μ-Bicomplete Categories and Parity Functors

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In this talk we will introduce to the theory of categories having the following completeness properties: they have finite *sums*, finite *products* and all the *initial algebras* and *final coalgebras* of the functors which can be constructed out of these four operations. These categories generalize μ-lattices [San00] and μ-algebras [Niw85] on one side, and on the other side they partially generalize bicomplete categories [Joy95]. For this reason we call them *μ-bicomplete*.

The focus of this talk will be on the structure of objects of μ-bicomplete categories, complementing previous work [San01] where we have analyzed the structure of arrows. The principal goal will be to exemplify the way the category of sets is μ-bicomplete, a case that in our opinion is of primary importance. It is well known that several set-theoretic structures of relevance to computer science (i.e., finite or infinite trees, finite or infinite terms, languages, inductive types, coinductive types, etc.) can be described either by using the language of initial algebras or by using the language of final coalgebras. But what happens if we use both languages? We provide an answer to this question by translating the algebraic language of μ-bicomplete categories into the combinatorial language of parity games. This is made possible by endowing parity games with an algebraic meaning, which is shown to be equivalent to the one of μ-bicomplete categories. On the combinatorial side, parity games can be considered as recognizers of infinite objects in a natural way: they recognize the set of deterministic winning strategies for a given player. The two different meanings of parity games, the algebraic one and the combinatorial one, are then shown to coincide. By means of this result we support the claim that the algebra of parity games is the one of μ-bicomplete categories and that the combinatorics of μ-bicomplete categories is the one of parity games.

1 μ-Bicomplete Categories

The obvious way to define μ-bicomplete categories mimics at the level of categories the definition of μ-lattices and of μ-algebras: μ-terms are defined and an algebra is a μ-algebra if it is possible to interpret μ-terms as expected. In a categorical context a μ-term will be interpreted as a functor instead of an order preserving function.

Definition 1.1 We define set of μ-terms in context as follows:

1. For each pair (X, x), where X is a finite set and $x \in X$, $x \in \mu\mathcal{T}(X)$.
2. If I is a finite set and $s : I \longrightarrow \mu\mathcal{T}(X)$, then $\bigwedge_I s, \bigvee_I s \in \mu\mathcal{T}(X)$.
3. If $s \in \mu\mathcal{T}(X)$ and $x \in X$, then $\mu_x.s, \nu_x.s \in \mu\mathcal{T}(X \setminus \{x\})$.

Definition 1.2 Let $\mathcal{C}$ be a category with finite products and finite coproducts. We define a partial interpretation of μ-terms in context $s \in \mu\mathcal{T}(X)$ as functors of the form $\| s \| : \mathcal{C}^X \longrightarrow \mathcal{C}$.

1. For $x \in \mathcal{T}(X)$, we let $\| x \| = \mathtt{pr}_x : \mathcal{C}^X \longrightarrow \mathcal{C}$.
2. We let $\| \bigwedge_I s \| = \prod_{i \in I} \| s_i \|$ and $\| \bigvee_I s \| = \coprod_{i \in I} \| s_i \|$, given that all the $\| s_i \|$ are defined.
3. We let $\| \mu_x.s \|$ be the parameterized initial algebra of

$$\| s \| : \mathcal{C} \times \mathcal{C}^{X \setminus \{x\}} \longrightarrow \mathcal{C}\,,$$

given that $\| s \|$ is defined. Similarly we let $\| \nu_x.s \|$ be the parameterized final coalgebra of $\| s \|$. If $\| s \|$ is not defined or if the desired initial algebras (final coalgebras) do not exist, then we leave $\| \mu_x.s \|$ ($\| \nu_x.s \|$) undefined.

Definition 1.3 A category with finite products and finite coproducts $\mathcal{C}$ is said to be *μ-bicomplete* if for each finite set of variables X and term $s \in \mu\mathcal{T}(X)$ the interpretation $\| s \|$ is defined.

In order to show that a category $\mathcal{C}$ is μ-bicomplete, it is enough to fix a class of functors of the form $\mathcal{C}^J \longrightarrow \mathcal{C}$, where J ranges on finite sets, and afterward show that this class contains the projections and is closed under finite products and coproducts, and formation of parameterized initial algebras and parameterized final coalgebras. It is well known that final coalgebras of λ-accessible unary functors exist in a locally presentable category [Bar93]. We refine ideas developed there to study how final coalgebras behave in a parameterized context. We obtain the next proposition which is the necessary step to show that locally presentable categories are μ-bicomplete.

Proposition 1.4 *Let $\mathcal{C}$ be a locally λ-presentable category, where $\lambda > \omega$ is a regular cardinal. The class of λ-accessible functors of the form $\mathcal{C}^J \longrightarrow \mathcal{C}$ is closed under formation of parameterized initial algebras and parameterized final coalgebras.*

Since it is well known that this class contains the projections and is closed under finite products and finite coproducts (cf. for example [AR94, §1.59]), it follows that

Corollary 1.5 *Every locally presentable category is μ-bicomplete.*

2 Parity Functors

We would like to have some kind of smoother terms. To this end, we generalize the usual notion of a parity game – cf. for example [AN01, Zie98] – in the following way:

Definition 2.1 A *parity game* is a tuple $G = \langle S, \rho, \kappa, \epsilon \rangle$, where

- $S = \langle \delta_0, \delta_1 : M \longrightarrow P \rangle$ is a finite graph of positions and moves. For each $p \in P$, we let M_p be $\delta_0^{-1}(p)$.
- $\rho : P \longrightarrow \{1, \ldots, n, \omega\}$ is a function such that, if $\rho(p) = \omega$, then $M_p = \emptyset$. We let $P_i = \rho^{-1}(i)$, $P_{<i} = \bigcup_{j<i} P_i$, for $i \in \{1, \ldots, n, \omega\}$.
- $\kappa : \{1, \ldots, n\} \longrightarrow \{\mu, \nu\}$.
- $\epsilon : P_{<\omega} \longrightarrow \{\sigma, \pi\}$.

We shall say that n is the height of G.

We interpret the above data as a two person game $G(E)$, parameterized in a choice of sets $E = \{E_p\}_{p \in P_\omega}$. The graph S is a board with a set of positions P and a set of allowed moves M. A move $m \in M$ is from position $\delta_0(m)$ to position $\delta_1(m)$; observe that we allow different moves relating the same pair of positions. From a position p the set of moves M_p is available and player $\epsilon(p)$ among players σ and π must choose how to move. If he cannot move, then he loses. On an infinite play $\gamma = \gamma_0 \to \gamma_1 \to \ldots \gamma_n \to \ldots$ we will be able to find regions among $P_1, \ldots, P_n$ which are visited infinitely often, and among them we will be able to pick a region P_k with k maximal. Then, this infinite path is a win for player σ if and only if k is colored by ν. More formally, if we let

$$\mathrm{In}\,\gamma \quad = \quad \{\, i \in \{1, \ldots, n\} \,|\, \mathrm{card}\,\{\, n \,|\, \rho(\gamma_n) = i \,\} = \omega \,\} \,,$$

then γ is a win for player σ if and only if

$$\epsilon(\,\max \mathrm{In}\,\gamma\,) \quad = \quad \nu \,.$$

If a play ends in a position $p \in P_\omega$, then player σ must choose an element $e \in E_p$, and then he wins. If $E_p = \emptyset$, then he loses.

Definition 2.2 Let $G = \langle S, \rho, \kappa, \epsilon \rangle$ be a parity game of height $n > 0$, then its predecessor game $P(G)$, of height $n - 1$, is obtained from G by erasing all the moves from P_n. More precisely, $P(G) = \langle S', \rho', \kappa', \epsilon' \rangle$, where:

- $S' = \langle \delta_0, \delta_1 : \delta_0^{-1}(P_{<n}) \longrightarrow P \rangle$.
- $\rho'(p) = \rho(p)$ if $\rho(p) < n$, otherwise $\rho'(p) = \omega$.
- For $i \in \{1, \ldots, n-1\}$, we let $\kappa'(i) = \kappa(i)$.
- If $\rho'(p) < n$ we let $\epsilon'(p) = \epsilon(p)$.

Let $\mathcal{C}$ be a category with products and coproducts and let G be a parity game, which we assume to be of height n.

Definition 2.3 For each $p \in P_{<\omega}$, let

$$\begin{aligned}
\mathtt{pr}(\delta_1, p) &= \left\langle \mathtt{pr}_{\delta_1(m)} \right\rangle_{m \in M_p} && : \mathcal{C}^P \longrightarrow \mathcal{C}^{M_p} \\
E_p &= \begin{cases} \prod \circ \mathtt{pr}(\delta_1, p)\,, & \epsilon(p) = \pi \\ \coprod \circ \mathtt{pr}(\delta_1, p)\,, & \epsilon(p) = \sigma \end{cases} && : \mathcal{C}^P \longrightarrow \mathcal{C} \\
E_{P_n} &= \left\langle E_p \right\rangle_{p \in P_n} && : \mathcal{C}^P \longrightarrow \mathcal{C}^{P_n}\,.
\end{aligned}$$

Definition 2.4 Let $\mathcal{C}$ be a category with finite products and finite coproducts. We define a partial correspondence $\|-\|$, from the class of parity games to the class of functors of the form $\mathcal{C}^{P_\omega} \longrightarrow \mathcal{C}^{P_{<\omega}}$, by induction on the height, as follows. If $n = 0$, then $P_{<\omega} = \emptyset$ so that there is a unique choice of $\|\,G\,\|$. Suppose that $n > 0$ and that $\|\,P(G)\,\|$ is defined. Let

$$F \quad = \quad \|\,P(G)\,\| \circ \mathtt{pr}_{\mathcal{C}^{P_n} \times \mathcal{C}^{P_\omega}}\,,$$

and consider the functor

$$\mathcal{C}^{P_{<n}} \times \mathcal{C}^{P_n} \times \mathcal{C}^{P_\omega} \xrightarrow{\langle F, E_{P_n} \rangle} \mathcal{C}^{P_{<n}} \times \mathcal{C}^{P_n} .$$

If $\kappa(n) = \mu$, then we let $\| G \|$ be the parameterized initial algebra of the above functor, otherwise, if $\kappa(n) = \nu$, we let $\| G \|$ be its parameterized final coalgebra. If $\| P(G) \|$ is undefined or if the required initial algebras or final coalgebras do not exist, then $\| G \|$ is undefined. We say that $\mathcal{C}$ is *complete with respect to parity functors* if for each parity game G, the functor $\| G \| : \mathcal{C}^{P_\omega} \longrightarrow \mathcal{C}^{P_{<\omega}}$ is defined.

With the following proposition we generalize to categories the well known fact that a vectorial μ-calculus has no more expressive power of its scalar version [AN01, §2.7].

Proposition 2.5 *A category is complete with respect to parity functors if and only if it is μ-bicomplete.*

In order to prove the proposition we translate a μ-term into a parity game with a given starting position and then show that the given projection of the parity functor has the universal property defining the interpretation of the μ-term. In a similar way but in the opposite direction we represent parity functors by collections of μ-terms, making essential use of the Bekič property. To carry out the two translations we apply the usual equational properties of the least prefixed point, cf. [Ési97], which have been generalized to initial algebras of functors in several occasions [Fre91, BÉ93, ÉL98]. We emphasize that proposition 2.5 depends also on the fact that the existence of certain initial algebras is implied by the existence of other initial algebras, or, more precisely, that some initial algebras are created or preserved by suitable functors. The existence of canonical isomorphisms between two different representations of an initial algebra is then a consequence of sharing the same universal property. To illustrate this point, we state explicitly the Bekič property as follows:

Proposition 2.6 *Consider a functor $\langle F, G \rangle : \mathcal{C} \times \mathcal{D} \times \mathcal{E} \longrightarrow \mathcal{C} \times \mathcal{D}$ and suppose that*

- *The functors $F(-, d, e)$ admit initial algebras $\chi_{d,e} : F(\mathbf{x}_{d,e}, d, e) \longrightarrow \mathbf{x}_{d,e}$.*
- *The functors $\langle \mathbf{x}_{-} \circ \mathbf{pr}_{\mathcal{D}\times\mathcal{E}}(-, e), G(-, e) \rangle : \mathcal{C} \times \mathcal{D} \times \mathcal{E} \longrightarrow \mathcal{C} \times \mathcal{D}$ admit initial algebras $\xi_e : \mathbf{x}_{\mathbf{z}_e, e} \longrightarrow \mathbf{y}_e$ and $\zeta_e : G(\mathbf{y}_e, \mathbf{z}_e, e) \longrightarrow \mathbf{z}_e$.*

Then the pair

$$\begin{aligned} \chi_{\mathbf{z}_e, e} &: F(\mathbf{x}_{\mathbf{z}_e, e}, \mathbf{z}_e, e) \longrightarrow \mathbf{x}_{\mathbf{z}_e, e} \\ \zeta_e \circ G(\xi_e, \mathbf{z}_e) &: G(\mathbf{x}_{\mathbf{z}_e, e}, \mathbf{z}_e, e) \longrightarrow \mathbf{z}_e \end{aligned}$$

is an initial $\langle F(-, e), G(-, e) \rangle$-algebra.

3 Parity Functors in the Category of Sets

In the category of sets a typical element of an inductive type is a kind of finite tree; on the other hand, a typical element of a coinductive type is a kind of infinite tree. We shall see that a similar tree-like representation is available for parity functors as well.

Definition 3.1 A *tree* over a pointed graph $\langle S, p_0 \rangle$ is a non empty, prefix closed collection of paths γ in S such that $\mathrm{dom}\,\gamma = p_0$.

Definition 3.2 Let G be a parity game and let $E = \{E_p\}_{p \in P_\omega}$ be a collection of sets. A *deterministic winning strategy* for player σ from position $p_0 \in P$ in the game $G(E)$ is a pair $\langle T, \lambda \rangle$ where T is a tree over the pointed graph $\langle S, p_0 \rangle$ such that:

- If $\epsilon(\mathrm{cod}\,\gamma) = \pi$ and $m \in M_{\mathrm{cod}\,\gamma}$, then $\gamma \star m \in T$.
- If $\epsilon(\mathrm{cod}\,\gamma) = \sigma$, then there exists a unique $m \in M_{\mathrm{cod}\,\gamma}$ such that $\gamma \star m \in T$,
- Every infinite path in the tree T is a win for player σ.

On the other hand, λ is a labeling of paths $\gamma \in T$ such that $\mathrm{cod}\,\gamma \in P_\omega$ by an element $e \in E_{\mathrm{cod}\,\gamma}$. We let $\mathcal{S}_{G_{p_0}}(\,E\,)$ be the set of deterministic winning strategies for player σ in the game $G(E)$ from position p_0.

Proposition 3.3 *We have the equality*

$$\|\,G\,\|(E) \quad = \quad \left\langle\, \mathcal{S}_{G_p}(E)\,\right\rangle_{p \in P_{<\omega}} .$$

The above equality means that the sets of deterministic winning strategies satisfy the universal properties involved in the definition of parity functors, so that they can be taken to be a concrete representation of those functors. This equality is reminiscent of the well known formula of the Propositional Modal μ-Calculus which describes the set winning position for player σ in a parity game, cf. [Wal96] for example. This formula has been a motivation to develop the ideas presented here, however we are not able yet to give a mathematical account of this analogy.
We end with a series of examples. The first two are meant to make more concrete the ideas presented. The last two are meant to suggest some connections with related contexts.

Example 3.4 We shall consider the set of finite lists over a set of symbols E, actually the initial algebra of the functor $1 + Y \times E$. This is the denotation of the μ-term $\mu_y.(\top \vee (y \wedge E))$, which we translate in figure 1 into a parity game, according to proposition 2.5. For convenience of exposition, we have labeled transitions in the figure, even if this is not strictly necessary. Positions of the games, labeled by σ or π, are grouped within boxes according to their height. The height is on the right of the boxes, the color is on the left. It is immediate to realize that there is a bijection between lists and deterministic winning strategies in the parity game. If we let $E = \{0, 1\}$, we have written the list $\mathrm{cons}(\mathrm{cons}(\mathrm{nil}, 0), 1)$ in the form of a winning strategy (the tree over the game). Observe that we cannot obtain infinite lists since every infinite path on a tree would be a loss for player σ.

Example 3.5 We want to calculate an algebraic expression describing the set of infinite trees with the following properties: 1) every node is labeled by an element of a given set E, 2) every node has a finite (possibly empty) list of sons. According to experience, this set could be expressed as the greatest solution of the equation

$$X \quad = \quad E \times X^*\,,$$

that is, the final coalgebra of the functorial expression on the right. On the other hand, we know that X^* is the least solution of

$$Y \quad = \quad 1 + Y \times X\,.$$

Hence we guess that the desired algebraic expression is given by the μ-term $\nu_x.(E \wedge \mu_y.(\top \vee (y \wedge x))$. We can verify that the guess is right by transforming the μ-term into a parity game,

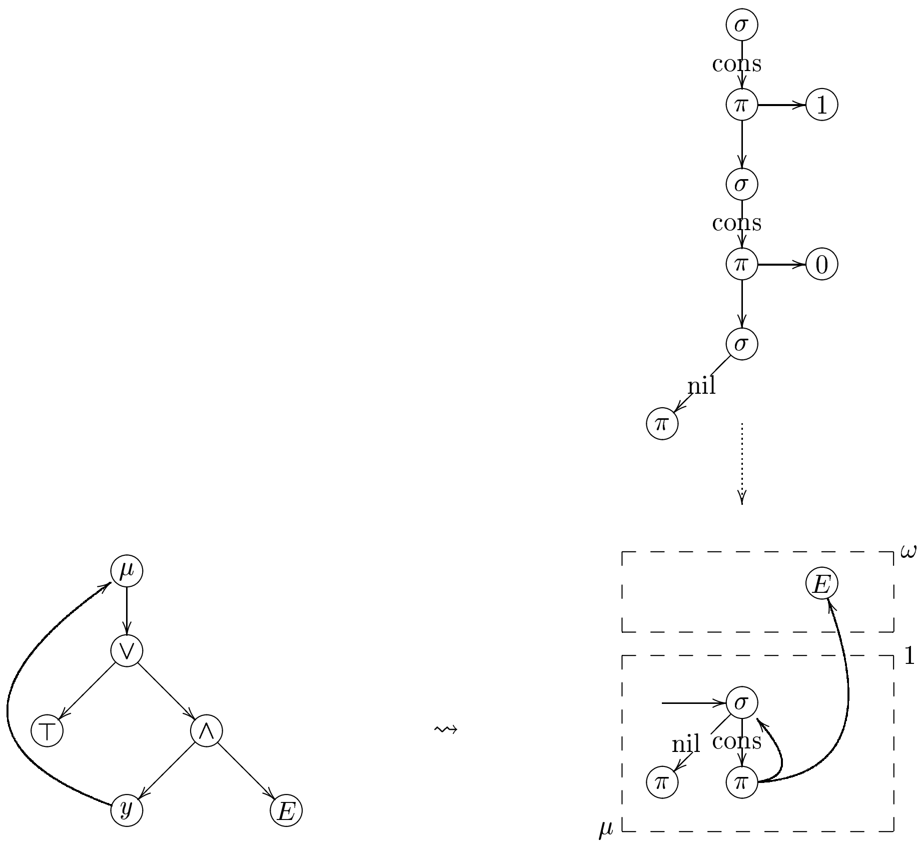

Figure 1: Lists as winning strategies

according to proposition 2.5, the result being the game on the right of figure 3.5. It is possible to convince ourself that given a labeled tree with those properties we have available a deterministic winning strategy for σ in the game – try to answer opponent's questions – and, conversely, that every such strategy has to come from a tree of this kind.

It is worth understanding what is going on with infinite paths. Player σ cannot answer that a node has an infinite list of sons: this would be done by answering infinitely often "cons" to the question "What tail ?", without being asked the question "What list is down ?". The maximal region visited infinitely often in such a play is colored by μ, hence it is a loss for player σ. On the other hand, player σ can answer infinitely often "cons" provided the play is going down in examining the tree, that is, provided this answer is alternating with the question "What list is down ?". The maximal region visited infinitely often in such a play is colored by ν, hence it is a win for player σ.

Example 3.6 It is well known that infinite finitely branching trees can be encoded as infinite binary trees. Proposition 3.3 can be taken to be a generalization of this fact, in that it shows that the elements of every nullary parity functor can be encoded as infinite trees with a bounded out-degree.

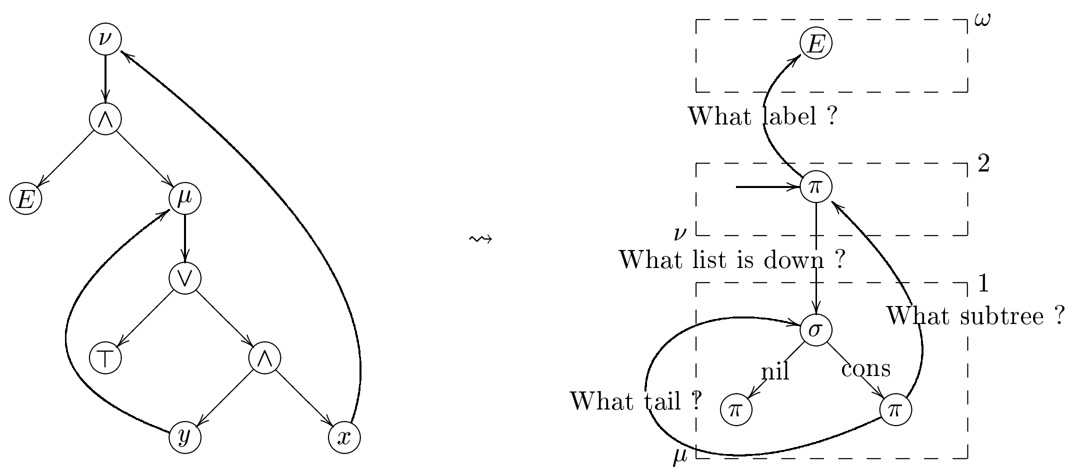

Figure 2: Infinite trees as winning strategies

Example 3.7 Charity [CS95] is a programming language designed out of categorical principles. In this context recursion and corecursion are derived from the properties of initial and final coalgebras. An important principle of this programming language states that it is possible to define an arrow $f : \mu_x.T(x) \times B \longrightarrow C$ from an algebra in context $g : T(C) \times B \longrightarrow C$, provided T is a strong categorical datatype [CS92]. This means that T comes with a natural transformation (a strength)

$$\theta^T_{A,B} : T(A) \times B \longrightarrow T(A \times B)$$

satisfying associativity and unitary constrains. The explicit characterization of (set-theoretic) parity functors allows to compute directly a strength. If A and B are two collections of sets indexed by P_ω, then we can associate to a strategy $\langle T, \lambda \rangle \in \mathcal{S}(G_p)(A)$ and to a collection $b = \{b_p\}_{p \in P_\omega}$ the strategy $\langle T, \lambda^b \rangle \in \mathcal{S}(G_p)(A \times B)$, where if $\gamma \in T$ and $\operatorname{cod}\gamma \in P_\omega$ then $\lambda^b(\operatorname{cod}\gamma) = (\lambda(\operatorname{cod}\gamma), b_{\operatorname{cod}\gamma})$.

References

[AN01] A. Arnold and D. Niwinski. *Rudiments of mu-calculus.* Number 146 in Studies in Logic and the Foundations of Mathematics. Elsevier (North-Holland), 2001.

[AR94] J. Adámek and J. Rosický. *Locally presentable and accessible categories.* Cambridge University Press, Cambridge, 1994.

[Bar93] M. Barr. Terminal coalgebras in well-founded set theory. *Theoret. Comput. Sci.*, 114(2):299–315, 1993.

[BÉ93] S. L. Bloom and Z. Ésik. *Iteration theories.* Springer-Verlag, Berlin, 1993. The equational logic of iterative processes.

[CS92] J. R. B. Cockett and D. Spencer. Strong categorical datatypes. I. In *Category theory 1991 (Montreal, PQ, 1991)*, pages 141–169. Amer. Math. Soc., Providence, RI, 1992.

[CS95] J. R. B. Cockett and D. Spencer. Strong categorical datatypes. II. A term logic for categorical programming. *Theoret. Comput. Sci.*, 139(1-2):69–113, 1995.

[ÉL98] Z. Ésik and A. Labella. Equational properties of iteration in algebraically complete categories. *Theoret. Comput. Sci.*, 195(1):61–89, 1998. Mathematical foundations of computer science (Cracow, 1996).

[Ési97] Z. Ésik. Completeness of Park induction. *Theoret. Comput. Sci.*, 177(1):217–283, 1997. Mathematical foundations of programming semantics (Manhattan, KS, 1994).

[Fre91] P. Freyd. Algebraically complete categories. In *Category theory (Como, 1990)*, pages 95–104. Springer, Berlin, 1991.

[Joy95] A. Joyal. Free bicomplete categories. *C. R. Math. Rep. Acad. Sci. Canada*, 17(5):219–224, 1995.

[Niw85] D. Niwiński. Equational μ-calculus. In *Computation theory (Zaborów, 1984)*, pages 169–176. Springer, Berlin, 1985.

[San00] L. Santocanale. Free μ-lattices. Research Series RS-00-28, BRICS, Department of Computer Science, University of Aarhus, November 2000.

[San01] L. Santocanale. A calculus of circular proofs and its categorical semantics. Research Series RS-01-15, BRICS, Department of Computer Science, University of Aarhus, May 2001. 30 pp.

[Wal96] Igor Walukiewicz. Pushdown processes: Games and model checking. Research Series RS-96-54, BRICS, Department of Computer Science, University of Aarhus, December 1996. 31 pp.

[Zie98] W. Zielonka. Infinite games on finitely coloured graphs with applications to automata on infinite trees. *Theoret. Comput. Sci.*, 200(1-2):135–183, 1998.