# CRAMER-RAO LOWER BOUND FOR FREQUENCY ESTIMATION IN MULTIPATH RAYLEIGH FADING CHANNELS

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## ABSTRACT

This paper concerns the estimation of a frequency offset of a known (pilot) signal propagated through a slowly fading multipath channel, such that channel parameters are considered to be constant over the observation interval. We derive a Cramer-Rao Lower Bound (CRLB) and maximum likelihood (ML) frequency estimation algorithm for additive Gaussian noise and path amplitudes having complex zero-mean Gaussian distribution when covariance matrices of the fading and noise are known. In particular, we consider the scenarios with white noise, independent fading of path amplitudes and pilot signals with a diagonal correlation matrix. We compare simulation results for the ML estimator with the CRLB. We also show that the results obtained can be extended to scenarios with fast fading channels.

## 1. INTRODUCTION

Frequency estimation in multipath fading channels has been widely addressed in the literature [1] [2] [3] [4] [5] [6]. In most publications frequency selectivity is considered as a distortion and multipath diversity is not used to improve the accuracy performance of the algorithms. However, similar to the way in which a Rake receiver improves the detection performance in multipath channels [7], frequency estimators can also exploit multipath diversity to improve the estimation accuracy. Below we derive a CRLB for the frequency estimation problem and propose a ML frequency estimator exploiting multipath diversity and knowledge of channel statistics to improve the accuracy performance. We will restrict attention to slowly fading channels, assuming the channel parameters to be constant over the observation interval. However, the results obtained can be extended to fast fading channels and we discuss such an extension.

The paper is organised as follows. Section 2 describes channel and signal models. In section 3 we derive the ML frequency estimator for multipath fading channels with known statistics and section 4 presents the derivation of the CRLB. Section 5 gives simulation results and section 6 contains conclusions. In the Appendix we derive the Fisher information for the frequency estimation problem.

## 2. SIGNAL AND CHANNEL MODELS

Using complex-envelope notation, the observed signal can be modelled as

$$z(t) = A(t)e^{j\omega t} + n(t),\ t = 0, \dots, N-1, \qquad (1)$$

where $\omega$ is an unknown frequency offset to be estimated and $N$ is the number of samples observed. The additive complex zero-mean Gaussian noise samples $n(t)$ have the covariance matrix $\mathbf{R_n}$ with elements $[\mathbf{R_n}]_{t_1,\,t_2} = E\{n(t_1)n^*(t_2)\}$ where $E\{\cdot\}$ denotes the statistical expectation and $(\cdot)^*$ denotes complex conjugate. For stationary white noise $\mathbf{R_n} = \sigma^2\mathbf{I}$ where the noise variance is $\sigma^2 = E\{|n(t)|^2\}$ and $\mathbf{I}$ is an $N \times N$ identity matrix. The complex envelope $A(t)$ can be represented as $A(t) = \sum_{m=1}^{M} a_m \varphi_m(t)$ where $\varphi_m(t) = s(t-\tau_m)$, $s(t)$ is the pilot signal, $M$ is the number of paths in the channel, $\{a_m\}_{m=1}^{M}$ and $\{\tau_m\}_{m=1}^{M}$ are amplitudes and delays of the paths, respectively. Note that the functions $\{\varphi_m(t)\}_{m=1}^{M}$ make up a basis for expansion of the complex envelope $A(t)$.

We consider $\{\tau_m\}_{m=1}^{M}$ as an admissible set of delays, for example, $\tau_m = (m-1)/f_s$ where $f_s$ is a sampling frequency; for the sake of simplicity, we consider that $f_s = 1$. For derivation of the ML frequency estimation algorithm we assume the delays to be known.

The model (1) is based on the assumption that all the paths have the same frequency offset. For example, this assumption is valid when there exists a frequency offset between carrier frequencies of the transmitter and the receiver, and the frequency offsets due to the Doppler effect are negligible.

The signal model (1) can be arranged in matrix form as

$$\mathbf{z} = \mathbf{\Psi a} + \mathbf{n}, \quad \mathbf{\Psi} = \mathbf{\Lambda\Phi} \qquad (2)$$

where $\mathbf{z}$ and $\mathbf{n}$ are $N \times 1$ column vectors with elements $z(t)$ and $n(t)$, respectively, $\mathbf{a}=[a_1, \dots, a_M]^T$ is an $M \times 1$ column vector of path amplitudes, $\mathbf{\Phi}$ is an $N \times M$ matrix with elements $[\mathbf{\Phi}]_{tm} = \varphi_m(t)$ and $\mathbf{\Lambda} = \mathrm{diag}(1, e^{j\omega 1}, \dots, e^{j\omega(N-1)})$. Then the probability density function (PDF) of the received signal vector $\mathbf{z}$ can be written as

$$\mathbf{p(z|\omega, a)} = \pi^{-N}|\mathbf{R_n}|^{-1} exp\{-(\mathbf{z} - \mathbf{\Psi a})^{\mathbf{H}}\mathbf{R_n^{-1}}(\mathbf{z} - \mathbf{\Psi a})\} \qquad (3)$$

where $(\cdot)^H$ denotes Hermitian transposition. The amplitudes of the paths $\{a_m\}_{m=1}^{M}$ are complex-valued zero-mean random variables with the Gaussian PDF

$$\mathbf{f(a)} = \pi^{-M}|\mathbf{R_a}|^{-1} exp\{-\mathbf{a^H R_a^{-1} a}\} \qquad (4)$$

where $\mathbf{R_a}$ is an $M \times M$ covariance matrix. The function $\mathbf{f(a)}$ (4) defines the multipath fading channel. If the complex amplitudes $\{a_m\}_{m=1}^{M}$ of the paths are uncorrelated, the covariance matrix is diagonal

$$\mathbf{R_a} = \mathrm{diag}(\sigma_1^2, \sigma_2^2, \dots, \sigma_M^2) \qquad (5)$$

with the path amplitude variances $\sigma_m^2$, $m = 1, \dots, M$.

For the given model the signal-to-noise ratio (SNR) is

$$SNR = \frac{E\left\{\mathbf{a}^H \mathbf{\Psi}^H \mathbf{\Psi} \mathbf{a}\right\}}{E\left\{\mathbf{n}^H \mathbf{n}\right\}} = \frac{tr[\mathbf{\Gamma} \mathbf{R_a}]}{tr[\mathbf{R_n}]} \tag{6}$$

where $tr[\cdot]$ is the trace operator and $\mathbf{\Gamma} = \mathbf{\Phi}^H \mathbf{\Phi}$. For example, for the additive white noise and pilot signals with a diagonal correlation matrix $\mathbf{\Gamma} = diag(\gamma_{11}, \dots, \gamma_{MM})$ we have

$$SNR = \frac{1}{N\sigma^2} \sum_{m=1}^{M} \gamma_{mm} \sigma_m^2. \tag{7}$$

## 3. ML FREQUENCY ESTIMATION

We now consider estimating the parameter $\omega$ provided that the path delays $\{\tau_m\}_{m=1}^M$ are known and amplitudes $\{a_m\}_{m=1}^M$ have the PDF (4) with a known covariance matrix $\mathbf{R_a}$. To get the frequency estimator we use the Bayesian approach by integrating out the nuisance parameters $\{a_m\}_{m=1}^M$[8]. The ML frequency estimator is derived by maximising the function $\mathbf{p}(\mathbf{z}|\omega)$:

$$\hat{\omega} = arg \max_{\omega \in \Omega} \{\mathbf{p}(\mathbf{z}|\omega)\} \tag{8}$$

where $\Omega$ is the frequency acquisition range (e.g., $\Omega = [-\pi, \pi]$),

$$\mathbf{p}(\mathbf{z}|\omega) = \int \mathbf{p}(\mathbf{z}|\omega, \mathbf{a}) \mathbf{f}(\mathbf{a}) Re[\mathbf{a}] Im[\mathbf{a}] \tag{9}$$

where $Re[\cdot]$ and $Im[\cdot]$ denote the real and imaginary parts of a complex-valued number respectively, the PDF $\mathbf{f}(\mathbf{a})$ is defined by (4) and the PDF $\mathbf{p}(\mathbf{z}|\omega, \mathbf{a})$ is defined by (3). Integrating in (9) gives

$$\mathbf{p}(\mathbf{z}|\omega) = \frac{e^{-\mathbf{z^H R_n^{-1} z}}}{\pi^N |\mathbf{R_n}|} \times \frac{e^W}{|\mathbf{R_a} \mathcal{T} + \mathbf{I}|}, \tag{10}$$

where

$$W = \mathcal{L}^H (\mathcal{T} + \mathbf{R_a^{-1}})^{-1} \mathcal{L}, \tag{11}$$

$$\mathcal{L} = \mathbf{\Psi}^H \mathbf{R_n^{-1} z}, \quad \mathcal{T} = \mathbf{\Psi}^H \mathbf{R_n^{-1}} \mathbf{\Psi}. \tag{12}$$

Assuming the noise covariance matrix $\mathbf{R_n}$ to be known we can omit the first factor in (10) and rewrite (8) as

$$\hat{\omega} = arg \max_{\omega \in \Omega} \{-ln(|\mathbf{R_a} \mathcal{T} + \mathbf{I}|) + W\}. \tag{13}$$

The estimate (13) is the ML estimate of the frequency $\omega$ when the covariance matrix of the complex amplitudes of the paths are known.

*(a) Additive white noise*

For the additive white noise we have $\mathbf{R_n} = \sigma^2 \mathbf{I}$ where $\mathbf{I}$ is an $N \times N$ identity matrix. Then from (12), using the identity $\mathbf{\Lambda^H \Lambda} \equiv \mathbf{I}$, we can write

$$\mathcal{L} = \sigma^{-2} \mathbf{F}, \quad \mathbf{F} = \mathbf{\Psi^H z}, \quad \mathcal{T} = \sigma^{-2} \mathbf{\Phi}^H \mathbf{\Phi}. \tag{14}$$

Note that now the matrix $\mathcal{T}$ does not depend on the frequency $\omega$ and, hence, the first additive in (13) is independent of the frequency $\omega$ as well. Thus, the frequency estimate is

$$\hat{\omega} = arg \max_{\omega \in \Omega} \{\mathbf{F^H} (\mathcal{T} + \mathbf{R_a^{-1}})^{-1} \mathbf{F}\}. \tag{15}$$

Elements $F_{\omega,m}$ of the vector $\mathbf{F}$ are the Discrete Fourier Transforms (DFTs) of the product $z(t)s^*(t - \tau_m)$:

$$F_{\omega,m} = \sum_{t=0}^{N-1} z(t) s^*(t - \tau_m) e^{-j\omega t} \tag{16}$$

i.e. $F_{\omega,m}$ is the output of a matched filter that correlates the received signal $z(t)$ with a delayed and frequency shifted complex conjugate version of the pilot signal $s(t)$.

*(b) Additive white noise, pilot signals with a diagonal correlation matrix, and independent fadings of path amplitudes*

In the case of independent fluctuations of the path amplitudes, the covariance matrix $\mathbf{R_a}$ is diagonal and defined by (5). For pilot signals with a diagonal correlation matrix $\mathbf{\Gamma}$, $\mathcal{T}$ is a diagonal matrix

$$\mathcal{T} = \sigma^{-2} \mathbf{\Gamma} = \sigma^{-2} diag(\gamma_{11}, \gamma_{22}, \dots, \gamma_{MM}) \tag{17}$$

where $\gamma_{mn} = (\mathbf{\Phi^H \Phi})_{mn} = \sum_{t=0}^{N-1} s^*(t - \tau_m) s(t - \tau_n)$. The relationship (17) means that $\gamma_{mn} = \delta_{mn} \gamma_{mm}$ where $\delta_{mn}$ is the Dirac delta function. In this case, the ML estimate is

$$\hat{\omega} = arg \max_{\omega \in \Omega} \left\{ \sum_{m=1}^{M} \xi_m |F_{\omega,m}|^2 \right\} \tag{18}$$

where $\xi_m = (\gamma_{mm} + 1/\sigma_m^2)^{-1}$. This shows that the frequency estimator combines the periodograms $|F_{\omega,m}|^2$ of the paths with weights depending on the signal-to-noise ratio in these paths. This is similar to a Rake receiver combining multipath components of a received signal [7]. Note that the property $\gamma_{mn} = \delta_{mn} \gamma_{mm}$ can be satisfied only approximately, for example, for large $N$ and pseudo-noise pilot signals.

*(c) Additive white noise and frequency-flat fading*

For a one-path channel with a known delay the algorithm transforms to the well-known ML frequency estimator based on maximising the periodogram [9]

$$\hat{\omega} = arg \max_{\omega \in \Omega} \left\{ |F_{\omega,1}|^2 \right\}. \tag{19}$$

We can summarise results of this section as follows.

(1) Provided that the statistics of fading and noise are known, i.e. the covariance matrices $\mathbf{R_a}$ and $\mathbf{R_n}$ are specified, the ML frequency estimate is defined by (13) with $\mathcal{L}$ and $\mathcal{T}$ from (12), and $W$ from (11).

(2) If the additive noise is white, the ML frequency estimate is defined by (15) with $\mathcal{L}$ and $\mathcal{T}$ from (14).

(3) For multipath channels with independent Rayleigh fading of path amplitudes, additive white noise and pilot signals possessing a diagonal correlation matrix, the ML

frequency estimate is defined by (18) with $F_{\omega,m}$ from (16) and $\mathcal{T}$ from (17).

(4) Finally, for a one-path (frequency-flat) Rayleigh fading channel the ML frequency estimate is a maximiser of the periodogram (19).

## 4. CRLB

The CRLB ($CRB_\omega$) for frequency estimation in slowly fading multipath channels with arbitraty covariances of the noise $\mathbf{R_n}$ and fading $\mathbf{R_a}$ is $CRB_\omega = I_{\omega\omega}^{-1}$ where the value $I_{\omega\omega}$ derived in Appendix is

$$\begin{aligned} I_{\omega\omega} &= 2Re\{tr[\Upsilon \mathbf{R}_a \Upsilon \mathcal{N}^{-1} + \mathcal{W}\mathbf{R}_a \mathcal{T}\mathcal{N}^{-1} \\ &- \Upsilon \mathbf{R}_a \mathcal{T}\mathcal{N}^{-1}\dot{\mathcal{T}}\mathcal{N}^{-1}]\} \end{aligned} \tag{20}$$

where $\mathcal{N} = \mathcal{T} + \mathbf{R}_a^{-1}$, $\mathcal{T} = \mathbf{\Psi}^H \mathbf{R}_n^{-1}\mathbf{\Psi}$, $\Upsilon = \mathbf{\Psi}^H \mathbf{R}_n^{-1}\dot{\mathbf{\Psi}}$, $\mathcal{W} = \dot{\mathbf{\Psi}}^H \mathbf{R}_n^{-1}\dot{\mathbf{\Psi}}$ and we use the following notation for derivatives with respect to $\omega$: $\dot{\mathbf{\Psi}} = \frac{\partial}{\partial\omega}\mathbf{\Psi}$, $\dot{\mathcal{T}} = \frac{\partial}{\partial\omega}\mathcal{T}$, etc.

We now consider several particular cases of the channel and signal models.

*(a) Additive white noise*

In this case we have $\mathbf{R}_n = \sigma^2\mathbf{I}$, $\mathcal{T} = \sigma^{-2}\mathbf{\Phi}^H\mathbf{\Phi} = \sigma^{-2}\mathbf{\Gamma}$ and $\dot{\mathcal{T}} = 0$. Then we get from (20)

$$\begin{aligned} CRB_\omega^{-1} &= 2tr[\mathbf{\Phi}^H\mathbf{K}^2\mathbf{\Phi}\mathcal{R}_\mathbf{a}\mathbf{\Gamma}(\mathbf{\Gamma} + \mathcal{R}_\mathbf{a}^{-1})^{-1} \\ &- \mathbf{\Phi^H K \Phi}\mathcal{R}_\mathbf{a}\mathbf{\Phi^H K \Phi}(\mathbf{\Gamma} + \mathcal{R}_\mathbf{a}^{-1})^{-1}] \end{aligned} \tag{21}$$

where $\mathcal{R}_\mathbf{a} = \sigma^{-2}\mathbf{R_a}$ and $\mathbf{K} = diag(0, 1, \ldots, N-1)$. Here we used the relations $\mathbf{\Lambda}^H\mathbf{\Lambda} = \mathbf{I}$, $\mathbf{\Psi}^H\dot{\mathbf{\Psi}} = j\mathbf{\Phi}^H\mathbf{K}\mathbf{\Phi}$, $\dot{\mathbf{\Psi}}^H\dot{\mathbf{\Psi}} = \mathbf{\Phi}^H\mathbf{K}^2\mathbf{\Phi}$ and the fact that $\mathbf{R}_a$ and $\mathbf{\Gamma}$ are Hermitian matrices.

*(b) Additive white noise, pilot signals with a diagonal correlation matrix, and independent fadings of path amplitudes*

In this case, we have $\mathbf{R_a} = diag(\sigma_1^2, \ldots, \sigma_M^2)$ and $\mathcal{T} = \sigma^{-2}diag(\gamma_{11}, \ldots, \gamma_{MM})$. From (21) we get the CRLB

$$\begin{aligned} CRB_\omega^{-1} &= 2\sum_{m=1}^{M} \frac{\gamma_{mm}\left(\frac{\sigma_m^2}{\sigma^2}\right)^2}{\gamma_{mm}\left(\frac{\sigma_m^2}{\sigma^2}\right)+1} \sum_{t=0}^{N-1} t^2|\varphi_m(t)|^2 \\ &- 2\sum_{m=1}^{M}\sum_{l=1}^{M} \frac{\frac{\sigma_m^2}{\sigma^2}\cdot\frac{\sigma_l^2}{\sigma^2}}{\gamma_l\frac{\sigma_l^2}{\sigma^2}+1} \left|\sum_{t=0}^{N-1} t\varphi_m^*(t)\varphi_l(t)\right|^2 \end{aligned} \tag{22}$$

If, additionally, $\mathbf{R}_a = \sigma_a^2\mathbf{I}$ and $\mathbf{\Gamma} = \gamma\mathbf{I}$, then

$$\begin{aligned} CRB_\omega^{-1} &= 2\frac{\gamma\left(\frac{\sigma_a^2}{\sigma^2}\right)^2}{\gamma\frac{\sigma_a^2}{\sigma^2}+1} \left[\sum_{m=1}^{M}\sum_{t=0}^{N-1} t^2|\varphi_m(t)|^2 \right. \\ &\left. - \frac{1}{\gamma}\sum_{m=1}^{M}\sum_{l=1}^{M}\left|\sum_{t=0}^{N-1} t\varphi_m^*(t)\varphi_l(t)\right|^2\right]. \end{aligned} \tag{23}$$

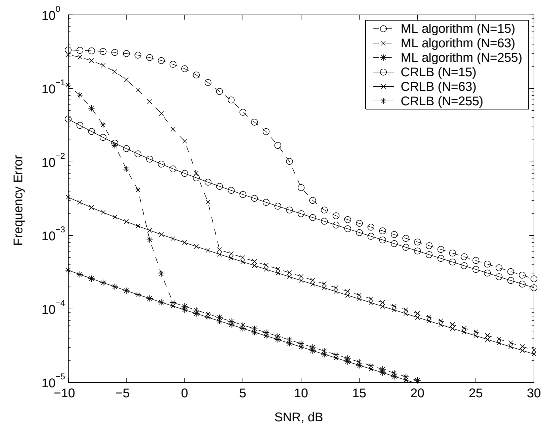


**Fig. 1**. *Dependence of the frequency error on SNR.*

*(c) Additive white noise and frequency-flat fading*

For the case of single path propagation (frequency-flat fading) when $z(t) = as(t)e^{j\omega t} + n(t)$ we get from (23)

$$CRB_\omega^{-1} = 2\frac{\gamma\left(\frac{\sigma_a^2}{\sigma^2}\right)^2}{\gamma\frac{\sigma_a^2}{\sigma^2}+1}\left[\sum_{t=0}^{N-1} t^2|s(t)|^2 - \frac{1}{\gamma}\left(\sum_{t=0}^{N-1} t|s(t)|^2\right)^2\right] \tag{24}$$

where $\gamma = \sum_{t=0}^{N-1}|s(t)|^2$. When $|s(t)| \equiv 1$, taking into account that $\gamma = N$, we get

$$CRB_\omega^{-1} = \frac{N\left(\frac{\sigma_a^2}{\sigma^2}\right)^2}{N\frac{\sigma_a^2}{\sigma^2}+1}\frac{N(N^2-1)}{6}. \tag{25}$$

When $SNR \gg (1/N)$, the CRLB is approximately

$$CRB_\omega^{-1} = SNR\frac{N(N^2-1)}{6} \tag{26}$$

where $SNR = \sigma_a^2/\sigma^2$.

## 5. SIMULATION RESULTS

We consider the following simulation scenario. A binary maximum length sequence (m-sequence), representing a pilot signal, is transmitted through a Rayleigh multipath channel with independent fluctuations of path amplitudes and with additive white noise. The length of the sequence is $N = 15$, $N = 63$ or $N = 255$. We simulate $M = 6$ of paths with the amplitude variances equal to each other. We calculate the frequency error $\Delta f = \sqrt{\frac{1}{N_{mc}}\sum_{i=1}^{N_{mc}}(f - \hat{f}_i)^2}$ where $f = \omega/2\pi$ is an analysed frequency and $\hat{f}_i = \hat{\omega}_i/2\pi$ is a frequency estimate in the $i$-th simulation trial; $N_{mc} = 10000$ is the total number of simulation trials. Fig.1 shows the dependence of the frequency error on SNR for $f = 0.2$ for the ML frequency estimator (15) and the CRLB (21). It can be seen that at high SNRs the frequency error approaches the CRLB for all the sequence lengths.

## 6. CONCLUSIONS

We have considered the frequency estimation in slowly fading multipath channels. We have derived a ML algorithm and CRLB for a multipath fading channel with additive Gaussian noise and path amplitudes having Rayleigh distribution when covariance matrices of the fading and noise are known. We have also considered particular cases of the additive white noise, independent fading of path amplitudes, pilot signals with a diagonal correlation matrix and frequency flat fading. The ML frequency estimator exploits multipath diversity by combining periodograms of multipath signal components and searching for the maximum of the combined statistic. The simulation has shown that at high SNRs the variance of the frequency error for the ML frequency estimator approaches the CRLB.

For the sake of simplicity we have restricted attention to slowly fading channels, assuming the channel parameters to be constant over the observation interval. However, the results obtained are also applicable to fast fading multipath channels. Note that for such channels the complex envelope $A(t)$ can be represented as [10]

$$A(t) = \sum_{p=1}^{P}\sum_{m=1}^{M} a_{mp} s(t-\tau_m) e^{-j\omega_p t} = \sum_{q=1}^{Q} a_q \varphi_q(t) \quad (27)$$

where $q = (m,p)$ is a multiindex, $Q = MP$, $\varphi_q(t) = s(t-\tau_m)e^{-j\omega_p t}$ are basis functions for expansion of the complex envelope, $\omega_p \in \Omega_d$ and $\Omega_d$ is the frequency range of Doppler spreading. Using the basis functions $\{\varphi_q(t)\}_{q=1}^{Q}$ instead of $\{\varphi_m(t)\}_{m=1}^{M}$ for calculation of the matrix $\mathbf{\Phi}$ and substituting $Q$ instead of $M$ we get ML frequency estimation algorithms and the CRLB for fast fading channels.

## 7. REFERENCES


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## 8. APPENDIX

Let the complex-valued vector $\mathbf{z}$ in (2) have the Gaussian distribution with zero mean and covariance matrix $\mathbf{R_z}$ and $\omega$ is an unknown frequency. In our case the Fisher information matrix contains only one element [8]

$$I_{\omega\omega} = tr\left\{\left(\mathbf{R_z^{-1}}\frac{\partial \mathbf{R_z}}{\partial\omega}\right)^2\right\} = -tr\left\{\frac{\partial \mathbf{R_z}}{\partial\omega}\frac{\partial \mathbf{R_z^{-1}}}{\partial\omega}\right\}. \quad (28)$$

Since the vectors $\mathbf{a}$ and $\mathbf{n}$ are statistically independent we have

$$\mathbf{R_z} = \mathbf{\Psi}\mathbf{R_a}\mathbf{\Psi}^H + \mathbf{R_n}. \quad (29)$$

For the sake of simplicity, we use the notation: $\dot{\mathbf{\Psi}} = \frac{\partial}{\partial\omega}\mathbf{\Psi}$. Using (29) and relations $\mathbf{R_z^{-1}} = \mathbf{R_n^{-1}} - \mathbf{R_n^{-1}}\mathbf{\Psi}\mathcal{N}^{-1}\mathbf{\Psi}^H\mathbf{R_n^{-1}}$, $\mathcal{N} = \mathcal{T} + \mathbf{R_a^{-1}}$, $\mathcal{T} = \mathbf{\Psi}^H\mathbf{R_n^{-1}}\mathbf{\Psi}$, $\Upsilon = \mathbf{\Psi}^H\mathbf{R_n^{-1}}\dot{\mathbf{\Psi}}$ and $\mathcal{W} = \dot{\mathbf{\Psi}}^H\mathbf{R_n^{-1}}\dot{\mathbf{\Psi}}$ we obtain the following expressions for derivatives:

$$\frac{\partial \mathbf{R_z}}{\partial\omega} = \dot{\mathbf{\Psi}}\mathbf{R_a}\mathbf{\Psi}^H + \mathbf{\Psi}\mathbf{R_a}\dot{\mathbf{\Psi}}^H, \quad (30)$$

$$\begin{aligned}\frac{\partial \mathbf{R_z^{-1}}}{\partial\omega} &= -\mathbf{R_n^{-1}}\dot{\mathbf{\Psi}}\mathcal{N}^{-1}\mathbf{\Psi}^H\mathbf{R_n^{-1}} - \mathbf{R_n^{-1}}\mathbf{\Psi}\mathcal{N}^{-1}\dot{\mathbf{\Psi}}^H\mathbf{R_n^{-1}} \\ &+ \mathbf{R_n^{-1}}\mathbf{\Psi}\mathcal{N}^{-1}\dot{\mathcal{T}}\mathcal{N}^{-1}\mathbf{\Psi}^H\mathbf{R_n^{-1}}. \end{aligned} \quad (31)$$

Then, substituting (29)-(31) in (28) we get

$$\begin{aligned}
&\mathbf{I}_{\omega\omega} = tr\{[\dot{\mathbf{\Psi}}\mathbf{R_a}\mathbf{\Psi}^H + \mathbf{\Psi}\mathbf{R_a}\dot{\mathbf{\Psi}}^H][\mathbf{R_n^{-1}}\dot{\mathbf{\Psi}}\mathcal{N}^{-1}\mathbf{\Psi}^H\mathbf{R_n^{-1}} \\
&+ \mathbf{R_n^{-1}}\mathbf{\Psi}\mathcal{N}^{-1}\dot{\mathbf{\Psi}}^H\mathbf{R_n^{-1}} - \mathbf{R_n^{-1}}\mathbf{\Psi}\mathcal{N}^{-1}\dot{\mathcal{T}}\mathcal{N}^{-1}\mathbf{\Psi}^H\mathbf{R_n^{-1}}]\} \\
&= tr\{[\dot{\mathbf{\Psi}}\mathbf{R_a}\mathbf{\Psi}^H\mathbf{R_n^{-1}}\dot{\mathbf{\Psi}}\mathcal{N}^{-1}\mathbf{\Psi}^H\mathbf{R_n^{-1}} + \mathbf{\Psi}\mathbf{R_a}\dot{\mathbf{\Psi}}^H\mathbf{R_n^{-1}}\dot{\mathbf{\Psi}}\mathcal{N}^{-1}\mathbf{\Psi}^H\mathbf{R_n^{-1}} \\
&+ \dot{\mathbf{\Psi}}\mathbf{R_a}\mathbf{\Psi}^H\mathbf{R_n^{-1}}\mathbf{\Psi}\mathcal{N}^{-1}\dot{\mathbf{\Psi}}^H\mathbf{R_n^{-1}} + \mathbf{\Psi}\mathbf{R_a}\dot{\mathbf{\Psi}}^H\mathbf{R_n^{-1}}\mathbf{\Phi}\mathcal{N}^{-1}\dot{\mathbf{\Psi}}^H\mathbf{R_n^{-1}} \\
&- \dot{\mathbf{\Psi}}\mathbf{R_a}\mathbf{\Psi}^H\mathbf{R_n^{-1}}\mathbf{\Psi}\mathcal{N}^{-1}\dot{\mathcal{T}}\mathcal{N}^{-1}\mathbf{\Psi}^H\mathbf{R_n^{-1}} \\
&- \mathbf{\Psi}\mathbf{R_a}\dot{\mathbf{\Psi}}^H\mathbf{R_n^{-1}}\mathbf{\Psi}\mathcal{N}^{-1}\dot{\mathcal{T}}\mathcal{N}^{-1}\mathbf{\Psi}^H\mathbf{R_n^{-1}}]\} \\
&= tr\{[\mathbf{\Upsilon}\mathcal{N}^{-1}\mathbf{\Upsilon}\mathbf{R_a} + \mathcal{W}\mathcal{N}^{-1}\mathcal{T}\mathbf{R_a} + \mathbf{R_a}\mathcal{T}\mathcal{N}^{-1}\mathcal{W} \\
&+ \mathbf{\Upsilon^H}\mathcal{N}^{-1}\mathbf{\Upsilon^H}\mathbf{R_a} - \mathcal{T}\mathcal{N}^{-1}\dot{\mathcal{T}}\mathcal{N}^{-1}\mathbf{\Upsilon}\mathbf{R_a} - \mathbf{\Upsilon^H}\mathcal{N}^{-1}\dot{\mathcal{T}}\mathcal{N}^{-1}\mathcal{T}\mathbf{R_a}]\} \\
&= 2Re\{tr[\mathbf{\Upsilon}\mathcal{N}^{-1}\mathbf{\Upsilon}\mathbf{R_a} + \mathbf{R_a}\mathcal{T}\mathcal{N}^{-1}\mathcal{W} - \mathbf{\Upsilon}\mathbf{R_a}\mathcal{T}\mathcal{N}^{-1}\dot{\mathcal{T}}\mathcal{N}^{-1}]\} \\
&= 2Re\{tr[\mathbf{\Upsilon}\mathbf{R_a}\mathbf{\Upsilon}\mathbf{R_a}(\mathcal{T}\mathbf{R_a}+\mathbf{I})^{-1} + \mathbf{R_a}\mathcal{T}\mathbf{R_a}(\mathcal{T}\mathbf{R_a}+\mathbf{I})^{-1}\mathcal{W} \\
&- \mathbf{\Upsilon}\mathbf{R_a}\mathcal{T}\mathbf{R_a}(\mathcal{T}\mathbf{R_a}+\mathbf{I})^{-1}\dot{\mathcal{T}}\mathbf{R_a}(\mathcal{T}\mathbf{R_a}+\mathbf{I})^{-1}]\}.
\end{aligned}$$

Here we used the following relations:

$\mathcal{T}\mathcal{N}^{-1}\mathbf{R_a^{-1}} = (\mathbf{R_a}(\mathcal{T}+\mathbf{R_a^{-1}})\mathcal{T}^{-1})^{-1} = (\mathbf{R_a}+\mathcal{T}^{-1})^{-1}$,
$\mathbf{R_a^{-1}}\mathcal{N}^{-1}\mathcal{T} = (\mathcal{T}^{-1}(\mathcal{T}+\mathbf{R_a^{-1}})\mathbf{R_a})^{-1} = (\mathbf{R_a}+\mathcal{T}^{-1})^{-1}$,
$\mathcal{N}^{-1} = (\mathcal{T}+\mathbf{R_a^{-1}})^{-1} = ((\mathcal{T}\mathbf{R_a}+\mathbf{I})\mathbf{R_a^{-1}})^{-1} = \mathbf{R_a}(\mathcal{T}\mathbf{R_a}+\mathbf{I})^{-1}$,
$\mathbf{R_a}\mathcal{T}\mathcal{N}^{-1} = \mathbf{R_a}\mathcal{T}\mathbf{R_a}(\mathcal{T}\mathbf{R_a}+\mathbf{I})^{-1}$.