Localisation for random perturbations of periodic Schrödinger operators with regular Floquet eigenvalues*

Ivan Veselić

Fakultät für Mathematik,
Ruhr-Universität Bochum,
D-44780 Bochum, Germany,
and SFB 237
ivan@mathphys.ruhr-uni-bochum.de

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Abstract

We prove a localisation theorem for continuous ergodic Schrödinger operators $H_\omega := H_0 + V_\omega$, where the random potential V_ω is a nonnegative Anderson-type random perturbation of the periodic operator H_0. We consider a lower spectral band edge of $\sigma(H_0)$, say $E = 0$, at a gap which is preserved by the perturbation V_ω. Assuming that all Floquet eigenvalues of H_0, which reach the spectral edge 0 as a minimum, have there a positive definite Hessian, we conclude that there exists an interval I containing 0 such that H_ω has only pure point spectrum in I for almost all ω.

1 Introduction and results

Localisation

Already in the fifties Anderson [1] concluded by physical reasoning that some random quantum Hamiltonians on a lattice should exhibit *localisation* in certain energy regions. That is to say that the corresponding self-adjoint operator has pure point spectrum in these energy intervals.

Since then mathematical physicists developed a machinery to prove rigorously this phenomenon from solid state physics. Most of them used the so-called

*Up to some minor modifications this is preprint no. 98-569 (August 21,1998) of http://www.ma.utexas.edu/mp_arc *multiscale analysis* (MSA) introduced in a paper by Fröhlich and Spencer [14] to prove a weaker form of localisation at low energies for the discrete analogue of the Schrödinger operator. This quite complicated reasoning was streamlined by von Dreifus and Klein [41]. The underlying lattice structure made the MSA easier to apply to discrete Hamiltonians but soon adaptations for continuous Schrödinger operators followed [28, 22, 6, 24]. We prove in Theorem 1.1 a localisation result for energies near internal spectral edges of a periodic Schrödinger operator H_0 which is perturbed by an Anderson-type potential V_ω. Unlike [2, 20] our results are not restricted to a special disorder regime of the random coupling constants in V_ω. Instead we assume that the periodic operator H_0 has regular Floquet eigenvalues. This behaviour is commonly assumed among physicists. Recently Klopp and Ralston proved that it is generic [26].

In the remainder of this section we introduce our model, state the main Theorem 1.1 and the technical Proposition 1.2 on which it is based. Section 2 explains how to deduce Theorem 1.1 from Proposition 1.2, in Section 3 we describe the functional calculus with almost analytic functions, Section 4 contains a comparison lemma for the integrated density of states on finite cubes and on the whole of $\mathbb{R}^d$ and the last Section 5 completes the proof of Proposition 1.2.

The model

On the Hilbert space $L^2(\mathbb{R}^d)$ we consider a self-adjoint operator $H := H_\omega$ made up of a periodic Schrödinger operator H_0 and a random perturbation V_ω

$$H_\omega := H_0 + V_\omega \ . \tag{1}$$

Here $H_0 := -\Delta + V_0$ is the sum of the negative Laplacian and a $\mathbb{Z}^d$-periodic potential $V_0 \in L^p_{loc}(\mathbb{R}^d)$ with $p = 2$ if $d \leq 3$, $p > 2$ if $d = 4$ and $p \geq d/2$ if $d \geq 5$. Such a potential is an infinitesimal perturbation of $-\Delta$ so the sum is self-adjoint with domain $D(-\Delta) = W_2^2(\mathbb{R}^d)$, the Sobolev space of L^2-functions whose second derivative is also in L^2 (cf. [33, 34]). The random perturbation is of *Anderson type*

$$V_\omega(x) := \sum_{k\in\mathbb{Z}^d} \omega_k \, u(x-k) \ , \tag{2}$$

where $(\omega_k)_{k\in\mathbb{Z}^d}$ is a collection of independent identically distributed (i.i.d.) random variables, called *coupling constants*. Their distribution has a bounded density with support $[0, \omega_{\max}]$ for some $\omega_{\max} > 0$. The non-negative *single site potential* u has to decay exponentially and have an uniform lower bound on some open subset of $\mathbb{R}^d$, more precisely

$$u \geq \delta_1 \chi_\Lambda, \ \delta_1 > 0 \text{ where } \Lambda := \Lambda_s := \{x \in \mathbb{R}^d | \ \|x\|_\infty < s/2\}, \ s > 0$$

and

$$\|\chi_{\Lambda_1} u(\cdot - l)\|_{L^p} \leq \delta_2 \, e^{-\delta_3 l}, \ \delta_2, \delta_3 > 0 \ . \tag{3}$$

H_ω is an ergodic operator and we infer from [18, 4] or [32] that there exists a set $\sigma \subset \mathbb{R}$ such that $\sigma = \sigma(H_\omega)$ for almost all $\omega \in \Omega$, i.e. the spectrum of H_ω is almost surely non-random. In the same sense σ_{ac}, σ_{sc} and σ_{pp} are ω-independent subsets of the real line.

Under some mild assumptions the periodic background operator H_0 has a spectrum with *band structure*, i.e. $\sigma(H_0) = \bigcup_{n\in\mathbb{N}}[E_n^-, E_n^+]$, $E_1^- \le E_1^+ \le E_2^- \le \ldots$, where for some n we have open spectral gaps, i.e. $E_n^+ < E_{n+1}^-$ (cf. [9, 38, 34]). We assume that there exist positive numbers a, b and b' with

$$[0,a] \subset \sigma(H_0), \; [-b,0[\subset \rho(H_0) \text{ and } [-b',0[\subset \rho(H_\omega)\,.$$

Since 0 is in the support of the density of ω_0 it follows that $0 \in \sigma(H_\omega)$. In this case we say that 0 is a *lower band edge* of the periodic operator, which is preserved by the positive random perturbation V_ω.

H_0 can be decomposed into a *direct integral* via an unitary transformation U (cf. [38, 34])

$$U H_0 U^* = \int_{[-\pi,\pi]^d}^{\oplus} H_0|_{\Lambda_1}^{\theta}\, d\theta \;.$$

Here $H_0|_{\Lambda_1}^{\theta}$ is the same formal differential expression as H_0 acting on functions $f \in W_2^2(\Lambda_1)$ with *θ-boundary conditions*, i.e. for all $j = 1, \ldots, d$ we have a phase shift in the corresponding direction: $f(x+e_j) = e^{i\theta_j} f(x)$ where $x_j = -1/2$. It is an operator with discrete spectrum, which consists of the so-called *Floquet eigenvalues*

$$E_1(\theta) \le \ldots \le E_n(\theta) \le \ldots \qquad n \in \mathbb{N} \;.$$

These are Lipshitz-continuous on $[-\pi,\pi]^d$. In fact they "generate" the bands of the spectrum of H_0

$$\sigma(H_0) = \bigcup_{n\in\mathbb{N}} \bigcup_{\theta\in[-\pi,\pi]^d} E_n(\theta) \;.$$

There is a finite set of indices $\mathcal{N} \subset \mathbb{N}$ (cf. [38]) such that

$$E_n(\theta) = 0 \text{ for some } \theta \in [-\pi,\pi]^d \Longrightarrow \; n \in \mathcal{N} \;.$$

Since 0 is a lower band edge of $\sigma(H_\omega)$, $E_n(\theta) = 0$ has to be a minimum of $E_n(\cdot)$. If for all $n \in \mathcal{N}$, $E_n(\cdot)$ has only quadratic minima at 0 (i.e. the Hessian of $E_n(\cdot)$ at any minimum with value 0 is positive definite) we say that H_0 has *regular Floquet eigenvalues at* 0.

Results

Our result on localisation at an upper internal spectral band edge is the following

Theorem 1.1 *If H_0 has regular Floquet eigenvalues at 0 and H_ω is constructed as above, then there exists a number $E_0 > 0$ such that*

$$\sigma(H_\omega) \cap [0, E_0] \subset \sigma_{pp}(H_\omega) \ .$$

The proof of the theorem is based on the following proposition.

Proposition 1.2 *If H_0 has regular Floquet eigenvalues at 0 and H_ω is constructed as above, we have for all $q > 0$:*
There exists a $l_0 := l_0(q) \in \mathbb{N}$ and an $\alpha \in]0, 1/4[$ such that for all $l \geq l_0$

$$\mathbb{P}\{\omega | \ \sigma(H_\omega|^\theta_{\Lambda_l}) \cap [0, l^{-\alpha}[\neq \emptyset \ \} \leq l^{-q}$$

The proof of Proposition 1.2 is given in sections 3 to 5. It uses the existence of *Lifshitz-tails* of the *integrated density of states* (IDS) of the ergodic operator H_ω if H_0 has regular Floquet eigenvalues, which was proved by Klopp in [25], who also noted that his result could be used for a localisation proof.

Theorem 1.1 is proved using the MSA. Since this technique is well understood by now [2, 20] we only sketch it to show how Proposition 1.2, which is the main technical novelty of this paper, enters. This is done in Section 2, where also a discussion of previous results can be found.

Remark 1.3 At any lower band edge one can prove localisation under the analogous assumptions. Here $E = 0$ was chosen only for notational simplicity. If the Anderson-type perturbation V_ω is negative our theorem can be used to establish localisation on any upper band edge with regular Floquet eigenvalues.

If the underlying $\mathbb{Z}^d$ is replaced by some other Euclidean lattice

$$\Gamma := \{\gamma \in \mathbb{R}^d | \ \gamma = \sum_{j=1}^{d} \beta_j a_j, \beta \in \mathbb{Z}^d\} \ ,$$

where $\{a_j\}_{j=1}^d$ is a basis of $\mathbb{R}^d$, the same theorem and proposition are valid by a simple modification of the proofs.

In any case we will use the maximum norm when considering lattice points k or γ in $\mathbb{Z}^d$ or Γ, i.e. $|\gamma| := \|\gamma\|_\infty := \max\{|\gamma_j|, \ j = 1, \ldots, d\}$, where $(\gamma_1, \ldots, \gamma_d) \in \mathbb{R}^d$ are the components of γ.

An inspection of our proofs and the papers [25, 21] and [20] shows that Proposition 1.2 and Theorem 1.1 extend to single site potentials u with sufficiently fast polynomial decay (in L^p-sense).

Example 1.4 Finally we give an example of a periodic operator which has only regular Floquet eigenvalues at all band edges. Thus we know that our condition in the above theorem is fulfilled and we can prove localisation at any lower band edge. Let V_0 satisfy the conditions posed above on the periodic potential and let it be a sum of potentials V_j which are periodic in the jth coordinate direction and constant in all the others; more precisely

$$V_0(x) := \sum_{j=1}^{d} V_j(x_j)$$

where $V_j : \mathbb{R} \to \mathbb{R}$ is a periodic function and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. Then both H_0 and $H_0|_{\Lambda_1}^{\theta}$ can be decomposed into a direct sum of one-dimensional operators. For these it is known that all Floquet eigenvalues are regular [9, 25]. As the eigenvalues of the direct sum are just sums of the eigenvalues of the one-dimensional operators it is clear that the former also have to be regular.

Corollary 1.5 *Let the ergodic operator $H_\omega := -\Delta + V_0 + V_\omega$ be constructed as above and the periodic potential be decomposable, i.e.*

$$V_0(x) := \sum_{j=1}^{d} V_j(x_j) \ .$$

Let E be a lower spectral band edge of the periodic operator $H_0 := -\Delta + V_0$ at a spectral gap which is not closed by the perturbation V_ω. Then there exists an interval $I \ni E$ such that

$$\sigma(H_\omega) \cap I \subset \sigma_{pp}(H_\omega) \ .$$

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2 Multiscale analysis and associated ideas

In this section we explain how Theorem 1.1 is deduced from Proposition 1.2 and discuss previous localisation results.

An intermediary step in the proof of localisation is the establishing of the exponential decay of the resolvent

$$\sup_{\epsilon \neq 0} \|\chi_x R(\epsilon) \chi_y\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq C \exp(-c|x-y|) \text{ for almost all } \omega \ , \tag{4}$$

where $R := R(\epsilon) := (H_\omega - E - i\epsilon)^{-1}$ is the resolvent of H_ω near an energy value E in the energy interval $I \in \mathbb{R}$ for which we want to prove localisation. The χ_x and χ_y are characteristic functions of unit cubes centered at x, respectively at y. This bound can be used to rule out absolutely continuous spectrum [29] and is interpreted as absence of diffusion [14, 28] in the energy region I if (4) holds for all $E \in I$.

It turns out that the finite size resolvent $R_\Lambda(\epsilon) := (H_\omega|_\Lambda - E - i\epsilon)^{-1}$ is easier approachable than $R(\epsilon)$ on the whole space. Here $H_\omega|_\Lambda$ is the restriction of H_ω to $L^2(\Lambda)$ with some appropriate boundary conditions (b.c.); the use of Dirichlet or periodic b.c. is most common. However the operator $H_\omega|_\Lambda$ is not ergodic and for its resolvent an estimate like (4) can be expected to hold only with a probability strictly smaller than one. This is the place where MSA enters. It is an induction argument over increasing length scales l_j. They are defined recursively by $l_{j+1} := [l_j^\zeta]_3$, where $[l_j^\zeta]_3$ is the greatest multiple of 3 smaller than l_j^ζ. The scaling exponent ζ has to be from the interval $]1,2[$. On each scale one considers the box resolvent $R_j(\epsilon) := R_{\Lambda_j}(\epsilon)$ and proves its exponential decay with a probability which tends to 1 as $j \to \infty$. We outline briefly the ingredients of the MSA as it is given in [6, 20] or [4].

First we explain some notation which is used afterwards. Let $\delta > 0$ be a small constant independent of the length scale l_j and $\phi_j(x) \in C^2$ a function which is identically equal to 0 for x with $\|x\|_\infty > l_j - \delta$ and identically equal to one for x with $\|x\|_\infty < l_j - 2\delta$. The commutator $W(\phi_j) := [-\Delta, \phi_j] := -(\Delta\phi_j) - 2(\nabla\phi_j)\nabla$ is a local operator acting on functions which live on a ring of width δ near the boundary of $\Lambda_j := \Lambda_{l_j}$. We say that a pair $(\omega, \Lambda_j) \in \Omega \times \mathcal{B}(\mathbb{R}^d)$ is *m-regular*, if

$$\sup_{\epsilon \neq 0} \|W(\phi_j) R_j(\epsilon) \chi_{l_j/3}\|_{\mathcal{L}} \leq e^{-ml_j} \quad . \tag{5}$$

Here $\|\cdot\|_{\mathcal{L}}$ is the operator norm on $L^2(\Lambda_j)$ and $\chi_{l_j/3}$ the characteristic function of $\Lambda_{l_j/3} := \{y | \, \|y\|_\infty \leq l_j/6\}$. Thus the distance of the supports of $\nabla\phi_j$ and $\chi_{l/3}$ is at least $l_j/3 - 2\delta \geq l_j/4$.

Let $q_0 > 0$. The starting point of the MSA is the estimate

$$\text{(H1)}(l_0, m_0, q_0) \qquad \mathbb{P}\{\omega | \, (\omega, \Lambda_0) \text{ is } m_0\text{-regular}\} \geq 1 - l_0^{q_0}$$

which serves as the base clause of the induction. The induction step consists in proving

$$\text{(H1)}(l_j, m_j, q_j) \Longrightarrow \text{(H1)}(l_{j+1}, m_{j+1}, q_{j+1}) \tag{6}$$

For the mass of decay m_{j+1} and the probability exponent q_{j+1} on the scale l_{j+1} the following estimates are valid

$$\forall \xi > 0 \; \exists c_1, c_2, c_3 \text{ independent of } j \text{ such that}$$

$$m_{j+1} \geq m_j \left(1 - \frac{4l_j}{l_{j+1}}\right) - \frac{c_1}{l_j} - c_2 \frac{\log l_{j+1}}{l_{j+1}} \tag{7}$$

$$l_{j+1}^{q_{j+1}} \leq c_3 \left(\frac{l_{j+1}}{l_j}\right)^{2d} l_j^{2q_j} + \frac{1}{2} l_{j+1}^{-\xi} \; . \tag{8}$$

For the recursion clause (6) a *Wegner estimate* [42] is needed:

$$\text{(H2)} \qquad \mathbb{P}\{\omega | \, d(\sigma(H_\omega|_\Lambda), E) \leq \eta\} \leq C_W \, \eta |\Lambda|^2$$

for all boxes $\Lambda \subset \mathbb{R}^d$ and all $\eta > 0$, such that $[E-\eta, E+\eta]$ is contained in a suitable small energy interval near the spectral band edge (cf. Theorem 3.1 in [20]). Here $|\Lambda|$ stands for the Lebesgue measure of the cube Λ.

The deterministic part of the induction step uses the *geometric resolvent formula* [6, 16]

$$\phi_\Lambda (H_{\Lambda'} - z)^{-1} = (H_\Lambda - z)^{-1}\phi_\Lambda + (H_\Lambda - z)^{-1} W(\phi_\Lambda)(H_{\Lambda'} - z)^{-1} \tag{9}$$

for $z \in \rho(H_{\Lambda'}) \cap \rho(H_\Lambda)$ and $\phi_\Lambda \in C^2$ with support in $\Lambda \subset \Lambda'$. It gives the estimate

$$\|\chi_{l/3}(\cdot - x) R_{3l'}(\epsilon) \chi_{l/3}(\cdot - y)\|_{\mathcal{L}} \le (3^d e^{-ml})^{3|x-y|l^{-1}-4} \|R_{3l'}(\epsilon)\|_{\mathcal{L}} \tag{10}$$

if no two disjoint non-regular boxes $\Lambda_l \subset \Lambda_{l'}$ with center in $\frac{l}{3}\mathbb{Z}^d \cap \Lambda_{3l'}$ exist for ω. In our case $l := l_j$ is the length scale on which the exponential decay of the resolvent is already known and $l' := l_{j+1}$ the scale on which we want to prove it. By the estimates (H1),(H2) we have with probability $1 - l_{j+1}^{q_{j+1}}$ (bounded by the inequality (8)) exponential decay on the length scale l_{j+1} with mass m_{j+1} (bounded as in (7)).

We stated above the ingredients of the MSA as they are valid if u is supported in Λ_1. If the single site potential is of long range type as in (3) one has to use the adapted MSA from the paper [20].

Once the estimate (H1) is established on all length scales $l_j, j \in \mathbb{N}$, one infers an exponential decay estimate for the resolvent on the whole of $\mathbb{R}^d$. Afterwards one uses a spectral averaging technique (cf.[6]) based on ideas of Kotani, Simon, Wolf and Howland to conclude localisation [27, 37, 17].

Recent papers concentrate on proofs for the Wegner estimate and the initial length scale decay of the resolvent. At the same time adaptations of the MSA for various random Schrödinger operators, as well as Hamiltonians governing the motion in classical physics appeared [10, 11, 7, 39]. Recently Najar [31] obtained analog results to [25] and the present paper concerning Lifshitz tails and localisation for acoustic operators.

We discuss briefly some results for quantum mechanical Hamiltonians.

In [24] Klopp proved a Wegner lemma for energies at the infimum of the spectrum which applies to an Anderson perturbation V_ω with single site potentials u that are allowed to change sign. For V_ω a Gaussian random field a Wegner estimate was shown in [12]. Its main feature is that no underlying lattice structure of V_ω is needed. This result allows one to conclude localisation for the corresponding Schrödinger operator at low energies [13]. Kirsch, Stollmann and Stolz proved in [20] a Wegner estimate with minimal decay conditions on the single site potential u and deduced a localisation result for Hamiltonians with long range interactions. They require

$$|u(x)| \le C(|x|+1)^{-m} \text{ for some } m > 4d\ . \tag{11}$$

The resolvent decay estimate (H1) for some initial length scale can be proved with semiclassical techniques. Using the Agmon metric one can achieve rigorously decay bounds with what is called among physicists WKB-method [6, 16]. However this reasoning is only applicable for energies near the bottom of the spectrum.

The so-called *Combes-Thomas argument* [5] allows one to infer the following inequality

$$\|\chi_x(H-z)^{-1}\chi_y\|_{\mathcal{L}} \leq C\ d(\sigma(H),z)^{-1}\ e^{-c\,d(\sigma(H),z)|x-y|} \tag{12}$$

where H is a self-adjoint Schrödinger operator on $L^2(\mathbb{R}^d)$ and $z \in \rho(H)$. It was first applied to multiparticle Hamiltonians [5], but it is also useful in our case, as soon as we get a lower bound on $d(\sigma(H_\omega|_\Lambda),z)$. Thus it is sufficient to prove an estimate like

$$\mathbb{P}\{\omega|\ d(\sigma(H_\omega|_{\Lambda_l},I) < \frac{1}{2}\,l^{-\alpha}\ \} \leq l^{-q} \tag{13}$$

for some $\alpha \in]0,1/4]$. Such a bound follows immediately from Proposition 1.2 with $I := [0, \frac{1}{2}\,l^{-\alpha}[$, for $l > (2b')^{-1/\alpha}$. Now Inequality (12) implies the initial scale estimate (H1) with $m_0 \geq c\,l^{-1/4}$ for l large and $E \in I$, cf. [20, Lemma 5.5]. The constant c depends on the energy and the potential, but not on l and m_0.

Two possibilities were used to deduce (13). The first is to assume a special disorder regime, more precisely to demand a sufficiently fast decay of the density g of the distribution of ω near the endpoints 0 and $\omega_{\max}$ of $\operatorname{supp} g$:

$$\begin{aligned} &\exists \tau > d/2\ :\ \forall \text{ small } \epsilon > 0 \\ &\int_0^\epsilon g(s)ds \leq \epsilon^\tau, \text{ respectively } \int_{\omega_{\max}-\epsilon}^{\omega_{\max}} g(s)ds \leq \epsilon^\tau \end{aligned}$$

depending on whether one wants to consider a lower or upper band edge. This approach was used in [6, 20]. Its shortcoming is that it excludes quite a few distributions, e.g. the uniform distribution on $[0,\omega_{\max}]$.

The other way to prove (13), which we pursue, is to use the existence of Lifshitz tails of the integrated density of states at the edges of the spectrum. One defines the IDS usually as follows:

$$N(E) \ := \lim_{\Lambda\nearrow\mathbb{R}^d} N(H_\omega|_\Lambda^D, E) \tag{14}$$

$$:= \lim_{\Lambda\nearrow\mathbb{R}^d} |\Lambda|^{-1}\#\{\text{ eigenvalues of } H_\omega|_\Lambda^D \text{ below } E\}\ , \tag{15}$$

i.e. as the limit of the normed counting function of eigenvalues of a box Hamiltonian. Here $H_\omega|_\Lambda^D$ is the restriction of H_ω to $L^2(\Lambda)$ with Dirichlet b.c. As $H_\omega|_\Lambda^D$ has compact resolvent and hence discrete spectrum, definition (14) makes sense. $N(E)$ is almost surely ω-independent and the use of Dirichlet b.c. in its definition implies [19]

$$N(E) := \sup_{\Lambda\nearrow\mathbb{R}^d} N(H_\omega|_\Lambda^D, E)\ . \tag{16}$$

One says that $N(\cdot)$ exhibits Lifshitz tails at some spectral edge $\mathcal{E}$ if

$$\lim_{E\to\mathcal{E}} \frac{\log|\log|N(E)-N(\mathcal{E})||}{\log|E-\mathcal{E}|} = -\frac{d}{2}\ . \tag{17}$$

At the infimum of the spectrum, i.e. for $\mathcal{E} = \inf \sigma(H_\omega)$, (16) and (17) imply

$$\#\{\text{eigenvalues of } H_\omega|_\Lambda^D \text{ in } [\mathcal{E}, E]\} \le |\Lambda| N(E) \le |\Lambda| \exp(-cE^{-d/4})$$

since $N(\mathcal{E}) = 0$. This estimate was used in [24] together with a Čebišev inequality to prove (H1) at the bottom of the spectrum.

If one considers an internal band edge $\mathcal{E}$, Lifshitz asymptotics are not so easy to exploit since (16) cannot be directly used to bound

$$|N(H_\omega|_\Lambda, \mathcal{E}) - N(H_\omega|_\Lambda, E)| \ .$$

Therefore a comparison technique between $N(\cdot)$ and $N(H_\omega|_\Lambda, \cdot)$ is needed. In the one-dimensional case Mezincescu [30] proved Lifshitz tails at internal band edges as well as a comparison lemma for the IDS (Lemma 2, in Section 4). This proof relies on the delicate analysis of Dirichlet eigenfunctions of $H_\omega|_\Lambda$ and their roots. The results in [30] make a localisation proof in the one-dimensional case possible [40].

We prove in Section 4 an approximation lemma for the IDS of the multi-dimensional operator H_ω, which enables us to prove Proposition 1.2. In our case however periodic b.c. seem to be more efficient than Dirichlet b.c. since H_ω is a perturbation of a periodic operator.

In [25] it was proved that the IDS of H_ω exhibits Lifshitz asymptotics at a lower band edge $\mathcal{E}$ if before the perturbation V_ω the Floquet eigenvalues of the periodic background operator H_0 at $\mathcal{E}$ were regular. Thus our approximation lemma can be applied to conclude localisation.

3 Functional calculus with almost analytic functions

In this section we introduce the Helffer-Sjöstrand formula (18) which is exploited in Section 4 to prove the IDS approximation lemma.

For an self-adjoint operator on $L^2(\mathbb{R}^d)$ and a complex-valued measurable function $f : \mathbb{R} \to \mathbb{C}$ one can define the operator

$$f(A) \text{ with domain } D(f(A)) := \{\psi \in L^2(\mathbb{R}^d) |\ f(A)\psi \in L^2(\mathbb{R}^d)\}$$

via the spectral theorem. The latter is normally proved using Riesz' representation theorem for $C(K)^*$, where K is a compact metric space, and the Cayley-transform if A is unbounded. Helffer and Sjöstrand [15] proved the following representation formula

$$f(A) := \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}}(z)(z - A)^{-1} dz \wedge d\tilde{z} \tag{18}$$

if f is smooth and compactly supported. Here $\tilde{f} : \mathbb{C} \to \mathbb{C}$ denotes an *almost analytic extension* of $f : \mathbb{R} \to \mathbb{C}$. Davies [8] uses equation (18) as a starting point to develop systematically a functional calculus equivalent to the standard one. For further details on the material of this section see his book.

Definition 3.1 *For $n \in \mathbb{N}$ and $f \in C_0^n(\mathbb{R}, \mathbb{C})$ define the almost analytic extension (of order n) $\tilde{f} : \mathbb{C} \to \mathbb{C}$ by*

$$\tilde{f}(x,y) := \tilde{f}_n(x,y) := \left(\sum_{r=0}^{n} f^{(r)}(x) \frac{(iy)^r}{r!} \right) s(x,y) \ , \tag{19}$$

where we used the convention $z := x + iy := (x,y) \in \mathbb{C}$. The cutoff function s is defined with the abbreviation $\langle x \rangle := \sqrt{x^2+1}$ by the formula

$$s(x,y) := t\left(\frac{y}{\langle x \rangle} \right) , \ t \in C_0^\infty(\mathbb{R})$$

with $t(x) = 0$ for $|x| > 2$, $t(x) = 1$ for $|x| < 1$ and $\|t'\|_\infty \leq 2$.

With this choice of the almost analytic extension formula (18) holds true. If the support of f is contained in $[-R, R]$, $\tilde{f}$ vanishes outside the set $\{z \in \mathbb{C} | \ x \in \operatorname{supp} f, |y| < 2R + 2\}$. A calculation of the derivatives shows

$$\begin{aligned} \frac{\partial \tilde{f}_n}{\partial \bar{z}}(z) &= \frac{1}{2} \left(\frac{\partial \tilde{f}_n}{\partial x} + i \frac{\partial \tilde{f}_n}{\partial y} \right)(z) \\ &= \frac{1}{2} f^{(n+1)}(x) \frac{(iy)^n}{n!} \ s(x,y) + \frac{1}{2}(s_x(x,y) + i s_y(x,y)) \ \sum_{r=0}^{n} f^{(r)}(x) \frac{(iy)^r}{r!} \ . \end{aligned} \tag{20}$$

By calculating the partial derivatives of s we see

$$|s_x + i s_y| \leq \frac{6}{\langle x \rangle} \chi_{\{\langle x \rangle < |y| < 2\langle x \rangle\}} \ , \tag{21}$$

which shows that they vanish for $|y| \leq 1$ since always $\langle x \rangle = \sqrt{x^2+1} \geq 1$. Putting the bounds together we get

$$\left| \frac{\partial \tilde{f}_n}{\partial \bar{z}}(x,y) \right| \leq \frac{1}{2n!} |f^{(n+1)} s| \ |y|^n + \frac{3}{\langle x \rangle} \chi_{\{\langle x \rangle < |y| < 2\langle x \rangle\}} \sum_{r=0}^{n} |f^{(r)}| \frac{|y|^r}{r!} \ . \tag{22}$$

Later on f will be an approximation of the characteristic function $\chi_{[0,E]}$. It is going to have support inside $[-E/2, 2E]$ and be equal to 1 on $[0, E]$. One can choose f in such a way that $\|f^{(n)}\|_\infty \leq C E^{-n}$ and

$$|||f|||_n := \sum_{i=1}^{n} \|f^{(n)}\|_\infty \leq \tilde{C} E^{-n} \tag{23}$$

for sufficiently small E. The constants $C, \tilde{C}$ are independent of E.

4 IDS approximation lemma

In this section we bound the difference of the IDS of the ergodic operator H_ω and its periodic approximation $H_{\omega,l}$ defined by

$$H_{\omega,l}(x) := H_0(x) + \sum_{k \in \mathbb{Z}^d} \omega_{\tilde{k}}\, u(x-k) \tag{24}$$

where $\tilde{k} := k \,(\mathrm{mod}(2l+1)\mathbb{Z}^d)$. For any $l \in \mathbb{N}$ and $\omega \in \Omega$ it is a (2l+1)$\mathbb{Z}^d$-periodic operator. Our assumptions on u and ω ensure that it is an infinitesimally small perturbation of H_0, uniformly in l and ω. Hence it is a lower bounded symmetric operator which is self-adjoint on the domain $W_2^2(\mathbb{R}^d)$. Its IDS is defined by (cf. [34, 35, 38])

$$N_{\omega,l}(E) := N(H_{\omega,l}, E) := (2\pi)^{-d} \sum_{n \in \mathbb{N}} \int_{B_l} \chi_{\{E_n(\theta)<E\}}\, d\theta\ . \tag{25}$$

Here $E \in \mathbb{R}$ is an energy value, $E_n(\theta)$ is the n-th eigenvalue of $H_{\omega,l}|^{\theta}_{\Lambda_{2l+1}}$ and

$$B_l := \left[\frac{-\pi}{2l+1}, \frac{\pi}{2l+1}\right]^d$$

if H_ω is $\mathbb{Z}^d$-ergodic. For some other Euclidean lattice it has to be replaced by the basic cell of the corresponding dual lattice $\Gamma^* := \{\gamma^* \in (\mathbb{R}^d)^* = \mathbb{R}^d \,|\, \forall \gamma \in \Gamma : \gamma^* \cdot \gamma \in 2\pi\mathbb{Z}\}$. We prove the following approximation result:

Theorem 4.1 *Let H_ω be defined as in Section 1 and $H_{\omega,l}$ as above. Denote by N respectively $N_{\omega,l}$ the corresponding IDS'. For a real valued function $g \in C_0^{n+1}$ with support in $[-1/2, 1/2]$ we have*

$$\left| \mathbb{E}\left(\int_{\mathbb{R}} g(x) dN_{\omega,l}(x) \right) - \int_{\mathbb{R}} g(x) dN(x) \right| \le \text{const.}\ |\operatorname{supp} g|\ |||g|||_{n+1}\, l^{d^2+1-n}$$

for sufficiently large $l \in \mathbb{N}$.

The proof is split into several lemmata. Remark 4.2 and Lemma 4.3 are taken from Section 5.2 of [25]. We denote with χ_l the characteristic function of the periodicity cell $\Lambda_{2l+1} := \{x \in \mathbb{R}^d |\ \|x\|_\infty \le l + 1/2\}$ of $H_{\omega,l}$ and by $\chi_{l,\gamma}(x) := \chi_l(x-\gamma)$ its translation by $\gamma \in \mathbb{Z}^d$.

Remark 4.2 Note that one can infer from [3, 4],[32] and [25] the following equalities

$$\int_{\mathbb{R}} g(x)\, dN(x) = \mathbb{E}(\mathrm{Tr}\, \chi_0\, g(H_\omega)\chi_0) \tag{26}$$

respectively

$$\int_{\mathbb{R}} g(x)\, dN_{\omega,l}(x) = (2l+1)^{-d} (\mathrm{Tr}\, \chi_l\, g(H_{\omega,l})\chi_l) \ . \tag{27}$$

Using the decomposition

$$\chi_l = \sum_{k \in \mathbb{Z}^d, |k|<2l+1} \chi_{0,k} \ ,$$

the $(2l+1)\mathbb{Z}^d$-periodicity of $H_{\omega,l}$ and the i.i.d. property of $(\omega_k)_{k\in\mathbb{Z}^d}$ one gets

$$\mathbb{E}\left(\int_{\mathbb{R}} g(x)\, dN_{\omega,l}(x) \right) = \mathbb{E}\ (\mathrm{Tr}\, \chi_0\, g(H_{\omega,l})\chi_0) \ . \tag{28}$$

Since $H_{\omega,l}$ is uniformly lower bounded there exists a $\lambda \geq 0$ such that $\mathrm{Id} \leq \lambda + H_{\omega,l}$ and $\mathrm{Id} \leq \lambda + H_\omega$ for all l, ω. From [36] we know that the operator $\chi_l(\lambda + H_{\omega,l})^{-q}(z - H_{\omega,l})^{-1}$ is trace class for all $q > d/2$. Using results from the appendix of [23] we infer

$$\|\chi_{0,\beta}(z-H_\omega)^{-1}(\lambda+H_\omega)^{-q}\chi_0\|_{\mathrm{Tr}} \quad \leq \quad \frac{\tilde{C}_1}{|y|} \exp(-|y|\,|\beta|/\tilde{C}_1) \tag{29}$$

for some $\tilde{C}_1 \geq 1$ independent of ω. This estimate is in fact a sophisticated version of the Combes-Thomas argument which we encountered already in Section 2. A simple resolvent estimate gives

$$\|\chi_0(z-H_{\omega,l})^{-1} T_\gamma\, u\, \chi_{0,\beta+\gamma}\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq \frac{\tilde{C}_1}{|y|} \|\chi_{0,\beta+\gamma} u\|_{L^p} \ , \tag{30}$$

where T_γ is the translation by $\gamma \in \mathbb{Z}^d$. As the single site potential u decays exponentially, inequality (30) gives a exponential bound in $-|\gamma+\beta|$. If one assumes that u decays polynomially with a sufficiently negative exponent, one still can carry trough the proof of Theorem 4.1.

Lemma 4.3 *If $g \in C_0^{n+1}$ and $\tilde{f}$ is an almost analytic extension of $f(x) := (\lambda + x)^q g(x)$, one has*

$$\begin{aligned} &\left| \mathbb{E}\left(\int_{\mathbb{R}} g(x)\, dN_{\omega,l}(x) \right) - \int_{\mathbb{R}} g(x)\, dN(x) \right| \\ &\quad \leq \quad \frac{C_1}{2\pi} \int_{\mathbb{C}} |y|^{-2} \left| \frac{\partial \tilde{f}}{\partial \bar{z}}(x,y) \right| \left(\sum_{\substack{\beta \in \mathbb{Z}^d \\ \gamma \in \mathbb{Z}^d, |\gamma|>l}} \|\chi_{0,\gamma+\beta} u\|_{L^p} \exp(-|y|\,|\beta|/C_1) \right) dx\, dy \end{aligned}$$

Proof:

We use without explicit reference the equations collected in the above Remark 4.2 and the Helffer-Sjöstrand formula (18). Let $\mathbb{N} \ni q > d/2$. If we multiply

$$g(H_{\omega,l}) = \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}}(z)(z - H_{\omega,l})^{-1}(\lambda + H_{\omega,l})^{-q}\, dz \wedge d\bar{z} \tag{31}$$

by the characteristic function χ_0 of Λ_1 we get a trace-class operator and consequently

$$\mathrm{Tr}(\chi_0\, g(H_{\omega,l})\chi_0) = \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}}(z)\, \mathrm{Tr}(\chi_0\, (z - H_{\omega,l})^{-1}(\lambda + H_{\omega,l})^{-q}\chi_0)\, dz \wedge d\bar{z}\ . \tag{32}$$

The same formula holds with H_ω substituted for $H_{\omega,l}$. To bound the trace of $\chi_0\, (H_{\omega,l} - H_\omega)\chi_0$ in mean we estimate

$$\|\chi_0\, (z - H_{\omega,l})^{-1}(\lambda + H_{\omega,l})^{-q}\chi_0 - \chi_0\, (z - H_\omega)^{-1}(\lambda + H_\omega)^{-q}\chi_0\|_{\mathrm{Tr}} \leq \Sigma_1 + \Sigma_2$$

by the two summands

$$\begin{aligned}
\Sigma_1 &= \|\chi_0 \left((z - H_{\omega,l})^{-1} - (z - H_\omega)^{-1}\right)(\lambda + H_\omega)^{-q}\chi_0\|_{\mathrm{Tr}} \\
&= \left\|\chi_0 \left((z - H_{\omega,l})^{-1} \left(\sum_{\gamma \in \mathbb{Z}^d, |\gamma| > l} (\omega_{\tilde{\gamma}} - \omega_\gamma) u(x - \gamma)\right)(z - H_\omega)^{-1}\right)\right. \\
&\quad \left. \times\, (\lambda + H_\omega)^{-q}\chi_0\right\|_{\mathrm{Tr}}
\end{aligned}$$

and

$$\begin{aligned}
\Sigma_2 &= \|\chi_0(z - H_{\omega,l})^{-1} \left((\lambda + H_{\omega,l})^{-q} - (\lambda + H_\omega)^{-q}\right)\chi_0\|_{\mathrm{Tr}} \\
&= \left\|\chi_0(z - H_{\omega,l})^{-1} \sum_{m=1}^{q} (\lambda + H_{\omega,l})^{m-q-1} \left(\sum_{\gamma \in \mathbb{Z}^d, |\gamma| > l} (\omega_{\tilde{\gamma}} - \omega_\gamma) u(x - \gamma)\right)\right. \\
&\quad \left. \times\, (\lambda + H_\omega)^{-m}\chi_0\right\|_{\mathrm{Tr}}\ ,
\end{aligned}$$

where in the last equality we used an iterated resolvent formula. Since $|\omega_{\tilde{\gamma}} - \omega_\gamma| \leq \omega_{\max}$ and by standard bounds for the trace norm $\|\cdot\|_{\mathrm{Tr}}$ we have

$$\begin{aligned}
\Sigma_1 &\leq \omega_{\max} \sum_{\beta \in \mathbb{Z}^d} \sum_{\gamma \in \mathbb{Z}^d, |\gamma| > l} \left\|\chi_0(z - H_{\omega,l})^{-1} u(x - \gamma)\chi_{0,\beta}\right\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \\
&\quad \times \|\chi_{0,\beta}(z - H_\omega)^{-1}(\lambda + H_\omega)^{-q}\chi_0\|_{\mathrm{Tr}} \\
&\leq \frac{C_1}{|y|^2} \sum_{\beta \in \mathbb{Z}^d} \sum_{\gamma \in \mathbb{Z}^d, |\gamma| > l} \|\chi_{0,\gamma+\beta} u\|_{L^p} \exp(-|y|\, |\beta|/C_1)
\end{aligned}$$

As Σ_2 can be bounded in the same way, our lemma is proved.

q.e.d.

Up to now we followed the proof of Theorem 5.1 of [25] almost literally. From now on we need sharper and more explicit estimates because later we will have to take the limit $l \to \infty$ simultaneously with an approximation $g \to \chi_{[0,E]}$. Special care is needed because the parameters E and l are functions of each other.

Lemma 4.4 *If we choose the constant C sufficiently large and c sufficiently small, there exists a $l_1 := l_1(d, \delta_3) < \infty$ such that:*

$$\sum_{\beta \in \mathbb{Z}^d} \sum_{\substack{\gamma \in \mathbb{Z}^d \\ |\gamma| > l}} \|\chi_{0,\gamma+\beta} u\|_{L^p} \exp(-|y|\,|\beta|/C_1) \quad \leq \quad C\, e^{-c|y|l}\, |y|^{-d^2}$$

for all y with $|y| \leq 3$ and $l \geq l_1$.

Proof:

Since the single site potential u decays exponentially in L^p sense, we know

$$\begin{aligned} & \sum_{\beta \in \mathbb{Z}^d} \sum_{\substack{\gamma \in \mathbb{Z}^d, \\ |\gamma| > l}} \|\chi_{0,\beta+\gamma} u\|_{L^p} \; \exp(-|y|\,|\beta|/C_1) \\ \leq \quad & \delta_2 \sum_{\substack{\beta \in \mathbb{Z}^d, \\ |\beta| \geq l/4}} \sum_{\substack{\gamma \in \mathbb{Z}^d, \\ |\gamma| > l}} \exp\left(-\delta_3 |\beta + \gamma|\right) \; \exp(-|y|\,|\beta|/C_1) \\ + \quad & \delta_2 \sum_{\substack{\beta \in \mathbb{Z}^d, \\ |\beta| < l/4}} \sum_{\substack{\gamma \in \mathbb{Z}^d, \\ |\gamma| > l}} \exp\left(-\delta_3 |\beta + \gamma|\right) \; \exp(-|y|\,|\beta|/C_1) \; . \end{aligned} \tag{33}$$

Using $|\beta| = 1/2|\beta| + 1/2|\beta|$, we have

$$\begin{aligned} \exp(-|y|\,|\beta|/C_1) \quad &= \quad \exp(-|y|\,|\beta|/2C_1) \; \exp(-|y|\,|\beta|/2C_1) \\ &\leq \quad \exp(-|y|\,|\beta|/2C_1) \; \exp(-|y|\, l/8C_1) \; , \end{aligned}$$

for $|\beta| \geq l/4$. If $|\beta| < l/4$, we have the relation :

$$|\gamma + \beta| \geq |\gamma| - |\beta| = 1/2|\gamma| + (1/2|\gamma| - |\beta|) \geq 1/2|\gamma| + l/4 \; ,$$

which implies $\exp(-|\gamma + \beta|) \leq \exp(-1/2|\gamma|) \exp(-l/4)$. This inequalities allow us to bound the two double sums above by

$$\begin{aligned} & \delta_2 \sum_{\substack{\beta \in \mathbb{Z}^d, \\ |\beta| \geq l/4}} \sum_{\substack{\gamma \in \mathbb{Z}^d, \\ |\gamma| > l}} \exp(-\delta_3 |\beta + \gamma|) \; \exp(-|y|\,|\beta|/2C_1) \; \exp(-|y| l/8C_1) \\ + \quad & \delta_2 \sum_{\substack{\beta \in \mathbb{Z}^d, \\ |\beta| < l/4}} \sum_{\substack{\gamma \in \mathbb{Z}^d, \\ |\gamma| \geq l}} \exp(-\delta_3 |\gamma|/2) \; \exp(-\delta_3 l/4) \; \underbrace{\exp(-|y|\,|\beta|/C_1)}_{\leq 1} \\ =: \quad & \delta_2 S_1 + \delta_2 S_2 \end{aligned}$$

We first look for a bound of the sum S_1

$$\begin{aligned} S_1 &\leq \exp(-|y|l/8C_1) \sum_{\substack{\beta\in\mathbb{Z}^d,\\ |\beta|\geq l/4}} \exp(-|y|\,|\beta|/2C_1) \underbrace{\sum_{\substack{\gamma\in\mathbb{Z}^d,\\ |\gamma|\geq l}} \exp(-\delta_3|\beta+\gamma|)}_{=\sum_{\gamma\in\mathbb{Z}^d}\exp(-\delta_3|\gamma|)} \\ &\leq \exp(-|y|l/8C_1) \left(\sum_{\gamma\in\mathbb{Z}^d} \exp(-\delta_3|\gamma|) \right) \sum_{n\geq l/4} e^{-|y|\,n/2C_1} 2^d (2n+1)^{d-1} \;, \end{aligned}$$

where we used polar coordinates for β. We have to bound the last factor as a function of $|y|$ and l.

$$\begin{aligned} \sum_{n\geq l/4} \exp(-|y|\,n/2C_1) 2^d(2n+1)^{d-1} &\leq \sum_{n\geq l/4}^{n_0} \exp(-|y|\,n/2C_1)2^d(2n+1)^{d-1} \\ &+ \sum_{n>n_0} \exp(-|y|\,n/4C_1) \;, \end{aligned}$$

where we have to choose n_0 in such a way, that for all $n \geq n_0 := n_0(|y|, d, C_1)$ the relation

$$\begin{aligned} \exp(-|y|\,n/2C_1)2^d(2n+1)^{d-1} &\leq \exp(-|y|\,n/4C_1) \\ \Leftrightarrow 2^d(2n+1)^{d-1} &\leq \exp(|y|\,n/4C_1) \end{aligned}$$

becomes valid. To this end we take $n_0 := [2^d 3^{d-1} d! \left(\frac{|y|}{4C_1}\right)^{-d} + 1]_{\text{Gauss}} \leq C_2|y|^{-d}$ (since $|y| \leq 3$), which yields

$$\begin{aligned} 2^d(2n+1)^{d-1} &\leq 2^d 3^{d-1} n^{d-1} \leq \left(\frac{|y|}{4C_1}\right)^d \frac{n^d}{d!} \\ &\leq \sum_{l=0}^{\infty} \frac{1}{l!} \left(|y| \frac{n}{4C_1}\right)^l = \exp(|y|\,n/4C_1) \;\; \forall n \geq n_0 \end{aligned}$$

as we wanted. Herewith we are able to calculate the sum in polar coordinates.

$$\begin{aligned} \sum_{n\geq l/4}^{n_0} \exp(-|y|\,n/2C_1)2^d(2n+1)^{d-1} &\leq C_2|y|^{-d} e^{-|y|l/8C_1}\, 2^d(2C_2|y|^{-d}+1)^{d-1} \\ &\leq C_3|y|^{-d^2} \end{aligned}$$

For the other part of the sum we use the inequality

$$\begin{aligned} \sum_{n>n_0} \exp(-|y|\,n/4C_1) &\leq \sum_{n\geq 0} \exp(-|y|\,n/4C_1) = \frac{1}{1-\exp(-|y|/4C_1)} \\ &\leq \frac{4C_1}{|y|} \exp\left(\frac{|y|}{4C_1}\right) \;. \end{aligned}$$

Consequently we get

$$S_1 \le \exp(-|y|l/8C_1) \left(\sum_{\gamma\in\mathbb{Z}^d} \exp(-\delta_3|\gamma|) \right) \underbrace{\left(C_3|y|^{-d^2} + \frac{4C_1}{|y|} \exp\left(\frac{|y|}{4C_1}\right)\right)}_{\le C_4|y|^{-d^2}} ,$$

since $|y| \le 3$. This finally gives for the first sum the bound

$$S_1 \le C_5 \exp\left(-\frac{|y|l}{8C_1}\right) |y|^{-d^2} . \tag{34}$$

Now we turn to the second sum S_2.

$$\begin{aligned} S_2 &\le \exp(-\delta_3 l/4)(1+l/2)^d \sum_{\substack{\gamma\in\mathbb{Z}^d,\\ |\gamma|\ge l}} \exp(-\delta_3|\gamma|/2) \\ &\le \exp(-\delta_3 l/4)(1+l/2)^d \sum_{\gamma\in\mathbb{Z}^d} \exp(-\delta_3|\gamma|/2) \\ &\le \exp(-\delta_3 l/8) , \end{aligned}$$

if we choose $l \ge l_1 := l_1(d,\delta_3)$ sufficiently large. Putting the two sums together we arrive at the following estimate

$$\delta_2(S_1+S_2) \le \delta_2 \left(C_5 \exp\left(-\frac{|y|l}{8C_1}\right) |y|^{-d^2} + \exp(-\delta_3 l/8)\right) \quad \forall l \ge l_0 . \tag{35}$$

As the second summand is tame, we subsume it into the first one. We set $C_6 := \min(\delta_3/24, 1/8C_1) < 1$, $C_7 := C_5 + 3^{d^2}$ and use $|y| \le 3$ to calculate

$$\begin{aligned} C_5 \exp(-|y|l/8C_1)|y|^{-d^2} &\le C_5|y|^{-d^2} \exp(-C_6|y|l) \\ \exp(-\delta_3 l/8) &\le 3^{d^2}|y|^{-d^2} \exp(-C_6|y|l) \end{aligned}$$

$$\Longrightarrow C_5 \exp(-|y|l/8C_1)|y|^{-d^2} + \exp(-\delta_3 l/8) \le C_7|y|^{-d^2} \exp(-C_6|y|l) .$$

The final estimate

$$\delta_2(S_1+S_2) \le \delta_2\, C_7|y|^{-d^2} e^{-C_6|y|l}$$

proves the lemma with $C = C(d,\delta_2,\delta_3,C_1) := \delta_2 C_7$ and $c = c(\delta_3, C_1) := C_6$.

q.e.d.

Lemma 4.5 *Let f be in $C^{n+1}([-1/2,1/2])$ and $\tilde f$ its almost analytic extension of order n. There exists a $l_2 := l_2(n,d,c) < \infty$ such that we have for all $l \ge l_2$:*

$$\int_{\mathbb{C}} \left| \frac{\partial \tilde f}{\partial \bar z}(x,y) \right| |y|^{-d^2-2} e^{-c|y|l}\, dx\, dy \le 2c^{-n+d^2+2} |||f|||_{n+1}\, |\operatorname{supp} f|\, l^{-n+d^2+1} .$$

Proof:

The constant c in the statement of the lemma corresponds to C_6 in the proof of Lemma 4.4. Since (cf. equation (22))

$$\left|\frac{\partial \tilde{f}_n}{\partial \bar{z}}(x,y)\right| \quad \leq \quad \frac{1}{2n!}|f^{(n+1)}s|\ |y|^n + \frac{3}{\langle x\rangle}\chi_{\{\langle x\rangle<|y|<2\langle x\rangle\}}\sum_{r=0}^{n}|f^{(r)}|\frac{|y|^r}{r!}$$

we have to bound the sum of two integrals. The second one is:

$$\int_{\mathbb{C}} dxdy|y|^{-d^2-2}\exp(-C_6|y|l)\frac{3}{\langle x\rangle}\chi_{\{\langle x\rangle<|y|<2\langle x\rangle\}}\sum_{r=0}^{n}|f^{(r)}|\frac{|y|^r}{r!}\ .$$

The properties of $\langle x\rangle$, s and f ensure $\langle x\rangle \geq 1, 1 < |y| < 3$, thus we estimate the above line by:

$$6\int_{x\in\operatorname{supp} f}\int_{y>0} dxdy\ e^{-C_6 l}\sum_{r=0}^{n}|f^{(r)}|\frac{3^r}{r!}\,\chi_{[1,3]}(y) \leq 60\ |\operatorname{supp} f|\ |||f|||_n\, e^{-C_6 l}$$

since $3^r \leq 5r!$ and $|||f|||_n := \sum_{r=0}^{n}\|f^{(r)}\|_\infty$. Now we turn our attention to:

$$\begin{aligned}
&\frac{1}{2n!}\int_{\operatorname{supp} f} dx\int dy\ |y|^{n-d^2-2}\exp(-C_6|y|l)\ |f^{(n+1)}(x)| \\
&\leq\ \frac{|||f|||_{n+1}}{n!}\int_{\operatorname{supp} f} dx\int_{y>0} dy\ y^{n-d^2-2}\exp(-C_6|y|l) \\
&=\ \frac{|||f|||_{n+1}}{ln!}l^{-n+d^2+2}\int_{\operatorname{supp} f} dx\int_{t>0} dt\ \exp(-C_6 t)\ t^{n-d^2-2} \\
&=\ \frac{|||f|||_{n+1}}{n!}\ l^{-n+d^2+1}|\operatorname{supp} f|\ C_6^{-n+d^2+2}(n-d^2-2)! \\
&\leq\ C_6^{-n+d^2+2}|||f|||_{n+1}\ |\operatorname{supp} f|\ l^{-n+d^2+1}\ .
\end{aligned}$$

Since $60\exp(-C_6 l) \leq C_6^{-n+d^2+2}l^{-n+d^2+1}$ for sufficiently large l, i.e. $l \geq l_2(d, C_6, n)$, we have

$$\int_{\mathbb{C}}\left|\frac{\partial \tilde{f}}{\partial \bar{z}}(x,y)\right||y|^{-d^2-2}e^{-C_6|y|l}\,dx\,dy \leq 2C_6^{-n+d^2+2}|||f|||_{n+1}\,|\operatorname{supp} f|\ l^{-n+d^2+1}\ ,$$

which ends the proof of the lemma.

q.e.d.

We have to bound the derivatives of $f := (\lambda+\cdot)^q g$ in terms of the derivatives of g itself. A simple calculation using Leibniz' formula shows $|||f|||_{n+1} \leq C_8\,|||g|||_{n+1}$, where C_8 depends only on n, q and λ.

By now we can write down the needed inequalities for our difference of integrals with respect to N and $N_{\omega,l}$.

$$
\begin{aligned}
& \left| \mathbb{E}\left(\int g(x) dN_{\omega,l}(x) \right) - \int g(x) dN(x) \right| \\
= \ & \left| \mathbb{E}(\mathrm{Tr}\, \chi_0\, g(H_{\omega,l})\chi_0)) - \mathbb{E}(\mathrm{Tr}\, \chi_0\, g(H_\omega)\chi_0)) \right| \\
\leq \ & \mathbb{E}\left(\frac{1}{2\pi} \int_{\mathbb{C}} dxdy \ \left| \frac{\partial \tilde{f}}{\partial \bar{z}}(x,y) \right| \right. \\
& \left. \times \ \| \chi_0 (H_{\omega,l} + \lambda)^{-q} (H_{\omega,l} - z)^{-1} \chi_0 - \chi_0 (H_\omega + \lambda)^{-q} (H_\omega - z)^{-1} \chi_0 \|_{\mathrm{Tr}} \right) \\
\leq \ & \frac{1}{2\pi} \int_{\mathbb{C}} dxdy \ \frac{C_1}{|y|^2} \left| \frac{\partial \tilde{f}}{\partial \bar{z}}(x,y) \right| \left(\sum_{\beta \in \Gamma} \sum_{\substack{\gamma \in \Gamma, \\ |\gamma| > l}} \| \tau_{\beta+\gamma} V \|_{L^p(C_0)} \exp(-|y|\, |\beta| / C_1) \right) \\
\leq \ & \frac{1}{2\pi} \int_{\mathbb{C}} dxdy \ \frac{C_1}{|y|^2} \left| \frac{\partial \tilde{f}}{\partial \bar{z}}(x,y) \right| \delta_2 (S_1 + S_2) \\
\leq \ & \frac{1}{2\pi} \int_{\mathbb{C}} dxdy \ \delta_2 C_1 \left| \frac{\partial \tilde{f}}{\partial \bar{z}}(x,y) \right| C_7 |y|^{-d^2-2} \exp(-C_6 |y| l) \\
\leq \ & \frac{\delta_2 C_1 C_7}{2\pi} \\
& \times \ \left(60\, |\operatorname{supp} f|\ |||f|||_n \exp(-C_6 l) + C_6^{-n+d^2+2} |||f|||_{n+1}\ |\operatorname{supp} f|\ l^{-n+d^2+1} \right) \\
\leq \ & \frac{\delta_2 C_1 C_7}{\pi C_6^{n-d^2-2}}\ |\operatorname{supp} f|\ |||f|||_{n+1} l^{-n+d^2+1} \\
\leq \ & C_9\ |\operatorname{supp} f|\ |||g|||_{n+1} l^{-n+d^2+1}
\end{aligned}
$$

if we choose $l \geq l_3 := \max(l_1, l_2) = l_3(d, n, \delta_3, C_1)$ and set $C_9 := \frac{\delta_2 C_1 C_7 C_8}{\pi C_6^{n-d^2-2}}$. This proves Theorem 4.1 with C_9 as the constant on the rightern side.

q.e.d.

Now we want to compare the "number" of states of H_ω and $H_{\omega,l}$ in the energy interval $[0, E]$. To this end take $g \in C_0^{n+1}(\mathbb{R}, [0,1])$ with $g(x) = 1$ for all $x \in [0, E]$ and support in $[-E/2, 2E]$. Moreover let g have minimal derivative in the sense of inequality (23). We estimate

$$
\begin{aligned}
\mathbb{E}[N_{\omega,l}(E) - N_{\omega,l}(0)] - [N(E) - N(0)] &\leq \mathbb{E}\left(\int_0^E g\ dN_{\omega,l} \right) \\
&\leq \left| \mathbb{E}\left(\int_0^E g\ dN_{\omega,l} \right) - \int_0^E g\ dN \right| + \int_0^E g\ dN\ . \quad (36)
\end{aligned}
$$

As $0 \leq g \leq 1$ one has for $l \geq l_3$

$$\mathbb{E}[N_{\omega,l}(E) - N_{\omega,l}(0)] \leq 2(N(E) - N(0)) + C_{10}\, E^{-n}\, l^{-n+d^2+1} \ , \qquad (37)$$

where we used Theorem 4.1 and equation (23). If N has Lifshitz asymptotics at the lower band edge 0, as defined in equation (17), there exists an energy value E_1 such that

$$N(E) - N(0) \leq \exp(-E^{-d/4}) \quad \forall E \in [0, E_1] \ . \qquad (38)$$

Together with (37) this gives

$$\mathbb{E}[N_{\omega,l}(E) - N_{\omega,l}(0)] \leq 2\exp(-E^{-d/4}) + C_{10}\, E^{-n}\, l^{-n+d^2+1} \ \forall E \in [0, E_1] \ . \qquad (39)$$

For $\alpha \in]0,1[$ we set $E := l^{-\alpha}$. This implies

$$\begin{aligned} \mathbb{E}[N_{\omega,l}(E) - N_{\omega,l}(0)] &\leq 2\exp(-(l^{-\alpha})^{-d/4}) + C_{10}\,(l^{-\alpha})^{-n}\, l^{-n+d^2+1} \\ &= 2\exp(-l^{\alpha d/4}) + C_{10}\, l^{-n(1-\alpha)+d^2+1} \\ &\leq 2\, C_{10}\, l^{-n(1-\alpha)+d^2+1} \qquad (40) \end{aligned}$$

if $l \geq \tilde{l}_4 := \tilde{l}_4(d, n, \alpha, C_{10})$. Now we can formulate the IDS approximation theorem.

Theorem 4.6 *Let N and $N_{\omega,l}$ be the IDS of H_ω and $H_{\omega,l}$ respectively and $\alpha \in]0,1[$. If N has a Lifshitz tail at the lower band edge 0, the bound*

$$\mathbb{E}[N_{\omega,l}(l^{-\alpha}) - N_{\omega,l}(0)] \leq 2\, C_{10}\, l^{-n(1-\alpha)+d^2+1} \qquad (41)$$

is valid for sufficiently large l.

In fact we have to choose $l \geq l_5$ and $l_5 := \max(\tilde{l}_4, E_1^{-1/\alpha})$ (cf. (40) and (38)).

5 Sparsity of states near the lower band edge

We want to estimate the probability of finding an eigenvalue of $H_{\omega,l}(\theta)$ in a small energy interval $I \ni 0$. Here $H_{\omega,l}(\theta) := H_{\omega,l}|^{\theta}_{\Lambda_{2l+1}} = H_\omega|^{\theta}_{\Lambda_{2l+1}}$ is the operator $H_{\omega,l}$ restricted to $L^2(\Lambda_{2l+1})$ with θ-boundary conditions. The following lemma allows to bound this probability using the IDS of $H_{\omega,l}$.

Lemma 5.1

$$\int_{\theta \in B_l} d\theta\, \mathbb{P}(\{\omega|\ \sigma(H_{\omega,l}(\theta)) \cap [0, E[\neq \emptyset\}) \leq (2\pi)^d\ \mathbb{E}(N_{\omega,l}(E) - N_{\omega,l}(0)) \ .$$

Proof:

$$
\begin{aligned}
& \int_{\theta \in B_l} d\theta \, \mathbb{P}(\{\omega | \ \sigma(H_{\omega,l}(\theta)) \cap [0, E[\neq \emptyset\}) \\
\leq \quad & |\Lambda_{2l+1}| \int_{\theta \in B_l} d\theta \ \mathbb{E}(N(H_{\omega,l}(\theta), E) - N(H_{\omega,l}(\theta), 0)) \quad \text{Čebyšev inequality} \\
= \quad & |\Lambda_{2l+1}| \ \mathbb{E}(\int_{\theta \in B_l} d\theta \, (N(H_{\omega,l}(\theta), E) - N(H_{\omega,l}(\theta), 0)) \quad \text{Fubini's theorem} \\
= \quad & (2\pi)^d \ \mathbb{E}(N_{\omega,l}(E) - N_{\omega,l}(0)) \quad \text{equations (14,25)}
\end{aligned}
$$

q.e.d.

Since the MSA works with specific boundary conditions, e.g. periodic ones,we have to get rid of the average over $\theta \in B_l$ in the last bound. This is possible using the Lipshitz-continuity in θ of the eigenvalues of $H_{\omega,l}(\theta)$.

Lemma 5.2 *For any fixed* $\theta_0 \in B_l$, $E < 1$ *and some* $\xi > 0$ *we have*

$$
\mathbb{P}(\{\omega | \, \sigma(H_{\omega,l}(\theta_0)) \cap [0, E[\neq \emptyset\}) \leq \frac{(2\pi)^d}{|B_l|} \mathbb{E}(N_{\omega,l}(E + C_{12} l^{-\xi}) - N_{\omega,l}(0)) \ . \quad (42)
$$

Proof:

The eigenvalues of $H_{\omega,l}(\theta)$ are Lipshitz continuous in θ, so we have :

$$
|E_j(H_{\omega,l}(\theta)) - E_j(H_{\omega,l}(\theta'))| \leq \Xi_{j,l} |\theta - \theta'|
$$

for some $\Xi_{j,l} > 0$. One can choose the $\Xi_{j,l}$ independent of j and l only as a function of $E_j(H_{\omega,l}(\theta))$. As we consider only eigenvalues in the energy interval $[0, E[\subset [0, 1[$ even this dependence can be eliminated. Thus we can find $\Xi > 0$ such that

$$
\Xi \geq \Xi_{j,l} \ \forall l, j \ .
$$

Now we can estimate :

$$
\begin{aligned}
\mathbb{P}(\{\omega | \ \sigma(H_{\omega,l}(\theta_0)) \cap [0, E[\neq \emptyset\}) &= \mathbb{P}(\{\omega | \ \exists j \in \mathbb{N} : \ E_j(H_{\omega,l}(\theta_0)) \in [0, E[\ \}) \\
&= \int_{\theta \in B_l} \frac{d\theta}{|B_l|} \mathbb{P}(\{\omega | \ \exists j \in \mathbb{N} : \ E_j(H_{\omega,l}(\theta_0)) \in [0, E[\ \}) \quad (43)
\end{aligned}
$$

$$
\begin{gathered}
E_j(H_{\omega,l}(\theta_0)) \in [0, E[\implies E_j(H_{\omega,l}(\theta)) \in [0, E + \Xi \, \mathrm{diam}(B_l)[\quad \forall \theta \in B_l \\
\text{and } \ \mathrm{diam}(B_l) \leq C_{11} \ l^{-1}
\end{gathered}
$$

$$
\begin{aligned}
\dots \quad &\leq \quad \int_{\theta \in B_l} \frac{d\theta}{|B_l|} \, \mathbb{P}(\{\omega| \ \exists j \in \mathbb{N}: \ E_j(H_{\omega,l}(\theta)) \in [0, E + C_{12} l^{-1}[\ \}) \\
&= \quad \int_{\theta \in B_l} \frac{d\theta}{|B_l|} \, \mathbb{P}(\{\omega| \ \sigma(H_{\omega,l}(\theta)) \cap [0, E + C_{12} l^{-1}[\neq \emptyset \ \}) \\
&\leq \quad (2\pi)^d \ |B_l|^{-1} \ \mathbb{E}(N_{\omega,l}(E + C_{12} l^{-1}) - N_{\omega,l}(0))
\end{aligned}
$$

q.e.d.

We choose now $0 < \alpha < 1$ and $E := l^{-\alpha}$ as before. Thus for $l \geq l_6$ the bound $E + C_{12}\, l^{-1} \leq 2l^{-\alpha}$ is valid, with l_6 depending on α and C_{12}. As the IDS is monotone increasing in the energy, this implies

$$N_{\omega,l}(E + C_{12}\, l^{-1}) \leq N_{\omega,l}(2l^{-\alpha}) \ .$$

If N has Lifshitz tails, we estimate as in (40):

$$\mathbb{E}(N_{\omega,l}(2l^{-\alpha}) - N_{\omega,l}(0)) \leq 2\, C_{10}\, l^{-n(1-\alpha)+d^2+1}$$

for $l \geq l_4(d, n, \alpha, C_{10})$. In this way we obtain from Lemma 5.2

$$\mathbb{P}(\{\omega|\, \sigma(H_{\omega,l}(\theta_0)) \cap [0, l^{-\alpha}[\neq \emptyset\}) \leq C_{13} l^{-n(1-\alpha)+d^2+d+1} \tag{44}$$

since $|B_l|^{-1} \leq \tilde{C}_{13} l^d$ where $\tilde{C}_{13}$ depends only on the dimension. We sat $C_{13} := 2\, C_{10}\, \tilde{C}_{13}\, (2\pi)^d$. The probability in (44) can be bounded by l^{-q} for arbitrary $q > 0$ if only $l \geq l_7$ is sufficiently large and

$$
\begin{aligned}
-n(1-\alpha) + d^2 + d + 1 \quad &< \quad -q \\
\iff -n(1-\alpha) \quad &< \quad -q - d^2 - d - 1 \ .
\end{aligned}
\tag{45}
$$

It is obvious that for any $0 < \alpha < 1$ we can choose n in such a way that the relation (45) is valid. Similarly, for any fixed $n > q + d^2 + d + 1$ it is possible to choose α sufficiently small, so that (45) holds. Particularly we can choose α from $]0, 1/4[$.

Recall that if H_0 has regular Floquet eigenvalues at the lower spectral band edge 0, the IDS N of $H_\omega := H_0 + V_\omega$ exhibits Lifshitz asymptotics at 0. Thus we proved Proposition 1.2 with $l_0 := \max_{i=1}^{7} l_i$.

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