Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 373–381.

On Asymptotic Decompositions for Solutions of Systems of Differential Equations in the Case of Multiple Roots of the Characteristic Equation

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This paper is a surwey of results on asymptotic expansions of solutions of linear systems $\varepsilon\frac{dx}{dt} = A(t)x$, when the roots of matrix $A(t)$ are multiple. The results, obtained by the author, as well as by other mathematicians, are briefly reviewed, and some open problems are listed.

1 A short historical review

The paper presents the results on investigation of linear differential systems with coefficients depending on "slow" time $\tau = \varepsilon t$ ($\varepsilon > 0$ is a small parameter). Fundamental results on investigation of such systems were obtained by S.F. Feshchenko, S.G. Krein, Yu.L. Daletskii, I.Z. Shtokalo, I.M. Rapoport. The works of these authors appeared under direct influence of asymptotic methods developed by N.M. Krylov, N.N. Bogoliubov, Yu.A. Mitropolskii.

The sources of construction of asymptotic decompositions for solutions of systems of differential equations containing a parameter, can be found in the papers by Liouville, Birkhoff, Schlesinger, Tamarkin.

In particular, Liouville considered the issue of decomposition of arbitrary functions on fundamental functions of the equation

$$\frac{d^2y}{dx^2} + (\lambda q(x) - r(x))y = 0. \tag{1.1}$$

The fundamental functions obtained by Liouville for the equation (1.1) in the case of large values of the parameter λ, possess the property of orthogonality. For this reason the form of decomposition of a given function with respect to fundamental functions of the equation (1.1) can be determined directly. It is necessary only to show that 1) the constructed series converges; 2) it represents the given function. Liouville showed the convergence of the series by means of asymptotic formulae for fundamental functions that he obtained. The proof of the statement 2 was obtained by means of certain Sturm's results.

After the papers of Sturm and Liouville the theory of asymptotic representation of functions begun to develop quickly.

However, all these studies were concerned with self-conjugate differential equations. These limitations were removed in the investigations by Schlesinger, Birkhoff, Tamarkin.

Birkhoff considered construction of an asymptotic solution for the differential equation

$$\frac{d^ny}{dx^n} + \rho a_{n-1}(x,\rho)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + \rho^n a_0(x,\rho)y = 0, \tag{1.2}$$

where $a_i(x,\rho)$, $(i = 0, 1, \ldots, n-1)$ are analytical functions with respect to the complex parameter ρ on infifnity and have derivatives of all orders by real variable $x \in [a;b]$. Unlike Schlesinger who proved the asymptotic property of solutions only on some fixed ray $\arg\rho = \alpha$ for large $|\rho|$, Birkhoff proves the same properties for the area $\theta < \arg\rho < \psi$.

Tamarkin generalized Birkhoff's results for systems of linear differential equations

$$\frac{dy_i}{dx} = \sum_{k=1}^{n} a_{ik}(x,\rho)y_k, \qquad i = 1,\ldots,n, \tag{1.3}$$

where $a_{ik}(x,\rho)$ are single-value functions of complex parameter ρ, analytical near the point $\rho = \infty$ but having singularities with $\rho = \infty$ (a pole of the order $r \geq 1$). The asymptotic expressions for solutions of the system (1.3), derived by Birkhoff, contain as particular cases similar formulae, established by other methods by Schlesinger for systems of the form (1.3) and by Birkhoff for a differential equation of the order n (the latter considered the case $r = 1$).

In 1936 the paper by Trzitzinsky appeared where he gave a complete exposure of the issue of asymptotic representation for solutions of systems of ordinary differential equations with generalization of the Schlesinger–Birkhoff–Tamarkin theory for the case of linear integral-differential equations.

During the period of 1940–1945 a series of V.S. Pugachiov's papers appeared in which, unlike the previous researchers, the author presented the asymptotic representation for solutions in more general form.

We can also speak about papers by G.L. Turritin and M. Hukuchara as papers on asymptotic issues, where the asymptotic decomposition of a system of linear differential equations, with coefficients depending on a parameter, into lower-order systems.

At the end of a short historical review of classical papers on asymptotic representation for solutions of linear differential equations, we shall note that these methods were comprehensively and fruitfully developed in the following. The extensive lists of references related to these investigations are given in the books [1, 2].

As we have mentioned above, under the influence of asymptotic methods of Krylov–Bogoliubov–Mitropolskii the investigations on linear differential equations containing a small parameter in a singular way, started to develop extensively.

S.F. Feshchenko obtained the first results in this direction in 1948–1949. For the equation

$$\frac{d^2y}{dt^2} + \varepsilon\rho(\tau,\varepsilon)\frac{dy}{dt} + q(\tau,\varepsilon)y = \varepsilon f(\tau,\varepsilon)\cdot e^{i\theta(t,\varepsilon)}, \tag{1.4}$$

where $\rho(\tau,\varepsilon)$, $q(\tau,\varepsilon)$, $f(\tau,\varepsilon)$ are slowly changing functions, allowing decomposition by degress of the small parameter ε. The case when the function $\nu(\tau)$ $\left(\nu(\tau) = \frac{d\theta(t,\varepsilon)}{dt}\right)$ with certain τ from the area of its variation coincides with one of the simple roots of the characteristic equation, constructed for the equation (1.4) was considered, that is very important from mathematical physics applications perspective, and also from the theoretical side. This case was named "resonance" by the author.

The theorems proved by S.F. Feshchenko allow to construct an asymptotic solution for the equation (1.4) in the "resonance" and "non-resonance" (when $\nu(\tau)$ for any τ does not coincide with any root of the characteristic equation) cases.

The similar theorems were obtained by S.F. Feshchenko for the system of linear differential equations of the form (1.4).

Then S.F. Feshchenko obtained very important results on asymptotic decomposition of systems of linear differential equations of the form

$$\frac{dx}{dt} = A(\tau,\varepsilon)x, \tag{1.5}$$

where x is an n-dimensional vector, $A(\tau,\varepsilon)$ is real square matrix of the order n allowing the representation

$$A(\tau,\varepsilon)=\sum_{s=0}^{\infty}\varepsilon^{s}A_{s}(\tau). \tag{1.6}$$

In particular he proved the following theorems.

Theorem 1.1. *Let us assume that the roots of the characteristic equation*

$$\det\|A_0(\tau)-\lambda\cdot E\|=0 \tag{1.7}$$

(E is a unit matrix) can be splitted into two groups $\lambda_1(\tau),\ldots,\lambda_r(\tau)$ and $\lambda_{r+1}(\tau),\ldots,\lambda_n(\tau)$ so that no root from the first group for all $\tau\in[0;L]$ is equal to roots from the second group. Then, if $A(\tau,\varepsilon)$ on the interval $[0,L]$ has derivatives on τ of all orders, the system of differential equations (1.5) has a formal solution of the form

$$x=U_1(\tau,\varepsilon)\xi_1+U_2(\tau,\varepsilon)\xi_2, \tag{1.8}$$

where $U_1(\tau,\varepsilon)$, $U_2(\tau,\varepsilon)$ are rectangular matrices of the size correspondingly $(n\times r)$, $(n\times n-r)$ and ξ_1 is an r-dimensional vector, ξ_2 is a $n-r$-dimensional vector, determined by systems of differential equations

$$\frac{d\xi_1}{dt}=W_1(\tau,\varepsilon)\xi_1,\qquad \frac{d\xi_2}{dt}=W_2(\tau,\varepsilon)\xi_2 \tag{1.9}$$

of the order correspondingly r and $n-r$.

Theorem 1.2. *If $A(\tau,\varepsilon)$ satisfies the conditions of the theorem 1.1 and eigenvalues of the matrices*

$$\Delta_i(\tau)=\frac{1}{2}\left(W_i(\tau)+W_i^{*}(\tau)\right),\qquad i=1,2,$$

where $W_1(\tau)$, $W_2(\tau)$ are diagonal cells of the matrix $T^{-1}(\tau)A_0(\tau)T(\tau)$ ($T(\tau)$ is a matrix of transformation, $T^{-1}(\tau)$ is the inverse of $T(\tau)$), $W_1^{*}(\tau)$, $W_2^{*}(\tau)$ are matrices conjugate respectively to the matrices $W_1(\tau)$, $W_2(\tau)$ and are non-positive, then for any $L>0$ and $0<\varepsilon\le\varepsilon_0$ it is possible to find such constant $c>0$ not depending on ε, that if only $x\mid_{t=0}=x_m\mid_{t=0}$ (x_m is an m-approximation), then

$$\|x-x_m\|\le\varepsilon^{m}c. \tag{1.10}$$

Using of Theorems 1.1, 1.2 it is possible to asymptotically lower the order of the system (1.5). In particular, if all roots of the equation (1.7) are distinct at the interval $[0,L]$, then these theorems allow to obtain an asymptotic solution for the system (1.5).

However, by means of theorems on asymptotic decomposition it is possible mainly only to lower the order of the initial system. In the case of multiple roots of the characteristic equation it is impossible to get a solution of the initial system differential equations by means of these theorems. Though this case is frequently encountered both in investigation of theoretical issues and in solution of practical problems. Even in investigation of one of the simplest equations – the Sturm–Liouville equation – we encounter a multiple root. These roots are also encountered in investigation of systems of differential equations with a small parameter at certain derivatives in the problems of optimal control. Let us note that the case of multiple roots, especially when multiple elementary divisors correspond to multiple roots, is rather complicated. It is the consequence of the fact that the initial system of differential equations in general does not have solutions allowing decomposition by integer degrees of the parameter ε. Such solutions, unlike the case of simple roots, are represented by formal series by different fractional orders of this parameter, and these orders depend not only on multiplicity of a root of the characteristic equation, but also on corresponding elementary divisors and on some relations among coefficients of the system under consideration.

The case of multiple roots of the characteristic equation was comprehensively studied by M.I. Shkil. These results are partially presented in the following paragraphs.

2 Asymptotic decomposition in the case of multiple roots of the characteristic equation

Let us consider the system of the form (1.5). We assume that the characteristic equation (1.7) has at least one root $\lambda = \lambda_0(\tau)$ of the constant multiplicity $k, (2 \le k < n)$, with the corresponding elementary divisor of the same multiplicity.

Theorem 2.1. *If $A(\tau, \varepsilon)$ has at the interval $[0; L]$ derivatives by τ of all orders and the matrix*

$$C(\tau) = T^{-1}(\tau)\left(\frac{dT(\tau)}{d\tau} - A_1(\tau) \cdot T(\tau)\right), \tag{2.1}$$

where $T(\tau)$ is the matrix transforming $A_0(\tau)$ to the Jordan form, and $T^{-1}(\tau)$ is inverse of $T(\tau)$, such that for every $\tau \in [0; L]$ its element

$$c_{k1} \neq 0, \tag{2.2}$$

then the system of differential equations (1.5) has a formal solution of the form

$$x = u(\tau, \mu) \exp\left(\int_0^t \lambda(\tau, \mu) dt\right), \tag{2.3}$$

where an n-dimensional vector $u(\tau, \mu)$ and a scalar function $\lambda(\tau, \mu)$ allow decompositions

$$u(\tau, \mu) = \sum_{S=0}^{\infty} \mu^s u_s(\tau), \qquad \lambda(\tau, \mu) = \lambda_0(\tau) + \sum_{S=1}^{\infty} \mu^s \lambda_s(\tau), \tag{2.4}$$

where

$$\mu = \varepsilon^{1/k}. \tag{2.5}$$

Let us note that if $c_{k1}(\tau) \equiv 0$ but at the same time $c_{k-1,1}(\tau) + c_{k2}(\tau) \neq 0$, then the initial system has a formal solution of the form (2.3), where $u(\tau, \mu)$, $\lambda(\tau, \mu)$ can be represented by formal series by degrees of the parameter $\mu = \varepsilon^{\frac{1}{k-1}}$.

Let us adduce a more general result.

Let the following conditions be fulfilled:

1) the matrix $A(\tau, \varepsilon)$ has derivatives by τ of all orders at the interval $[0, L]$;

2) the characteristic equation (1.7) has one root of constant multiplicity k;

3) there are $r \ge 1$ elementary divisors, corresponding to the root $\lambda_0(\tau)$, of the form

$$(\lambda - \lambda_0(\tau))^{k_1}, \dots, (\lambda - \lambda_0(\tau))^{k_r};$$

4) one of the following conditions is satisfied:

$$a)\ k_1 = k_2 = \dots = k_r = k, \qquad b)\ k_1 > k_2 > \dots > k_r.$$

Then for the case a) the following theorem is true:

Theorem 2.2. *If the conditions 1)–4) are fulfilled, then for the vector*

$$x = u(\tau,\mu)\exp\left(\int_0^t \lambda(\tau,\mu)dt\right), \tag{2.6}$$

where an n-dimensional vector $u(\tau,\mu)$ and a scalar function $\lambda(\tau,\mu)$ can be represented by formal series of the form

$$u(\tau,\mu) = \sum_{S=0}^{\infty} \mu^s u_s(\tau), \qquad \lambda(\tau,\mu) = \sum_{S=0}^{\infty} \mu^s \lambda_s(\tau), \tag{2.7}$$

where $\mu = \varepsilon^{\frac{1}{k}}$, to be a formal vector solution of the system (1.5), it is necessary and sufficient that the function $(\lambda_1(\tau))^k$ for every $\tau \in [0;L]$ be a root of the equation

$$\det \begin{Vmatrix} \rho + c_{k1}(\tau) & c_{k\,k+1}(\tau) & \cdots & c_{k\,l_{r-1}+1}(\tau) \\ c_{2\,k1}(\tau) & \rho + c_{2k\,k+1}(\tau) & \cdots & c_{2k\,l_{r-1}+1}(\tau) \\ \vdots & \vdots & \cdots & \vdots \\ c_{n1}(\tau) & c_{n\,k+1}(\tau) & \cdots & \rho + c_{n\,l_{r-1}+1}(\tau) \end{Vmatrix} = 0, \tag{2.8}$$

where $c_{k1}(\tau),\ldots,c_{n\,l_{r-1}+1}(\tau)$, $l_{r-1} = (r-1)k$ are elements of the matrix (2.1).

Let us note that the proof of the sufficient condition of this theorem simultaneously gives a method for construction of coefficients of the formal series (2.7).

The similar theorem is true for the case b). It was proved also that for the both cases formal solutions are asymptotic decompositions by the parameter ε of the true solutions of the system (1.5).

3 Turning points

The theorems, adduced in the Section 1.2, hold true under the condition that the roots of the characteristic equation and the corresponding elementary divisors preserve the constant multiplicity for all $\tau \in [0;L]$. If these conditions are violated (turning points appear, see [4]), then the construction of asymptotic solutions for solutions of the systems under study is rather difficult. Some results for the cases with turning points were obtained only for one second-order differential equation [3], and for systems of two second-order differential equations [4]. Investigation of this case by different authors was carried out with the use of Airy functions or by reduction of the differential equations under study to certain model equations. One of these equations is e.g. the Airy equation. More details can be found in the book [4].

The author of the present paper was the first to attempt constructing of formal decompositions in elementary functions for solutions of the systems of differential equations (1.5) [5].

Theorem 3.1. *Let the following conditions be fulfilled for the system of differential equations (1.5):*

1. *The matrix $A(\tau,\varepsilon)$ admits a decomposition*

$$A(\tau,\varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s A_s(\tau).$$

2. *The matrices $A_s(\tau)$ $(s = 0,1,\ldots)$ are infinitely differentiable at the interval* $[0,L]$.

3. There exists such integer number $k \geq 1$ that the roots of the equation

$$\det \|A_0(\tau) + \varepsilon A_1(\tau) + \cdots + \varepsilon^k A_k(\tau) - \lambda E\| = 0 \tag{3.1}$$

are simple for all $\tau \in [0; L]$.

Then there exists a formal vector that is a solution of the system (1.5) such as

$$x(\tau, \varepsilon) = U(\tau, \varepsilon) \exp \left(\frac{1}{\varepsilon} \int\limits_0^\tau \Lambda(\sigma, \varepsilon) d\sigma \right) \cdot a, \tag{3.2}$$

where $U(\tau, \varepsilon)$ is $(n \times n)$-matrix which allows a formal decomposition

$$U(\tau, \varepsilon) = \sum_{S=0}^{\infty} \varepsilon^s U_s(\tau, \varepsilon), \tag{3.3}$$

$\Lambda(\tau, \varepsilon)$ is a diagonal matrix, constructed of the roots of the equation (1.7)*, a is a constant n-dimensional vector.*

Let us note that unlike formal decompositions, adduced in the paragraph 1.2, coefficients in the decomposition (3.3) depend on ε, what presents considerable difficulties for investigation of asymptotic properties of these decompositions. Some results in this direction were obtained in the papers of the author and his students [6, 7].

4 Simplification of formal decompositions. Problems

The proof of existence of formal solution for the system (1.5) can be simplified considerably by means of consideration of another algebraic equation, related to the system (1.5). However in this case new and rather difficult problems appear, related to substantiation of asymptotic properties of formal solutions, obtained by means of this method. We will illustrate the above statements by consideration of the simplified system of the form

$$\frac{dx}{dt} = A(\tau)x, \tag{4.1}$$

where $n \times n$-matrix $A(\tau)$ is differentiable sufficient number of times at the interval $[0, L]$ ($\tau = \varepsilon t$, $\varepsilon > 0$ is a small parameter).

We shall assume that the characteristic equation

$$\det \|A(\tau) - \lambda E\| = 0 \tag{4.2}$$

at the interval $[0, L]$ has only one identically multiple root $\lambda = \lambda_0(\tau)$ of the multiplicity n, with corresponding elementary divisor of the same multiplicity.

Then by means of the substitution

$$x = V(\tau)y, \tag{4.3}$$

where $V(\tau)$ is a matrix, reducing the matrix $A(\tau)$ to the Jordan form, the system (4.1) can be reduced to the form

$$\frac{dy}{dt} = B(\tau, \varepsilon)y, \tag{4.4}$$

where

$$B(\tau, \varepsilon) = W(\tau) - \varepsilon V^{-1}(\tau)V'(\tau), \tag{4.5}$$

$W(\tau)$ is a Jordan cell, corresponding to the root $\lambda_0(\tau)$, $V^{-1}(\tau)$ is the inverse of $V(\tau)$, $V'(\tau)$ is a derivative of $V(\tau)$.

Let us construct an equation

$$\det \|B(\tau,\varepsilon) - \rho E\| = 0. \tag{4.6}$$

We will assume that the roots $\rho_1(\tau,\varepsilon),\ldots,\rho_n(\tau,\varepsilon)$ of equation (4.6) are simple for $\forall\, x \in [0;L]$ and $\forall\, \varepsilon \in (0;\varepsilon_0]$, or that

$$\rho_i(\tau,\varepsilon) \neq \rho_j(\tau,\varepsilon), \qquad i \neq j, \quad \forall\, i,j = \overline{1,n}. \tag{4.7}$$

Then making the following substitution in the system (4.4)

$$y = U_m(\tau,\varepsilon,\varepsilon) z, \qquad U_m(\tau,\varepsilon,\varepsilon) = \sum_{S=0}^{m} \varepsilon^s U_s(\tau,\varepsilon), \tag{4.8}$$

($m \geq 1$ is a natural number) and defining the matrices $U_s(\tau,\varepsilon)$ ($s = \overline{0,m}$) by means of the method [2], we arrive at the system of differential equations of the form

$$U_m(\tau,\varepsilon,\varepsilon)\frac{dz}{dt} = U_m(\tau,\varepsilon,\varepsilon)\left(\Lambda_m(\tau,\varepsilon,\varepsilon) + \varepsilon^{m+1} C_m(\tau,\varepsilon)\right) z, \tag{4.9}$$

where a diagonal matrix

$$\Lambda_m(\tau,\varepsilon,\varepsilon) = \sum_{S=0}^{m} \varepsilon^s \Lambda_s(\tau,\varepsilon), \tag{4.10}$$

and an $n \times n$-matrix $C_m(\tau,\varepsilon)$ are determined by means of the formulae from [2].

Let for all $\tau \in [0;L]$ and for a sufficiently small $\varepsilon \in (0;\varepsilon_0]$ the following conditions are fulfilled:

1. The matrix $U_m(\tau,\varepsilon,\varepsilon)$ is non-singular. Then the system (4.9) can be written in the form

$$\frac{dz}{dt} = \left(\Lambda_m(\tau,\varepsilon,\varepsilon) + \varepsilon^{m+1} C_m(\tau,\varepsilon)\right) z, \tag{4.11}$$

2. $\mathrm{Re}(\rho_j(\tau,\varepsilon,\varepsilon))_{j=\overline{1,n}} \leq 0$,

$$C_m(\tau,\varepsilon) = O\left(\varepsilon^{-\alpha}\right) \qquad (\varepsilon \to 0), \tag{4.12}$$

where $0 \leq \alpha < m$, then the system (4.11) can be integrated by means of the method of sequential approximations (conditions 2 ensure the applicability of this method). Whence for the vector z we obtain an asymptotic formula by the parameter ε ($\varepsilon \to 0$):

$$z = \exp\left(\frac{1}{\varepsilon}\int_0^\tau \sum_{s=0}^{m} \varepsilon^s \Lambda_s(\sigma,\varepsilon)d\sigma\right) a + O\left(\varepsilon^{m-\alpha}\right), \tag{4.13}$$

where a is a constant n-dimensional vector.

3. The matrix $U_m(\tau,\varepsilon,\varepsilon)$ is limited by the norm. Then using (4.3), (4.8), (4.13) we obtain an asymptotic formula for the vector x.

Finally we note that the formula (4.13) was obtained in assumption of conditions 1–2, where coefficients of the system (4.1) do not appear explicitly. The following question a rises:

What should be the requirements for the matrix $A(\tau)$ for the conditions 1–3 to be fulfilled? The answer for this question presents the problems mentioned at the beginning of the Section 4.

The solution of these problems requires further research.

Example. Let us consider a scalar equation

$$\frac{d^2x}{dt^2} + \varepsilon p(\tau)x = 0, \tag{4.14}$$

where $p(\tau) \neq 0$ at the interval $[0; L]$ and has continuous derivatives up to the second order.

The equation (4.14) can be represented in the form of the system (4.4) where $y = (y_1, y_2)$ $\left(y_1 = x,\ y_2 = \frac{dx}{dt}\right)$ is a two-dimensional vector, $B(\tau, \varepsilon)$ is a square matrix of the form

$$B(\tau, \varepsilon) = \left\| \begin{matrix} 0 & 1 \\ -\varepsilon p(\tau) & 0 \end{matrix} \right\|. \tag{4.15}$$

Then according to the assumption the equation has simple roots

$$\rho_1(\tau, \varepsilon) = \sqrt{-\varepsilon p(\tau)}, \qquad \rho_2(\tau, \varepsilon) = -\sqrt{-\varepsilon p(\tau)}. \tag{4.16}$$

We apply the transformation (4.8) to the system (4.4) with the matrix (4.15), putting $m = 1$. Then we obtain a system of the form (4.9) where the matrices $U_1(\tau, \varepsilon, \varepsilon)$, $\Lambda_1(\tau, \varepsilon, \varepsilon)$ are:

$$U_1(\tau, \varepsilon, \varepsilon) = \left\| \begin{matrix} 1 + \dfrac{\varepsilon p'(\tau)}{8p(\tau)\sqrt{-\varepsilon p(\tau)}} & 1 - \dfrac{\varepsilon p'(\tau)}{8p(\tau)\sqrt{-\varepsilon p(\tau)}} \\ \sqrt{-\varepsilon p(\tau)} - \dfrac{\varepsilon p'(\tau)}{8p(\tau)} & -\dfrac{\varepsilon p'(\tau)}{8p(\tau)} \end{matrix} \right\|,$$

$$\Lambda_1(\tau, \varepsilon, \varepsilon) = \operatorname{diag}\left(\sqrt{-\varepsilon p(\tau)} + \frac{\varepsilon p'(\tau)}{4p(\tau)},\ -\sqrt{-\varepsilon p(\tau)} + \frac{\varepsilon p'(\tau)}{4p(\tau)}\right), \tag{4.17}$$

$$U_1^{-1}(\tau, \varepsilon, \varepsilon) = \frac{1}{a(\tau, \varepsilon)} \left\| \begin{matrix} -\dfrac{\varepsilon p'(\tau)}{8p(\tau)} & \dfrac{\varepsilon p'(\tau)}{8p(\tau)\sqrt{-\varepsilon p(\tau)}} - 1 \\ \dfrac{\varepsilon p'(\tau)}{8p(\tau)} - \sqrt{-\varepsilon p(\tau)} & 1 + \dfrac{\varepsilon p'(\tau)}{8p(\tau)\sqrt{-\varepsilon p(\tau)}} \end{matrix} \right\|,$$

$$a(\tau, \varepsilon) = \frac{\varepsilon p'(\tau)}{8p(\tau)} - \sqrt{-\varepsilon p(\tau)} - \frac{\varepsilon^2 (p'(\tau))^2}{32p^2(\tau)\sqrt{-\varepsilon p(\tau)}}.$$

The matrix $C_1(\tau, \varepsilon)$ is determined by means of the formula

$$C_1(\tau, \varepsilon) = -U_1^{-1}(\tau, \varepsilon, \varepsilon)(U_1(\tau, \varepsilon)\Lambda_1(\tau, \varepsilon) + U_1'(\tau, \varepsilon)), \tag{4.18}$$

where

$$U_1(\tau, \varepsilon) = \frac{p'(\tau)}{8p(\tau)\sqrt{-\varepsilon p(\tau)}} \left\| \begin{matrix} 1 & -1 \\ -\sqrt{-\varepsilon p(\tau)} & -\sqrt{-\varepsilon p(\tau)} \end{matrix} \right\|,$$

$$\Lambda_1(\tau, \varepsilon) = \operatorname{diag}\left(\frac{p'(\tau)}{4p(\tau)}, \frac{p'(\tau)}{4p(\tau)}\right). \tag{4.19}$$

The direct computation of the elements of the matrix $C_1(\tau, \varepsilon)$ (we will omit it, as it is very cumbersome) shows that they have the order $O\left(\varepsilon^{-\frac{1}{2}}\right)$ in the neighbourhood of the point $\varepsilon = 0$ for all $\tau \in [0; L]$. So, having required the fulfilment of the condition

$$\operatorname{Re} \Lambda_1(\tau, \varepsilon, \varepsilon) \leq 0 \tag{4.20}$$

(this condition will be satisfied when the function $p(\tau) > 0$ for $\forall\, \tau \in [0; L]$), we get the following asymptotic formula for the vector z:

$$z = \exp\left(\frac{1}{\varepsilon}\int_0^\tau \Lambda_1(\sigma, \varepsilon, \varepsilon)d\sigma\right)a + O\left(\varepsilon^{\frac{1}{2}}\right), \tag{4.21}$$

References

[1] Feshchenko S.F., Shkil M.I. and Nikolenko l.D., Asymptotic Methods in the Theory of Linear Differential Equations, Kyiv, Naukova Dumka, 1966.

[2] Shkil M.I., On Asymptotic Methods in the Theory of Linear Differential Equations and their Application, Kyiv, KSU, part I, 1996; part II, 1997.

[3] Cesari L., Asymptotic Behaviour and Stability of Solutions of Ordinary Differential Equations, Moscow, Mir, 1964.

[4] Vazov V., Asymptotic Decompositions of Solutions of Ordinary Differential Equations, Moscow, Mir, 1968.

[5] Shkil M.I., On periodic solutions of a system of second-order differential equations, *Arhivum mathematicum (Brno)*, 1987, V.23, N 1, 1987.

[6] Shkil M.I. and Karpenko Yu.I., An asymptotic solution of a singularly perturbed system of second-order differential equations with point resonance, *Proc. of the Academy of Sciences of Ukraine*, 1991, N 5.

[7] Shkil M.I. and Zavision G.V., Asymptotic representation for solutions of systems of linear differential equations with turning points, in the book "Differential-Functional Equations", Kyiv Polytechnic Institute, 1991.