamma)}}{h_{s(\gamma)}^{(l)}} \le C\|w\|^2_{H_{00}^{1/2}(\gamma)}.$$

Now, following the abstract Schwarz framework, see [8], we show that the spaces $\mathcal{V}_k^{(l)}$ provide a stable splitting of $\mathcal{V}_h$:

Lemma 4.3 *There exists a decomposition of $v \in \mathcal{V}_h$ with $v = \sum_{k=1}^K \sum_{l=0}^L Z_k x_k^{(l)}$, $x_k^{(l)} \in \mathcal{X}_k^{(l)}$ and a constant $C = 1/c_0$ independent of H and L such that*

$$\sum_{k=1}^{K}\sum_{l=0}^{L} b_k^{(l)}(x_k^{(l)}, x_k^{(l)}) \le C^2 H^{-2} a(v,v).$$

Proof. Let $v \in \mathcal{V}_h \subset \mathcal{X}_h$ be decomposed as $\sum_{k=1}^K \sum_{l=0}^L v_k^{(l)}$, where $v_k := v|_{\bar{\Omega}_k} \in \mathcal{X}_k$ and $v_k^{(l)} := (P_k^{(l)} - P_k^{(l-1)}) v_k \in \mathcal{X}_k^{(l)}$. In view of Lemma 3.1, the operator $Z := \sum_{k=1}^K Z_k$ restricted to $\mathcal{V}_h$ is the identity, and thus we get $v = \sum_{k=1}^K \sum_{l=0}^L Z_k v_k^{(l)}$. Using the definition of $\mathcal{V}_k^{(l)}$ and a discrete norm equivalence, we find by using Lemma 4.1

$$\begin{aligned}\sum_{k=1}^K \sum_{l=0}^L b_k^{(l)}(v_k^{(l)}, v_k^{(l)}) &= \sum_{k=1}^K \sum_{l=0}^L \sum_{p \in \mathcal{N}_k^{(l)}} \left(v_k^{(l)}(p)\right)^2 \\ &\leq C \sum_{k=1}^K \sum_{l=0}^L \frac{\|v_k^{(l)}\|^2_{L^2(\Omega_k)}}{(h_k^{(l)})^2} \leq C \sum_{k=1}^K \|v_k\|^2_{H^1(\Omega_k)}.\end{aligned}$$

The use of the ellipticity of $a(\cdot,\cdot) : \mathcal{V}_h \times \mathcal{V}_h \longrightarrow \mathbb{R}$, and the scaling of the H^1-norm completes this proof. $\square$

We note, that for $x_k^{(l)} \in \mathcal{X}_k^{(l)}$, the function $Z_k x_k^{(l)}$ is nonzero only in Ω_k and its adjacent subdomains. We apply a standard coloring argument to obtain a strengthened Cauchy-Schwarz inequality, see [10].

Lemma 4.4 *There exists a constant C independent of L and H such that $\rho(\mathcal{E}) \leq C(1+L)$, where $\rho(\mathcal{E})$ is the spectral radius of $\mathcal{E} = (\epsilon_{(k;l),(i;j)})_{1 \leq k,i \leq K, 0 \leq l,j \leq L}$ with*

$$a(v_k^{(l)}, v_i^{(j)}) \leq \epsilon_{(k;l),(i;j)} a(v_k^{(l)}, v_k^{(l)})^{\frac{1}{2}} a(v_i^{(j)}, v_i^{(j)})^{\frac{1}{2}}, \quad v_k^{(l)} \in \mathcal{V}_k^{(l)},\ v_i^{(j)} \in \mathcal{V}_i^{(j)}.$$

In a next step, we give a kind of one-sided measure of the approximation properties of the bilinear forms $b_k^{(l)}(\cdot,\cdot)$.

Lemma 4.5 *There exists a constant c_1 independent of H and L such that*

$$a(Z_k x_k^{(l)}, Z_k x_k^{(l)}) \leq c_1 (1+L) b_k^{(l)}(x_k^{(l)}, x_k^{(l)}), \quad x_k^{(l)} \in \mathcal{X}_k^{(l)}.$$

Proof. For $x_k^{(l)} \in \mathcal{X}_k^{(l)}$, we find that

$$\begin{aligned}a(Z_k x_k^{(l)}, Z_k x_k^{(l)}) = a_k &\left(x_k^{(l)} - \sum_{\gamma \in \Gamma_k^s} Z_\gamma x_k^{(l)} \ ,\ x_k^{(l)} - \sum_{\gamma \in \Gamma_k^s} Z_\gamma x_k^{(l)} \right) + \\ &+ \sum_{\gamma \in \Gamma_k^m} a_{s(\gamma)}(Z_\gamma x_k^{(l)}, Z_\gamma x_k^{(l)}).\end{aligned} \tag{4.2}$$

A standard inverse estimate shows that the first term is bounded in terms of the bilinear forms $b_k^{(l)}(\cdot,\cdot)$

$$a_k(x_k^{(l)}, x_k^{(l)}) \leq C b_k^{(l)}(x_k^{(l)}, x_k^{(l)}).$$

Moreover, by a strengthened Cauchy-Schwarz inequality, an inverse inequality and Lemma 4.2, we get for any interface $\gamma \in \Gamma_k^s$

$$\begin{aligned}a_k(Z_\gamma x_k^{(l)}, Z_\gamma x_k^{(l)}) &\leq C \sum_{i=0}^L \|E_\gamma^{(i)} \left(P_\gamma^{(i)} - P_\gamma^{(i-1)}\right) \Pi_\gamma x_k^{(l)}\|^2_{H^1(\Omega_k)} \\ &\leq C \sum_{i=0}^L \frac{1}{(h_k^{(i)})^2} \|E_\gamma^{(i)} \left(P_\gamma^{(i)} - P_\gamma^{(i-1)}\right) \Pi_\gamma x_k^{(l)}\|^2_{L^2(\Omega_k)} \\ &\leq C \sum_{i=0}^L \frac{1}{h_k^{(i)}} \| \left(P_\gamma^{(i)} - P_\gamma^{(i-1)}\right) \Pi_\gamma x_k^{(l)}\|^2_{L^2(\gamma)} \\ &\leq C \|\Pi_\gamma x_k^{(l)}\|^2_{H_{00}^{1/2}(\gamma)}.\end{aligned} \tag{4.3}$$

To obtain an upper bound for the $H_{00}^{1/2}$-norm of $\Pi_\gamma x_k^{(l)}$, we decompose $x_k^{(l)}$ into two parts. With p_s and p_e denoting the two endpoints of γ and $\varphi_s^{(l)}$ and $\varphi_e^{(l)}$ the nodal basis functions associated with p_s and p_e at the l-level, we introduce a function χ_∂ by

$$\chi_\partial := x_k^{(l)}(p_s)\varphi_s^{(l)} + x_k^{(l)}(p_e)\varphi_e^{(l)}.$$

Now, $x_k^{(l)}$ can be written on γ as

$$x_k^{(l)} = \chi_\partial + \chi_i,$$

where $\chi_i \in H_{00}^{1/2}(\gamma)$, and thus

$$\|\Pi_\gamma x_k^{(l)}\|_{H_{00}^{1/2}(\gamma)} \leq \|\Pi_\gamma \chi_\partial\|_{H_{00}^{1/2}(\gamma)} + \|\Pi_\gamma \chi_i\|_{H_{00}^{1/2}(\gamma)}. \tag{4.4}$$

The $H_{00}^{1/2}$-stability of Π_γ, (2.4), in combination with an inverse inequality yields

$$\|\Pi_\gamma \chi_i\|^2_{H_{00}^{1/2}(\gamma)} \leq C\|\chi_i\|^2_{H_{00}^{1/2}(\gamma)} \leq C \sum_{p\in\mathcal{N}_k^{(l)}} \left(x_k^{(l)}(p)\right)^2.$$

To give an estimate of the second term in (4.4), we assume that γ is the segment $(0, H_k)$. By the definition of the $H_{00}^{1/2}(\gamma)$ norm,

$$\begin{aligned}\|\Pi_\gamma \chi_\partial\|^2_{H_{00}^{1/2}(\gamma)} &= |\Pi_\gamma \chi_\partial|^2_{H^{1/2}(\gamma)} + \\ &+ \int_0^{H_k} \frac{|\Pi_\gamma \chi_\partial|^2}{x}\,\mathrm{d}x + \int_0^{H_k} \frac{|\Pi_\gamma \chi_\partial|^2}{H_k - x}\,\mathrm{d}x\end{aligned} \tag{4.5}$$

Using [6][Lemma 5], we get

$$\begin{aligned}|\Pi_\gamma \chi_\partial|^2_{H^{1/2}(\gamma)} &\leq C\left\{\left(x_k^{(l)}(p_s)\right)^2 |\Pi_\gamma \varphi_s^{(l)}|^2_{H^{1/2}(\gamma)} + \left(x_k^{(l)}(p_e)\right)^2 |\Pi_\gamma \varphi_e^{(l)}|^2_{H^{1/2}(\gamma)}\right\} \\ &\leq C\left\{\left(x_k^{(l)}(p_s)\right)^2 + \left(x_k^{(l)}(p_e)\right)^2\right\}.\end{aligned}$$

Since the last two terms of (4.5) are very similar, we concentrate on the first. We split the integral into two, over $(0, h_k^{(L)})$ and $(h_k^{(L)}, H_k)$, respectively. A standard analysis yields

$$\int_{h_k^{(L)}}^{H_k} \frac{|\Pi_\gamma \chi_\partial|^2}{x}\,\mathrm{d}x \leq C\left(1 + \log\left(\frac{H_k}{h_k^{(L)}}\right)\right) \|\Pi_\gamma \chi_\partial\|^2_{L^\infty(\gamma)},$$

and

$$\int_0^{h_k^{(L)}} \frac{|\Pi_\gamma \chi_\partial|^2}{x}\,\mathrm{d}x \leq C\|\Pi_\gamma \chi_\partial\|^2_{L^\infty(\gamma)}.$$

Again using [6][Lemma 5], we obtain

$$\|\Pi_\gamma \chi_\partial\|^2_{L^\infty(\gamma)} \leq C\left\{\left(x_k^{(l)}(p_s)\right)^2 + \left(x_k^{(l)}(p_e)\right)^2\right\}.$$

Recalling that $h_k^{(l)} = 2^{-l}h_k^{(0)}$ and using $h_k^{(0)} \sim H_k$, we find

$$\|\Pi_\gamma x_k^{(l)}\|_{H_{00}^{1/2}(\gamma)} \leq C(1 + L) \sum_{p\in\mathcal{N}_k^{(l)}} \left(x_k^{(l)}(p)\right)^2,$$

and thus

$$a_k(Z_\gamma x_k^{(l)}, Z_\gamma x_k^{(l)}) \leq C(1+L) b_k^{(l)}(x_k^{(l)}, x_k^{(l)}).$$

Proceeding as before, we get for $\gamma \in \Gamma_k^m$

$$a_{s(\gamma)}(Z_\gamma x_k^{(l)}, Z_\gamma x_k^{(l)}) \leq C(1+L) b_k^{(l)}(x_k^{(l)}, x_k^{(l)}).$$

Finally, substituting the above results into (4.2) gives

$$a(Z_k x_k^{(l)}, Z_k x_k^{(l)}) \leq C(1+L) b_k^{(l)}(x_k^{(l)}, x_k^{(l)}).$$

□

Remark 4.1 *In the case of two subregions the factor L disappears from the estimate in Lemma 4.5. This follows immediately from the proof since $\chi_\partial = 0$ in this case.*

Relying on the abstract Schwarz framework, we have shown the following estimate for our preconditioned system:

Theorem 4.1 *The operator $T := \sum_{k=1}^K \sum_{l=0}^L T_k^{(l)}$ satisfies, for any $v \in \mathcal{V}_h$,*

$$c_0 H^2 a(v,v) \leq a(Tv, v) \leq c_1 (1+L)^2 a(v,v),$$

where the constants c_0 and c_1 are independent of L and H.

Remark 4.2 *We remark that by introducing a coarse space based on a continuous vertex basis function for each subdomain vertex, we could eliminate the dependence of H in Theorem 4.1.*

Remark 4.3 *It can be proved that in the case of two subregions, the factor $(1+L)^2$ disappears in the estimate of Theorem 4.1; see also the numerical results in Section 6.*

5. Implementation. Our aim is to give a simple matrix representation of our preconditioner

$$\mathbf{C}_\mathcal{V} = \sum_{k=1}^K \sum_{l=0}^L \mathbf{Z}_k \mathbf{R}_k^{(l)} \left(\mathbf{Z}_k \mathbf{R}_k^{(l)}\right)^T = \sum_{k=1}^K \mathbf{Z}_k \left(\sum_{l=0}^L \mathbf{R}_k^{(l)} \left(\mathbf{R}_k^{(l)}\right)^T\right) \mathbf{Z}_k^T.$$

In most iterative methods, the preconditioner has to be applied to the residual $\mathbf{r}_\mathcal{V} \in \mathbb{R}^{|\mathcal{N}_\mathcal{V}|}$. The vector $\mathbf{r}_k := \mathbf{Z}_k^T \mathbf{r}_\mathcal{V}$ is in $\mathbb{R}^{n_k}$, and we have to compute

$$\mathbf{C}_k \mathbf{r}_k := \sum_{l=0}^L \mathbf{R}_k^{(l)} \left(\mathbf{R}_k^{(l)}\right)^T \mathbf{r}_k.$$

But this is a standard mapping since $\mathbf{C}_k$ is of the same type as a BPX-preconditioner for the subdomain Ω_k, c.f. [8].

We now consider $\mathbf{Z}_k$ in more detail. A matrix representation of the operator Π_γ can easily be given in terms of the mass matrices $\mathbf{M}_\gamma$, $\mathbf{S}_\gamma$ and $\mathbf{C}_\gamma$. We introduce the prolongation matrix $\mathbf{I}_\gamma^{(l)}$ from $W_{h;0}^{(l)}$ onto $W_{h;0}^{(L)}$ and the prolongation matrices $\mathbf{i}_\gamma^{(l)}$ from $W_{h;0}^{(l)}$ onto $W_{h;0}^{(l+1)}$ associated with the natural embedding operators and the mass matrices

$$\mathbf{G}_\gamma^{(l)} := \left(\left(\varphi_i^{(l)}, \varphi_j^{(l)}\right)_{L^2(\gamma)}\right)_{i=1;j=1}^{N_l \quad N_l} \in \mathbb{R}^{N_l \times N_l}, \; 0 \leq l \leq L, \tag{5.1}$$

where N_l denotes the number of interior nodes of the interface triangulation on γ inherited from $\mathcal{T}_{s(\gamma)}^{(l)}$. We obtain a matrix representation of $P_\gamma^{(l)}$ by

$$\mathbf{P}_\gamma^{(l)} := \left(\mathbf{G}_\gamma^{(l)}\right)^{-1} \left(\mathbf{I}_\gamma^{(l)}\right)^T \mathbf{G}_\gamma^{(L)}. \tag{5.2}$$

We are now able to give a matrix representation of the operator Z_γ

$$\mathbf{Z}_\gamma \mathbf{v} = \sum_{l=0}^{L} \mathbf{R}_{s(\gamma)}^{(l)} \mathbf{E}_\gamma^{(l)} \left(\left(\mathbf{G}_\gamma^{(l)}\right)^{-1} - \mathbf{i}_\gamma^{(l-1)} \left(\mathbf{G}_\gamma^{(l-1)}\right)^{-1} \left(\mathbf{i}_\gamma^{(l-1)}\right)^T \right) \times$$
$$\times \left(\mathbf{I}_\gamma^{(l)}\right)^T \mathbf{G}_\gamma^{(L)} \mathbf{S}_\gamma^{-1} \left(\mathbf{S}_\gamma \,|\, \mathbf{C}_\gamma \,|\, \mathbf{M}_\gamma \,|\, 0\right) \begin{pmatrix} \nu_s \\ \nu_c \\ \nu_m \\ \nu_i \end{pmatrix}.$$

Here, $\mathbf{E}_\gamma^{(l)}$ is a matrix representation of the operator $E_\gamma^{(l)}$, the trivial extension from γ to $\Omega_{s(\gamma)}$ at the l-level.
We note, that most of the work in applying the mapping Z_k can be done in parallel for two adjacent subdomains. To see this, let γ be an interface and let $\mathbf{v}_{m(\gamma)}$ and $\mathbf{v}_{s(\gamma)}$ be the vectors on the master and slave subdomain, respectively. On the master subdomain there is no change of the vector $\mathbf{v}_{m(\gamma)}$, while on the slave subdomain we have to calculate an update for the interface of the form $\mathbf{Z}_\gamma \mathbf{v}_{m(\gamma)} - \mathbf{Z}_\gamma \mathbf{v}_{s(\gamma)}$. This can be done by first calculating

$$\mathbf{x} := \mathbf{M}_\gamma \nu_m - \left(\mathbf{S}_\gamma \,|\, \mathbf{C}_\gamma\right) \begin{pmatrix} \nu_s \\ \nu_c \end{pmatrix},$$

see (2.8), and then

$$\sum_{l=0}^{L} \mathbf{R}_{s(\gamma)}^{(l)} \mathbf{E}_\gamma^{(l)} \left(\left(\mathbf{G}_\gamma^{(l)}\right)^{-1} - \mathbf{i}_\gamma^{(l-1)} \left(\mathbf{G}_\gamma^{(l-1)}\right)^{-1} \left(\mathbf{i}_\gamma^{(l-1)}\right)^T \right) \left(\mathbf{I}_\gamma^{(l)}\right)^T \mathbf{G}_\gamma^{(L)} \mathbf{S}_\gamma^{-1} \mathbf{x}.$$

The cost of applying $\mathbf{Z}_k$ is of order $\mathcal{O}(\sqrt{n_k})$. It is well known [5] that applying the BPX preconditioner $\mathbf{C}_k$ is of order $\mathcal{O}(n_k)$. Altogether we get a computational cost of order $\mathcal{O}(|\mathcal{N}|)$ for the preconditioner $\mathbf{C}_\mathcal{V}$.

Remark 5.1 *We remark that the computational cost can be further reduced if we use dual Lagrange multipliers. Then $\mathbf{S}_\gamma$ is a diagonal matrix. Moreover, we can replace $P_\gamma^{(l)}$ by a quasi-projection operator which gives rise to a diagonal matrix.*

6. Numerical results. In this section, we present some numerical test examples illustrating the performance of our mortar multilevel additive Schwarz method. We use standard conforming P_1-Lagrange finite elements in each subdomain Ω_k and nonmatching simplicial triangulations. Starting with an initial triangulation, we decompose each element into four subelements in each refinement step. To solve the algebraic system $\mathbf{A}_\mathcal{V} \mathbf{u}_\mathcal{V}^* = \mathbf{f}_\mathcal{V}$, we apply a preconditioned conjugate gradient method where the preconditioner is defined in Section 3. If the decomposition into subdomains is fixed, Theorem 4.1 yields a condition number of $\mathbf{C}_\mathcal{V} \mathbf{A}_\mathcal{V}$ which is bounded by CL^2, and thus the number of iterations which are necessary to reach a given tolerance is $\mathcal{O}(L)$.

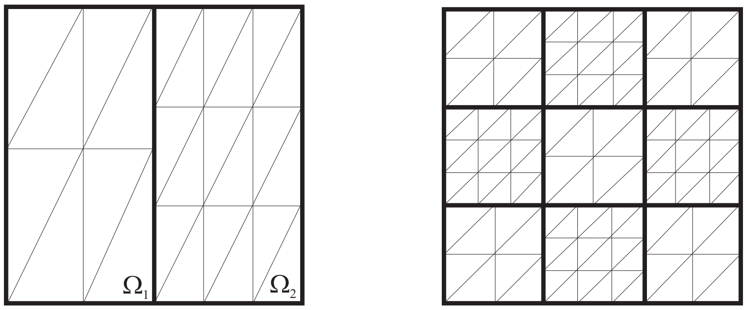
We start with the following test example:

$$\begin{aligned} -\Delta u &= f, && \text{in } \Omega := (-1,1)^2, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

and the right-hand side f is chosen to be

$$f = 2\pi^2 \sin(\pi x) \sin(\pi y).$$

Then, the weak solution is given by $u(x,y) = \sin(\pi x)\sin(\pi y)$. In a first step (Example 1), we decompose the unit square into two subdomains. The initial nonmatching triangulation and the decomposition into two subdomains are shown in the left picture of Figure 6.1.

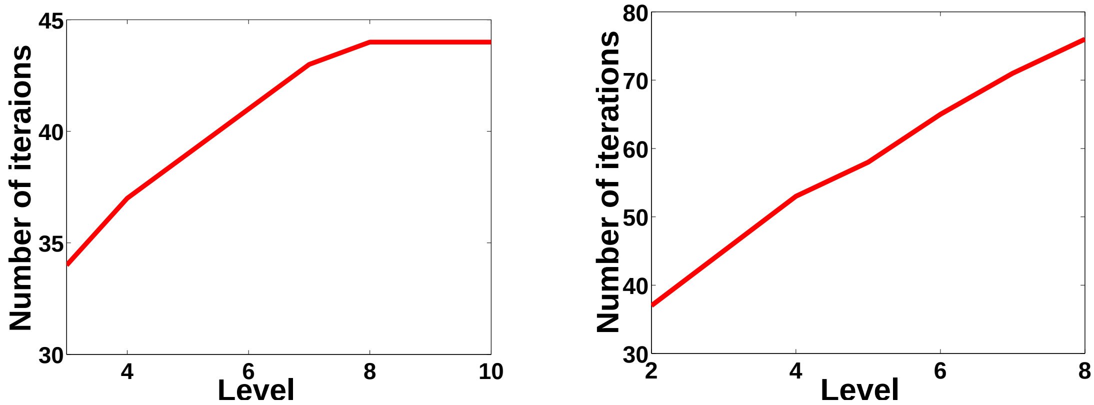


(a) Example 1 (b) Example 2

FIGURE 6.1. *Decomposition into subdomains and initial triangulation*

The number of iteration steps to obtain the given tolerance 10^{-8} is given in the left part of Figure 6.2. Asymptotically, we obtain level independent convergence rates for our preconditioner. The numerical results for this test setting are better than predicted by Theorem 4.1. This fact is due to our special decomposition; we have no crosspoints, c.f. Remark 4.3. The condition number estimates for this setting are in the following table:

Level No.	3	4	5	6	7	8	9	10
Condition No.	19.86	24.52	27.63	30.17	31.95	33.05	33.54	33.61



(a) Example 1 (b) Example 2

FIGURE 6.2. *Number of iterations*

For our second example, we use the same boundary value problem but decompose the unit square into nine subdomains. Here, we have four interior crosspoints, see the right part of Figure 6.1.

The numerical results are shown in the right picture of Figure 6.2. For this example, we do not obtain level independent convergence rates. The number of iteration steps increase with the number of levels, and a linear growth can be observed.

Using a continuous coarse space in our second example improves considerably the condition numbers. Without a coarse space, we observe a quadratic growth of the condition of the preconditioned system. The numerical results confirm our theoretical results. Adding a coarse space yields qualitative better numerical results.

Level No.	2	3	4	5	6	7	8
Condition No. without coarse space	126.9	190.4	267.7	358.3	462.0	578.8	708.1
Condition No. with coarse space	69.14	91.06	137.9	196.0	263.8	341.2	428.1

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