COMMUTATIVE SEMIGROUPS WITH FEW INVARIANT CONGRUENCES

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Abstract. Simple objects in the class of commutative semigroups with a group of automorphisms are studied. Among others, the following result is proven: Let S be a semilattice possessing no smallest element and such that S is simple over a commutative automorphism group G. Then (up to isomorphism) S is a subchain of the real line $\mathbb{R}$, G is a subgroup of $\mathbb{R}(+)$ and $G + S \subseteq S$.

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Simple objects in the class of chains with an automorphism group were classified in [6], [12] and [13], those in the class of semilattices with a two-generated commutative automorphism group in [7] and, finally, simple objects in the class of commutative semigroups with a two-generated commutative automorphism group were described in [8] (see also [9]). In the present note, using partially different approach, we obtain a similar description for arbitrarily many commuting automorphisms and, moreover, we get some basic information on the non-commuting case. This information is more opulent in case when the given automorphism group G does not contain free subsemigroup of rank 2 (i.e., G satisfies hereditarily the Ore conditions).

1. Semimodules – introduction

Let G be a group. By a (unitary left $G-$) semimodule we shall mean a commutative semigroup S ($= S(+)$) together with a scalar multiplication $G \times S \to S$ such that $a(x+y) = ax + ay$, $(ab)x = a(bx)$ and $1x = x$ for all $a, b \in G$ and $x, y \in S$.

A semimodule S will be called

- ip–semimodule (idempotent) if $x + x = x$ for every $x \in S$;
- up–semimodule (unipotent) if $x + x = y + y$ for all $x, y \in S$;
- zp–semimodule (zeropotent) if $x + x + y = x + x$ for all $x, y \in S$;
- zs–semimodule if S is zeropotent and $S + S = S$;
- za–semimodule if $x + y = x + z$ for all $x, y, z \in S$;
- qza–semimodule if $x + y = x + z$ for all $x, y, z \in S$ such that $y \neq x \neq z$;
- cn–semimodule (cancellative) if $x + y \neq x + z$ for all $x, y, z \in S$ such that $y \neq z$;
- module if S is an (abelian) group.

An element $w \in S$ will be called neutral if $w + x = x$ for every $x \in S$ and, if it exists, it will usually be denoted by the symbol 0. If $0 \in S$, then S^* denotes the set of invertible elements of S (it is a submodule) and $S^{\square} = S \setminus \{0\}$. If $0 \notin S$, then $S^{\square} = S$.

An element $w \in S$ will be called absorbing if $w + x = w$ for every $x \in S$ and, if it exists, it will be denoted by the symbol o. If $o \in S$, then $S^{\circ} = S \setminus \{o\}$. If $o \notin S$, then $S^{\circ} = S$. An idempotent semimodule with an absorbing element will also be called an ipa–semimodule.

By an ideal of a semimodule S we mean a non–empty subset I of S such that $GI \subseteq I$ and $S + I \subseteq I$. A semimodule S will be said

- ideal–simple if it is non–trivial and $I = S$, whenever I is an ideal of S and $\mathrm{card}(I) \geq 2$;
- faithful if for every $a \in G, a \neq 1$, there exists $x \in S$ with $ax \neq x$;
- congruence–simple (or only simple) if it is non–trivial and $\mathrm{id}_S, S \times S$ are the only congruences of S.

Let S be a semimodule. The basic quasiordering $\leq_S$ is defined on S by $x \leq_S y$ if and only if $x = y + z$ for some $z \in S \cup \{0\}$. Clearly, $\leq_S$ is compatible with respect to the operations of S, and hence $\eta_S = \ker(\leq_S)$ is a congruence of S. Now, $\eta_S = S \times S$ if and only if S is a module and $\eta_S = \mathrm{id}$ in each of the following five cases:

(1) S is idempotent;
(2) S is zeropotent;

(3) S is cancellative and $0 \notin S$;
(4) S is cancellative, $0 \in S$ and $S^* = 0$;
(5) S is simple but not a module.

2. Simple semimodules – introduction and examples

2.1 Theorem. Let S be a simple semimodule. Then just one of the following four cases takes place:

(1) S is a two–element za–semimodule;
(2) S is a zs–semimodule;
(3) S is an ip–semimodule;
(4) S is a cn–semimodule.

Proof. First, assume that S is neither unipotent nor idempotent. The endomorphism $x \to 2x$ of S is injective and $\varrho = S \times S$, where ϱ is the congruence of S defined by $(x, y) \in \varrho$ if and only if $2^i x \leq_S y$ and $2^i y \leq_S x$ for some $i \geq 0$. Consequently, if $x + y = x + z$ for some $x, y, z \in S$, then there are $i \geq 0$ and $w \in S \cup \{0\}$ such that $2^i y = x + w$; we have $y + 2^i y = y + x + w = z + x + w = z + 2^i y$ and $2y = y + z$ for $i = 0$. However, if $i \geq 1$, then $2(y + 2^{i-1}y) = 2(z + 2^{i-1}y)$, $y + 2^{i-1}y = z + 2^{i-1}y$ and $2y = z + y$ by induction. Quite similarly, $y + z = 2z$, and hence $y = z$.

Now, assume that S is unipotent but not zeropotent. Then $x \to 3x$ is an injective endomorphism of S and, if $x+y = x+z$, then $3y = y+o = y+x+x = z+x+x = 3z$ and $y = z$.

Finally, assume that S is zeropotent and put $T = S + S$ and $r = (T \times T) \cup \mathrm{id}_S$. If $r = S \times S$, then S is a zs–semimodule. On the other hand, if $r = \mathrm{id}_S$, then $s = (S^\circ \times S^\circ) \cup \mathrm{id}_S$ is a congruence of S, $s = \mathrm{id}_S$ and S is a two–element za–semimodule. ▲

2.2 Proposition. Let S be a simple semimodule possessing a neutral element. Then just one of the following two cases takes place:

(1) S is a two–element ip–semimodule;
(2) S is a module.

Proof. It follows immediately from 2.1 that S is either idempotent or cancellative. In the former case, $r = (S^\square \times S^\square) \cup \mathrm{id}_S$ is a congruence of S, $r = \mathrm{id}_S$ and $\mathrm{card}(S) = 2$. On the other hand, if S is cancellative, then the relation s defined on S by $(x, y) \in s$ if and only if $x + S^* = y + S^*$ is a congruence of S, $s = S \times S$ and $S^* = S$. ▲

2.3 Proposition. Let S be a finite simple semimodule. Then just one of the following three cases takes place:

(1) S is a two–element za–semimodule;
(2) S is an idempotent qza–semimodule;
(3) S is a module.

Proof. Applying 2.1 and using the easy observations that every non–trivial zs–semimodule is infinite and that every finite cn–semimodule is a module, we can assume that S is idempotent. Then, being finite and simple, S possesses an absorbing element and every element of S° is an atom. Thus S is a qza–semimodule. ▲

2.4 Proposition. Let G be a periodic group and S a simple semimodule. Then just one of the following three cases takes place:

(1) S is a two–element za–semimodule;
(2) S is an idempotent qza–semimodule;
(3) S is a module.

Proof. Assume that S is neither a module nor a za–semimodule. Then $\eta_S = \mathrm{id}_S$ and, if $x, y \in S$ are such that $x + y \neq o$, then $x \leq_S a(x+y)$ for some $a \in G$ (since S is ideal–simple). Now, $x + y \leq_S a(x+y) \leq_S a^2(x+y) \leq \dots$ and, since $a^n = 1$ for some $n \geq 1$, we have $x + y = a(x+y)$ and $x = x + y$. Quite similarly, $y = x + y$, so that $x = y$ and the rest is clear from 2.3. ▲

2.5 Proposition. Let S be a simple qza–semimodule. Then just one of the following three cases takes place:

(1) S is a two–element za–semimodule;
(2) S is a two–element module;
(3) S is an ipa–semimodule.

Proof. An easy consequence of 2.1. ▲

2.6 Example. For every subgroup H of G, let $S_H = \{xH; x \in G\} \cup \{o\}$, $xH \oplus xH = xH$, $xH \oplus yH = o$ if $x^{-1}y \notin H$ and $a * xH = axH$. Then $S_H = S_H(\oplus, *)$ is a simple idempotent qza–semimodule and it is faithful if and only if $\mathrm{Core}_G(H) = 1$. Clearly, $S_{H_1} \cong S_{H_2}$ if and only if the subgroups H_1, H_2 are conjugate and every simple idempotent qza–semimodule is isomorphic to S_H for a subgroup H.

2.7 Example. Let S denote the set of ordered pairs of integers equipped with a semilattice operation $\oplus$, where $(n,m) \oplus (k,l) = (\min(n,k), \min(m,l))$. Now, the mappings $\alpha : (n,m) \to (m,n)$ and $\beta : (n,m) \to (n, m+1)$ are automorphisms of $S(\oplus)$ and G will be the automorphism group generated by α and β (see 3.3). Then S is a simple idempotent G–semimodule without absorbing element (notice also that $S(\oplus)$ is not a chain).

2.8 Example. Let T denote the set of countable infinite subsets of $\aleph_1$ and let $S = T \cup \{\aleph_1\}$. Now, define an operation $\oplus$ on S by $x \oplus y = \aleph_1$ if $x \cap y \neq \varnothing$ and $x \oplus y = x \cup y$ otherwise. Then $S(\oplus)$ is a zeropotent commutative semigroup without irreducible elements, and hence S is also a G–zs–semimodule, $G = \mathrm{Aut}(S(\oplus))$. It is easy to see that S is a simple semimodule.

2.9 Example. (i) Let S be the additive semigroup and G the multiplicative group of positive rationals. Then S is a simple cancellative G–semimodule, but not a module. (Notice that G is a free abelian group of rank $\aleph_0$.) More generally, if F is a subfield of the field $\mathbb{R}$ of real numbers and if $S = F^+(+)$ and $G = F^+(\cdot)$, then S is a simple cancellative G-semimodule.

(ii) Let $F = \mathbb{Q}(\sqrt{2}) \subseteq \mathbb{R}$, $T = \{a_1^2 + \dots + a_n^2\ ;\ n \geq 1, a_i \in F^+\}$, $S = T(+)$ and $G = T(\cdot)$. Then S is a simple cancellative G-semimodule and $S \neq F^+$.

2.10 Proposition. Suppose that at least one of the following three conditions is satisfied:

(1) G is periodic;
(2) G is finite;
(3) G is a finitely generated abelian group.

Then every simple cn–semimodule S is a module. Moreover, if (2) or (3) is true, then S is finite.

Proof. If G is periodic, then S is a module by 2.4. If, moreover, G is finite and $0 \neq x \in S$, then the abelian group $S(+)$ is generated by the finite set Gx, and hence S is finite, since $S(+)$ possesses no non–trivial characteristic subgroups.

Now, assume that G is a finitely generated abelian group and let M be a module such that S is a subsemimodule of M and $M = \{x - y; x, y \in S\}$. Then M is a simple module and, using the well known fact that a field finitely generated as a ring is finite, we conclude that M is finite; then $S = M$. ▲

3. Hereditarily uniform groups

We shall say that a semigroup S is left/right uniform if any two left/right ideals of S possess a non-empty intersection. Such semigroups are called left/right reversive in [4] and they are also known as those satisfying the Ore condition(s) (the Axiom M_V "Existence of common multiplum" of O. Ore - see [15]). Further, we shall say that a semigroup S is hls/hrs–uniform if every subsemigroup of S is left/right uniform.

3.1 Lemma. (i) A semigroup S is hls–uniform if and only if $A \cap B \neq \varnothing$, whenever A, B are subsemigroups of S such that $AB \subseteq B$ and $BA \subseteq A$.

(ii) A right cancellative semigroup S is hls–uniform if and only if no subsemigroup of S is a free semigroup of rank 2.

(iii) A cancellative semigroup is hls–uniform if and only if it is hrs–uniform.

(iv) A group is hls–uniform if and only if it is hrs–uniform.

(v) A group G is hs–uniform if and only if $G/Z(G)$ is so.

(vi) If H is a normal subgroup of G such that G/H is periodic, then G is hs–uniform if and only if H is so.

3.2 Corollary (i) Every periodic group is hs–uniform.

(ii) (B. H. Neumann - c.f. [16]) Every locally nilpotent group is hs–uniform.

3.3 Example. Let G be a group possessing two generators a, b such that $a^2 = 1$ and $abab = baba$. Then G is metabelian and hs–uniform (3.1(vi)).

3.4 Example. Let $R = \mathbb{Z}[x, x^{-1}, y, y^{-1}]$, $\mathbb{Z}$ being the ring of integers, and let R^* denote the multiplicative group of invertible elements. Put $G = R \times R^*$ and define a multiplication on G by $(f, a)(g, b) = (f + ag, ab)$ for all $f, g \in R$ and $a, b \in R^*$. Then G is a metabelian group that is not hs–uniform.

3.5 Remark. (i) Let S be a right cancellative semigroup that is not hls–uniform and let A, B be disjoint subsemigroups of S such that $AB \subseteq B$ and $BA \subseteq A$. Now, take $a \in A, b \in B$ and put $C = A \cup B$ and $C_k = Cab^k, k \geq 1$. Then C_k are pair–wise disjoint left ideals of C and $C_i C_j \subseteq C_j$ for all $i, j \geq 1$. The relation $r = \bigcup (C_k \times C_k)$ is a congruence of the semigroup $D = \bigcup C_k$ and D/r is an infinite semigroup of right zeros.

(ii) We shall say that a subset P of a semigroup S is power–separable if $a^n \neq b^m$ for $a, b \in P, a \neq b$ and all $n \geq 1, m \geq 1$. Further, we denote by $\theta(S)$ the smallest cardinal number such that $\theta(S) \geq \mathrm{card}(P)$ for every power–separable subset P of S.

Now, by (i), if S is a right cancellative semigroup possessing no infinite power–separable subset, then S is hls–uniform.

3.6 Example. Let G be a group possessing two generators a, b such that $a^2 = b^2$. Then G is metabelian, $\theta(G) \leq 3$ and G is hs–uniform (3.5(ii)).

3.7 Theorem. Let G be an hs–uniform group and S a simple semimodule possessing an absorbing element. Then just one of the following two cases takes place:

(1) S is a two–element za–semimodule;
(2) S is an idempotent qza–semimodule.

Proof. In view of 2.1, we can assume that S is either a zs–semimodule or an ip–semimodule.

First, let S be a zs–semimodule and let $x, y, z \in S$ be such that $x = y + z \neq o$. Put $A = \{a \in G; y \leq_S ax\}$ and $B = \{b \in G; z \leq_S bx\}$. Since S is ideal–simple, we have $A \neq \varnothing \neq B$ and, moreover, one checks easily that $AA \cup AB \subseteq A$ and $BB \cup BA \subseteq B$. Consequently, since G is hs–uniform, we have $A \cap B \neq \varnothing$ and, if $a \in A \cap B$, then $o \neq x = y + z \leq_S ax + ax = o$, a contradiction.

Now, let S be idempotent but not a qza–semimodule; then we can find $x, z \in S^\circ$ such that $x \leq_S z$ and $x \neq z$. Further, since S is simple, we can find also an element $y' \in S$ such that $y' + x = o \neq y' + z = y$. Put $A = \{a \in G; z \leq_S ax\}$ and $B = \{b \in G; z \leq_S by\}$. Again, $A \neq \varnothing \neq B$, $AA \cup BA \subseteq A, BB \cup AB \subseteq B$, $a \in A \cap B \neq \varnothing$ and, finally, $z = z + z \leq_S ax + ay = a(x + y) = o$, a contradiction. ▲

3.8 Remark. The following conditions are equivalent for a group G:

(i) G is hs-uniform.
(ii) Every simple ipa-semimodule is a qza-semimodule.
(iii) There exist no simple zs-semimodule.

Indeed, (i) implies both (ii) and (iii) by 3.7 and the fact that (i) follows from (ii) will be shown in the next section (see 4.2 and 8.2, 8.3). As concernes the last implication ((iii) implies (i)), we may proceed in the following way (sketched only):

Let R be a subsemigroup of G and let $\mathcal{A}$ denote the set of non-empty subsets A of G such that $AR \subseteq A$. Define an addition on $\mathcal{A}$ by $A + B = A \cup B$ if $A \cap B = \emptyset$ and $A + B = G$ otherwise. Then $\mathcal{A}(+)$ is a commutative zp-semigroup (where G plays the role of an absorbing element) and ρ is a congruence of $\mathcal{A}(+)$, where $(A, B) \in \rho$ if and only if $\{C\ ;\ A \cap C = \emptyset\} = \{C\ ;\ B \cap C = \emptyset\}$. Furthermore, $(aG, G) \in \rho$ and $(a(A + B), aA + aB) \in \rho$ for all $a \in G$ and $A, B \in \mathcal{A}$; if $(A, B) \in \rho$, then also $(aA, aB) \in \rho$. Now, $\mathcal{B} = \mathcal{A}/\rho$ becomes a G-zp-semimodule (via $a(A/\rho) = (aA)/\rho$ for all $A \in \mathcal{A}$ and $a \in G$) and $\alpha = \mathcal{B} \times \mathcal{B}$ whenever α is a congruence of $\mathcal{B}$ such that $(R/\rho, G/\rho) \in \alpha$. In particular, if $(R, G) \in \rho$, then $\rho = \mathcal{A} \times \mathcal{A}$ and R is right uniform. On the other hand, if R is not right uniform, then $(R, G) \notin \rho$ and there exists a congruence τ of $\mathcal{B}$ maximal with respect to $(R/\rho, G/\rho) \notin \tau$. Now, the factor-semimodule $\mathcal{B}/\tau$ is a simple zs-semimodule.

3.9 Remark. It seems to be a well known open problem whether, given a cancellative hs-uniform semigroup, the corresponding group of fractions is also hs-uniform.

4. The semimodule $\mathcal{J}(G, R)$ of fractional left ideals

Let R be a subsemigroup of a group G such that $1 \in R$ and $R \neq G$. We denote by $\mathcal{I}^\circ$ $(= \mathcal{I}(R)^\circ)$ the set of left ideals of R and by $\mathcal{I}$ $(= \mathcal{I}(R))$ the set $\mathcal{I}^\circ \cup \{\varnothing\}$.

Further, we denote by $\mathcal{J}^\circ$ $(= \mathcal{J}(G,R)^\circ)$ the set of fractional left ideals of R in G (i.e., the set of subsets $Ix, I \in \mathcal{I}^\circ, x \in G$) and by $\mathcal{J}$ $(= \mathcal{J}(G,R))$ the set $\mathcal{J}^\circ \cup \{\varnothing\}$. It is easy to see that $\mathcal{J}$ is closed under arbitrary intersections and, in particular, $\mathcal{J}$ is a non-trivial semilattice with respect to the operation of intersection and $\varnothing$ is an absorbing element of $\mathcal{J}$. Now, we make $\mathcal{J}$ an ipa–semimodule by setting $a * A = Aa^{-1}$ for all $a \in G$ and $A \in \mathcal{J}$ (to avoid confusion, the scalar multiplication is denoted by $*$). In the sequel, we shall need the following relation σ defined on $\mathcal{J} : (A,B) \in \sigma$ if and only if $\{C \in \mathcal{I}; A \cap C = \varnothing\} = \{C \in \mathcal{I}; B \cap C = \varnothing\}$; clearly, σ is a congruence of the semimodule $\mathcal{J}$.

4.1 <u>Proposition</u>. (i) The semimodule $\mathcal{J}$ is ideal–simple.

(ii) If R is not a group, then $M(R) = R \backslash R^* \in \mathcal{I}^\circ$, $(M(R), R) \in \sigma \neq \mathrm{id}_{\mathcal{J}}$ and $\mathcal{J}$ is not simple.

(iii) $\mathcal{J}$ is a qza–semimodule if and only if R is a group.

(iv) $\mathcal{J}$ is faithful if and only if $\mathrm{Core}_G(R^*) = 1$.

Proof. (i) Let $\mathcal{A} \neq \{\varnothing\}$ be an ideal of $\mathcal{J}$ and let $\varnothing \neq A \in \mathcal{A}$ and $a \in A$. Then $Ra \in \mathcal{A}$ and, for every $x \in G, Rx = Ra \cdot a^{-1}x = x^{-1}a * Ra \in \mathcal{A}$.

(ii), (iii) and (iv). These observations are easy. ▲

Let $\mathcal{P}$ $(= \mathcal{P}(G,R))$ denote the subsemimodule of $\mathcal{J}$ generated by $\varnothing$ and R and put $(\mathcal{K}(G,R) =)$ $\mathcal{K} = \mathcal{J}/\sigma$ and $(\mathcal{G}(G,R) =)$ $\mathcal{G} = \mathcal{P}/\sigma$.

4.2 <u>Proposition</u>. (i) The semimodule $\mathcal{K}$ is simple.

(ii) $\mathcal{K}$ is a qza–semimodule if and only if R is left uniform.

(iii) $\mathcal{K}$ is faithful if and only if $\mathrm{Core}_G(W) = 1$, where $W = \{w \in G; wR^{-1}R = R^{-1}R\}$ (W is a subgroup of G).

(iv) If $\mathcal{L}$ is a subsemimodule of $\mathcal{K}$ such that $\mathcal{G} \subseteq \mathcal{L}$, then $\mathcal{L}$ is a simple semimodule.

Proof. (i) First, $\mathcal{K}$ is ideal–simple by 4.1(i). Now, let r be a congruence of $\mathcal{K}$ such that $(x,y) \in r, x \neq y$; we can assume that $x \leq_S y$. Then there is $z \in \mathcal{K}$ with $x + z = o \neq y + z, y + z \in I = \{w \in \mathcal{K}; (o,w) \in r\}, I = \mathcal{K}$ and $r = \mathcal{K} \times \mathcal{K}$.

(ii) Let R be left uniform and let $A, B \in \mathcal{J}$ be such that $(A,B) \notin \sigma$; we are going to show that $A \cap B = \varnothing$. To that purpose, we can assume that $A = Ix, B = Jy, I, J \in \mathcal{I}^\circ, x, y \in G$, and that $A \cap C = \varnothing \neq B \cap C$ for some $C \in \mathcal{J}$. Now, $Ixy^{-1} \cap Cy^{-1} = \varnothing \neq J \cap Cy^{-1} = L$ and, if $K = Ixy^{-1} \cap J = \varnothing$, then $A \cap B = \varnothing$. If $K \neq \varnothing$, then $\varnothing \neq K \cap L \subseteq Ixy^{-1} \cap Cy^{-1} = \varnothing$, a contradiction.

Conversely, if $\mathcal{K}$ is a qza–semimodule and if $I, J \in \mathcal{I}^\circ$, then $(I, I \cup J) \in \sigma$, and hence $I \cap J \neq \varnothing$.

(iii) Easy.

(iv) It is sufficient to show that $\mathcal{L}$ is ideal–simple (see the part (i) of this proof). If $\mathcal{A}$ is an ideal of $\mathcal{L}$, $\mathcal{A} \neq \{o\}$, then $\mathcal{A} \cup (\mathcal{A} + \mathcal{K})$ is an ideal of $\mathcal{K}$, and hence $\mathcal{A} \cup (\mathcal{A} + \mathcal{K}) = \mathcal{K}$. The rest is clear. ▲

Let $\mathcal{S}$ be a subsemilattice of the semilattice $\mathcal{J}(\cap)$. For $A \in \mathcal{J}$, put $(A : \mathcal{S}) = \{x \in G; P \subseteq Ax$ for some $P \in \mathcal{S}\}$ and define a congruence $\lambda_{\mathcal{S}}$ of $\mathcal{J}$ by $(A,B) \in \lambda_{\mathcal{S}}$ if and only if $(A : \mathcal{S}) = (B : \mathcal{S})$. The following lemma is obvious:

4.3 <u>Lemma</u>. (i) If $\mathcal{T}$ is a subset of $\mathcal{J}$ such that $\mathcal{S} \subseteq \mathcal{T}$ and $\mathcal{T} \times \mathcal{T} \subseteq \lambda_{\mathcal{S}}$, then $\mathcal{S}$ is downwards–cofinal in T.

(ii) If $(R : \mathcal{S})$ is a subsemigroup of G, then $(R \cap Rx, R) \in \lambda_{\mathcal{S}}$ for every $x \in (R : \mathcal{S})$.

5. The semimodule $\mathcal{N}(G,R)$

Let R be a subsemigroup of a group G such that $1 \in R$ and R is not a subgroup of G. We put $N(R) = \{x \in G; Rx = yR$ for some $y \in G\}$ and $M(R) = R \setminus R^*$. Further, let $H(R) = N(R)/R^*$ (the factor–group) and let $\varphi_R : N(R) \to H(R)$ denote the natural projection. We can define a compatible quasiordering μ on $N(R)$ by $(a,b) \in \mu$ if and only if $Ra \subseteq Rb$ and μ induces a compatible ordering $\leq^R$ of the group $H(R)$.

For $a \in N(R)$, let $\mathcal{S}_a$ denote the subsemilattice of $\mathcal{J}(\cap)$ generated by the elements $Ra^i, i \geq 0$.

5.1 Lemma. (i) $a^i \in (R : \mathcal{S}_a)$ for every $i \geq 0$.

(ii) $(R : \mathcal{S}_a)$ is a subsemigroup of G.

Proof. (i) Obvious.

(ii) If $y \in (R : \mathcal{S}_a)$, then $R \cap Ra \cap \cdots \cap Ra^n \subseteq Ry, Ra = bR, Ra \cap Ra^2 \cap \cdots \cap Ra^{n+1} = bR \cap bRa \cap \cdots \cap bRa^n \subseteq bRy = Ray$ and it follows by induction that $a^iy \in (R : \mathcal{S}_a)$ for every $i \geq 0$. Now, if $R \cap Ra \cap \cdots \cap Ra^m \subseteq Rx$, then $Ry \cap Ray \cap \cdots \cap Ra^my \subseteq Rxy$ and $xy \in (R : \mathcal{S}_a)$. ▲

Let $(\mathcal{N}(G,R) =)\ \mathcal{N} = \{Ix; I \in \mathcal{I}^\circ(R), x \in N(R)\} \cup \{\varnothing\}$. Then $\mathcal{N}$ is an ideal of $\mathcal{J}(\cap)$ and $a * A = Aa^{-1} \in \mathcal{N}$ for all $a \in N(R)$ and $A \in \mathcal{N}$; thus $\mathcal{N}$ is an $N(R)$–semimodule. Further, if $A, B \in \mathcal{N}$, then $AB \in \mathcal{N}$, and so $\mathcal{N}$ is also a semigroup under the multiplication, $\varnothing$ is an absorbing element and R is a left neutral element of $\mathcal{N}$. Moreover, $\mathcal{N}^\circ = \mathcal{N} \setminus \{\varnothing\}$ is a subsemigroup of $\mathcal{N}$ and the ordering of inclusion is a compatible ordering of the semigroup $\mathcal{N}$.

Now, define a mapping $\epsilon_R : H(R) \to \mathcal{N}^\circ$ by $\epsilon_R(a/R^*) = Ra$ for every $a \in N(R)$. Then ϵ_R is an injective homomorphism of the group $H(R)$ into the (multiplicative)semigroup $\mathcal{N}^\circ$, $\epsilon_R(1) = R$ and ϵ_R preserves the orderings.

In the remaining part of this section, we shall assume that $R \subseteq N(R)$. The following lemma is obvious:

5.2 Lemma. (i) $Ra = aR$ for every $a \in N(R)$.

(ii) $M(R)a = aM(R)$ for every $a \in N(R)$.

(iii) If I is a proper left ideal of R, $a \in I$ and $x \in N(R)$, then either $IxRx^{-1} \neq IxR$ or $IxRa \neq IxR$.

(iv) If $x \in N(R)$ is such that $RxRx^{-1} = RxR$, then $x \in R^*$.

(v) R is a (two–sided) neutral element of the multiplicative semigroup $\mathcal{N}$.

5.3 Proposition. The following conditions are equivalent:

(i) $H(R)$ is an abelian group.

(ii) $N(R)' \subseteq R^*$.

(iii) The semigroup $\mathcal{N}^\circ$ is commutative.

Proof. (ii) implies (iii). Let $I, J \in \mathcal{I}^\circ$ and $a, b \in N(R)$. Then $Iaba^{-1}b^{-1} = I, Ia = \bigcup_{b \in I} Rba = a \bigcup Rb = aI, IJ = \bigcup_{x \in I, y \in J} RxRy = \bigcup RyRx = JI$ and, finally, $IaJb = JbIa$. ▲

From now on, we shall assume that $\mathcal{N}$ is a chain. Then $\mathcal{N}^\circ$ is a subsemimodule of the $N(R)$–semimodule $\mathcal{N}$. For $A \in \mathcal{N}^\circ$, define a relation ν_A on $\mathcal{N}^\circ$ by $(B,C) \in \nu_A$ if and only if $AB = AC$.

5.4 Lemma. (i) ν_A is a congruence of the $N(R)$–semimodule $\mathcal{N}^\circ$.

(ii) ν_A is right compatible with respect to the multiplication of $\mathcal{N}^\circ$.

(iii) If $B, C, D \in \mathcal{N}^\circ$ are such that $(B, C) \in \nu_A$ and $B \subseteq D \subseteq C$, then $(B, D) \in \nu_A$ and $(C, D) \in \nu_A$.

(iv) $\nu_R = \mathrm{id}$.

(v) There are $a, b \in N(R)$ such that $(Ra, Rb) \notin \nu_A$.

Proof. Use the fact that $\mathcal{N}$ is a chain (and also 5.2). ▲

Finally, assume that $\nu_A \lceil \{Ra; a \in R\} = \mathrm{id}$ for every $A \in \mathcal{N}^\circ$.

5.5 Lemma. Let $A \in \mathcal{N}^\circ$ be such that $\nu_A \neq \mathrm{id}$. Then:

(i) $A = Iw$, where $w \in N(R)$ and I is an ideal of R such that $I \subseteq M(R)$ and $IM(R) = I$.

(ii) If $B, C \in \mathcal{N}^\circ$ are such that $B \subseteq C$ and $B \neq C$, then $(B, C) \in \nu_A$ if and only if $B = M(R)v$ and $C = Rv$ for some $v \in N(R)$.

(iii) $M(R)M(R) = M(R)$ and $M(R)$ is not a principal ideal of R.

(iv) $M(R)K = K = KM(R)$ for every non–principal ideal K of R.

(v) ν_A is a congruence of the (multiplicative) semigroup $\mathcal{N}^\circ$.

Proof. (i) and (ii). Let $J, K \in \mathcal{I}^\circ$ and $u, v \in N(R)$ be such that $Ju \subseteq Kv$, $Ju \neq Kv$ and $AJu = AKv$; put $L = Juv^{-1}$. Then $AL = AK$, $L \subseteq K$ and $L \neq K$; in particular, $L \in \mathcal{I}^\circ$ and $Ju = Lv$.

Suppose, for a moment, that K is not a principal ideal and take $x \in K \backslash L$. Then $L \subseteq Rx \neq K$, $Rx \neq Ry$, $y \in K \backslash Rx$ and $L \subseteq Rx \subseteq Ry \subseteq K$. But $ARx = ARy$ implies $Rx = Ry$, a contradiction.

We have proved that $K = Rz$ is a principal ideal of R. Now, $L_1 = Lz^{-1} \subseteq R$, $L_1 \neq R$, $Ju = L_1 zv$, $Kv = Rzv$ and, proceeding similarly as above, we show that $R = Rt$ for every $t \in R \backslash L_1$. It follows that $R \backslash L_1 \subseteq R^*$ and $M(R) \subseteq L_1$. Since $L_1 \neq R$, we have $L_1 = M(R)$ and $Ju = M(R)zv$, $Kv = Rzv$.

Finally, $IwM(R)zv = AJu = AKv = IwRzv$. But then $IM(R)wzv = Iwzv$ and $IM(R) = I$.

(iii) We have $AM(R)^2 = IwM(R)^2 = IM(R)^2 w = Iw = IM(R)w = IwM(R) = AM(R)$. If $M(R) \neq M(R)^2$, then $M(R) = Rv$ is a principal ideal by (ii), and hence $M(R)^2 = Rv^2$ and $(Rv^2, Rv) \in \nu_A$. But then $Rv^2 = Rv$ and $M(R) = Rv = R$, a contradiction.

(iv) By (ii), $(M(R), R) \in \nu_A$, and so $(M(R)K, K) \in \nu_A$. If $M(R)K \neq K$, then K is a principal ideal by (ii), a contradiction.

(v) In view of 5.4(ii), we have to show that ν_A is left compatible. Let $B, C, D \in \mathcal{N}^\circ$ be such that $B \subseteq C$, $B \neq C$ and $(B, C) \in \nu_A$. By (ii), $B = M(R)w$ and $C = Rw$, $w \in N(R)$. Further, $D = Kz$, $K \in \mathcal{I}^\circ$, $z \in N(R)$ and, if K is not principal, then $DB = Kzw = DC$. If $K = Rv$ is principal, then $DB = M(R)vzw$, $DC = Rvzw$ and $(DB, DC) \in \nu_A$ by (ii). ▲

5.6 Proposition. There exists a congruence ν of the (multiplicative) semigroup $\mathcal{N}^\circ$ such that the following is true:

(i) If $\nu_A = \mathrm{id}$ for every $A \in \mathcal{N}^\circ$, then $\nu = \mathrm{id}$ and $\mathcal{N}^\circ$ is a left cancellative semigroup.

(ii) If $A_1, A_2 \in \mathcal{N}^\circ$ are such that $\nu_{A_1} \neq \mathrm{id} \neq \nu_{A_2}$, then $\nu_{A_1} = \nu = \nu_{A_2}$.

(iii) The factor–semigroup $\mathcal{N}^\circ/\nu$ is left cancellative.

(iv) ν is a congruence of the $N(R)$–semimodule $\mathcal{N}^\circ$.

Proof. The existence of ν is clear from the preceding lemma and it is also easy to see that ν is left cancellative. ▲

Now, put $(\mathcal{M}(G,R) =)\ \mathcal{M} = \mathcal{N}^\circ/\nu$ and denote by $\Phi_R : \mathcal{N}^\circ \to \mathcal{M}$ the natural projection. Then $\mathcal{M}$ is a left cancellative semigroup with a neutral element and, simultaneously, $\mathcal{M}$ is a chain compatible with the multiplication; in fact, $\mathcal{M}$ is an $N(R)$–semimodule. For $A, B \in \mathcal{N}^\circ$ we have $A/\nu \le B/\nu$ if and only if $A \subseteq B$ and for $a \in N(R), a * (A/\nu) = Aa^{-1}/\nu = A/\nu \cdot Ra^{-1}/\nu$. Furthermore, $\Phi_R \epsilon_R(a/R^*) = Ra/\nu, a \in N(R)$. The image $\mathcal{H} = \Phi_R\epsilon_R(H(R))$ is a subgroup of $\mathcal{M}$ and $\mathcal{H}$ is both downwards– and upwards–cofinal in the chain $\mathcal{M}$.

6. Simple subsemimodules of $\mathcal{J}(G,R)$

Let R be a subsemigroup of a group G such that $1 \in R$ and R is not a subgroup and let $\mathcal{R}$ be a $(G-)$ subsemimodule of $\mathcal{J} = \mathcal{J}(G,R)$ such that $\mathcal{R}$ is simple, $R \in \mathcal{R}$ and $\mathcal{R}$ possesses no absorbing element.

6.1 Lemma. (i) If $a \in N(R)\backslash R^{-1}$, then $\mathcal{S}_a \subseteq \mathcal{R}$ and $\mathcal{S}_a$ is downwards–cofinal in $\mathcal{R}$.

(ii) $N(R) \subseteq R \cup R^{-1}$.

(iii) The set $\{Ra; a \in N(R)\}$ is a chain.

Proof. (i) Put $\tau = \lambda_{\mathcal{S}_a}\lceil\mathcal{R}$, so that τ is a congruence of the semimodule $\mathcal{R}$. If $\tau = \mathcal{R} \times \mathcal{R}$, then the result follows from 4.3(i), and hence assume $\tau = \mathrm{id}$. By 5.1, $(R : \mathcal{S}_a)$ is a subsemigroup of G and $a^i \in (R : \mathcal{S}_a)$ for every $i \ge 0$. By 4.3(ii), $(R \cap Ra, R) \in \tau$, and therefore $R = R \cap Ra$ and $a \in R^{-1}$, a contradiciton.

(ii) Let $a \in N(R)\backslash(R \cup R^{-1})$. Then, by (i), both $\mathcal{S}_a$ and $\mathcal{S}_{a^{-1}}$ are downwards–cofinal in $\mathcal{R}$. In particular, $T = R \cap Ra \cap \cdots \cap Ra^n \subseteq Ra^{-1}$ for some $n \ge 0$ and we have $Ta = Ra \cap Ra^2 \cap \cdots \cap Ra^{n+1} \subseteq R, Ta \subseteq R \cap Ta = R \cap Ra \cap \cdots \cap Ra^{n+1} = T \cap Ra^{n+1} \subseteq T$. Consequently, $T \subseteq Ta^{-i} \subseteq Ra^{-i}$ by induction. Now, we conclude that T is a smallest element of $\mathcal{R}$, a contradiction.

(iii) This follows easily from (ii). ▲

6.2 Lemma. (i) The quasiordering μ of $N(R)$ is linear.

(ii) If $N(R) \ne R^*$, then $N(R)\backslash R \ne \varnothing \ne N(R)\backslash R^{-1}$.

(iii) If $a \in N(R)\backslash R^{-1}$, then the set $\{a^i; i \ge 0\}$ is downwards–cofinal in $N(R)(\mu)$.

(iv) If $a \in N(R)\backslash R$, then the set $\{a^i; i \ge 0\}$ is upwards–cofinal in $N(R)(\mu)$.

Proof. (i) follows immediately from 6.1(iii) and (ii) is obvious.

(iii) By 6.1(i), if $b \in N(R)$, then $R \cap Ra \cap \cdots \cap Ra^n \subseteq Rb$ for some $n \ge 0$ and the rest is clear.

(iv) If $b \in N(R)$, then $Ra^{-n} \subseteq Rb^{-1} = xR$ and $Rb = x^{-1}R \subseteq Ra^n$. ▲

6.3 Corollary. (i) $H(R)$ $(\le^R)$ is a linearly ordered group.

(ii) If $a \in N(R)\backslash R^{-1}$ and $w = \varphi_R(a) \in H(R)$, then the set $\{w^i; i \ge 0\}$ is downwards–cofinal in $H(R)$.

(iii) If $a \in N(R)\backslash R$ and $w = \varphi_R(a) \in H(R)$, then the set $\{w^i; i \ge 0\}$ is upwards–cofinal in $H(R)$.

6.4 Proposition. (i) If $N(R) \ne R^*$, then $H(R)$ $(\le^R)$ is an archimedean linearly ordered group.

(ii) If $N(R) = R^*$, then $H(R) = 1$.

Proof. (i) The set $\varphi_R(N(R)\backslash R)$ is just the set of positive elements of $H(R)$ and the rest is clear from 6.3. ▲

Now, using the well known Hölder (-Baer-Cartan-Loonstra-Conrad) Theorem [5] ([1], [2], [10], [3]), we get the following consequence:

6.5 Corollary. $H(R)$ is an abelian group and $N(R)' \subseteq R^*$.

6.6 Proposition. Assume that $R \subseteq N(R)$. Then:

(i) $N(R) = R \cup R^{-1} = < R >_G$.
(ii) $\mathcal{N}$ is a chain.
(iii) The (multiplicative) semigroup $\mathcal{N}^\circ$ is commutative.

Proof. (i) Combine 6.1(ii) and 6.5.
(ii) Use 6.1(iii).
(iii) See 5.3. ▲

In the remaining part of this section, assume that $N(R) = G$. Then $\mathcal{N} = \mathcal{J}, \mathcal{N}^\circ = \mathcal{J}^\circ, G' \subseteq R^*, \mathcal{J}$ is a chain and $\mathcal{J}^\circ$ is a commutative monoid under the multiplication.

6.7 Lemma. (i) ν is a congruence of the G–semimodule (chain) $\mathcal{J}^\circ$.
(ii) ν is a cancellative congruence of the monoid $\mathcal{J}^\circ$.
(iii) $\nu\lceil\mathcal{R} = \mathrm{id}$.

Proof. (i) and (ii) see 5.4 and 5.6.
(iii) See 5.4(v). ▲

Now, consider again the linearly ordered cancellative commutative monoid $\mathcal{M} = \mathcal{J}^\circ/\nu$ (see the preceeding section). Then $\mathcal{M}$ is also a G–semimodule, $\mathrm{Ker}(\Phi_R) = \nu$ and $\nu\lceil\mathcal{R} = \mathrm{id}$. In particular, $\Phi_R\lceil\mathcal{R}$ is an injective homomorphism of the semimodule $\mathcal{R}$ into the semimodule $\mathcal{M}$ and $\mathcal{L} = \Phi_R(\mathcal{R})$ is both upwards– and downwards–cofinal in $\mathcal{M}$.

6.8 Proposition. The linearly ordered commutative monoid $\mathcal{M}$ is archimedean.

Proof. The operation of $\mathcal{M}$ will be denoted again multiplicatively. Let $\alpha, \beta \in \mathcal{M}, 1 < \alpha, \alpha = Ix/\nu, I \in \mathcal{I}^\circ(R), x \in G, R \subseteq Ix$ and $R \neq Ix$. Then $x \notin R$ and $bx \notin R$ for some $b \in I$; we have $R \subseteq Rbx \subseteq Ix, R \neq Rbx$ and $a = bx \notin G\backslash R$. Now, by 6.3(iii), $\mathcal{H} = \Phi_R\epsilon_R(H(R))$ is upwards–cofinal in $\mathcal{M}$ and the set $\{(Ra/\nu)^i; i \geq 0\}$ is also upwards–cofinal in $\mathcal{M}$. But $Ra/\nu = \gamma$, where $1 < \gamma \leq \alpha$ and $\gamma^i \leq \alpha^i$. In particular, $\beta \leq \alpha^n$ for some $n \geq 0$. ▲

6.9 Let $\mathcal{G}$ be an (abelian) group of fractions of the cancellative commutative monoid $\mathcal{M}$. The linear order on $\mathcal{M}$ is uniquely extended to a linear order $\leq$ of $\mathcal{G}$ and, again, $\mathcal{G}$ is archimedean and $\mathcal{L}$ is both upwards– and downwards–cofinal in $\mathcal{G}$. Furthermore, $\mathcal{M}$ is a G–semimodule $(x * (A/\nu) = A/\nu \cdot Rx^{-1}/\nu = Ax^{-1}/\nu)$ and we extend the G–semimodule structure to the whole $\mathcal{G}$ by setting $x * (\alpha^{-1}\beta) = \alpha^{-1}\beta \cdot Rx^{-1}/\nu = \alpha^{-1}(x * \beta), \alpha, \beta \in \mathcal{M}, x \in G$. In this way, $\mathcal{L}$ becomes a subsemimodule of $\mathcal{G}$. Now, by the Hölder Theorem, there exists an injective group homomorphism Ψ of the group $\mathcal{G}$ into the additive group $\mathbb{R}(+)$ of real numbers and this homomorphism is preserving the linear orders. The G–scalar multiplication on $\mathcal{G}$ is transferred in the following way: $\Psi(x * \alpha) = \Psi(\alpha) - \Psi(Rx/\nu)$ for all $\alpha \in \mathcal{G}$ and $x \in G$. The mapping $\psi : G \to \mathbb{R}(+)$, $\psi(x) = -\Psi(Rx/\nu)$ is a group homomorphism and $\Psi(x * \alpha) = \psi(x) + \Psi(\alpha)$.

7. Simple subchains of $\mathbb{R}$

Let A be a non–zero subgroup of the additive group $\mathbb{R}(+)$ of real numbers and let S be a non–empty subset of R such that $A+S \subseteq S$. Now, $S = S(\oplus,*) = Ch(A,S)$ is an A–semimodule, where $x \oplus y = \min(x,y)$ and $a * x = a + x$ for all $a \in A$ and $x, y \in S$.

7.1 Proposition. (i) If A is not cyclic, then S is a simple semimodule.

(ii) If A is cyclic, then S is simple if and only if $S = A + r$ for some (and then for every) $r \in S$.

Proof. We can assume that $0 \in S$. Then it is easy to see that S is simple if and only if the following condition is satisfied:

For all $x, y \in S, x < y$, there exist $a, b \in A$ such that $x \leq a < b \leq y$.

Now, if A is not cyclic, then A is dense in $\mathbb{R}$, and hence S is simple. On the other hand, if A is cyclic and the above condition is satisfied, then, apparently, $S = A$. ▲

7.2 Remark. (i) Let $A = < a >$ be cyclic and let S be simple. Then S is isomorphic to the following A–semimodule $\mathbb{Z}(\oplus,*)$ defined on $\mathbb{Z}$: $x \oplus y = \min(x,y)$ and $na * x = n + x$ for all $n, x, y \in \mathbb{Z}$.

(ii) Let A be a non–cyclic finitely generated subgroup of $\mathbb{R}(+)$. Then A is a free abelian group of finite rank $n \geq 2$; let $\{a_1, \dots, a_n\}$ be a free basis of A. Then at most one of the numbers $a_1, \dots, a_n$ is rational, the mapping $x \to a_1^{-1}x$ is an automorphism of $\mathbb{R}(+)$ and the set $\{1, b_2, \dots, b_n\}, b_i = a_1^{-1}a_i$, is a free basis of $B = a_1^{-1}A$; the numbers $b_2, \dots, b_n$ are all irrational and the set $\{1, b_2, \dots, b_n\}$ is linearly independent over the field $\mathbb{Q}$ of rationals.

7.3 Remark. Let A be a non–zero subgroup of $\mathbb{R}(+)$ and S_1, S_2 two non–empty subsets of $\mathbb{R}$ such that $A + S_1 \subseteq S_1$ and $A + S_2 \subseteq S_2$.

(i) If A is cyclic and both $Ch(A, S_1)$ and $Ch(A, S_2)$ are simple, then these A–semimodules are isomorphic.

(ii) If A is not cyclic, then both $Ch(A, S_1)$ and $Ch(A, S_2)$ are simple and these A–semimodules are isomorphic if and only if $S_2 = S_1 + r$ for some $r \in \mathbb{R}$.

7.4 Let G be a group and let $\psi : G \to \mathbb{R}(+)$ be a non–zero homomorphism. If S is a non–empty subset of $\mathbb{R}$ such that $\psi(G) + S \subseteq S$, then we get the corresponding G–semimodule $S = S(\oplus,*) = Ch(\psi, S), x \oplus y = \min(x,y)$ and $a * x = \psi(a) + x$ for all $a \in G$ and $x, y \in S$. Notice, that if $0 \in S_1$ and $0 \in S_2$, then $Ch(\psi_1, S_1) \cong Ch(\psi_2, S_2)$ if and only if $\mathrm{Ker}(\psi_1) = \mathrm{Ker}(\psi_2)$ and there is $r \in \mathbb{R}, r > 0$ such that $S_2 = rS_1$ and $\psi_2(a) = r\psi_1(a)$ for every $a \in G$.

8. Simple idempotent semimodules

Let G be a group and S a simple idempotent G–semimodule. If $w \in S^\circ$, then $R_w = \{a \in G; w \leq_S aw\}$ is a subsemigroup of G and $1 \in R_w$ and, for every $x \in S$, we have $\Phi_w(x) = \{a \in G; w \leq_S ax\} \in \mathcal{J}(G, R_w)$. Further, it is easy to check that the following is true:

8.1 Proposition. The mapping $\Phi_w : S \to \mathcal{J}(G, R_w)$ is an injective semimodule homomorphism of S into $\mathcal{J}$. Moreover:

(i) $\Phi(o) = \varnothing$ (if $o \in S$).

(ii) $\Phi(S^\circ) \subseteq \mathcal{J}^\circ$.
(iii) $\Phi(ax) = \Phi(x)a^{-1}$ for all $a \in G, x \in S$.
(iv) $\Phi(a^{-1}w) = R_w a$ for every $a \in G$.
(v) The set $\Phi(S^\circ)$ is both upwards– and downwards–cofinal in $\mathcal{J}^\circ$.

8.2 Theorem. Let G be a group. Then the subsemimodules $\mathcal{L}$ of $\mathcal{K}(G,R)$ such that $\mathcal{G}(G,R) \subseteq \mathcal{L}$ (R running through all proper subsemigroups of G with $1 \in R$) are up to isomorphism the only simple ipa–semimodules.

Proof. Combine 8.1 and 4.2(iv). ▲

8.3 Remark. It follows readily from 8.2 (or from 3.7 and 4.2) that G is hs–uniform if and only if every simple ipa–semimodule is a qza–semimodule.

8.4 Theorem. Let G be an abelian group and S a simple idempotent semimodule without absorbing element. Then S is a chain and there exist a non–zero homomorphism $\psi : G \to \mathbb{R}(+)$ and a subset T of $\mathbb{R}$ such that $0 \in T, \psi(G) + T \subseteq T$ and S is isomorphic to the semimodule $Ch(\psi, T)$ (see the preceding section).

Proof. Combine 8.1, 6.9 and 7.4. ▲

The description of simple idempotent semimodules over abelian groups (see 8.2 and 8.4) was independently achieved by M. Maróti (see [11]).

8.5 Example. Put $G = \mathbb{Z}(+) \times \mathbb{Z}(+)$.

(i) For every pair (r,s) of relatively prime integers define a semimodule $Z_{(r,s)}$ $(= Z_{(r,s)}(\oplus,*))$ on the set $\mathbb{Z}$ of integers by $x \oplus y = \min(x,y)$ and $(n,m) * x = nr + ms + x$. Now, $Z_{(r,s)}$ are pair–wise non–isomorphic simple semimodules.

(ii) For $u \in \{1,-1\}$, every irrational $q \in \mathbb{R}$ and every subset T of $\mathbb{R}$ such that $0 \in T, 1+T = T$ and $q + T = T$, define a semimodule $W_{(u,q,T)}$ $(= W_{(u,q,T)}(\oplus,*))$ on the set T by $x \oplus y = \min(x,y)$ and $(n,m) * x = nu + mq + x$. Again, $W_{(u,q,T)}$ are pair–wise non–isomorphic simple semimodules.

(iii) Every simple ip–semimodule possessing no absorbing element is isomorphic to one of the semimodules $Z_{(r,s)}, W_{(u,q,T)}$.

8.6 Remark. (cf.[14]) If G is an hs–uniform group, then $\mathrm{card}(S) \leq \max(\aleph_0, 2^{\mathfrak{a}}), \mathfrak{a} = \mathrm{card}(G)$, for every simple semimodule S. If G is an abelian group, then $\mathrm{card}(S) \leq \max(2^{\aleph_0}, \mathfrak{a})$.

9. Simple cancellative semimodules over abelian groups

9.1 Remark. (i) Let G be a group and M a non-trivial G–module. Then M is simple as a semimodule if and only if M is simple as a module and this is equivalent to the fact that 0 and M are the only submodules of M. If it is so, then M is also a simple R–module, $R = \mathbb{Z}G$, and ${}_RM \cong R/I$ for a maximal left ideal I of R.

(ii) Let S be a simple cn–semimodule, not a module. By 2.2, S possesses no neutral element and consequently $x + y \neq x$ for all $x, y \in S$.

9.2 Let G be an abelian group and M a simple G–module. By 9.1(i), every non–zero endomorphism of M is an automorphism, and hence the endomorphism ring F of M is a division ring. Moreover, the mapping $\varphi : G \to F$, defined by $(\varphi(a))(x) = ax$, $a \in G$, $x \in M$, is a homomorphism of the group G into the multiplicative group (of non–zero elements) of the division ring F.

Choose $w \in M$, $w \neq 0$, and put $\psi(f) = f(w)$ for every $f \in F$. Then ψ is an isomorphism of the group $F(+)$ onto the group $M(+)$, $\psi(1_F) = w$ and $\psi(\varphi(a)) = aw$,

$a \in G$. Moreover, F is a G–module (via φ) and $\psi : {}_GF \to {}_GM$ is an isomorphism of the G–modules.

Let $E = \{\varphi(a_1) + \cdots + \varphi(a_n); n \geq 1, a_i \in G\}$. Then E is a subsemiring of F and $F = \{f - g; f, g \in E\}$. Now, it is clear that F is a field. Notice that if $0 \in E$, then $0 \in N$ for every subsemimodule N of M.

9.3 Theorem. (i) Let φ be a homomorphism of an abelian group G into the multiplicative group of non–zero elements of a field F such that $F = \{f - g; f, g \in E\}$, where $E = \{\varphi(a_1) + \cdots + \varphi(a_n); n \geq 1, a_i \in G\}$. Then F becomes a simple G–module ($ax = \varphi(a)x, a \in G, x \in F$).

(ii) Every simple G–module is of the form described in (i).

Proof. See 9.2. ▲

9.4 Let G be an abelian group and S a cn–semimodule possessing no neutral element. Clearly, S is a subsemimodule of a module M such that $M = \{u - v; u, v \in S\}$. Now, assume that M is a simple module (e.g., if S is simple). Then, by 9.2, we get a homomorphism $\varphi : G \to F = \mathrm{End}({}_GM)$ and a module isomorphism $\psi : {}_GF \to {}_GM$, $\psi(f) = f(w)$, $w \in M$, $w \neq 0$. Moreover, $F = \{f - g; f, g \in E\}$, where $E = \{\varphi(a_1) + \cdots + \varphi(a_n); n \geq 1, a_i \in G\}$.

Put $T = \psi^{-1}(S)$. Then T is a subsemimodule of ${}_GF$, ${}_GT \cong {}_GS$, $0 \notin T$ and $F = \{r - s; r, s \in T\}$.If $w \in S$, then $H = \varphi(G) \subseteq T$.

9.5 Theorem. Let G be an abelian group and S a non-trivial cn–semimodule possessing no neutral element. Let M be a module such that S is a subsemimodule of M and $M = \{x - y; x, y \in S\}$. Then S is a simple semimodule if and only if the following three conditions are satisfied:

(1) M is a simple module (see 9.3);
(2) $Gx + S = S$ for every $x \in S$;
(3) For all $x, y \in S$ there exist $z \in S$ and a positive integer n such that $x + z = ny$.

Proof. (i) Let S be a simple semimodule. Then M is simple (see 9.1(i) and 9.4) and $Gx + S = S$, since the relation $((Gx + S) \times (Gx + S)) \cup \mathrm{id}_S$ is a congruence of S. Further, let $w \in S$ and, for any $x \in S$, let P_x denote the set of $a \in G$ such that $ax + u = nw$ for some $u \in S$ and $n \geq 1$. Then ε is a congruence of S, where $(x, y) \in \varepsilon$ if and only if $P_x = P_y$, and we have $(w, 2w) \in \varepsilon$. But $2w \neq w$ by 9.1(ii), hence $\varepsilon = S \times S$, $P_x = P_w$ and $1_G \in P_x$. The rest is clear.

(ii) Now, assume that the three conditions (1), (2) and (3) are satisfied. With regard to 9.4, we can also assume that G is a subgroup of the multiplicative group (of non-zero elements) of a field F and that S is a subsemimodule of ${}_GF$ such that $G \subseteq S$. Moreover, $F = \{r - s; r, s \in E\}$, $E = \{a_1 + \cdots + a_n; n \geq 1, a_i \in G\}$. The rest of the proof is divided into three parts:

(a) Let τ be a congruence of ${}_GS$ such that $\tau \neq \mathrm{id}_S$ and the factor–semimodule S/τ is cancellative. We claim that $\tau = S \times S$.

Put $N = \{x - y; (x, y) \in \tau\}$. Then N is a non–zero submodule of ${}_GF$ and, since ${}_GF$ is a simple module, we have $N = F$. Now, if $u, v \in S$ are arbitrary, then $u - v = x - y$ for some $(x, y) \in \tau$ and consequently $x + v = y + u$ and $(u, v) \in \tau$, since S/τ is cancellative.

(b) Let τ be a congruence of S such that $(w, 2w) \in \tau$ for some $w \in S$. We claim that $\tau = S \times S$.

Denote by ξ the natural projection of S onto $T = S/\tau$. Due to the condition (3), $\xi(w)$ is the only idempotent of $T(+)$, and therefore, $G\xi(w) = \xi(w)$ and $\xi(w)+T = T$ by (2). Now, it is clear that $\xi(w)$ is a neutral element of $T(+)$ and that $T(+)$ is a group (use (3) again). Thus T is a module and $\tau = S \times S$ by (a).
(c) Let $\tau \neq S\times S$ be a congruence of S and let $w \in S$. According to (b), $(w, 2w) \notin \tau$ and we can consider a congruence σ of S maximal with respect to $\tau \subseteq \sigma$ and $(w, 2w) \notin \sigma$. It follows from (b) again that σ is a maximal congruence of S, hence $T = S/\sigma$ is a simple semimodule such that $T(+)$ possesses no idempotent element. Using 2.1, we conclude that T is a cn–semimodule. Thus $\tau = \sigma = \mathrm{id}_S$ by (a) and we have proven that S is simple. ▲

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