The Problem of Absolute Stability: A Dynamic Programming Approach ⋆

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Abstract

We consider the problem of absolute stability of a feedback system composed of a linear plant and a single sector-bounded nonlinearity. Pyatnitskiy and Rapoport used a variational approach and the Maximum Principle to derive an implicit characterization of the "most destabilizing" nonlinearity. In this paper, we address the same problem using a dynamic programming approach. We show that the corresponding value function is composed of simple building blocks which are the generalized first integrals of appropriate linear systems. We demonstrate how the results can be used to design stabilizing switched controllers.

Key words: Hamilton-Jacobi-Bellman equation; switched linear systems; hybrid systems; differential inclusions.

1 Introduction

Consider the system

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + \mathbf{b}\phi(t, y(t)) \\ y(t) &= \mathbf{c}^T\mathbf{x}(t) \end{aligned} \tag{1}$$

where $\mathbf{b}, \mathbf{c}, \mathbf{x}(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}$, A is an asymptotically stable matrix, and ϕ belongs to S_k, the set of scalar time-varying functions in the sector $[0, k]$. [1] Note that we can view (1) as the feedback connection of a linear system and a single nonlinear function from S_k.

Problem 1 (Absolute stability [19, Ch. 5]) *Find the value* $k^* := \inf\{k \geq 0 : \exists \phi^* \in S_k$ *for which (1) is not asymptotically stable*$\}$.

In other words, for $k \in [0, k^*)$, (1) is asymptotically stable for *any* $\phi \in S_k$. The problem is difficult because S_k contains an infinite number of functions and, therefore, a solution must actually entail the characterization of the "most destabilizing" nonlinearity ϕ^*.

Applying the idea of *global linearization* [5] we can restate Problem 1 in a more convenient form. Since $\phi \in S_k$, we have $\phi(t, y) = a(t, y)y$, with $0 \leq a(t, y) \leq k$, so (1) becomes

$$\dot{\mathbf{x}} \in \text{co}\{A, B_k\}\mathbf{x}, \quad B_k := A + k\mathbf{b}\mathbf{c}^T \tag{2}$$

where co denotes convex hull. We can now restate Problem 1.

Problem 2 (Absolute stability) *Find the value* $k^* := \inf\{k \geq 0 : (2)$ *is not asymptotically stable*$\}$.

Specifying ϕ^* in (1) is equivalent to specifying the "most unstable trajectory" $\mathbf{x}^*(t)$ of (2).

Note that (2) is the *relaxed* version [20, Ch. 2] of the switched linear system

$$\dot{\mathbf{x}}(t) \in \{A\mathbf{x}(t), B_k\mathbf{x}(t)\}. \tag{3}$$

Stability analysis of switched linear systems is a very active research area (see, e.g., [12]). For our purposes, (2) and (3) are equivalent since it is well-known [15] that (2) is asymptotically stable if and only if (3) is.

Pyatnitskiy and Rapoport [16][17] introduced the idea of using a variational approach to describe the "most

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[1] i.e., $\phi(t, 0) = 0$ and $0 \leq z\phi(t, z) \leq kz^2$ for all $t \geq 0$.

destabilizing" nonlinearity ϕ^*. Applying the *Maximum Principle*, they derived an *implicit* characterization of ϕ^* in terms of a two-point boundary value problem.

A different approach to optimal control problems is based on dynamic programming and the Hamilton-Jacobi-Bellman (HJB) equation. Unlike the *Maximum Principle*, a solution to the HJB equation provides an *explicit* formula for the optimal control (unfortunately, such a solution is usually unattainable).

In this paper we address the same variational problem studied by Pyatnitskiy and Rapoport using dynamic programming. We show that the corresponding value function can be constructed by concatenating the contours of two *generalized first integrals*, and provide a recipe for explicitly constructing these generalized first integrals.

This new approach also provides a geometrically intuitive characterization of the "most unstable" trajectory. Furthermore, for the special case $n = 2$, we can actually derive an *explicit* solution to the HJB equation by showing how the contours of the two *generalized first integrals* are concatenated. This yields a deeper understanding of the solution of the second-order absolute stability problem given in [13].

The remainder of this paper is organized as follows. In Section 2, we recall several known results that will be used later on. In Section 3, we formulate the optimal control problem and study it using a dynamic programming approach. In Section 4, we relate the generalized first integrals of the subsystems to the solution of the HJB equation. In Section 5, we apply our results to the problem of designing a stabilizing switched controller. The final section summarizes the paper. All the proofs are placed in Appendix A.

2 Preliminaries

We say that (3) is *locally asymptotically stable* [2] if the following two conditions hold for *any* solution $\mathbf{x}(t)$: (1) $\forall \epsilon > 0$, $\exists \delta > 0$ such that $||\mathbf{x}(0)|| \leq \delta$ implies $||\mathbf{x}(t)|| \leq \epsilon$, $\forall t \geq 0$; and (2) $\exists c > 0$ such that $||\mathbf{x}(0)|| < c$ implies $\lim_{t\to\infty} ||\mathbf{x}(t)|| = 0$.

We assume from here on that $A + k^*\mathbf{bc}^T$ is an asymptotically stable matrix. [3] We also assume that the pair $(A, \mathbf{b})$ $[(A, \mathbf{c})]$ is controllable [observable]. This assumption guarantees that if there is an unbounded trajectory $\mathbf{x}(t)$ for some $\mathbf{x}(0)$, then there exists an unbounded trajectory for *any* $\mathbf{x}(0) \in (\mathbb{R}^n \setminus \{\mathbf{0}\})$ (see [16, Lemma 1]).

For a function $V : \mathbb{R}^n \to (-\infty, +\infty)$, and $\mathbf{q} \in \mathbb{R}^n$, let $\frac{\partial V(\mathbf{x})}{\partial \mathbf{q}} := \lim_{h\downarrow 0} \frac{V(\mathbf{x}+h\mathbf{q})-V(\mathbf{x})}{h}$ be the (one-sided) derivative of V at $\mathbf{x}$ in the direction $\mathbf{q}$. If V is convex, then this derivative always exists [18] and, furthermore, since a convex function is differentiable almost everywhere, $\frac{\partial V(\mathbf{x})}{\partial \mathbf{q}} = V_{\mathbf{x}}(\mathbf{x})\mathbf{q}$ for almost all $\mathbf{x}$.

3 An optimal control problem

Instead of studying all the possible trajectories of (3), we would like to analyze the single "most unstable" trajectory. Following the pioneering work of Pyatnitskiy and Rapoport [16][17], we use a variational approach to characterize this trajectory.

It follows from Filippov's Selection Theorem [20, Theorem 2.3.13] that (2) can be written as

$$\dot{\mathbf{x}}(t) = (A + u(t)\mathbf{bc}^T)\mathbf{x}(t), \quad u \in \mathbb{U}^k \tag{4}$$

where $\mathbb{U}^k$ is the set of all measurable functions mapping $[0, \infty)$ to $[0, k]$.

Fixing $t_f > 0$ and an initial position $\mathbf{x}(0) = \mathbf{x_0}$, we consider the following (Mayer type) optimal control problem.

Problem 3 *Find an admissible control that maximizes the cost-functional $J(u, t_f, \mathbf{x_0}) := ||\mathbf{x}(t_f)||$ along the solutions of (4).*

The intuitive interpretation of Problem 3 is clear: find the control that "pushes" $\mathbf{x}(t_f)$ as far away from the origin as possible [4]. It is possible to show that the set of solutions of (4), equipped with the supremum norm topology, is compact [20, Ch. 2] and, therefore, maximizing J amounts to maximizing a continuous functional on a compact set. Hence, there exists a $\tilde{u} \in \mathbb{U}^k$ such that $J(\tilde{u}, t_f, \mathbf{x_0}) = \sup_{u\in\mathbb{U}^k} J(u, t_f, \mathbf{x_0})$. We refer to $\tilde{u}$ as the *worst case switching law* (WCSL) and to the corresponding trajectory $\tilde{\mathbf{x}}$ as the *worst case trajectory.*

The HJB equation [20, Ch. 12] associated with Problem 3 is

$$V_t(t, \mathbf{x}) = -\max_{r\in[0,k]} \left\{V_{\mathbf{x}}(t, \mathbf{x})(A + r\mathbf{bc}^T)\mathbf{x}\right\} \tag{5}$$
$$V(t_f, \mathbf{x}) = ||\mathbf{x}|| \text{ for all } \mathbf{x} \in \mathbb{R}^n$$

[2] For switched linear systems local asymptotic stability implies global (and exponential) asymptotic stability, so from now on we use the term asymptotic stability.

[3] This is the "interesting" case where (3) is not asymptotically stable although we are still switching between two asymptotically stable subsystems.

[4] For other optimal control problems for switched systems, see [2] [3, Ch. 3] [22] and the references therein.

where $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is the *value function*, $V_t = \frac{\partial V}{\partial t}$, and $V_{\mathbf{x}}$ is the row vector $(V_{x_1}, \ldots, V_{x_n})$.

For our purposes, it is sufficient to study the infinite-horizon case, that is, when $t_f \to \infty$. In this case, (5) formally simplifies to

$$\max_{r \in [0,k]} \left\{ V_{\mathbf{x}}(\mathbf{x})(A + r\mathbf{b}\mathbf{c}^T)\mathbf{x} \right\} = 0. \qquad (6)$$

The next result relates (6) to the value k^*.

Definition 1 *We say that a function $Q : \mathbb{R}^n \to [0, +\infty)$ is class CH if Q is convex; $Q(\mathbf{x}) \geq 0$ with equality only for $\mathbf{x} = \mathbf{0}$; and $Q(c\mathbf{x}) = cQ(\mathbf{x})$ for all $c > 0$ and all $\mathbf{x} \in \mathbb{R}^n$.*

Theorem 1 *There exists a class CH function V satisfying*

$$\max_{r \in [0,k]} \left\{ \frac{\partial V(\mathbf{x})}{\partial((A + r\mathbf{b}\mathbf{c}^T)\mathbf{x})} \right\} = 0 \qquad (7)$$

if and only if $k = k^*$.

Note that (7) is nothing but (6) with the directional derivative replacing the classical one, as V is not necessarily differentiable. However, it follows from (7) that at any point $\mathbf{x}$ where $V_{\mathbf{x}}(\mathbf{x})$ does exist, (6) holds, so (6) holds for all $\mathbf{x} \in (\mathbb{R}^n \setminus E)$, with $\text{meas}(E) = 0$.

The next result provides a characterization of $\tilde{u}$ in terms of V, and shows that, almost everywhere, $\tilde{u}$ is unique, bang-bang, state-feedback [i.e., $\tilde{u}(t) = \tilde{u}(\mathbf{x}(t))$], and without singular arcs.

Theorem 2 *For $k = k^*$, let V be the function in Theorem 1. Define $p : (\mathbb{R}^n \setminus E) \to \mathbb{R}$ by $p(\mathbf{x}) := V_{\mathbf{x}}(\mathbf{x})(\mathbf{b}\mathbf{c}^T)\mathbf{x}$. Then, the optimal control $\tilde{u}$ satisfies*

$$\tilde{u}(t) = \begin{cases} 0, & \text{if } p(\tilde{\mathbf{x}}(t)) < 0 \\ k, & \text{if } p(\tilde{\mathbf{x}}(t)) > 0. \end{cases} \qquad (8)$$

Furthermore, the zeros of $p(\tilde{\mathbf{x}}(t))$ are isolated.

By substituting (8) in (6), we get

Theorem 3 *For $k = k^*$, let V be the function in Theorem 1. For any $\mathbf{x}$ such that $V_{\mathbf{x}}(\mathbf{x})\mathbf{b}\mathbf{c}^T\mathbf{x} < 0$, we have $V_{\mathbf{x}}(\mathbf{x})A\mathbf{x} = 0$. For any $\mathbf{x}$ such that $V_{\mathbf{x}}(\mathbf{x})\mathbf{b}\mathbf{c}^T\mathbf{x} > 0$, we have $V_{\mathbf{x}}(\mathbf{x})(A + k^*\mathbf{b}\mathbf{c}^T)\mathbf{x} = 0$.*

In other words, the value function V is non-increasing, and it is constant along the optimal trajectory.[5]

We now show that these results provide an intuitive interpretation of $\tilde{u}$ and V. Consider the case $n = 2$.

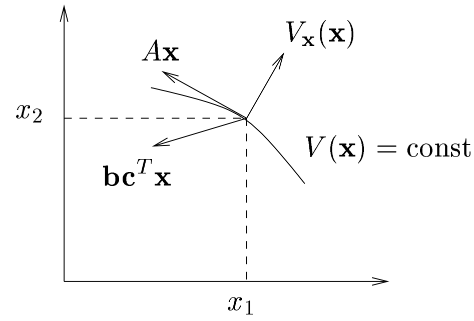

Fig. 1. Geometrical explanation of $\tilde{u}$ when $V_{\mathbf{x}}(\mathbf{x})A\mathbf{x} = 0$ and $V_{\mathbf{x}}(\mathbf{x})\mathbf{b}\mathbf{c}^T\mathbf{x} < 0$.

Fix a point $\mathbf{x} = (x_1, x_2)^T$ such that $V_{\mathbf{x}}(\mathbf{x})\mathbf{b}\mathbf{c}^T\mathbf{x} < 0$, so $V_{\mathbf{x}}(\mathbf{x})A\mathbf{x} = 0$ (that is, $V_{\mathbf{x}}(\mathbf{x})$ is orthogonal to $A\mathbf{x}$) (see Fig. 1). Then, a solution of $\dot{\mathbf{x}} = A\mathbf{x}$ follows the contour $V(\mathbf{x}) = \text{const}$, whereas the solution of $\dot{\mathbf{x}} = (A + r\mathbf{b}\mathbf{c}^T)\mathbf{x}$, for any $r > 0$, crosses this contour going toward the origin. The WCSL is $\tilde{u}(\mathbf{x}) = 0$ which indeed corresponds to setting $\dot{\tilde{\mathbf{x}}} = A\tilde{\mathbf{x}}$. Thus, $\tilde{u}$ indeed "pushes" the trajectory as far away from the origin as possible, where "far" is with respect to the function V.

The geometrically intuitive interpretation of the "most destabilizing" switching-law for the case $n = 2$ already appeared in the work of Filippov [7][6] and, more recently, in [6] and [21] (see also [4]). In particular, the solution of the absolute stability problem for $n = 2$ in [13], can be viewed as a special case of Theorems 1–3. The variational approach shows that this geometric intuition carries over to *any* n.

The following result shows that the value of the cost-functional J is indeed closely related to the stability of (2). We use the notation $S^{n-1} := \{\mathbf{x} \in \mathbb{R}^n : ||\mathbf{x}|| = 1\}$.

Proposition 4 *Fix $\epsilon > 0$. If $k < k^*$, then there exists a $T_1 = T_1(k)$ such that*

$$J(u, t_f, \mathbf{x_0}) < \epsilon \qquad \forall\, \mathbf{x_0} \in S^{n-1}, \forall\, u \in \mathbb{U}^k, \forall\, t_f \geq T_1. \qquad (9)$$

Conversely, for any $k > k^*$ there exists a state-feedback control $\tilde{u} = \tilde{u}(\mathbf{x}) \in \mathbb{U}^k$ and a $T_2 = T_2(k) > 0$ such that

$$J(\tilde{u}, t_f, \mathbf{x_0}) > \epsilon \qquad \forall\, \mathbf{x_0} \in S^{n-1}, \forall\, t_f \geq T_2. \qquad (10)$$

4 The generalized first integral

It follows from (6) that for $\tilde{u} = 0$ ($\tilde{u} = k^*$), we have $V_{\mathbf{x}}(\mathbf{x})A\mathbf{x} = 0$ ($V_{\mathbf{x}}(\mathbf{x})B_{k^*}\mathbf{x} = 0$). Hence, $V(\mathbf{x})$ is composed of two basic *building-blocks*, namely, a

[5] this is, of course, a particular case of Bellman's Principle of Optimality.

[6] Filippov's characterization was presented using polar coordinates $(r, \theta) := (\sqrt{x_1^2 + x_2^2}, \arctan(x_2/x_1))$ but it is equivalent to our characterization (see [10]).

function $H^A(\mathbf{x})$ that satisfies $H^A_{\mathbf{x}}(\mathbf{x})A\mathbf{x} = 0$ and a function $H^B(\mathbf{x})$ that satisfies $H^B_{\mathbf{x}}(\mathbf{x})B\mathbf{x} = 0$ (for convenience of notation, we use B to denote B_{k^*}). In this section we show how to construct such functions.

Definition 2 ([13]) *Consider the system*

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)). \tag{11}$$

A function $H^{\mathbf{f}}(\cdot) : \mathbb{R}^n \to \mathbb{R}_+$ is a generalized first integral of (11) if $H^{\mathbf{f}}(\mathbf{x})$ is not constant on any open subset of $\mathbb{R}^n$, $H^{\mathbf{f}}(\mathbf{x}(t))$ is piecewise constant *along the trajectories of (11), and it attains a finite set of values for all $t \in [0, T]$ (T finite).*

Note that if (11) is Hamiltonian [8], then its *classical* first integral is also a generalized first integral. Note also that the definition implies that we can characterize the trajectory $\mathbf{x}(t)$, $t \in [0, T]$, of (11) as the concatenation of a finite number of contours $H^{\mathbf{f}}(\mathbf{x}) = c_1$, $H^{\mathbf{f}}(\mathbf{x}) = c_2, \ldots$

It was shown in [13] ([10]) how to explicitly construct generalized first integrals for *second-order* linear (homogeneous) systems. We now discuss the construction of $H^A(\mathbf{x})$ for the system $\dot{\mathbf{x}}(t) = A\mathbf{x}(t)$, where $\mathbf{x}(t) \in \mathbb{R}^n$, and $n > 2$. For the sake of simplicity, we assume that $n = 2k$ and that A is *pseudo-diagnolizable* [11], that is, there exists an invertible matrix T such that $T^{-1}AT = diag(D^1, \ldots, D^k)$, where each D^i is a real 2×2 matrix in one of two canonical forms: $\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$ or $\begin{pmatrix} \lambda & 0 \\ 0 & \zeta \end{pmatrix}$. Defining $\mathbf{y} = T^{-1}\mathbf{x}$, we get

$$\dot{\mathbf{y}} = diag(D^1, \ldots, D^k)\mathbf{y}, \tag{12}$$

so if $H^i : \mathbb{R}^2 \to \mathbb{R}_+$ is the generalized first integral of the *second-order* linear system $\dot{\mathbf{z}} = D^i\mathbf{z}$, then $H^D(\mathbf{y}) := H^1(y_1, y_2) + \cdots + H^k(y_{n-1}, y_n)$ is a generalized first integral of (12). Thus, $H^A(\mathbf{x}) := H^D(T\mathbf{y})$ is a generalized first integral of $\dot{\mathbf{x}}(t) = A\mathbf{x}(t)$.

Note that this provides a simple theoretical recipe for constructing the generalized first integral for a linear system. However, in order to solve the absolute stability problem we need to construct the generalized first integral of $\dot{\mathbf{x}} = (A + k\mathbf{b}\mathbf{c}^T)\mathbf{x}$, and find for what value k the contours of $H^A(\mathbf{x})$ and $H^{B_k}(\mathbf{x})$ can be concatenated to form the function V. However, in general, there does not exist a closed-form formula for the eigenvectors of B_k in terms of k and, therefore, we cannot expect to have a closed-form expression for $H^{B_k}(\mathbf{x})$ as a function of k.

5 Designing a switched controller

Recently, the problem of designing a control algorithm that relies on switching between several possible controllers has attracted a great deal of attention.[7] In this section, we use an example to demonstrate how our results can be used to design a switching controller.

Consider the system

$$\begin{aligned} \dot{\mathbf{x}} &= F\mathbf{x} + \mathbf{b}u \\ y &= \mathbf{c}^T\mathbf{x} \end{aligned} \tag{13}$$

where $F = \begin{pmatrix} 0 & 1 \\ -1 & \epsilon \end{pmatrix}$, with $\epsilon \geq 0$, $\mathbf{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and $\mathbf{c}^T = (1, 0)$. Our goal is to design an output-feedback controller that stabilizes (13).

Note that the system cannot be stabilized by a *static* output-feedback controller $u = f(y)$. Indeed, letting $R(\mathbf{x}) := x_1^2 + x_2^2 - 2\int_0^{x_1} f(p)dp$, we get that the derivative of R, along the trajectories of the closed-loop system, is $\dot{R}(\mathbf{x}(t)) = 2\epsilon x_2^2 \geq 0$. Since $R(\mathbf{0}) = 0$, this implies that for any $\mathbf{x}(0)$ such that $R(\mathbf{x}(0)) > 0$, $\mathbf{x}(t) \not\to \mathbf{0}$. Nevertheless, Artstein [1] provided examples of *switched* controllers in the form

$$u(t) = -q(t)y, \quad q(t) \in [0, k] \tag{14}$$

that stabilize (13) for the particular case $\epsilon = 0$.[8]

We show that our results can be used to solve

Problem 4 *Consider the closed-loop system given by (13) and (14), namely,*

$$\dot{\mathbf{x}} = F\mathbf{x} - q(t)\mathbf{b}\mathbf{c}^T\mathbf{x}, \quad q(t) \in [0, k].$$

Determine for what values of $k \geq 0$ a stabilizing controller exits.

Using the transformation $\tau = -t$, we see that a stabilizing switched controller exists if and only if

$$\dot{\mathbf{x}} \in \text{co}\{A, B_k\}\mathbf{x} \tag{15}$$

with $A := -F$, and $B_k := A + k\mathbf{b}\mathbf{c}^T$, admits an unbounded trajectory, so we need to solve the absolute stability problem for (15). The generalized first integral of $\dot{\mathbf{x}} = B_k\mathbf{x}$ is[9] $H^{B_k}(\mathbf{x}) = ((k+1)x_1^2 - \epsilon x_1 x_2 + x_2^2)w(\mathbf{x}; k)$, with $w(\mathbf{x}; k) := \exp(\frac{-2\epsilon}{\sqrt{4k+4-\epsilon^2}} \arctan(\frac{x_1\sqrt{4k+4-\epsilon^2}}{2x_2 - \epsilon x_1}))$, and since $A = B_0$, $H^A(\mathbf{x}) = H^{B_0}(\mathbf{x})$. Computing, we get $H^A_{\mathbf{x}}(\mathbf{x})B_k\mathbf{x} = 2kx_1x_2w(\mathbf{x}; 0)$, so $\text{sgn}(H^A_{\mathbf{x}}(\mathbf{x})B_k\mathbf{x}) =$

[7] see, e.g., [9] [14] [12] and the references therein.

[8] that is, when (13) is the harmonic oscillator with the control being the external force.

[9] For details on calculating H^{B_k}, see [13]. Note that it is easy to verify, by direct calculation, that $H^{B_k}_{\mathbf{x}}(\mathbf{x})B_k\mathbf{x} = 0$.

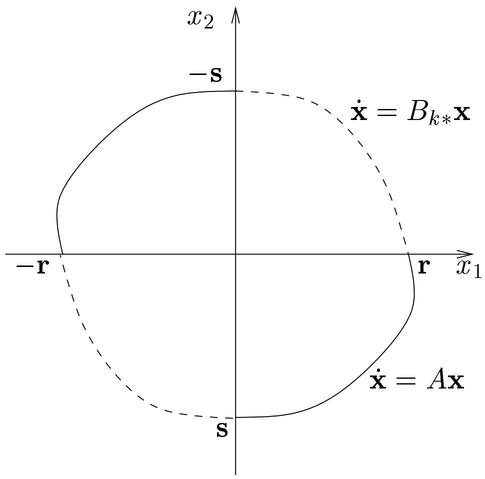

Fig. 2. A contour of the function $V(\mathbf{x})$.

$\text{sgn}(x_1x_2)$. Now the results in Section 3 imply that for $k = k^*$ the WCSL for (15) is

$$\tilde{u} = \begin{cases} 0, & \text{if } x_1x_2 < 0 \\ 1, & \text{if } x_1x_2 > 0. \end{cases} \tag{16}$$

Hence, $k = k^*$ iff there exists a CH function $V(\mathbf{x})$ whose contours are scaled versions of the contour shown schematically in Fig. 2 with $\mathbf{s} = (0, s_2)$ and $\mathbf{r} = (r_1, 0)$.

Since V is homogeneous, we can choose s_2 arbitrarily, say $s_2 = -1$. This yields: $\exp(At_1)(0, -1)^T = (r_1, 0)^T$, and $\exp(B_{k^*}t_2)(r_1, 0)^T = (0, 1)^T$, which together imply

$$\exp(B_{k^*}t_2)\begin{pmatrix} \exp(\frac{-\epsilon}{\sqrt{4-\epsilon^2}}\arccos(\epsilon/2)) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{17}$$

Given ϵ, Eq. (17) can be easily solved numerically to get the two unknowns t_2 and k^*. Fig. 3 depicts k^* as a function of ϵ. It may be seen that k^* increases with ϵ. This is reasonable, since as ϵ increases, (13) becomes "less stable", so a larger control effort is required to stabilize it.

For concreteness, consider the case $\epsilon = 0.1$. Then, (17) yields $k^* = 0.3373$ (to 4 decimals), so (15) has an unbounded trajectory if and only if

$$k > k^* = 0.3373, \tag{18}$$

and in this case the control (16) yields an unbounded trajectory. Hence, we obtain a complete solution to Problem 4: a stabilizing switched controller in the form (14) exists *if and only if* (18) holds, and is given by

$$q(t) = \begin{cases} 0, & \text{if } x_1(t)x_2(t) < 0 \\ k, & \text{if } x_1(t)x_2(t) > 0. \end{cases}$$

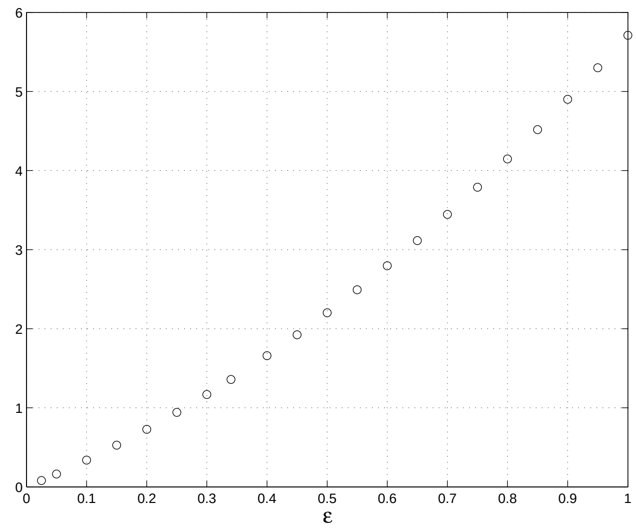

Fig. 3. k^* for various values of ϵ.

Finally, note that the expressions for $H^A(\mathbf{x})$ and $H^{B_{k^*}}(\mathbf{x})$ provide an *explicit* expression for the value function $V(\mathbf{x})$ as a concatenation of contours $H^A(\mathbf{x}) = \text{const}$ and $H^{B_{k^*}}(\mathbf{x}) = \text{const}$.

6 Summary

We characterized the "most unstable" nonlinearity in the absolute stability problem using a dynamic programming approach. We showed that the value function is composed of two simple building-blocks which are the *generalized first integrals* of the sub-systems. We also demonstrated how these results can be applied to the dual problem of designing switched controllers.

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A Proofs

PROOF OF THEOREM 1. Suppose that (7) holds. It follows from [20, Theorem 12.2.1] that there exists a $\mathbf{x_0} \neq \mathbf{0}$, and a trajectory $\mathbf{x}(t)$ of (2) satisfying $V(\mathbf{x}(t)) \equiv V(\mathbf{x_0})$ for all $t \geq 0$. Hence, $\mathbf{x}(t) \not\to \mathbf{0}$ and, therefore,

$$k \geq k^*. \tag{A.1}$$

Fix $\epsilon, \delta > 0$ such that $l := 1 - \epsilon\delta > 0$, and let $A_\epsilon := A - \epsilon I$. Then,

$$\begin{aligned} V(\mathbf{x} + \delta A_\epsilon \mathbf{x}) &= lV(\mathbf{x} + \frac{\delta}{l}A\mathbf{x}) \\ &= -\epsilon\delta V(\mathbf{x} + \frac{\delta}{l}A\mathbf{x}) + V(\mathbf{x} + \frac{\delta}{l}A\mathbf{x}), \end{aligned}$$

and it is easy to see that this implies $\frac{\partial V(\mathbf{x})}{\partial A_\epsilon \mathbf{x}} < \frac{\partial V(\mathbf{x})}{\partial A\mathbf{x}}$.

Similarly, for $B_\epsilon := A + k\mathbf{b}\mathbf{c}^T - \epsilon I$, we have $\frac{\partial V(\mathbf{x})}{\partial B_\epsilon \mathbf{x}} < \frac{\partial V(\mathbf{x})}{\partial B\mathbf{x}}$. It follows from (7) that V strictly decreases along the trajectories of the system $\dot{\mathbf{x}} \in \{A_\epsilon \mathbf{x}, B_\epsilon \mathbf{x}\}$, for all $\epsilon > 0$. Hence, $k \leq k^*$. Combining this with (A.1), we conclude that $k = k^*$.

The reverse implication follows from [16, Theorem 1]. □

Proof of Theorem 2. Eq. (8) follows immediately from (6), so we have only to prove the last statement of the theorem. Denote $m_1(t) := V_{\mathbf{x}}(\tilde{\mathbf{x}}(t))\mathbf{b}$ and $m_2(t) := \mathbf{c}^T\tilde{\mathbf{x}}(t)$. We will show that both m_1 and m_2 have isolated zeros and, therefore, so does $p(\tilde{\mathbf{x}}(t))$. Seeking a contradiction, assume that $m_2(t) \equiv 0$ on some interval of time. Differentiating the absolutely continuous function m_2, we get

$$0 = \dot{m}_2(t) = \mathbf{c}^T(A + r\mathbf{b}\mathbf{c}^T)\tilde{\mathbf{x}}(t) = \mathbf{c}^T A\tilde{\mathbf{x}}(t). \qquad \text{(A.2)}$$

It follows from (A.2) that $\dot{m}_2(t)$ is absolutely continuous, so that we can differentiate it again. Continuing in this fashion, we get $\tilde{\mathbf{x}}^T(t)[\mathbf{c}; A^T\mathbf{c}; \ldots; (A^T)^{n-1}\mathbf{c}] = \mathbf{0}$, and this contradicts the observability of $(A, \mathbf{c})$.

Using a similar reasoning, based on the fact that the function $\lambda(t) := V_{\mathbf{x}}(\tilde{\mathbf{x}}(t))$ satisfies, almost everywhere, the adjoint equation $\dot{\lambda} = -(A + \tilde{u}\mathbf{b}\mathbf{c}^T)^T\lambda$, we get that $m_1(t)$ has isolated zeros. □

Proof of Proposition 4. If $k < k^*$, then there exist $a, b > 0$ such that any trajectory of (3), with $\mathbf{x}(0) \in S^{n-1}$, satisfies $||\mathbf{x}(t)|| \leq ae^{-bt}$ for all $t \geq 0$. Hence, (9) holds for $T_1 := \max\{0, \frac{1}{b}\ln(a/c)\}$.

To prove the second part of the theorem, fix $k > k^*$. For any $e > 0$, denote $A_e := A - eI$. Consider the absolute stability problem (with δ replacing k) for the system $\dot{\mathbf{y}} \in \{A_e\mathbf{y}, (A_e + \delta\mathbf{b}\mathbf{c}^T)\mathbf{y}\}$. It follows from continuity arguments that we can find a $e > 0$ such that the solution is $\delta^* \in (k^*, k)$. Hence, there exists a control $\tilde{u}(\mathbf{y}) \in \mathbb{U}^{\delta^*}$, and a function V such that $V(\mathbf{y}(t))$ is constant along the trajectories of $\dot{\mathbf{y}} = (A_e + \tilde{u}\mathbf{b}\mathbf{c}^T)\mathbf{y} = (A + \tilde{u}\mathbf{b}\mathbf{c}^T)\mathbf{y} - e\mathbf{y}$. This implies that along the trajectories of $\dot{\mathbf{x}} = (A + \tilde{u}\mathbf{b}\mathbf{c}^T)\mathbf{x}$, $V(\mathbf{x}(t))$ is an increasing function of t. Furthermore, for any $\beta \geq \alpha > 0$, there exists a $T = T(\alpha, \beta)$ such that $V(\mathbf{x}(0)) = \alpha$ implies $V(\mathbf{x}(t)) > \beta$ for all $t > T$. This implies the existence of T_2 such that (10) holds. □

References

[1] Z. Artstein. Examples of stabilization with hybrid feedback. In R. Alur, T. A. Henzinger, and E. D. Sontag, editors, *Hybrid systems III: Verification and Control*, pages 173–185. Lecture Notes in Computer Science, Vol. 1066, Springer, 1996.

[2] J. A. Ball, J. Chudoung, and M. V. Day. Robust optimal switching control for nonlinear systems. *SIAM J. Control Optim.*, 41(3):900–931, 2002.

[3] M. Bardi and I. Capuzzo-Dolcetta. *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations.* Birkhuaser, 1997.

[4] U. Boscain. Stability of planar switched systems: The linear single input case. *SIAM J. Control Optim.*, 41(1):89–112, 2002.

[5] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory.* SIAM Studies in Applied Mathematics, Vol. 15, SIAM, 1994.

[6] W. P. Dayawansa and C. F. Martin. A converse Lyapunov theorem for a class of dynamical systems which undergo switching. *IEEE Trans. Automat. Control*, 44(4):751–760, 1999.

[7] A. F. Filippov. Stability conditions in homogeneous systems with arbitrary regime switching. *Automat. Remote Control*, 41(8):1078–1085, 1980.

[8] H. Goldstein. *Classical Mechanics.* Addison-Wesley, 1980.

[9] J. P. Hespanha and A. S. Morse. Switching between stabilizing controllers. *Automatica*, 38(11):1905–1917, 2002.

[10] D. Holcman and M. Margaliot. Stability analysis of switched homogeneous systems in the plane. *SIAM J. Control Optim.*, 41(5):1609–1625, 2003.

[11] J. H. Hubbard and B. H. West. *Differential Equations: A Dynamical Systems Approach, Part II: Higher-Dimensional Systems.* Springer-Verlag, 1995.

[12] D. Liberzon. *Switching in Systems and Control.* Birkhäuser, Boston, 2003.

[13] M. Margaliot and G. Langholz. Necessary and sufficient conditions for absolute stability: the case of second-order systems. *IEEE Trans. Circuits Syst. I*, 50(2):227–234, 2003.

[14] N. H. McClamroch and I. Kolmanovsky. Performance benefits of hybrid control design for linear and nonlinear systems. *Proc. IEEE*, 88(7):1083–1096, 2000.

[15] A. P. Molchanov and Ye. S. Pyatnitskiy. Criteria of asymptotic stability of differential and difference inclusions encountered in control theory. *Systems Control Lett.*, 13:59–64, 1989.

[16] E. S. Pyatnitskiy and L. B. Rapoport. Criteria of asymptotic stability of differential inclusions and periodic motions of time-varying nonlinear control systems. *IEEE Trans. Circuits Syst. I*, 43(3):219–229, 1996.

[17] L. B. Rapoport. Asymptotic stability and periodic motions of selector-linear differential inclusions. In F. Garofalo and L. Glielmo, editors, *Robust Control via Variable Structure and Lyapunov Techniques*, pages 269–285. LNCIS 217, Springer, 1996.

[18] R. T. Rockafellar. *Convex Analysis.* Princeton University Press, 1970.

[19] M. Vidyasagar. *Nonlinear Systems Analysis.* Second Edition, Prentice Hall, 1993.

[20] R. Vinter. *Optimal Control.* Birkhuaser, 2000.

[21] X. Xu and P. J. Antsaklis. Stabilization of second-order LTI switched systems. *Int. J. Control*, 73(14):1261–1279, 2000.

[22] X. Xu and P. J. Antsaklis. Optimal control of switched systems based on parameterization of the switching instants. *IEEE Trans. Automat. Control*, 49(1):2–16, 2004.