The ellipticity principle for steady and selfsimilar polytropic potential flow*

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Abstract

We prove the *ellipticity principle* for selfsimilar potential flows for gas dynamics. We show that the interior of a pseudo-subsonic-sonic-region of a smooth solution must be pseudo-subsonic. In fact, the pseudo-Mach number is below that of a domain-dependent function which is < 1 in the interior and ≤ 1 on the boundary. Therefore the interior must stay pseudo-subsonic under homotopy of pseudo-subsonic-sonic boundaries. Self-similar flows represent time-asymptotic flow patterns for flows with self-similar solid boundary and far field values. Our result indicates that such a flow do not have pseudo-supersonic bubbles either away from boundary or along a straight solid boundary within a pseudo-subsonic region. This is in contrast with stationary flows, for which a supersonic bubble can arise within a subsonic region and eventually forms shocks. Our analysis is for polytropic gases. We give two examples showing that the ellipticity principle does not extend to the isentropic and full Euler equations, or to arbitrary pressure laws.

1 Introduction

Multidimensional gas dynamics is described, among other models, by the isentropic Euler equations of compressible gas dynamics, in d space dimensions,

$$\rho_t + \nabla \cdot (\rho \vec{v}) = 0 \tag{1}$$

$$\vec{v}_t + \nabla \vec{v} \cdot \vec{v} + \nabla(\pi(\rho)) = 0. \tag{2}$$

Hereafter, $\nabla = \nabla_{\vec{x}}$ denotes the gradient with respect to the space variables $\vec{x} = (x_1, x_2, \cdots, x_d)$. Here $\vec{v} = (v^1, v^2, \cdots, v^d)$ is the velocity of the gas, ρ the density, and π is defined in terms of the pressure $p = p(\rho)$ by $\pi_\rho = p_\rho/\rho$. We assume throughout that $p_\rho > 0$. We will be concerned mostly with *polytropic* pressure laws, that is, the pressure p is given as a function of the density ρ as follows:

$$p(\rho) = \rho^\gamma.$$

Throughout this paper, we restrict ourselves to $\gamma \in (1, \infty)$ (a typical value is $\gamma = \frac{7}{5}$ for air); however, some results remain true for other values, e.g. $\gamma = 1$ (isothermal case).

The Euler equations possess the highly unstable vortex sheets. To focus on the acoustic waves and shock waves, one often considers the *potential flow*, with the *irrotationality* assumption $v^i_j = v^j_i$, $i, j = 1, \ldots, d$, $v^i_j \equiv v^i_{x_j}$ etc, which leads to;

$$\vec{v} = \nabla_{\vec{x}} \phi$$

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for some scalar *potential* ϕ. For smooth flows, substituting this into (2) yields, for $i = 1, \dots, d$,

$$
\begin{aligned}
0 &= \phi_{it} + \nabla\phi_i \cdot \nabla\phi + \nabla(\pi(\rho)) \\
&= (\phi_t + \frac{|\nabla\phi|^2}{2} + \pi(\rho))_i.
\end{aligned}
$$

Thus, for some constant A,

$$
\rho = \pi^{-1}(A - \phi_t - \frac{|\nabla\phi|^2}{2}).
$$

Substituting this into (1) yields a single second-order quasilinear hyperbolic equation, *potential flow* equation, for a scalar field ϕ: using $c^2 = p_\rho$ and $\pi_\rho = \frac{\rho}{c^2}$, we get

$$
\phi_{tt} + 2\nabla\phi_t \cdot \nabla\phi + \nabla\phi^T \nabla^2\phi\nabla\phi - c^2\Delta\phi^2 = 0. \tag{3}
$$

Setting time derivatives to zero yields the *steady potential flow* equation,

$$
(c^2 - \nabla\phi\nabla\phi^T) : \nabla^2\phi = 0, \tag{4}
$$

For polytropic pressure laws, the sound speed c has a particularly simple form:

$$
c^2 = (\gamma - 1)(A - \phi_t - \frac{|\nabla\phi|^2}{2}). \tag{5}
$$

Due to the invariance of first-order systems of conservation laws under the scaling $x \leftarrow sx$, $t \leftarrow st$, there is an interesting class of solutions to (1) and (2), the *selfsimilar solutions*, for which the dependence variables are functions of the self-similar variables

$$
\frac{\vec{x}}{t} \Leftrightarrow \vec{\xi}
$$

Such a solution appears in many important physical situations, e.g. Riemann problems, Mach reflection, flow around solid corners. Thus

$$
(\rho(t, \vec{x}), u(t, \vec{x})) = (\rho(\xi), u(\xi)).
$$

The corresponding ansatz for the potential ϕ is

$$
\phi(t, \vec{x}) = t\psi(\vec{\xi})
$$

so that in these new variables we have

$$
\vec{v} = \nabla_\xi\psi.
$$

The potential flow equation becomes

$$
(c^2 I - (\nabla\psi - \vec{\xi})(\nabla\psi - \vec{\xi})^T) : \nabla^2\psi = 0 \tag{6}
$$

Here $\nabla = \nabla_{\vec{\xi}}$ is the gradient with respect to the self-similar variables. Often, a more convenient variable is

$$
\chi := \psi - \frac{|\vec{\xi}|^2}{2}
$$

so that the corresponding potential flow equation becomes autonomous:

$$
(c^2 I - \nabla\chi\nabla\chi^T) : \nabla^2\chi = |\nabla\chi|^2 - dc^2 \tag{7}
$$

Here,

$$
c = c(\chi, |\nabla\chi|^2) = \pi^{-1}(A - \chi - \frac{|\nabla\chi|^2}{2}).
$$

Clearly, adding a constant to ψ resp. χ does not matter in (6) resp. (7), so for simplicity we take $A = 0$ in (5):

$$c^2 = (1-\gamma)(\chi + \frac{|\nabla\chi|^2}{2}). \tag{8}$$

$\nabla\chi = \nabla\psi - \vec{\xi}$ is called *pseudo-velocity.* It is the velocity in selfsimilar coordinates.

(4) and (7) inherit a number of pleasant symmetries from (1),(2): it is invariant under translation, corresponding to $v \leftarrow v + v_0$, $x \leftarrow x - v_0 t$ for standard coordinates, and rotation.

For many interesting flow patterns, (4) and (7) are of mixed-types. The type of (4) is determined by the (local) *Mach number*

$$M := \frac{|\nabla\phi|}{c};$$

and that of (7) is determined by the (local) *pseudo-Mach number*

$$L := \frac{|\nabla\chi|}{c},$$

with $0 \le L < 1$ for elliptic, $L = 1$ for parabolic, $L > 1$ for hyperbolic regions.

Since the mathematical theory of hyperbolic and of elliptic equations is quite different, it is important to identify the supersonic and subsonic regions. The equations are highly nonlinear, and these regions are not known a priori from the flow data. For stationary flows, it is known that supersonic bubble can arise and shocks form within a subsonic region as certain flow parameters vary, such as the increase of the upstream Mach number, [Mor82]. Our main theorem, Theorem 2, says that this cannot happen for self-similar flows, that is a pseudo-supersonic-sonic region cannot arise in the interior of a continuously changing smooth pseudo-subsonic-sonic region. It cannot arise between a pseudo-subsonic region and a self-similar solid boundary either. The pseudo-Mach number L cannot attain 1 in the interior of a pseudo-subsonic-sonic, $L \le 1$, region of a C^3 solution. There is a strong geometric nature of the potential self-similar flows, which, for instance, forces the self-similar flows to be pseudo-supersonic for $|\xi|$ sufficiently large, as the pseudo-velocity $\nabla\chi = \nabla\psi - \vec{\chi}$ becomes large there, assuming that the the flow velocity $\vec{v} = \nabla\psi$ remains bounded. Our main theorem on the pseudo-subsonic regions should therefore be useful for further analysis on general flow patterns of self-similar gas flows. The study of this general research direction is left to the future.

These results are complemented by counterexamples in the case of isentropic and full self-similar Euler equations, as well as certain non-polytropic pressure laws.

A variety of maximum principles for fluid variables has appeared throughout the literature. An ellipticity principle is derived in [Zhe97], see also [LZY98, Zhe01], for the selfsimilar pressure-gradient equations.

In contrast to simplified equations such as pressure-gradient, zero-pressure or transonic small disturbance equation, the coefficients in the self-similar equation (7) depend on both the dependent variable χ as well as its *gradient* $\nabla\chi$. For this reason, classical methods for establishing the maximum principles, e.g. using the barrier functions, are not applicable. The classical gradient maximum principles for steady potential flows, see Section 15 of [GT83], apply to equation whose coefficients depend *only* on the gradient of the dependent variable. Thus our result is the consequence of different physics and requires new analysis. The lower order term, the RHS of (7), expresses the geometric effects and plays an essential role in our analysis.

2 Maximum principles for steady potential flow

As a comparison to our result in the next section for self-similar flows, we recall classical maximum principles for steady potential flow.

Theorem 1. *Let ϕ be a C^2 solution of the stationary potential equation* (4) *in an open domain where $\rho > 0$, $M \leq 1$. Then the density ρ satisfies a strong minimum principle; and the speed $|\vec{v}| = |\nabla\psi|$ and the Mach number M satisfy strong maximum principles.*

Proof. We may restrict ourselves to $M < 1$ domains since $M = 1$ determines $\rho, c, |\nabla\psi|$ uniquely. The coefficients in (4) depend only on $\nabla\phi$, but not on x or ϕ, so Theorem 15.1 in [GT83] immediately provides a strong maximum principle for $|\nabla\phi|$. $c^2 = (\gamma-1)(A - \frac{|\nabla\phi|^2}{2})$ for some constant A, and $M^2 = \frac{1}{A|\nabla\phi|^{-2} - \frac{1}{2}}$, so clearly we also have a strong maximum principle for M and a strong minimum principle for ρ. □

3 The ellipticity principle for selfsimilar polytropic potential flows

Theorem 2 (Ellipticity principle). *Let $\Omega \subset \mathbb{R}^d$ open. There exist positive constants δ, ϵ so that, for any $\hat{c} > 0$ and any $b \in C^2(\Omega)$ with $b = 0$ on $\partial\Omega$ and $|\nabla b| \leq \frac{\epsilon}{\hat{c}}$, $|\nabla^2 b| \leq \frac{\epsilon}{\hat{c}^2}$ and for any solution $\chi \in C^3(\Omega)$ of* (7) *with $L \leq 1$, $\rho > 0$ and $c \leq \hat{c}$, we have*

$$L^2 \leq \max\{1-b, (1-\delta)^2\}$$

in Ω.

Remark 1. *At its heart the bound on L depends only on the domain, since the velocities $c, \vec{v}, \vec{\xi}$ can be rescaled to allow $\hat{c} = 1$.*

Proof. Set $N := L^2 + b$. Assume that N has a maximum in some interior point with $L \geq 1-\delta$. In particular, $|\nabla\chi| = Lc$ is nonzero. Since the equations are rotation-invariant, we may, without loss of generality, choose $\chi_1 > 0$, $\chi_j = 0$ for some $j > 1$. The following quantities are meant to be evaluated that point:

$$N_1 = \frac{2\chi_1\chi_{11}c^2 + (\gamma-1)\chi_1^3(1+\chi_{11})}{c^4} + b_1 = 0$$
$$\Leftrightarrow \chi_{11} = \frac{-cb_1 - (\gamma-1)L^3}{L(2+(\gamma-1)L^2)} = \frac{1-\gamma}{\gamma+1} + O(\epsilon+\delta).$$

For $j > 1$,

$$N_j = \frac{2\chi_1\chi_{1j}c^2 + (\gamma-1)\chi_1^3\chi_{1j}}{c^4} + b_2 = 0$$
$$\Leftrightarrow \chi_{1j} = O(\epsilon).$$

From (7),

$$\sum_{j>1}\chi_{jj} = (L^2-1)(\chi_{11}+1) - (d-1) \leq -(d-1) + O(\delta+\epsilon) \tag{9}$$

Moreover, using $(f/g)'' = (f''g - fg'' + 2f(g')^2 - 2f'gg')/g^3$,

$$
\begin{aligned}
N_{11} &= \frac{2(\chi_1\chi_{111} + \chi_{11}^2 + \sum_{j>1}\chi_{1j}^2)c^4 + (\gamma-1)\chi_1^2c^2(\chi_{11}(1+\chi_{11}) + \sum_{j>1}\chi_{1j}^2 + \chi_1\chi_{111})}{c^6} \\
&+ \frac{2(\gamma-1)^2\chi_1^4(1+\chi_{11})^2 + 4(\gamma-1)\chi_1^2c^2\chi_{11}(1+\chi_{11})}{c^6} \leq 0 \\
&= c^{-1}L(2+(\gamma-1)L^2)\chi_{111} + c^{-2}(\chi_{11} + 2(\gamma-1)L^2(1+\chi_{11}))(2\chi_{11} + (\gamma-1)L^2(1+\chi_{11})) \\
&+ c^{-2}(2+(\gamma-1)L^2)\sum_{j>1}\chi_{1j}^2 + b_{11} \leq 0 \\
\Leftrightarrow \chi_{111} &\leq \frac{1}{cL(2+(\gamma-1)L^2)}\Big((\chi_{11} + 2(\gamma-1)L^2(1+\chi_{11}))\underbrace{(2\chi_{11} + (\gamma-1)L^2(1+\chi_{11}))}_{=O(\delta+\epsilon)} \\
&+ (2+(\gamma-1)L^2)\underbrace{\sum_{j>1}\chi_{1j}^2}_{=O(\epsilon)} + c^2\underbrace{b_{11}}_{=O(\epsilon)}\Big) \leq O(\delta+\epsilon)
\end{aligned}
$$

For $j > 1$,

$$
\begin{aligned}
N_{jj} &= \frac{2(\chi_1\chi_{1jj} + \sum_i\chi_{ij}^2)c^4 + (\gamma-1)\chi_1^2c^2(\sum_i\chi_{ij}^2 + \chi_1\chi_{1jj}) + 2(\gamma-1)^2\chi_1^4\chi_{1j}^2 + 4(\gamma-1)\chi_1^2\chi_{1j}^2c^2}{c^6} + b_{jj} \\
&= \frac{L(2+(\gamma-1)L^2)}{c}\chi_{1jj} + \frac{(2+(\gamma-1)L^2)(1+2(\gamma-1)L^2)}{c^2}\chi_{1j}^2 + \frac{2+(\gamma-1)L^2}{c^2}\sum_{k>1}\chi_{kj}^2 + b_{jj} \leq 0
\end{aligned}
$$

$$
\begin{aligned}
\Leftrightarrow c\chi_{1jj} &\leq \frac{-1}{L}\left((1+2(\gamma-1)L^2)\chi_{1j}^2 + \sum_{k>1}\chi_{kj}^2 + c^2b_{jj}\right) \\
&\leq \frac{-1}{L}\left(\chi_{jj}^2 + c^2b_{jj}\right) + O(\delta+\epsilon) \\
\Rightarrow c\sum_{j>1}\chi_{1jj} &\leq -(d-1) + O(\delta+\epsilon). \qquad (10)
\end{aligned}
$$

At this point it is necessary to take ∂_1 of (7):

$$
\begin{aligned}
& d(\gamma-1)\chi_1 + (2+d(\gamma-1))(\chi_1\underbrace{\chi_{11}}_{=\frac{1-\gamma}{\gamma+1}+O(\epsilon)} + \sum_{k>1}\underbrace{\chi_k}_{=0}\chi_{1k}) \\
&= \underbrace{(c^2-\chi_1^2)}_{\geq 0}\underbrace{\chi_{111}}_{\leq 0} + \sum_{j>1}(c^2 - \underbrace{\chi_j^2}_{=0})\chi_{1jj} - 2\sum_{j\neq k}\underbrace{\chi_j\chi_k}_{=0}\chi_{1jk} \\
&+ \underbrace{((1-\gamma)\chi_1 + (1-\gamma)(\chi_1\underbrace{\chi_{11}}_{=\frac{1-\gamma}{\gamma+1}+O(\delta+\epsilon)} + \sum_{j>1}\underbrace{\chi_j}_{=0}\chi_{1j}) - 2\chi_1\underbrace{\chi_{11}}_{=\frac{1-\gamma}{\gamma+1}+O(\delta+\epsilon)})}_{=O(\delta+\epsilon)\chi_1}\underbrace{\chi_{11}}_{=O(\delta+\epsilon)\chi_1} \\
&+ \sum_{k>1}((1-\gamma)\chi_1 + (1-\gamma)(\chi_1\underbrace{\chi_{11}}_{=\frac{1-\gamma}{\gamma+1}+O(\delta+\epsilon)} + \sum_{j>1}\underbrace{\chi_j}_{=0}\chi_{1j}) - 2\underbrace{\chi_k}_{=0}\chi_{1k})\chi_{kk} \\
&- 2\sum_{j>1}(\chi_{11}\chi_j + \chi_1\chi_{1j})\underbrace{\chi_{1j}}_{=O(\epsilon)} - 2\sum_{1<j\neq k>1}(\chi_{1j}\underbrace{\chi_k}_{=0} + \underbrace{\chi_j}_{=0}\chi_{1k})\chi_{jk}
\end{aligned}
$$

$$
\Leftrightarrow \quad O(\delta+\epsilon)\chi_1 + 2\frac{\gamma-1}{\gamma+1}\chi_1(\sum_{j>1}\chi_{jj} + d - 1) \leq c^2\sum_{j>1}\chi_{1jj}
$$

Using (9) we see

$$c \sum_{j>1} \chi_{1jj} \geq O(\delta + \epsilon) \tag{11}$$

Comparing (10) and (11) yields a contradiction (for $d \geq 2$) for δ and ϵ choosen sufficiently small. $\square$

Remark 2. *If $\partial\Omega$ is sufficiently smooth, one can always find $b \in C^3(\Omega)$ with $b = 0$ on $\partial\Omega$, $b > 0$ in Ω. Otherwise one can pick $b = 0$; then $L \leq 1 - \epsilon$ on the boundary implies $L < 1 - \epsilon$ in the interior (this is sufficient for doing homotopy in elliptic regularizations without losing ellipticity in the interior).*

Corollary 1. (7) *does not admit solutions $\chi \in C^3$ that are parabolic on some open set.*

Remark 3. *For region Ω with self-similar solid boundary, the solution can be extended by reflection and the theorem implies that no pseudo-sonic or pseudo-supersonic region can arise in Ω and next to the solid boundary by smooth homotopy.*

4 Counterexamples for non-irrotational flow

Example 1. *Let $d \geq 2$, $L_0 \in (0,1)$ arbitrary. There are smooth solutions of the isentropic and the full selfsimilar Euler equations so that L has a strict maximum at a point with $L = L_0$.*

Proof. Clearly it is sufficient to consider $d = 2$. We fix initial data as a quadratic polynomial on a short vertical line through the origin and extend it, using the Cauchy-Kovalevskaya theorem, to an analytic solution on a small ball that has a strict maximum for L in the origin.

The 2D isentropic selfsimilar Euler equations are

$$\begin{aligned} a\rho_1 + \rho a_1 \quad + b\rho_2 \quad + \rho b_2 &= \quad -2\rho \\ \frac{c^2}{\rho}\rho_1 + aa_1 \quad + ba_2 \quad &= -3\rho a \\ ab_1 + \frac{c^2}{\rho}\rho_2 \quad + bb_2 &= \quad -3\rho a \end{aligned} \tag{12}$$

where $(a,b)' = \vec{v} - \vec{x}/t$. Add the critical point conditions

$$(L^2)_1 = 0, \qquad (L^2)_2 = 0.$$

This yields a linear system

$$\begin{pmatrix} a & \rho & 0 & b & 0 & \rho \\ \frac{c^2}{\rho} & a & 0 & 0 & b & 0 \\ 0 & 0 & a & \frac{c^2}{\rho} & 0 & b \\ \sigma_\rho & \sigma_a & \sigma_b & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_\rho & \sigma_a & \sigma_b \end{pmatrix} \begin{pmatrix} \rho_1 \\ a_1 \\ b_1 \\ \rho_2 \\ a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} -2\rho \\ -3a \\ -3b \\ 0 \\ 0 \end{pmatrix}.$$

where $\sigma_\rho = \frac{(\gamma-1)c^4(a^2+b^2)}{\rho}$, $\sigma_a = 2ac^2$, $\sigma_b = 2bc^2$. Setting $\rho = 1$ (which is merely a matter of scaling) and $b = 0$ and fixing $b_1 = 0$ yields the system matrix

$$\begin{pmatrix} a & \rho & 0 & 0 & \rho \\ \frac{c^2}{\rho} & a & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 \\ \sigma_\rho & \sigma_a & 0 & 0 & 0 \\ 0 & 0 & \sigma_\rho & \sigma_a & 0 \end{pmatrix}.$$

with determinant

$$2c^6a^3((\gamma-1)a^2-2)$$

which is nonzero for any small nonzero a.

Taking ∂_1 and ∂_2 of (12) and adding the conditions

$$(L^2)_{11}=-1, \qquad (L^2)_{12}=0, \qquad (L^2)_{22}=-1$$

yields a system

$$\begin{pmatrix} a & \rho & 0 & b & 0 & \rho & 0 & 0 & 0 \\ \frac{c^2}{\rho} & a & 0 & 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & a & \frac{c^2}{\rho} & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & a & \rho & 0 & b & 0 & \rho \\ 0 & 0 & 0 & \frac{c^2}{\rho} & a & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & a & \frac{c^2}{\rho} & 0 & b \\ \sigma_\rho & \sigma_a & \sigma_b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_\rho & \sigma_a & \sigma_b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sigma_\rho & \sigma_a & \sigma_b \end{pmatrix} \begin{pmatrix} \rho_{11} \\ a_{11} \\ b_{11} \\ \rho_{12} \\ a_{12} \\ b_{12} \\ \rho_{22} \\ a_{22} \\ b_{22} \end{pmatrix} = \text{l.o. terms.}$$

Again we can use $b=0$ (and $\sigma_b=0$); using this fact the matrix is (after some reordering) almost lower-triangular (except for two 2×2 blocks); the determinant is

$$\frac{ac^2\sigma_a(a\rho\sigma_\rho-c^2\sigma_a)}{\rho}=\frac{2c^{12}a^4(a^2(\gamma-1)+2)^2}{\rho}.$$

For small nonzero a, the determinant is nonzero.

Hence we can pick derivatives with the required properties; this provides quadratic initial data on a vertical line. Using the fact that vertical lines are noncharacteristic for the selfsimilar Euler equations with $b=0$, $\rho=1$, and $a:=L_0c<c$, we obtain a local analytic solution which has a strict maximum of L in the origin. Moreover, $L=L_0$ in the origin by choice of a.

Using the isentropic assumption, one can derive a matching temperature field to obtain a full Euler counterexample. □

5 Counterexamples for nonpolytropic flow

Example 2. *Let $d\geq 2$ and $L_0\in(0,1)$, $\rho_0\in(0,\infty)$ arbitrary.*

$$\{p\in C^3(0,\infty): p_\rho>0\ \forall\rho\}$$

has a dense (in $C^2[0,\infty]$) subset of pressure laws p that admit a smooth solution χ of (7) *so that L has a strict local maximum with value L_0 at a point where $\rho=\rho_0$.*

Proof. We may restrict ourselves to $d=2$ as the other cases are treated by constant extension. Let $\epsilon>0$ and $I=[\underline{\rho},\overline{\rho}]\Subset(0,\infty)$ (where $\rho_0\in(\underline{\rho},\overline{\rho})$) arbitrary. In the origin, fix $\chi_2=0$ and pick χ,χ_1 so that $L=L_0$, $\rho=\rho_0$ and $\chi_1>0$. Some computation shows that, in the first- and second-order conditions for a strict maximum of L in the origin, $p_{\rho\rho\rho}(\rho_0)$ occurs *only* in $(L^2)_{11}$ (which depends linearly on it, with nonzero coefficient). Choose $D^2\chi, D^3\chi$ to satisfy the remaining conditions: $\nabla(L^2)=0$ and $(L^2)_{12}=0$, $(L^2)_{22}=0$ as well as (7) and its two derivatives (all at the origin). Now add

$$\alpha(\rho-\rho_0)^3\theta(\rho)$$

to p, where $\alpha \in \mathbb{R}$ is chosen so that $(L)_{11} < 0$ is satisfied as well, and where θ is a smooth cutoff function which is $= 1$ in $[\rho_0 - \delta, \rho_0 + \delta]$ and $= 0$ outside $(\rho_0 - 2\delta, \rho_0 + 2\delta)$, with $\delta > 0$ so small that $\|\alpha(\rho - \rho_0)^3\theta(\rho)\|_{C^2(I)} \leq \epsilon/2$. Finally, choose an analytic function $\tilde{p}$ so that

$$\|p + \alpha(\rho - \rho_0)^3\theta(\rho) - \tilde{p}\|_{C^2(I)} \leq \epsilon/2.$$

Given these values in the origin, we have a cubic polynomial as initial data on a short horizontal line; the Cauchy-Kovalevskaya theorem extends it to a local analytic solution of (7) which has a strict maximum for L in the origin.

$\epsilon > 0$ and I were arbitrary, so denseness follows. □

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