Width Restricted Layering of Acyclic Digraphs with Consideration of Dummy Nodes

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Abstract

This paper deals with the problem of finding graph layerings restricted to a given maximal width. However, other than previous approaches for width restricted layering, we take into account the space for dummy-nodes, which are introduced by edges crossing a layer. The main result is that the problem of finding a width-restricted layering under consideration of dummy nodes is NP-complete even when all regular nodes have the same constant width and all dummy nodes have the same constant width.

Keywords: Graph drawing; Width-restricted layering; Computational complexity

1 Introduction

The most popular heuristic for drawing directed acyclic graphs is the Sugiyama algorithm [5]. It proceeds by assigning each node of the graph to a layer (layering step), introducing a dummy node wherever an edge crosses a layer, and then ordering the nodes within each layer to reduce the number of crossings. The layering step determines the height and width of the resulting drawing. Usually, the height is defined as the number of layers created, while the width is defined by the maximal sum of the widths of regular nodes within one layer. This definition of width ignores the dummy nodes, i.e. the space needed for edges crossing a layer. However, if the regular nodes are relatively small, or if there are many dummy nodes, this should not be neglected. Algorithms for width-restricted layering like Coffman-Graham [1], which do not take dummy nodes into account, thus almost inevitably produce drawings that are too wide.

In this paper, we look at the problem of finding a layering given a maximal allowed width, taking into account the dummy nodes. We show that the problem of finding a width restricted layering is NP-complete even if all nodes have the same constant width. However, if the graph is connected and at most a constant number of nodes (regular plus dummy) can be placed in every single layer, the problem becomes solvable in polynomial-time.

Graph drawings based on width-restricted layerings have important practical applications, e.g. when a graph is too large to fit on a screen. In that case, it is much easier to navigate in one dimension when the drawing does not exceed the screen width, than to navigate in two dimensions, when the drawing exceeds width and height of the screen.

2 Definitions

Let $G = (N, E)$ be a directed acyclic graph with the set of nodes N and the set of directed edges E. A *Layering* of a graph is a mapping $f : N \mapsto \mathbb{N}$ of its nodes to positive integers, such that there exists a $v \in N$ with $f(v) = 1$ and $\forall u, v : (u, v) \in E \Rightarrow f(u) < f(v)$. The integer which a node is mapped to represents its layer. For each edge $(u, v) \in E$, an additional node is introduced in each layer k with $f(u) < k < f(v)$ to represent the space needed by an edge crossing a layer. These additional nodes are called *dummy nodes* as opposed to the *regular nodes* $v \in N$.
Let g be a width function that assigns each node $v \in N$ and each edge $e \in E$ a positive rational width. The *width of a dummy node* is then defined as the width of the corresponding edge. The *width of a layer* is the sum of the widths of all nodes (regular and dummy) on that layer. The *width of a layering* is the width of the widest layer. The *height of a layering* is $\max\{f(v) \mid v \in N\}$.
The considered problems are defined as follows:

Width-Restricted Graph-Layering with Dummy Nodes

> *Given:* A directed, acyclic graph G, width function g, integer k.
> *Question:* Is there a layering of width $\leq k$ for G?

Height-Width-Restricted Graph-Layering with Dummy Nodes

> *Given:* A directed, acyclic graph G, width function g, integers k_1, k_2.
> *Question:* Is there a layering of width $\leq k_1$ and height $\leq k_2$ for G?

3 Complexity Results

Finding a layering with minimal height (with or without consideration of dummy-nodes) is polynomial-time solvable by a simple longest path algorithm [3]. When dummy nodes are *not* taken into account, finding a layering with minimal width is trivial because a layering with only one regular node per layer solves the problem. Finding a width-restricted layering with minimum height without consideration of dummy nodes is NP-complete [2].

Theorem 1 *Width-Restricted Graph-Layering with Dummy Nodes is NP-hard even if every regular node has the same constant width r and every dummy node has the same constant width d.*

Proof Without loss of generality let the width of a regular node be $r = 1$. For the proof we assume $r > d$ (the proof for $r \leq d$ is similar). Choose integer k such that $1/k \leq d < 1/(k-1)$. We reduce the strongly NP-hard 3-Partition problem to our problem. 3-Partition is the problem to decide for given a set of integers $A = \{a_1, a_2, \ldots, a_{3m}\}$ and integer B with $B/4 < a_i < B/2$, $i \in [1 : 3m]$ and $\sum_{i=1}^{3m} a_i = mB$ if there exists a partition of A into sets $A_1, \ldots, A_m$ each of size 3 with $\sum_{a_j \in A_i} a_j = B$ (see [4]).

The basic idea of the proof is illustrated in Figure 1: A graph G is constructed containing a subgraph consisting of nodes $U \subset V$ which is so strongly connected that there exists only one feasible layering for this subgraph. This subgraph forms a skeletton in which the other nodes will have to be placed.
For every a_i of 3-Partition there are nodes in G (W-nodes) that form a block of width a_i in every feasible layering. Due to the skeletton defined by the U-nodes, the width restriction can be satisfied if and only if the blocks can be placed in groups of three, with the total width of every goup equaling B.

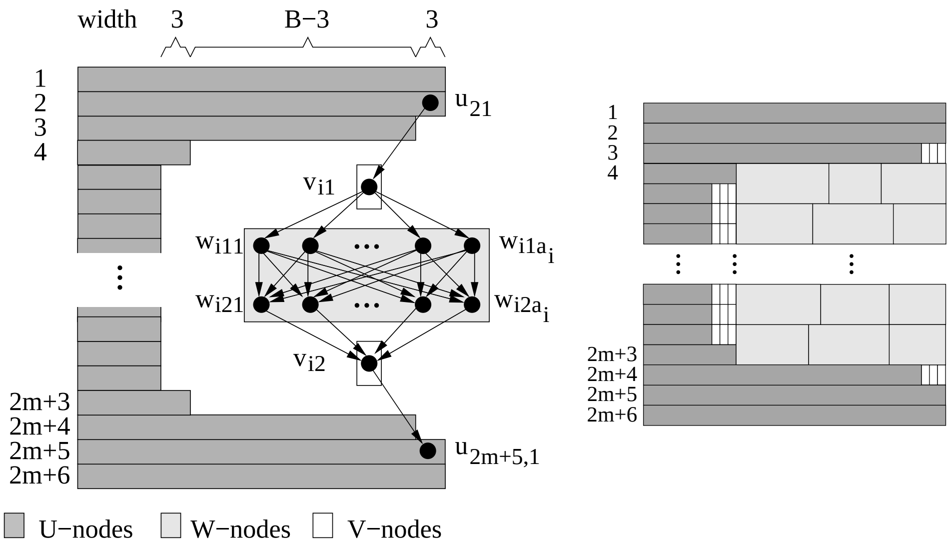

Figure 1: Construction for the NP-completeness proof: Skeletton of U-nodes with block of W- and V-nodes for one a_i (left), feasible drawing corresponding to a solution of 3-Partition (right).

Without loss of generality we consider an instance of 3-Partition with $B > 12(m+1)$ and $a_i > B/4 + m + 1$. Let $q = B + 3 + \lceil 3(m-2)d \rceil$. We construct a directed, acyclic graph $G = (N, E)$ with $N = U \cup V \cup W$ as follows: U consists of the nodes

- u_{ij} for $i \in [1 : 2m+6], j \in [1 : 2k+q]$,
- u_{ij} for $i \in \{1, 2, 2m+5, 2m+6\}, j \in [2k+q+1 : 2k+2q]$,
- u_{ij} for $i \in \{3, 2m+4\}, j \in [2k+q+1 : 2k+q+\lfloor q-(3+3(m-1)d) \rfloor]$,
- u_{ij} for $i \in \{4, 2m+3\}$, $j \in [2k+q+1 : 2k+q+\lfloor q-(B+3(m-1)d) \rfloor]$.

Let $V = \{v_{i1}, v_{i2} \mid i \in [1 : 3m]\}$ and $W = \{w_{ijl} \mid i \in [1 : 3m], j \in \{1, 2\}, l \in [1 : a_i]\}$.
The set of edges E consists of

- $(u_{ij}, u_{i+1,h})$ for every two nodes $u_{ij}, u_{i+1,h} \in U$,
- $(u_{2,1}, v_{i,1})$ and $(v_{i,2}, u_{2m+5,1})$ for $i \in [1 : 3m]$,
- $(v_{i,1}, w_{i,1,l})$ and $(w_{i,2,l}, v_{i,2})$ for $i \in [1 : 3m]$, $l \in [1 : a_i]$,
- $(w_{i,1,j}, w_{i,2,h})$ for every two nodes $w_{i,1,j}$, $w_{i,2,h} \in W$,
- $(u_{4,i}, u_{2m+3,i})$ for $i \in [1 : \lfloor \frac{\lceil 3(m-2)d \rceil - 3(m-2)d}{d} \rfloor]$,
- $(u_{2,i}, u_{5,i})$ and $(u_{2m+2,i}, u_{2m+5,i})$ for $i \in [1 : \lfloor \frac{\lceil 3(m-1)d \rceil - 3(m-1)d}{d} \rfloor]$.

We show that a solution of the instance of 3-Partition exists if, and only if there is a layering of G of width $\leq 2k+2q$. Assume that there exists such a layering.

Claim 1 *Every layering of G of width $\leq 2k+2q$ has $2m+6$ layers, all nodes u_{i*} are in layer i, for $i \in [1:2m+6]$ and the nodes in $V \cup W$ are in layers $[3:2m+3]$.*

Proof We show first that all nodes u_{i*} are in the same layer for $i \in [1:2m+6]$. Assume this is false for any $i \in [2:2m+5]$. Then there must be a layer l^* where nodes u_{i*} are on l^* and on a layer below it. Since there are nodes u_{i*} below l^* there must be a dummy node on the layer l^*+1 for each edge $(u_{ij}, u_{i+1,*})$ where u_{ij} is on or above l^* (Observe that each node $u_{i+1,*}$ is placed below l^*+1). If at least half of the nodes u_{i*} are on or above l^* these are at least $(2k+q)\frac{2k+q}{2}$ dummy nodes. These nodes need at least $d\frac{(2k+q)^2}{2} \geq \frac{1}{k}\frac{(2k+q)^2}{2}$ width in the layer l^*+1. But this contradicts our assumption on the width of the layering since $\frac{(2k+q)^2}{2k} = 2k+2q+\frac{q^2}{2k} > 2k+2q$. If less than half of the nodes u_{i*} are on or above l^* layer l^* itself would contain too many dummy nodes.
It remains to consider the cases $i=1$ and $i=2m+6$. Since both cases are symmetric, we only describe the case $i=1$. Assume that not all u_{1*} are in the same layer. Then there must be a lowest layer l_{low} of the layers containing nodes u_{1*}. For each node u_{1*} above l_{low} there must be $(2k+q)$ dummy nodes on l_{low} (since there are $(2k+q)$ nodes u_{2*}). If there are $x > 0$ nodes u_{1*} above l_{low}, the width of l_{low} must be at least $dx(2k+q)+(2k+2q-x) \geq \frac{1}{k}x(2k+q)+(2k+2q-x) > 2k+2q$. That contradicts the maximal width of $2k+2q$.
Now, it follows that the nodes u_{i*} and $u_{i+1,*}$ must be in successive layers. Otherwise there would be a layer in between which contained at least one dummy node for each edge $(u_{ij}, u_{i+1,h})$. These at least $(2k+q)^2$ dummy nodes would result in a width of $d \cdot (2k+q)^2 \geq (1/k)\cdot(2k+q)^2 = 4k+4q+(q^2/k) > 2k+2q$. No node $x \in V \cup W$ can be above layer 3 or below layer $2m-3$ because there is a directed path from $u_{2,1}$ to x and a directed path from x to u_{2m+4}. □

Claim 2 *The available space in all layers for the nodes in $V \cup W$ and dummy nodes of edges between nodes in $V \cup W$ is less than $6m+2B+6m(m-1)d+2(m+1)d$.*

Proof The following upper bounds hold for the amount of free space f_i in each layer i (i.e., after subtracting the width of all nodes U and the dummy nodes of edges between nodes in U from the width of the layer). For $i \in \{3, 2m+4\}$:

$$\begin{aligned} f_i &= q - \lfloor q-(3+3(m-1)d)\rfloor - \lfloor\frac{\lceil 3(m-1)d\rceil - 3(m-1)d}{d}\rfloor \cdot d \\ &= 3+3(m-1)d+\varepsilon \text{ for an } \varepsilon \in [0,d) \end{aligned}$$

For $i \in \{4, 2m+3\}$:

$$\begin{aligned} f_i &= q - \lfloor q-(B+3(m-1)d)\rfloor - \lfloor\frac{\lceil 3(m-1)d\rceil - 3(m-1)d}{d}\rfloor \cdot d \\ &= B+3(m-1)d+\varepsilon \text{ for an } \varepsilon \in [0,d) \end{aligned}$$

For $i \in [5:2m+2]$:

$$\begin{aligned} f_i &= q - \lfloor\frac{\lceil 3(m-2)d\rceil - 3(m-2)d}{d}\rfloor \cdot d \\ &= B+3+3(m-2)d+\varepsilon \text{ for an } \varepsilon \in [0,d) \end{aligned}$$

Altogether there is at most f_{max} free space available in layers 3 to $2m+4$ with

$$f_{max} = \sum_{n=3}^{2m+4} f_i \leq 6m + 2mB + 6m(m-1)d + 2(m+1)d$$

Since there are $6m + 2mB$ nodes in $V \cup W$ there is at most $6m(m-1)d + 2(m+1)d$ free space for the dummy nodes on edges that have at least one node in $V \cup W$. □

Claim 3 *For fixed $i \in [1:3m]$ and fixed $j \in \{1,2\}$ all w_{ij*} are in the same layer, and the layers containing w_{i1*} and w_{i2*} are consecutive.*

Proof For the edges on the path $u_{21}, v_{i1}, w_{i11}, w_{i21}, v_{i2}, u_{2m+5,1}$ there are at least $(2m+4) - 6 = 2(m-1)$ dummy nodes, $i \in [1:3m]$. Thus, following from Claim 2 there can be at most $2(m+1)$ additional dummy nodes. Would the nodes w_{i1*} be on more than one layer, then the lowest layer containing a node w_{i1*} would also contain at least $a_i - 1$ additional dummy nodes, since all w_{i1*} are connected to all w_{i2*}. Because $a_i - 1 > B/4$ and $B > 12(m+1)$ more than $2(m+1)$ additional dummy nodes would be necessary. A similar reasoning can be applied for w_{i2*}. Each layer between the two layers containing the nodes w_{i1*} and w_{i2*} would contain $(a_i - 1)^2$ additional dummy nodes since there is an edge from each w_{i1*} to each w_{i2*}. Since $(a_i - 1)^2 > B^2/4 > 2(m+1)$ this cannot be the case. The nodes v_{i*} must be in a layer next to the layers of the w_{i**} nodes. Otherwise there would be more than $a_i - 1 > 2(m+1)$ additional dummy nodes in every layer in between.□

We call the nodes w_{i**} and v_{i*} "block i". Every block is represented in each layer either by a dummy node (path from u_{21} to v_{i1} or from v_{i2} to $u_{2m+5,1}$) or by nodes of the block itself. Would there be W-nodes of more than 3 different blocks in any layer i then there would be at least $4 \cdot (B/4 + (m+1) + 1) \geq B + 9$ W-nodes in layer i (Recall, $a_i > B/4 + (m+1)$). Since each block that is not represented by its W-nodes in the layer is at least represented by a dummy node the layer would have width at least

$$B + 9 + (3m-4)d + 2k + q + \lfloor \frac{\lceil 3(m-2)d \rceil - 3(m-2)d}{d} \rfloor \cdot d > 2k + 2q$$

Clearly, there can not be a W-node above layer 4 or below layer $2m+3$, since otherwise one V-node belonging to any of those W-nodes would have to be placed on layer 2 or layer $2m+5$. Hence, exactly $2m$ layers are available for the W-nodes. Since each block requires 2 layers for its W-nodes and there are $3m$ blocks and no layer contains nodes of more than 3 blocks there must be W-nodes of exactly 3 layers in each layer $i \in [4:2m+3]$. This implies that in every layer $i \in \{3+2j \mid 0 \leq j < m\}$ there are exactly 3 blocks starting with their v_{i1} node and each layer $i \in \{6+2j \mid 0 \leq j < m\}$ contains exactly 3 nodes v_{i2}. Since there is less than $B + 3 + (3(m-2)+1)d$ space for nodes in $V \cup W$ and dummy nodes of edges belonging to any of those nodes in each layer $i \in [5:2m+2]$ and each of these layers contains exactly 3 V-nodes and $3(m-2)$ dummy nodes there can be no more than B W-nodes in the layer. Since the layers 4 and $2m+3$ must contain as many W-nodes as layers 5 and $2m+4$ there can be no more than $((2m+4)-4)B = 2mB$ W-nodes in the layers 5 to $2m+4$. Since only these layers contain W-nodes and there are exactly $2\sum_{i=1}^{3m} a_i = 2mB$ of them this implies that there must be exactly B W-nodes in each of these layers. Now, a partition of A according to the blocks starting in the same layers solves 3-Partition.
The other direction of the proof is easy and left to the reader. □

Theorem 2 *The Height-Width-Restricted Graph Layering with Dummy Nodes problem is polynomial time solvable if the given graph is connected and* $k_1 / \min\{\min\{g(v) \mid$

$v \in N\}, \min\{g(e) \mid e \in E\}\}$ *is constant (i.e. at most a constant number of regular nodes and dummy nodes can be placed on one layer).*

Proof We give only a sketch of the proof. Since any set of regular nodes and dummy nodes that can be placed on one layer is of constant size, the number of different layers occuring in all possible layerings is polynomial. Moreover, since G is connected the set of nodes and edges in a layer forms a cut in G. Then it can be decided in polynomial time for two layers whether one layer can be a possible (direct) successor of the other layer in a valid layering of the graph. Thus we obtain a directed graph were nodes are the possible layers and there is an edge from node u to node v when the layer corresponding to v is a possible successor of the layer corresponding to u. For finding a layering of width $\leq k_1$ and minimal height one has to find a shortest path from a source to a sink in the directed graph on the set of possible layers. □

Remark: It is necessary to require in Theorem 2 that the graph is connected because otherwise the problem to find a layering of restricted height and width becomes NP-complete even for maximal height 2 for all constant values r and d.

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