ZERO-FORCING FREQUENCY DOMAIN EQUALIZATION FOR DMT SYSTEMS WITH INSUFFICIENT GUARD INTERVAL

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ABSTRACT

A generalized discrete multitone (GDMT) scheme that not only uses the guard interval but also redundancy in the frequency domain for equalization has been introduced recently [1, 2, 3]. Instead of assigning all redundancy in the time domain it allows to take advantage of unused subcarriers to remove ISI and ICI. In this paper, we calculate the signal-to-noise ratio for each subcarrier for the zero-forcing equalizer proposed in [3]. From there, we derive a new adaptive loading algorithm that optimally chooses the length of the guard interval and the position of unused subcarriers. The cost function applied for the adaptive loading algorithm is the achievable data rate at a fixed error probability.

1. INTRODUCTION

Discrete Multi-Tone (DMT) systems are restricted by the fact that the guard interval (GI) introduced in form of a cyclic prefix has to be at least as long as the order of the channel impulse response (CIR). This mainly affects the achievable bandwidth efficiency and latency time of practical DMT systems. One approach to shorten the effective CIR length is to introduce a short FIR filter at the input of the receiver, the so called time-domain equalizer (TEQ), see [4] and references therein. In another approach the TEQ is transferred to the frequency domain, resulting in a separate complex frequency domain equalizer for each tone [5].

More recently, a different frequency domain equalizer, has been introduced under the name of generalized discrete multitone (GDMT) [1, 2]. Here the one tap frequency domain equalizer of a traditional DMT receiver is replaced by a block equalizer matrix and the guard interval is omitted. The equalizer takes advantage of inherent frequency domain redundancy in DMT due to unused tones, i.e. subcarriers to which the adaptive loading algorithm does not assign any data due to a too low signal-to-noise ratio (SNR). These subcarriers do not need to be equalized at the receiver but they contain information that can be exploited to obtain a better compensation of ISI and ICI in used subcarriers. In [3] GDMT has been extended to the case of an insufficient guard interval. It has been shown, that zero-forcing equalization with no remaining ISI and ICI is feasable if the length of the guard interval plus the number of unused subcarriers is at least as high as the CIR order. It thus allows to trade off time-domain redundancy for frequency domain redundancy.

We here evaluate the noise enhancement of this zero-forcing equalizer. This allows us to calculate the SNR in each subcarrier. As we will see, it will not only depend on the channel frequency response but also on the number and position of the unused subcarriers. The bit load per subcarrier k can then be derived as [6]

$$b_k = \log_2\left(1 + \frac{SNR_k}{\Gamma}\right). \qquad (1)$$

where Γ denotes the SNR gap. A new adaptive loading algorithm will be presented that maximizes the data rate at a given target error probability by finding the optimal length of the guard interval and the position of unused subcarriers.

2. THE GDMT TRANSCEIVER

The GDMT transceiver is depicted in Fig. 1. When compared with a traditional DMT transceiver, the only difference is that the one-tap frequency domain equalizer in DMT has been replaced by an $M \times M$ block equalizer $\mathbf{E}$ where M denotes the number of subcarriers. The relationship between the input symbol $\mathbf{u}(k)$ and the output symbol $\hat{\mathbf{u}}(k)$ in Figure 1 is given by [7, 3]:

$$\hat{\mathbf{u}}(k) = \mathbf{E}\,\frac{\mathbf{W}_M}{\sqrt{M}}\,\mathbf{Z}_R\begin{bmatrix}\mathbf{C}_0 & \mathbf{C}_1\end{bmatrix}\begin{bmatrix}\mathbf{Z}_T & \mathbf{0}\\ \mathbf{0} & \mathbf{Z}_T\end{bmatrix}\begin{bmatrix}\frac{\mathbf{W}_M^H}{\sqrt{M}} & \mathbf{0}\\ \mathbf{0} & \frac{\mathbf{W}_M^H}{\sqrt{M}}\end{bmatrix}\begin{bmatrix}\mathbf{u}(k-1)\\ \mathbf{u}(k)\end{bmatrix} + \mathbf{E}\,\frac{\mathbf{W}_M}{\sqrt{M}}\,\mathbf{Z}_R\,\mathbf{r}(k) \qquad (2)$$

where $\mathbf{W}_M/\sqrt{M}$ and $\mathbf{W}_M^H/\sqrt{M}$ describe the orthonormalized DFT and IDFT matrix, respectively, and $\mathbf{Z}_T$ and $\mathbf{Z}_R$ the introduction and removal of the guard interval, respectively. $\mathbf{C} = \begin{bmatrix}\mathbf{C}_0 & \mathbf{C}_1\end{bmatrix}$ is the size $(M+L)\times 2(M+L)$ channel matrix combining the P/S conversion at the transmitter, the convolution with the channel impulse response and the S/P conversion at the receiver, and $\mathbf{r}(k)$ is the additive channel noise after S/P conversion. We here assume that the channel impulse response $c(n)$ is of length L_c and shorter than M what is generally the case for ADSL and VDSL. The entries of the matrices are then given by:

$$[\mathbf{W}_M]_{k,l} = \exp(-j\frac{2\pi kl}{M}), \quad [\mathbf{W}_M^H]_{k,l} = \exp(j\frac{2\pi kl}{M}),$$
$$k,l = 0,\ldots,M-1,$$
$$\mathbf{Z}_T = \begin{bmatrix}\mathbf{0}_{L\times(M-L)}\ \mathbf{I}_L\\ \mathbf{I}_M\end{bmatrix}, \quad \mathbf{Z}_R = \begin{bmatrix}\mathbf{0}_{M\times L} & \mathbf{I}_M\end{bmatrix},$$

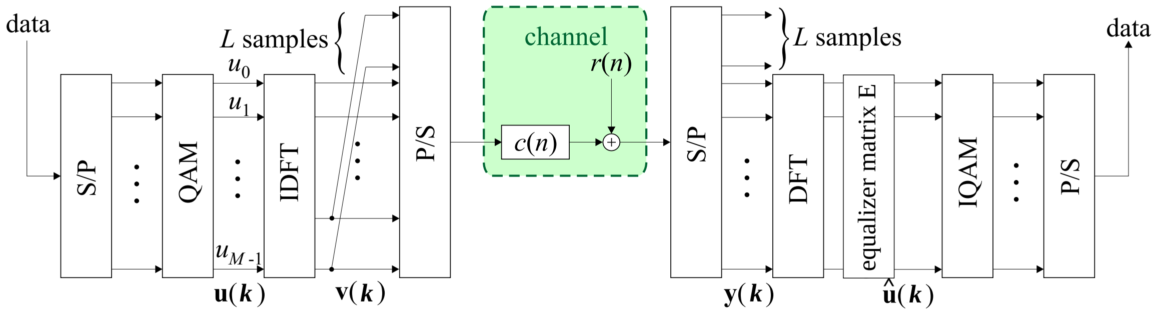

Fig. 1. GDMT transmission scheme

$$\mathbf{C}_0 = \begin{bmatrix} 0 & \cdots & c_{L_c-1} & \cdots & c_1 \\ \vdots & \ddots & & \ddots & \vdots \\ \vdots & & \ddots & \ddots & c_{L_c-1} \\ \vdots & \ddots & & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix},$$

$$\mathbf{C}_1 = \begin{bmatrix} c_0 & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ c_{L_c-1} & \cdots & c_0 & \ddots & \vdots \\ & \ddots & & \ddots & 0 \\ 0 & \cdots & c_{L_c-1} & \cdots & c_0 \end{bmatrix}.$$

3. ZERO FORCING EQUALIZATION

A zero-forcing equalizer removes the ISI and ICI introduced by the transmission channel. It is designed for the noise-free case and does not take noise enhancement into consideration. Given (2), ISI and ICI is removed if the following condition holds true:

$$\mathbf{E}\,\frac{\mathbf{W}_M}{M}\,\mathbf{Z}_R[\mathbf{C}_0 \quad \mathbf{C}_1]\begin{bmatrix}\mathbf{Z}_T & \mathbf{0}\\ \mathbf{0} & \mathbf{Z}_T\end{bmatrix}\begin{bmatrix}\mathbf{W}_M^H & \mathbf{0}\\ \mathbf{0} & \mathbf{W}_M^H\end{bmatrix}=[\mathbf{0}_M \quad \mathbf{I}_M] \quad (3)$$

To find the entries of $\mathbf{E}$, we introduce a matrix $\tilde{\mathbf{C}}$ that combines introducing the guard interval, the channel matrix $\mathbf{C}$, and removing the guard interval:

$$\tilde{\mathbf{C}} = [\tilde{\mathbf{C}}_0 \quad \tilde{\mathbf{C}}_1] = \mathbf{Z}_R\,[\mathbf{C}_0 \quad \mathbf{C}_1]\begin{bmatrix}\mathbf{Z}_T & \mathbf{0}\\ \mathbf{0} & \mathbf{Z}_T\end{bmatrix} \quad (4)$$

Introducing $\tilde{\mathbf{C}}$ into (3) and splitting it into two parts, we obtain the following constraints for zero-forcing equalization:

$$\mathbf{E}\,\mathbf{W}_M\,\tilde{\mathbf{C}}_0\,\mathbf{W}_M^H/M = \mathbf{0}_M, \quad (5)$$

$$\mathbf{E}\,\mathbf{W}_M\,\tilde{\mathbf{C}}_1\,\mathbf{W}_M^H/M = \mathbf{I}_M. \quad (6)$$

If the GI is of sufficient length $(L \geq L_c - 1)$ then $\tilde{\mathbf{C}}_0 = \mathbf{0}_M$, and thus (5) is always satisfied. Also $\tilde{\mathbf{C}}_1$ is circular such that $\mathbf{W}_M\,\tilde{\mathbf{C}}_1\,\mathbf{W}_M^H/M$ in (6) becomes a diagonal matrix. The zero-forcing equalizer $\mathbf{E}$ is identical to the DMT equalizer, namely:

$$[\mathbf{E}]_{k,k} = 1/C(e^{j2\pi k/M}), \quad k = 0,\ldots,M-1 \quad (7)$$

where $C(e^{j2\pi k/M})$ denotes the channel frequency response at the normalized frequencies $2\pi k/M$. If however the GI is of insufficient length $(L < L_c - 1)$ then $\tilde{\mathbf{C}}_0$ and $\tilde{\mathbf{C}}_1$ have the following form and it is no longer possible to solve (5) and (6) simultaneously:

$$\tilde{\mathbf{C}}_0 = \begin{bmatrix} 0 & \cdots & 0 & c_{L_c-1} & \cdots & c_{L+1} \\ \vdots & & & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & c_{L_c-1} \\ \vdots & & & & & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix}$$

$$\tilde{\mathbf{C}}_1 = \begin{bmatrix} c_0 & & & & c_L & \cdots & c_1 \\ \vdots & \ddots & & & \vdots & & \vdots \\ \vdots & & \ddots & & c_{L_c-1} & & \vdots \\ \vdots & & & \ddots & & \ddots & \vdots \\ c_{L_c-1} & & & & \ddots & & c_{L_c-1} \\ & \ddots & & & & \ddots & \\ & & c_{L_c-1} & \cdots & \cdots & \cdots & c_0 \end{bmatrix}$$

4. ZF EQUALIZATION FOR TRANSMISSION WITH UNUSED SUBCARRIERS

Assuming that K subcarriers are not used for data transmission, i.e. the value zero is transmitted in these subcarriers, the block equalizer $\mathbf{E}$ only needs to equalize the $N = M - K$ subcarriers used for data transmission, since there is no need to equalize unused subcarriers. In [3] it has been shown that perfect ZF equalization can be achieved for $K \geq L_c - L - 1$. The equalizer matrix can then be obtained from solving (5) and (6) for the used subcarriers only, yielding:

$$\mathbf{E} = \mathbf{S}_1\,\mathbf{C}_{\text{freq}}^{\dagger}\left(\mathbf{I}_M - \mathbf{W}_0((\mathbf{I}_M - \mathbf{S}_1)\mathbf{W}_0)^{\dagger}\right) \quad (8)$$

where † denotes the pseudo inverse. $\mathbf{C}_{\text{freq}}^{\dagger}$ is a diagonal matrix with

$$[\mathbf{C}_{\text{freq}}^{\dagger}]_{k,k} = \begin{cases} 1/C(e^{j2\pi k/M}), & \text{if } C(e^{j2\pi k/M}) \neq 0 \\ 0 & \text{otherwise} \end{cases} \quad (9)$$

$$k = 0,\ldots,M-1,$$

$\mathbf{S}_1 = \text{diag}(s_0,\ldots,s_{M-1})$ denotes a carrier selection matrix with

$$s_i = \begin{cases} 1 & \text{if subcarrier is used} \\ 0 & \text{if subcarrier is unused} \end{cases}$$

and $\mathbf{W}_0$ contains the first $L_c - L - 1$ columns of the DFT matrix $\mathbf{W}_M$. The nonzero entries of $\mathbf{E}$ are illustrated in Fig. 2. To equalize a used subcarrier, the signal is multiplied with the same scaling factor as in the original DMT scheme. In addition, a linear combination of the outputs of all unused subcarriers is added. The values received in the unused subcarriers contain ISI and ICI from used subcarriers as well as additive channel noise. The fact that the ISI and ICI component is not negligible is due to the low stopband attenuation of the IDFT at the transmitter that allows significant leakage into neighboring subcarriers.

$\mathbf{E} =$

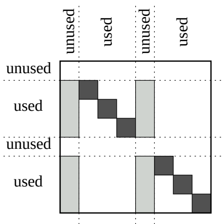

Fig. 2. Nonzero entries of equalizer matrix

Given the equalizer coefficients and the variance σ_r^2 of the additive channel noise, we can now calculate the noise variance at the output of the equalizer:

$$\text{diag}([\sigma_{n,0}^2, \sigma_{n,1}^2, \ldots, \sigma_{n,M-1}^2]) \tag{10}$$

$$= \sigma_r^2 \cdot \text{diag}\left(\mathbf{E}\frac{\mathbf{W_M}}{\sqrt{M}} \cdot \frac{\mathbf{W_M}^H}{\sqrt{M}}\mathbf{E}^H\right) = \sigma_r^2 \cdot \text{diag}\left(\mathbf{E} \cdot \mathbf{E}^H\right)$$

$$= \sigma_r^2\, \mathbf{C}_{\text{freq}}^{\dagger} (\mathbf{C}_{\text{freq}}^{\dagger})^H\, \mathbf{S}_1 \tag{11}$$

$$\cdot \left(\mathbf{I}_M + \text{diag}\left(\mathbf{W}_0\left(\mathbf{W}_0^H(\mathbf{I}_M - \mathbf{S}_1)\mathbf{W}_0\right)^{-1}\mathbf{W}_0^H\right)\right)$$

The derivation of (11) from (10) is described in [8]. The first noise term is the same as in a conventional DMT receiver with diagonal equalizer entries only. The second term arises from the non-zero non-diagonal entries in $\mathbf{E}$. It is proportional to the inverse of the squared channel magnitude response at the subcarrier frequency but in addition also depends on the position of the used and unused carriers since it contains the carrier selection matrix $\mathbf{S}_1$.

5. OPTIMAL SUBCARRIER SELECTION

The adaptive loading algorithm in a DMT system assigns bitrate and transmit energy to the subcarriers based on a specified bit error probability at the receiver and on the SNR per subcarrier. Since in GDMT, the equailzer noise enhancement not only depends on the channel frequency response but also on the placement of used and unused subcarriers, see (11), deciding which subcarriers to use becomes a more elaborate task than just choosing those with the highest channel magnitude response. In the following we will look at two special cases first before evaluating the general case.

5.1. Guard Interval is Too Short by One Tap

If $L_c - L - 1 = 1$ then the matrix $\mathbf{W}_0$ in (11) just consists of the first column of $\mathbf{W}_M$ and the inverse matrix in (11) is a scalar:

$$\left(\mathbf{W}_0^H\,(\mathbf{I}_M - \mathbf{S}_1)\,\mathbf{W}_0\right)^{-1} = \Big(\sum_{k=0}^{M-1}(1 - s_k)\Big)^{-1} = \frac{1}{K} \tag{12}$$

Substituting this result into (11) we obtain for the SNR at the output of a used subcarrier k :

$$SNR_k = \frac{\sigma_{u,k}^2}{\sigma_{n,k}^2} = \frac{\sigma_{u,k}^2 \cdot |C(e^{j\frac{2\pi k}{M}})|^2}{\sigma_r^2\left(1 + \frac{1}{K}\right)} \tag{13}$$

where $\sigma_{u,k}^2$ denotes the transmit power for subcarrier k. The SNR's in used subcarriers only depend on the channel magnitude frequency response and the number of unused subcarriers. Thus, once K has been chosen (and it has to be at least one since otherwise zero-forcing equalization is impossible) in order to select the subcarriers resulting in the highest data rate, we just have to choose those $N = M - K$ ones with the highest SNR's.

5.2. Unused Subcarriers are Spaced Equidistantly

The other special case that is easy to solve is where the inverse matrix in (11) is a scaled identity matrix. Remember that $\mathbf{W}_0$ consists of the first $L_c - L - 1$ columns of the M-point DFT matrix $\mathbf{W}_M$. Taking advantage of the fact that we can write $\mathbf{W}_0^H\,(\mathbf{I}_M - \mathbf{S}_1)\,\mathbf{W}_0$ as $\mathbf{W}_0^H\,(\mathbf{I}_M - \mathbf{S}_1)^H\,(\mathbf{I}_M - \mathbf{S}_1)\,\mathbf{W}_0$, we can conclude that the columns of $(\mathbf{I}_M - \mathbf{S}_1)\,\mathbf{W}_0$ must be orthogonal to each other. If the total number of subcarriers M is a power of two, then, if we choose K to be also a power of two, satisfying $K \geq L_c - L - 1$, and place the unused subcarriers M/K subcarriers apart from each other, the non-zero entries of $(\mathbf{I}_M - \mathbf{S}_1)\,\mathbf{W}_0$ form the first $L_c - L - 1$ (rotated) column vectors of a size K DFT matrix and are thus orthogonal. Taking further into consideration that in DMT the data in subcarrier $M - k$ is the complex conjugate of the data in subcarrier k, with $k = 1, \ldots, M/2 - 1$, in order to guarantee a real valued data at the output of the transmitter, yields the following possibilities for the carrier selection matrix $\mathbf{S}_1$:

$$s_i = \begin{cases} 0 & \text{if } i = j + \ell M/K, \quad j = 0,\, M/2K \\ 1 & \text{otherwise.} \qquad\qquad\quad\; \ell = 0, \ldots, K-1 \end{cases} \tag{14}$$

For these solutions we obtain $\mathbf{W}_0^H\,(\mathbf{I}_M - \mathbf{S}_1)\,\mathbf{W}_0 = K \cdot \mathbf{I}_{L_c - L - 1}$ and thus for the SNR in a used subcarrier k:

$$SNR_k = \frac{\sigma_{u,k}^2}{\sigma_{n,k}^2} = \frac{\sigma_{u,k}^2 \cdot |C(e^{j\frac{2\pi k}{M}})|^2}{\sigma_r^2\left(1 + \frac{L_c - L - 1}{K}\right)} \tag{15}$$

The SNR's again depend on the channel magnitude frequency response and the number of unused subcarriers but also on the number of samples by which the guard interval is too short.

5.3. General Case

To determine the noise variance at the equalizer output for a general placement of K unused subcarriers, we apply the matrix inversion lemma [9] to (11):

$$\text{diag}([\sigma_{n,0}^2, \sigma_{n,1}^2, \ldots, \sigma_{n,M-1}^2]) = \sigma_r^2\, \mathbf{C}_{\text{freq}}^{\dagger} (\mathbf{C}_{\text{freq}}^{\dagger})^H\, \mathbf{S}_1$$

$$\cdot \left(\mathbf{I}_M + \text{diag}\left(\frac{\mathbf{W}_0\,\mathbf{W}_0^H}{M}\left(\mathbf{I}_M - \mathbf{S}_1\frac{\mathbf{W}_0\,\mathbf{W}_0^H}{M}\right)^{-1}\right)\right) \tag{16}$$

$$= \sigma_r^2\, \mathbf{C}_{\text{freq}}^{\dagger} (\mathbf{C}_{\text{freq}}^{\dagger})^H\left(\mathbf{S}_1 - \mathbf{I}_M + \text{diag}\left(\left(\mathbf{I}_M - \mathbf{S}_1\frac{\mathbf{W}_0\,\mathbf{W}_0^H}{M}\right)^{-1}\right)\right)$$

For $||\mathbf{S}_1\,\mathbf{W}_0\,\mathbf{W}_0^H/M|| < 1$ the inverse matrix can be expressed using the Neumann expansion [9] and thus be approximated through a finite series:

$$\left(\mathbf{I}_M - \mathbf{S}_1\frac{\mathbf{W}_0\,\mathbf{W}_0^H}{M}\right)^{-1} = \sum_{i=0}^{\infty}\left(\mathbf{S}_1\frac{\mathbf{W}_0\,\mathbf{W}_0^H}{M}\right)^i \tag{17}$$

6. SIMULATION RESULTS

The performance of the GDMT ZF equalizer was evaluated through simulation of a GDMT transceiver in Matlab for a lowpass channel of length $Lc = 14$ with impulse response

$$c(n) = \frac{\sin(.55\pi(n-4))}{4.17n}, \quad n = 0, \ldots, 13. \qquad (18)$$

The GI was varied from 0 to $L_c - 1$ (traditional DMT), and AWGN channel noise σ_r^2 with different variances was applied. Each used subcarrier was assigned the transmit power $\frac{M}{N}\sigma_u^2$ such that the total transmit power remained constant independently of N. The bitrate per used subcarrier was then calculated according to (1) assuming a SNR gap Γ of 0 dB. The SNR at the output of a used subcarrier was calculated using (16). An iterative strategy was applied to assign used subcarriers to the carrier selection matrix $\mathbf{S}_1$: starting with just one used subcarrier in the index range $k = 1, \ldots, M/2-1$ (assuming that no data can be transmitted at dc ($k = 0$) and the Nyquist frequency ($k = M/2$) and that the range $k = M/2+1, \ldots, M-1$ is reserved for complex conjugate data) the optimal position is found. Then the next subcarrier and its complex conjugate copy is added the same way, until the number of used subcarriers reaches $M - (L_c - L - 1)$. Although this strategy does not guarantee optimality, simulation results have shown that for $L = L_c - 2$ it produces the same results as applying (13) directly. Denoting the sampling rate at the transmitter output $f_s = 1/T$, the bitrate is calculated as:

$$\text{bitrate} = \frac{f_s}{M+L} \sum_{k=1}^{M/2-1} b_k \qquad (19)$$

This approximation neglects the fact that b_k can only take integer values. Figure 3 shows the bitrate normalized by the sampling period for $M = 32$. For a small number of used subcarriers, the higher bandwidth efficiency when using a small GI is contributing more than the noise enhancement that increases with decreasing L. The weights of both contributions change when increasing the number of used subcarriers and then results in higher bitrates for longer GI's. The lower the additive channel noise, the further this point is moved towards higher numbers of used subcarriers.

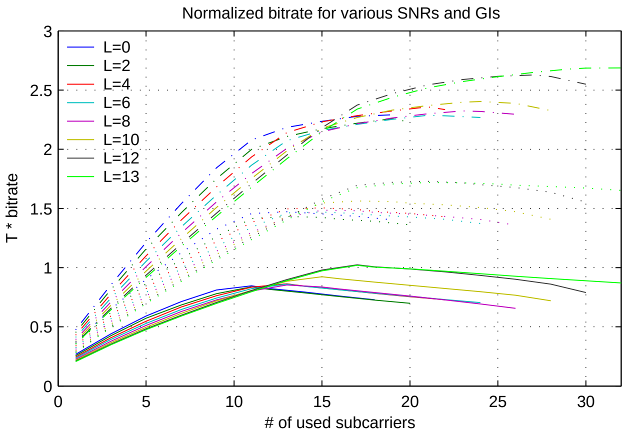

Fig. 3. Normalized bitrates for $20\log_{10}(\sigma_u^2/\sigma_r^2)$ = 10 dB (solid lines), 20 dB (dotted lines), and 30 dB (dashed lines).

Fig. 4 shows the normalized bitrate over the transceiver latency time that is proportional to $M + L$ for $M = 32$ and 64 and $L = 0$ to 13. For each value of L, the number of used subcarriers that resulted in the highest data rate was selected. GDMT allows to reduce the latency time at a moderate loss of data rate.

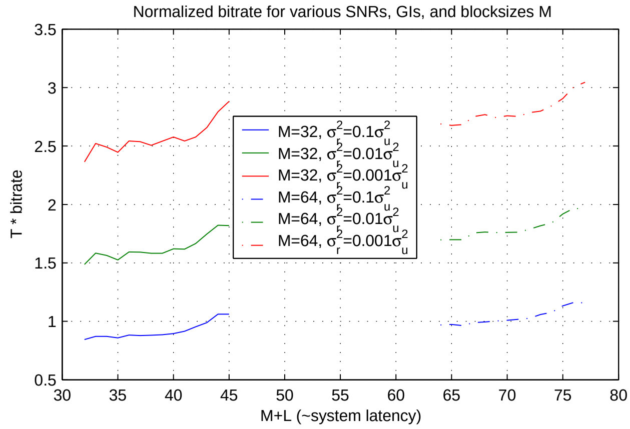

Fig. 4. Normalized bitrates for $20\log_{10}(\sigma_u^2/\sigma_r^2)$ = 10 dB, 20 dB, and 30 dB depending on system latency for $M = 32$ and 64.

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